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On the Existence of Periodic Solutions for a Kind of Second Order Neutral Functional Differential Equation

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Abstract By means of the continuation theorem of coincidence degree theory, some new results on the non-existence, existence and unique existence of periodic solutions for a kind of second order neutral functional differential equation are obtained.

Keywords Periodic solution, Continuation theorem, Neutral functional differential equation
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1 Introduction

The problems of periodic solutions for second order ordinary differential equations have been extensively studied. In recent papers, some results on the existence of periodic solutions of delay differential equations have appeared by applying the continuation theorem, see papers [1–5]. These papers were devoted mainly to studying the following types of equations:

$$x''(t) + g(x(t-\tau)) = p(t), \tag{1}$$

$$x''(t) + ax'(t) + bx(t) + g(x(t-1)) = p(t), \tag{2}$$

and

$$x''(t) = f(t, x(t), x(t-\tau(t)))x'(t) + \beta(t)g(x(t-\tau_1(t))) + p(t). \tag{3}$$

The growth condition imposed on the nonlinear function $g(x)$ of papers [2–5] is as follows:

$$\lim_{|x|\to\infty} \frac{|g(x)|}{|x|} = r. \tag{4}$$

But the work to get the existence of periodic solutions for the neutral functional differential equation (NFDE) by using the continuation theorem of coincidence degree rarely appeared. As far as we know, there were only two papers [6–7] devoted to studying the existence of periodic solutions to the first order NFDE. The reason for it lies in the following two respects: The first is that the criteria of the L-compactness of nonlinear operator N on the set $\bar{\Omega}$ is difficult to

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establish; the second is that the *a priori* bounds of periodic solutions is not easy to estimate. In paper [7], Enrico Serra studied a kind of first order NFDE of the following form:

$$x'(t) + ax'(t-\tau) = f(t, x(t)).$$

Under the condition: $|a| < 1$, and

$$\alpha(t) \le \liminf_{|s|\to\infty} \frac{f(t,s)}{s} \le \limsup_{|s|\to\infty} \frac{f(t,s)}{s} \le \beta(t), \text{ for a.e. } t \in P, \tag{5}$$

where $\alpha, \beta \in L^\infty$, $f : R \times R \to R$ is a Carathéodory function of period 2π in the first variable, P is a subset of $[0, 2\pi]$ with positive measure. The author obtained that the above equation has at least one periodic solution (Theorem 3.1). This article investigates the existence of a periodic solution for a kind of second order NFDE as follows:

$$\frac{d^2}{dt^2}(u(t) - ku(t-\tau)) = f(u(t))u'(t) + \alpha(t)g(u(t)) + \sum_{j=1}^{n} \beta_j(t)g(u(t-\gamma_j(t))) + p(t), \tag{6}$$

where $f, g \in C(R, R)$, $\alpha(t), p(t), \beta_j(t), \gamma_j(t)\,(j = 1, 2, \ldots, n)$ are continuous periodic functions defined on R with period $T > 0$, $k, \tau \in R$ are constants such that $|k| \ne 1$. By using the continuation theorem of coincidence degree theory and some new analysis techniques, we obtain some new results on the existence of the a periodic solution to Eq. (6). Even if for $k = 0$, the methods to estimate the *a priori* bounds of a periodic solution and to find the conditions imposed on $g(x)$ are different from the corresponding ones of the recent literatures. Meanwhile, we also obtain some other new results on the non-existence and unique existence of a periodic solution to Eq. (6), respectively.

2 Main Lemmas

Let $C_T = \{x | x \in C(R, R), x(t+T) \equiv x(t)\}$ with norm $|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|$, $\forall \varphi \in C_T$, and $C_T^1 = \{x | x \in C^1(R, R), x(t+T) \equiv x(t)\}$ with norm $\|\varphi\| = \max\{|\varphi|_0, |\varphi'|_0\}$. Clearly, C_T and C_T^1 are two Banach spaces. We also define operator A in the following form:

$$A : C_T \to C_T,\ (Ax)(t) = x(t) - kx(t-\tau).$$

Lemma 1 *If $|k| \ne 1$, then A has a continuous bounded inverse on C_T, and*:

(1) $||A^{-1}x|| \le \frac{||x||}{||k|-1|}$, $\forall x \in C_T$;

(2) $\int_0^T |(A^{-1}f)(t)|dt \le \frac{1}{|1-|k||} \int_0^T |f(s)|ds$, $\forall f \in C_T$;

(3) $\int_0^T |(A^{-1}f)(t)|^2 dt \le \frac{1}{(1-|k|)^2} \int_0^T |f(s)|^2 ds, \forall f \in C_T$.

