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Periodic Solutions of a Class of Second Order Functional Differential Equations

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Abstract We obtain sufficient conditions for the existence of periodic solutions of the following second order nonlinear differential equation:

$$a\ddot{x}(t) + b\dot{x}^{2k-1}(t) + cx^{2k-1}(t) + g(x(t-\tau_1), \dot{x}(t-\tau_2)) = p(t) = p(t+2\pi).$$

Our approach is based on the continuation theorem of the coincidence degree, and the priori estimate of periodic solutions.

Keywords Functional differential equation, Continuation theorem, Coincidence degree, Periodic solution

MR(2000) Subject Classification 34K13

1 Introduction and Main Results

The existence problem of periodic solutions for nonlinear differential equations with or without delay has been extensively investigated in the literature and many existence results have been obtained. Among the existence results of periodic solutions of second order differential equations with or without delay, there are some well-known solvability conditions, such as the sign condition [1, 2]; the monotonicity condition [3, 4]; the periodicity condition (see [4, 5] and their references); the unboundedness condition (see [6] and its references); the boundedness condition [7, 8]; the Landesman–Lazer type condition (see [2, 9] and their references); the Caratheodory condition [10–13]; and the growth condition [10, 14, 15]. In this paper, we show special interest in the growth condition. As far as the growth condition is considered, so far, most are linear growth and some functions growth condition. However, considerably less is known for the case where the nonlinear part has growth degree greater than one. In this direction, a second order delay differential equation and the complex-valued Rayleigh equation are considered in [15] and [16], respectively,

In [7] and [17], the authors studied the existence of periodic solutions of the following equations, respectively:

$$\ddot{x}(t) + m^2x(t) + f(x(t-\tau)) = p(t), \tag{1.1}$$

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and

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) + f(x(t-1)) = p(t), \tag{1.2}$$

where a, b, τ are constants, m is an integer, $f : R \to R$ is continuous, and $p : R \to R$ is a continuous and 2π-periodic function.

Motivated by Eqs. (1.1) and (1.2), in the present paper, we are concerned with the existence of 2π-periodic solutions of a the second order differential equation of a more general form

$$a\ddot{x}(t) + b\dot{x}^{2k-1}(t) + cx^{2k-1}(t) + g(x(t-\tau_1), \dot{x}(t-\tau_2)) = p(t) = p(t+2\pi), \tag{1.3}$$

where a, b, c, τ_1 and τ_2 are constants with $a \neq 0$, k is a positive integer, $g : R \times R \to R$ is continuous, and $p : R \to R$ is continuous and 2π-periodic.

Our goal is to obtain some sufficient conditions for the solvability of periodic problems of Eq. (1.3) under the growth condition in the case where the nonlinear part has growth degree greater than one, together with the restriction condition on a, b, c and k. Our approach is in the spirit of those successfully utilized for the study of a second order delay differential equation in [15] and the complex-valued Rayleigh equation in [16].

Namely, we combine the continuation theorem of Mawhin's coincidence degree [18] with the differential inequality technique for a priori bounds of periodic solutions of a parametrized second order equation. The following are our main results:

Theorem 1.1 *Assume that there exist a positive constant M and some nonnegative constants $\beta_1, \beta_2, \alpha_i$ $(i = 1, 2, \ldots, 2k-1)$ with $|c| > \alpha_{2k-1}$ such that*

$$|g(x_1, x_2)| \le M + \sum_{i=1}^{2k-1} \alpha_i |x_1|^i + \beta_1 |x_2|^{2k-1} + \beta_2 |x_2|^{k-\frac{1}{2}} \quad \forall (x_1, x_2) \in R^2 \tag{1.4}$$

and

$$(2\pi)^{(1-2k)} (|b| - \beta_1) > \left[2 \sum_{j=0}^{2k-1} \left(2\pi \sqrt[2k-1]{|c|}\right)^{-j} C_{2k-1}^{j} \right] \alpha_{2k-1} |b|. \tag{1.5}$$

Then Eq. (1.3) *has at least one 2π-periodic solution.*

Theorem 1.2 *Assume that there exist a positive constant M^* and some nonnegative constants $\beta_3, \beta_4, \gamma_i$ $(i = 1, 2, \ldots, 2k-1)$ with $|c| > \beta_3$ such that*

$$|g(x_1, x_2)| \le M^* + \sum_{i=1}^{2k-1} \gamma_i |x_2|^i + \beta_3 |x_1|^{2k-1} + \beta_4 |x_1|^{k-\frac{1}{2}} \quad \forall (x_1, x_2) \in R^2 \tag{1.6}$$

and

$$|bc| > \beta_3 \left(2|b| f_1(k) + f_2(k)\right) + \gamma_{2k-1} |c|, \tag{1.7}$$

where

$$f_1(k) = \sum_{j=0}^{2k-1} \frac{j}{2k-1} C_{2k-1}^{j} \left(\sqrt[2k]{2\pi}\right)^{(2k-1)(2k-1-j)},$$

$$f_2(k) = \sum_{j=0}^{2k-1} \frac{2k-1-j}{2k-1} C_{2k-1}^{j} \left(\sqrt[2k]{2\pi}\right)^{(2k-1)(2k-1-j)}.$$

Then Eq. (1.3) *has at least one 2π-periodic solution.*

2 The Proofs

We will need the notion of the coincidence degree on the continuation theorem formulated in [18].