Proof (1) According to paper [8], we get that $\forall x \in C_T$,

$$[A^{-1}x](t) = \begin{cases} \sum\limits_{j\ge 0} k^j x(t-j\tau), & \text{if } |k| < 1, \\ -\sum\limits_{j\ge 1} k^{-j} x(t+j\tau), & \text{if } |k| > 1. \end{cases}$$

Thus A has a continuous inverse A^{-1} on C_T, and

$$||A^{-1}x|| \le \frac{1}{||k|-1|}||x||.$$

(2) If $|k| < 1$, we have

$$\int_0^T |[A^{-1}f](s)|ds \le \sum_{j\ge 0} |k|^j \int_0^T |f(s-j\tau)|ds = \sum_{j\ge 0} |k|^j \int_{-j\tau}^{T-j\tau} |f(s)|ds$$
$$= \sum_{j\ge 0} |k|^j \int_0^T |f(s)|ds = \frac{1}{1-|k|} \int_0^T |f(s)|ds.$$

Similarly, if $|k| > 1$, again we have

$$\int_0^T |[A^{-1}f](s)|ds \le \frac{1}{|k|-1} \int_0^T |f(s)|ds.$$

Thus, the statement of Case (2) holds.

(3) For $f \in C_T$, $f(t) = \sum_{n\in Z} f_n \mathrm{e}^{i\mu_n t}$, where $f_n = \frac{1}{T}\int_0^T f(s)\mathrm{e}^{-i\mu_n s}ds$, $\mu_n = \frac{2n\pi}{T}$, $n \in Z$, Z is the set of integers. Let $x(t) = (A^{-1}f)(t)$, i.e., $x(t) - kx(t-\tau) = f(t)$. So $(A^{-1}f)(t) = \sum_{n\in Z} \frac{f_n}{1-k\mathrm{e}^{-i\mu_n\tau}} \mathrm{e}^{i\mu_n t}$. Thus, by using Parseval's inequality, we have

$$\frac{1}{T}\int_0^T |(A^{-1}f)(s)|^2 ds = \sum_{n\in Z} \frac{|f_n|^2}{|1-ke^{-i\mu_n\tau}|^2} = \sum_{n\in Z} \frac{|f_n|^2}{|1-k\cos\mu_n\tau - ik\sin\mu_n\tau|^2}$$
$$= \sum_{n\in Z} \frac{|f_n|^2}{1+k^2-2k\cos\mu_n\tau} \le \frac{1}{(1-|k|)^2} \sum_{n\in Z} |f_n|^2$$
$$= \frac{1}{(1-|k|)^2} \frac{1}{T} \int_0^T |f(s)|^2 ds,$$

which implies that

$$\int_0^T |(A^{-1}f)(s)|^2 ds \le \frac{1}{(1-|k|)^2} \int_0^T |f(s)|^2 ds.$$

Remark 1 By Hale's terminology [9], a solution $u(t)$ of Eq. (6) is $u \in C^1(R,R)$ such that $Au \in C^2(R,R)$ and Equation (6) is satisfied on R. In general, u does not belong to $C^2(R,R)$. But, under the condition $|k| \ne 1$, we see from the first part of Lemma 1 that $(Au)''(t) = (Au'')(t)$ and $(Au)'(t) = (Au')(t)$. So a solution u of Eq. (6) must belong to $C^2(R,R)$.

Now, we define an operator L in the following form:

$$L : D(L) \subset C_T^1 \to C_T,\ Lx = (Ax)'',$$

where $D(L) = \{x | x \in C^2(R,R), x(t+T) \equiv x(t)\}$. According to the first part of Lemma 1, we can easily get that $\mathrm{Ker}L = R$, and $\mathrm{Im}L = \{x | x \in C_T, \int_0^T x(s)ds = 0\}$. Therefore, L is a Fredholm operator with index zero. Let $P : C_T^1 \to \mathrm{Ker}L, Q : C_T \to C_T/\mathrm{Im}\,L$ defined by $Px = x(0), Qx = \frac{1}{T}\int_0^T x(s)ds$ and

$$L_P = L|_{C_T\cap\mathrm{Ker}P} : C_T \cap \mathrm{Ker}P \to \mathrm{Im}\,L.$$

Then L_P has a continuous inverse L_P^{-1} on $\mathrm{Im}\,L$ defined by

$$\left(L_P^{-1}y\right)(t) = (A^{-1}Fy)(t), \tag{7}$$

where

$$[Fy](t) = -\frac{1}{2}\int_0^T sy(s)ds + \int_0^T \frac{t}{T}sy(s)ds + \int_0^t (t-s)y(s)ds - \frac{1}{T}\int_0^T\int_0^u (u-s)y(s)dsdu.$$

Lemma 2 *Let $g \in C_T$, $\tau \in C_T^1$ with $\tau'(t) < 1, \forall t \in [0,T]$. Then $g(\mu(t)) \in C_T$, where $\mu(t)$ is the inverse function of $t-\tau(t)$.*

Proof We need to prove only that $\mu(a+T) = \mu(a)+T$, for arbitrary $a \in R$. By the condition $\tau'(t) < 1,\ \forall\ t \in [0,T]$, it is easy to see that the equation $t - \tau(t) = a$ has a unique solution t_0, and $t-\tau(t) = a+T$ has a unique solution t_1. That is,

$$t_0 - \tau(t_0) = a, \quad t_1 - \tau(t_1) = a+T,$$

i.e.,

$$\mu(a) = t_0 = a + \tau(t_0) \text{ and } \mu(a+T) = t_1. \tag{8}$$

Since

$$\begin{aligned} T + a + \tau(t_0) - \tau(T + a + \tau(t_0)) &= T + a + \tau(t_0) - \tau(a + \tau(t_0)) \\ &= T + a + \tau(t_0) - \tau(t_0) \\ &= T + a, \end{aligned}$$

it follows that $t_1 = T + a + \tau(t_0)$. So by (8), we have $\mu(a+T) = \mu(a) + T$, $\forall a \in R$.

Lemma 3 [10] *Let X and Y be two Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \forall \lambda \in (0,1)$;

(2) $Nx \notin \mathrm{Im}L, \forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(3) $\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$,

then equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

3 Main Results

Throughout this paper, we assume that $\gamma_j \in C_T^1$, and $\gamma_j'(t) < 1, \forall t \in [0,T], (j = 1,2,\ldots,n)$. So the function $t - \gamma_j(t)$ has a unique inverse denoted by $\mu_j(t), (j = 1,2,\ldots,n)$. We also denote

$$\bar{h} = \frac{1}{T}\int_0^T h(s)ds, \quad \widetilde{h} = \int_0^T |h(s)|ds, \quad \forall\ h \in C_T,$$

$$\Gamma(t) = \alpha(t) + \sum_{j=1}^n \frac{\beta(\mu_j(t))}{1-\gamma_j'(\mu_j(t))},$$

$$\Gamma_1(t) = |\alpha(t)| + \sum_{j=1}^n \left|\frac{\beta(\mu_j(t))}{1-\gamma_j'(\mu_j(t))}\right|.$$

For the sake of convenience, we list the following conditions which will be used for us to study Eq. (6) in this section:

[H_1] $g(x) > 0, \forall\, x \in R$, $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = 0$; or $\lim_{x\to+\infty} g(x) = 0$ and $\lim_{x\to-\infty} g(x) = +\infty$.

[H_2] $\bar{p}\Gamma(t) < 0,\ \forall\, t \in [0, T]$.

[H_3] $g(x)$ is a strictly monotonous function and satisfies

$$|g(x_1) - g(x_2)| \le L|x_1 - x_2| \text{ for all } x_1, x_2 \in R,$$

where $L > 0$ is a constant.

Theorem 1 *If assumptions* [H_1]–[H_2] *hold, furthermore,* $\sup_{x\in R} |f(x)| = \sigma_1 < +\infty$ *and*

$$\frac{|k|\sigma_1 T}{(1-|k|)^2} < 1,$$

then Eq. (6) *has at least one T-periodic solution.*

Proof It is easy to see that Eq. (6) has a T-periodic solution if and only if the following operator equation:

$$Lu = Nu,$$

has a T-periodic solution, where L is defined in Section 2, $N : C_T^1 \to C_T$,

$$(Nu)(t) = f(u(t))u'(t) + \alpha(t)g(u(t)) + \sum_{j=1}^{n} \beta_j(t)g(u(t-\gamma_j(t))) + p(t).$$

From (7), we see that N is L-compact on $\overline{\Omega}$, where Ω is any open, bounded subset of C_T^1. Take $\Omega_1 = \{x | x \in C_T^1 \cap D(L),\ Lu = \lambda Nu,\ \lambda \in (0,1)\}$. Then $\forall u \in \Omega_1$, u must satisfy

$$\frac{d^2}{dt^2}(u(t) - ku(t-\tau)) = \lambda f(u(t))u'(t) + \lambda\alpha(t)g(u(t)) + \lambda \sum_{j=1}^{n} \beta_j(t)g(u(t-\gamma_j(t))) + \lambda p(t). \quad (9)$$

Without loss of generality, we may assume that $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = 0$. By integrating the two sides of Eq. (9) on the interval $[0, T]$, we have

$$\int_0^T f(u(t))u'(t)dt + \int_0^T \alpha(t)g(u(t))dt + \sum_{j=1}^{n} \int_0^T \beta_j(t)g(u(t-\gamma_j(t)))dt = -\bar{p}T. \quad (10)$$

Since

$$\int_0^T \beta_j(t)g(u(t-\gamma_j(t)))dt = \int_{-\gamma_j(0)}^{T-\gamma_j(T)} \frac{\beta(\mu_j(s))}{1-\gamma_j'(\mu_j(s))} g(u(s))ds,$$

by applying Lemma 2, we know that $\frac{\beta(\mu_j(t))}{1-\gamma_j'(\mu_j(t))} \in C_T$. It follows that

$$\int_0^T \beta_j(t)g(u(t-\gamma_j(t)))dt = \int_0^T \frac{\beta_j(\mu_j(s))}{1-\gamma_j'(\mu_j(s))} g(u(s))ds,\ (j = 1, 2, \ldots, n),$$

which together with $\int_0^T f(u(t))u'(t)dt = 0$ yields from (10) that

$$\int_0^T \Gamma(t)g(u(t))dt = -\bar{p}T. \quad (11)$$

So by using the integral mean value theorem, we have that there is $t_1 \in [0, T]$ such that

$$g(u(t_1))\overline{\Gamma}\, T = -\bar{p}\, T,$$

i.e.,

$$g(u(t_1)) = \frac{-\bar{p}}{\overline{\Gamma}}.$$

By assumption [H_1] and [H_2], we obtain that there is a constant $M > 0$ such that $|u(t_1)| \le M$. Therefore,

$$|u|_0 \le M + \int_0^T |u'(t)|dt. \tag{12}$$

On the other hand, by multiplying the two sides of Eq. (9) by $(Au)(t)$ and integrating them on $[0, T]$, we get from Remark 1 that

$$\begin{aligned}
-\int_0^T [(Au')(t)]^2 dt = -\int_0^T [(Au)'(t)]^2 dt &= \lambda \int_0^T f(u(t))u'(t)[u(t) - ku(t-\tau)]dt \\
&\quad + \lambda \int_0^T \alpha(t) g(u(t))[u(t) - ku(t-\tau)]dt \\
&\quad + \lambda \sum_{j=1}^n \int_0^T \beta_j(t) g(u(t-\gamma_j(t)))[u(t) - ku(t-\tau)]dt \\
&\quad + \lambda \int_0^T p(t)[u(t) - ku(t-\tau)]dt.
\end{aligned} \tag{13}$$

Since $\int_0^T f(u(t))u'(t)u(t)dt = 0$, it follows from (13) that

$$\begin{aligned}
\int_0^T [(Au')(t)]^2 dt &\le |k|\sigma_1 |u|_0 \int_0^T |u'(t)|dt + (1+|k|)|u|_0 \tilde{p} \\
&\quad + (1+|k|)|u|_0 \Big[\int_0^T |\alpha(t)| g(u(t))dt + \sum_{j=1}^n \int_0^T |\beta_j(t)| g(u(t-\gamma_j(t)))dt \Big] \\
&= |k|\sigma_1 |u|_0 \int_0^T |u'(t)|dt + (1+|k|)|u|_0 \tilde{p} \\
&\quad + (1+|k|)|u|_0 \Big[\int_0^T |\alpha(t)| g(u(t))dt + \sum_{j=1}^n \int_0^T \frac{|\beta_j(\mu_j(t))|}{1-\gamma_j'(\mu_j(t))} g(u(t))dt \Big] \\
&= |k|\sigma_1 |u|_0 \int_0^T |u'(t)|dt + (1+|k|)|u|_0 \tilde{p} + (1+|k|)|u|_0 \int_0^T \Gamma_1(t) g(u(t))dt.
\end{aligned}$$

From (11), we have

$$\int_0^T (Au'(t))^2 dt \le |k|\sigma_1 |u|_0 \int_0^T |u'(t)|dt + (1+|k|)|u|_0 \left(1 + \left|\frac{\Gamma_1}{\Gamma}\right|_0\right) \tilde{p}. \tag{14}$$

By (12), we obtain from (14) that

$$\begin{aligned}
\int_0^T (Au'(t))^2 dt &\le |k|\sigma_1 \Big(\int_0^T |u'(t)|dt \Big)^2 \\
&\quad + \Big(|k|\sigma_1 M + (1+|k|)\tilde{p} \Big(1 + \left|\frac{\Gamma_1}{\Gamma}\right|_0 \Big) \Big) \int_0^T |u'(t)|dt
\end{aligned}$$

$$
\begin{aligned}
&+ (1+|k|)\tilde{p}M\left(1+\left|\frac{\Gamma_1}{\Gamma}\right|_0\right) \\
&\le |k|\sigma_1 T\int_0^T |u'(t)|^2dt \\
&\quad + \left(|k|\sigma_1 M + (1+|k|)\tilde{p}\left(1+\left|\frac{\Gamma_1}{\Gamma}\right|_0\right)\right) T^{1/2}\left(\int_0^T |u'(t)|^2dt\right)^{1/2} \\
&\quad + (1+|k|)\tilde{p}M\left(1+\left|\frac{\Gamma_1}{\Gamma}\right|_0\right). \qquad (15)
\end{aligned}
$$

By applying the third part of Lemma 1, we get

$$\int_0^T |u'(t)|^2dt = \int_0^T |(A^{-1}Au')(t)|^2dt \le \frac{\int_0^T |(Au')(t)|^2dt}{(1-|k|)^2}.$$

So it follows from (15) that

$$\int_0^T |u'(t)|^2dt \le \frac{|k|\sigma_1 T}{(1-|k|)^2}\int_0^T |u'(t)|^2dt + C_1\left(\int_0^T |u'(t)|^2dt\right)^{1/2} + C_2,$$

where $C_1 = \frac{\left(|k|\sigma_1 M+(1+|k|)\tilde{p}(1+|\frac{\Gamma_1}{\Gamma}|_0)\right)T^{1/2}}{(1-|k|)^2}$, $C_2 = \frac{(1+|k|)\tilde{p}M(1+|\frac{\Gamma_1}{\Gamma}|_0)}{(1-|k|)^2}$. Thus there is a constant $M_1 > 0$ such that

$$\int_0^T |u'(t)|^2dt \le M_1.$$

It follows from (12) that

$$|u|_0 \le M + T^{1/2}M_1^{1/2} := \overline{M}. \qquad (16)$$

By applying the second part of Lemma 1, we have from Remark 1 and (9) that

$$
\begin{aligned}
\int_0^T |u''(t)|dt &= \int_0^T |(A^{-1}Au'')(t)|dt \\
&\le \frac{1}{|1-|k||}\int_0^T |(Au'')(t)|dt = \frac{1}{|1-|k||}\int_0^T |(Au)''(t)|dt \\
&\le \frac{1}{|1-|k||}\left[\int_0^T |f(u(t))||u'(t)|dt + \int_0^T |\alpha(t)|g(u(t))dt\right. \\
&\quad \left. + \sum_{j=1}^n |\beta_j(t)|g(u(t-\gamma_j(t)))dt + \tilde{p}\right] \\
&\le \frac{1}{|1-|k||}\left[f_{\overline{M}}T^{1/2}M_1^{1/2} + |\alpha|_0 g_{\overline{M}} + \sum_{j=1}^n |\beta_j|_0 g_{\overline{M}} + \tilde{p}\right] \\
&:= \overline{M}_1,
\end{aligned}
$$

where $g_{\overline{M}} = \sup_{|x|\le\overline{M}} g(x)$, $f_{\overline{M}} = \sup_{|x|\le\overline{M}} |f(x)|$. Since $u(0) = u(T)$, it follows that there is a $\eta \in [0,T]$ such that $u'(\eta) = 0$. So

$$|u'|_0 \le \int_0^T |u''(t)|dt \le \overline{M}_1.$$

Thus Ω_1 is bounded.

Let $\Omega_2 = \{x|x \in \ker L,\ Nx \in \mathrm{Im}L\}$. $\forall x \in \Omega_2$, obviously, $u(t) \equiv \overline{C}$ ($\overline{C}$ is a constant), and

$$\int_0^T \left[\alpha(t)g(\overline{C}) + \sum_{j=1}^n g(\overline{C})\right] dt = -\bar{p}T,$$

i.e.,

$$g(\overline{C})\left(\bar{\alpha} + \sum_{j=1}^n \bar{\beta}\right) = -\bar{p}.$$

By the substitution $t = s - \gamma_j(s)$, i.e., $s = \mu_j(t)$, we have

$$\overline{\beta_j}T = \int_0^T \beta_j(s)ds = \int_{-\gamma_j(0)}^{T-\gamma_j(T)} \frac{\beta_j(\mu_j(t))}{1-\gamma_j'(\mu_j(t))}dt = \int_0^T \frac{\beta_j(\mu_j(t))}{1-\gamma_j'(\mu_j(t))}dt \quad (j = 1, 2, \dots, n).$$

So $\bar{\alpha} + \sum_{j=1}^n \bar{\beta} = \overline{\Gamma} \neq 0$. Thus $g(\overline{C}) = -\bar{p}[\bar{\alpha} + \sum_{j=1}^n \bar{\beta}]^{-1}$. From assumption [$\mathrm{H}_1$], we know that there is a constant $M_2 > 0$ such that $|\overline{C}| < M_2$. Thus Ω_2 is also bounded. Let $\Omega \supset \Omega_1 \cup \Omega_2$ be open and bounded. So Ω satisfies the conditions (1) and (2) of Lemma 3. Now, for $u \in \partial\Omega \cap \mathrm{Ker}L$, we take

$$H(u,\mu) = \begin{cases} \mu u + \dfrac{1-\mu}{T}\displaystyle\int_0^T \left[\alpha(t)g(u) + \sum_{j=1}^n \beta_j(t)g(u) + p(t)\right] dt \text{ for } \left(\bar{\alpha} + \sum_{j=1}^n \bar{\beta}\right) > 0, \\ -\mu u + \dfrac{1-\mu}{T}\displaystyle\int_0^T \left[\alpha(t)g(u) + \sum_{j=1}^n \beta_j(t)g(u) + p(t)\right] dt \text{ for } \left(\bar{\alpha} + \sum_{j=1}^n \bar{\beta}\right) < 0. \end{cases}$$

Clearly,

$$H(u,\mu) = \begin{cases} \mu u + \dfrac{1-\mu}{T}\left[g(u)\left(\bar{\alpha} + \displaystyle\sum_{j=1}^n \bar{\beta}\right) + \bar{p}\right] \text{for } \bar{\alpha} + \displaystyle\sum_{j=1}^n \bar{\beta} > 0, \\ -\mu u + \dfrac{1-\mu}{T}\left[g(u)\left(\bar{\alpha} + \displaystyle\sum_{j=1}^n \bar{\beta}\right) + \bar{p}\right] \text{for } \bar{\alpha} + \displaystyle\sum_{j=1}^n \bar{\beta} < 0. \end{cases}$$

Hence,

$$H(u,\mu) \neq 0 \text{ for } (u,\mu) \in \partial\Omega \cap \mathrm{Ker}L \times [0,1].$$

Therefore,

$$\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} = \deg\{H(u,0), \Omega \cap \mathrm{Ker}L, 0\} = \deg\{H(u,1), \Omega \cap \mathrm{Ker}L, 0\} \neq 0.$$

By applying Lemma 3, we obtain that Eq. (6) has at least one T-periodic solution.

Theorem 2 *If assumptions* [H_1]–[H_2] *hold and* $|k| < 1$*, then Eq.* (6) *has at least one* T*-periodic solution.*

Proof Consider the following equation:

$$\frac{d^2}{dt^2}(u(t) - ku(t-\tau)) = \lambda f(u(t))u'(t) + \lambda\alpha(t)g(u(t)) + \lambda\sum_{j=1}^n \beta_j(t)g(u(t-\gamma_j(t))) + \lambda p(t). \tag{17}$$

Let $u(t)$ be an arbitrary T-periodic solution of Eq. (17). Then, from the proof of Theorem 1, we know that

$$|u|_0 \leq M + \int_0^T |u'(t)|dt. \tag{18}$$

Multiplying the two sides of Eq. (17) by $u(t)$ and integrating them on the interval $[0,T]$, then

$$\int_0^T (u(t)-ku(t-\tau))''u(t)dt = \lambda\int_0^T f(u(t)u(t)u'(t))dt$$
$$+\lambda\sum_{j=1}^n\int_0^T \beta_j(t)g(u(t-\gamma_j(t)))u(t)dt+\lambda\int_0^T p(t)u(t)dt. \tag{19}$$

Since

$$\int_0^T (u(t)-ku(t-\tau))''u(t)dt = -\int_0^T (u'(t))^2dt + k\int_0^T u'(t)u'(t-\tau)dt$$

and $\int_0^T f(u(t)u(t)u'(t))dt = 0$, it follows from (19) and (11) that

$$\begin{aligned}\int_0^T (u'(t))^2dt &= k\int_0^T u(t)u'(t-\tau) - \lambda\int_0^T \alpha(t)g(u(t))u(t)dt\\ &\quad -\lambda\sum_{j=1}^n\int_0^T \beta_j(t)g(u(t-\gamma_j(t)))u(t)dt - \lambda\int_0^T p(t)u(t)dt\\ &\le |k|\left[\int_0^T (u'(t))^2dt\int_0^T (u'(t-\tau))^2dt\right]^{1/2} + |u|_0\int_0^T \Gamma_1(t)g(u(t))dt + |u|_0\int_0^T |p(t)|dt\\ &\le |k|\int_0^T (u'(t))^2dt + |u|_0\left(\left|\frac{\Gamma_1}{\Gamma}\right|_0 |\bar{p}T+\tilde{p}\right).\end{aligned}$$

In view of $|u|_0 \le M + \int_0^T |u'(t)|dt$ and $|k|<1$, one can easily find from the above formula that there is a constant $\widetilde{M}>0$ such that $\int_0^T (u'(t))^2dt \le \widetilde{M}$. The remainder can be proved in the same way as in Theorem 1.

In what follows, we will give another two results on the unique existence of T-periodic solution to Eq. (6).

Theorem 3 *Suppose that the assumptions of* $[\mathrm{H}_1]$–$[\mathrm{H}_3]$ *hold, and* $f(x)\equiv a$, *where* $a\in R$ *is a constant. Then* Eq. (6) *has a unique* T*-periodic solution, if*

$$|k| + LT\left(\tilde{\alpha}+\sum_{j=1}^n \widetilde{\beta}_j\right) < 1.$$

Proof By applying Theorem 2, we can easily obtain that Eq. (6) has a T-periodic solution. We suppose that $u_1(t)$ and $u_2(t)$ are two T-periodic solutions of Eq. (6), and also denote $z(t)=u_1(t)-u_2(t)$. Then $z(t)$ satisfies

$$\frac{d^2}{dt^2}(z(t)-kz(t-\tau)) = az'(t)+\alpha(t)[g(u_1(t))-g(u_2(t))]$$
$$+\sum_{j=1}^n \beta_j(t)[g(u_1(t-\gamma_j(t)))-g(u_2(t-\gamma_j(t)))]. \tag{20}$$

Integrating the two sides of Eq. (20), we have

$$\int_0^T \Gamma(t)[g(u_1(t))-g(u_2(t))]dt = 0,$$

which implies that there is a constant $\bar{t} \in [0,T]$ such that

$$[g(u_1(\bar{t})) - g(u_2(\bar{t}))]\overline{\Gamma}T = 0,$$

i.e.,

$$g(u_1(\bar{t})) = g(u_2(\bar{t})).$$

By [H$_3$], we get that $u_1(\bar{t}) = u_2(\bar{t})$, i.e., $z(\bar{t}) = 0$. It follows that

$$|z|_0 \le \int_0^T |z'(t)|dt. \tag{21}$$

On the other hand, by multiplying the two sides of Eq. (20) by $z(t)$, we have

$$\begin{aligned}&\int_0^T (z(t) - kz(t-\tau))''z(t)dt \\ &\quad = a\int_0^T z'(t)z(t)dt + \int_0^T \alpha(t)[g(u_1(t)) - g(u_2(t))]z(t)dt \\ &\quad\quad + \sum_{j=1}^n \int_0^T \beta_j(t)[g(u_1(t-\gamma_j(t))) - g(u_2(t-\gamma_j(t)))]z(t)dt. \end{aligned}\tag{22}$$

Since $\int_0^T (z(t) - kz(t-\tau))''z(t)dt = -\int_0^T (z'(t))^2dt + k\int_0^T z'(t)z'(t-\tau)dt$, $\int_0^T z(t)z'(t)dt = 0$ and $|g(u_1(t)) - g(u_1(t))| \le L|z(t)|$, $|g(u_1(t-\gamma_j(t))) - g(u_2(t-\gamma_j(t))| \le L|z(t-\gamma_j(t))|$, it follows from (22) that

$$\begin{aligned}\int_0^T z'^2(t)dt &= k\int_0^T z(t)z'(t-\tau)dt - \int_0^T \alpha(t)[g(u_1(t)) - g(u_2(t))]z(t)dt \\ &\quad - \sum_{j=1}^n \int_0^T \beta_j(t)[g(u_1(t-\gamma_j(t))) - g(u_2(t-\gamma_j(t)))]z(t)dt \\ &\le |k|\left[\int_0^T (z'(t))^2dt \int_0^T (z'(t-\tau))^2dt\right]^{1/2} + L|z|_0^2\int_0^T |\alpha(t)|dt \\ &\quad + L|z|_0^2\sum_{j=1}^n \int_0^T |\beta_j(t)|dt \\ &= |k|\int_0^T (z'(t))^2dt + \left[\tilde{\alpha} + \sum_{j=1}^n \widetilde{\beta}_j\right]L|z|_0^2. \end{aligned}\tag{23}$$

Considering (21), we see $|z|_0^2 \le T\int_0^T |z'(t)|^2dt$. So it follows from (23) that

$$\left[1 - |k| - LT\left(\tilde{\alpha} + \sum_{j=1}^n \widetilde{\beta}_j\right)\right]\int_0^T z'^2(t)dt \le 0. \tag{24}$$

In view of $1 - |k| - LT[\tilde{\alpha} + \sum_{j=1}^n \widetilde{\beta}_j] > 0$, it follows from (24) that $\int_0^T z'^2(t)dt = 0$, which together with (21) yields that $|z|_0 \le \int_0^T |z'(t)|dt \le T^{1/2}\big(\int_0^T z'^2(t)dt\big)^{1/2} = 0$, i.e., $|z|_0 = 0$, and then $u_1(t) \equiv u_2(t)$. Therefore, Eq. (6) has a unique T-periodic solution. Similarly, we have:

Theorem 4 *Suppose that assumptions* [H$_1$], [H$_2$] *and* [H$_3$] *hold, and* $f(x) \equiv a$, *where* $a \in R$

is a constant. Furthermore, we assume that

$$\frac{|k||a|T}{(1-|k|)^2}+\frac{LT(1+|k|)(\tilde{\alpha}+\sum_{j=1}^{n}\widetilde{\beta}_j)}{(1-|k|)^2}<1.$$

Then Eq. (6) *has a unique T-periodic solution.*

Finally, we give a new result on the non-existence of periodic solution to Eq. (6).

Theorem 5 *Suppose $g(x)\neq 0, \forall x\in R$, $\Gamma(t)\le 0$, $\forall t\in R$ or $\Gamma(t)\ge 0$, $\forall t\in R$, and $\overline{\Gamma}\neq 0$. If one of the following conditions holds*:

(1) $\bar{p}=0$; (2) $\operatorname{sgn}(g(x))\overline{\Gamma}\bar{p}>0$;

then Eq. (6) *has no T-periodic solution.*

Proof (1) If Eq. (6) has a T-periodic solution, by integrating the two sides of Eq. (6) on the interval $[0,T]$, we have

$$\int_0^T \alpha(t)g(u(t))dt+\sum_{j=1}^{n}\int_0^T \beta_j(t)g(u(t-\gamma_j(t)))dt=0. \tag{25}$$

From the proof of Theorem 1, we find that $\int_0^T \Gamma(t)g(u(t))dt=0$. It follows that there is a $\xi\in[0,T]$ such that $g(u(\xi))\overline{\Gamma}T=0$, i.e., $g(u(\xi))=0$, which contradicts $g(x)\neq 0, \forall x\in R$. So the conclusion holds.

(2) If Eq. (6) has a T-periodic solution, we can prove in the same way as in Case (1) that $\int_0^T \Gamma(t)g(u(t))dt=-\bar{p}T$, so there is $\xi_1\in[0,T]$ such that $g(u(\xi_1))\overline{\Gamma}T=-\bar{p}T$, i.e., $g^2(u(\xi_1))=-\frac{\bar{p}T}{\overline{\Gamma}}g(u(\xi_1))<0$, which also leads to a contradiction. Thus, Eq. (6) has no T-periodic solution.

Example 1 Consider the following equation:

$$\frac{d^2}{dt^2}(u(t)-ku(t-\tau))=(\sin t)\left[(u'(t))^{2n}+1\right] + \Big(1+\frac{1}{2}\sin t\Big)\left[\Big(u\Big(t-\frac{1}{2}\cos t\Big)\Big)^{2n}+1\right]+2\sin t, \tag{26}$$

where n is a positive integer, k,τ are constants such that $|k|\neq 1$. According to the definition of $\Gamma(t)$, we have

$$\Gamma(t)=\sin t+\frac{1+\frac{1}{2}\sin\mu(t)}{1+\frac{1}{2}\sin\mu(t)}=1+\sin t\ge 0,$$

and $\overline{\Gamma}=1\neq 0$, $g(x)=x^{2n}+1\ge 1$, $\bar{p}=\frac{1}{2\pi}\int_0^T \sin t dt=0$. So by applying Theorem 5, we obtain that Eq. (26) has no 2π-periodic solution.

Example 2 Let us consider the following equation:

$$(u(t)-9u(t+3))''=\frac{u^2(t)u'(t)}{1+u^2(t)}+\Big(\frac{1}{2}\sin t\Big)\mathrm{e}^{u(t)}+\Big(1-\frac{1}{3}\cos t\Big)\mathrm{e}^{u(t-\frac{1}{3}\sin t)}+\cos t-1. \tag{27}$$

Clearly, $T=2\pi, k=9$, $f(x)=\frac{x^2}{1+x^2}$, $g(x)=e^x$, $\alpha(t)=\frac{1}{2}\sin t$, $\beta_1(t)=1-\frac{1}{3}\cos t$, $\gamma_1(t)=\frac{1}{3}\sin t$, $p(t)=\cos t-1$. So $\sigma_1=1$, $\bar{p}=-1$, $\Gamma(t)=\frac{1}{2}\sin t+\frac{1-\frac{1}{3}\cos\mu_1(t)}{1-\frac{1}{3}\cos\mu_1(t)}=1+\frac{1}{2}\sin t>0, \forall t\in[0,2\pi]$, where $\mu_1(t)$ is the inverse of $t-\frac{1}{3}\sin t$, and then $\bar{p}\Gamma(t)<0, \forall t\in[0,2\pi]$. Furthermore,

$\lim_{x\to+\infty} g(x) = +\infty$, $\lim_{x\to-\infty} g(x) = 0$ and

$$\frac{|k|\sigma_1 T}{(1-|k|)^2} = \frac{18\pi}{64} < 1.$$

Thus, by applying Theorem 1, we have that Eq. (27) has at least one 2π-periodic solution.

Remark 2 From Example 2, we see that assumption $[H_1]$, imposed on $g(x) = e^x$, is different from the corresponding ones of (4) and (5) which are needed by papers [2–5,7].

Remark 3 Since Eq. (6) contains multiple deviating arguments, even if $k = 0$, the methods in papers [1–5] to estimate the *a priori* bounds of periodic solutions cannot be adapted for this paper.

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