Lemma 2.1 *Let X and Z be two given Banach spaces. Consider an operator equation*

$$Lx = \lambda Nx,$$

where $L : \mathrm{Dom}L \subset X \to Z$ is a Fredholm operator of index zero, $\lambda \in [0,1]$ is a parameter. Let P and Q denote two projectors such that

$$P : X \to \mathrm{Ker}L \quad and \quad Q : Z \to Z/\mathrm{Im}L.$$

Assume that $N : \overline{\Omega} \to Z$ is L-compact on $\overline{\Omega}$, where Ω is open and bounded in X. Furthermore, suppose that:

(a) *For each $\lambda \in (0,1)$ and $x \in \partial\Omega \cap \mathrm{Dom}L$, $Lx \neq \lambda Nx$;*

(b) *For each $x \in \partial\Omega \cap \mathrm{Ker}L$, $QNx \neq 0$;*

(c) $\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

We now present the proof for Theorem 1.1.

In order to use Lemma 2.1 for Eq. (1.3), we take $X = \{x \in C^1(R,R) : x(t+2\pi) = x(t)$ for all $t \in R\}$ and $Z = \{z \in C(R,R) : z(t+2\pi) = z(t)$ for all $t \in R\}$ and denote $|x|_0 = \max_{t\in[0,2\pi]} |x(t)|$ and $|x|_2 = \max\{|x|_0, |\dot{x}|_0\}$. Then X and Z are Banach spaces endowed with the norms $|\cdot|_2$ and $|\cdot|_0$, respectively.

Set

$$(Lx)(t) = a\ddot{x}(t),\ x \in X,\ t \in R;$$
$$(Nx)(t) = -b\dot{x}^{2k-1}(t) - cx^{2k-1}(t) - g(x(t-\tau_1),\ \dot{x}(t-\tau_2)) + p(t),\ x \in X,\ t \in R;$$
$$(Px)(t) = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt,\ x \in X,\ t \in R,\ (Qz)(t) = \frac{1}{2\pi}\int_0^{2\pi} z(t)dt,\ z \in Z,\ t \in R.$$

It is easy to prove that L is a Fredholm mapping of index 0, that $P : X \to \mathrm{Ker}L$ and $Q : Z \to Z/\mathrm{Im}L$ are projectors, and that N is L-compact on $\overline{\Omega}$ for any given open and bounded subset Ω in X.

The corresponding differential equation for the operator $Lx = \lambda Nx$, $\lambda \in (0,1)$, takes the form

$$a\ddot{x}(t) + \lambda b\dot{x}^{2k-1}(t) + \lambda cx^{2k-1}(t) + \lambda g\left(x(t-\tau_1),\ \dot{x}(t-\tau_2)\right) = \lambda p(t). \tag{2.1}$$

Let $x \in X$ be a solution of Eq. (2.1) for a certain $\lambda \in (0,1)$. Multiplying Eq. (2.1) by $\dot{x}$ and integrating over $[0,2\pi]$, we obtain

$$b\int_0^{2\pi} |\dot{x}(t)|^{2k}dt + \int_0^{2\pi} \dot{x}(t)g\left(x(t-\tau_1),\ \dot{x}(t-\tau_2)\right)dt = \int_0^{2\pi} \dot{x}(t)p(t)dt,$$

from which it follows that

$$|b|\int_0^{2\pi}|\dot{x}(t)|^{2k}dt \le \int_0^{2\pi}|\dot{x}(t)|\left[m+M+\sum_{i=1}^{2k-1}\alpha_i|x(t-\tau_1)|^i+\beta_1|\dot{x}(t-\tau_2)|^{2k-1}+\beta_2|\dot{x}(t-\tau_2)|^{k-\frac{1}{2}}\right]dt,$$

where $m=\max_{t\in[0,2\pi]}|p(t)|$.

By using the Hölder inequality, we can obtain from the above inequality

$$\begin{aligned}
&|b|\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\\
&\le\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{1}{2k}}\left[\left(\sqrt[2k]{2\pi}\right)^{2k-1}(m+M)+\sum_{i=1}^{2k-1}\alpha_i\left(\int_0^{2\pi}|x(t-\tau_1)|^{\frac{2ki}{2k-1}}dt\right)^{\frac{2k-1}{2k}}\right.\\
&\quad\left.+\beta_1\left(\int_0^{2\pi}|\dot{x}(t-\tau_2)|^{2k}dt\right)^{\frac{2k-1}{2k}}+\beta_2\left(\int_0^{2\pi}|\dot{x}(t-\tau_2)|^{k}dt\right)^{\frac{2k-1}{2k}}\right]\\
&=\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{1}{2k}}\left[\left(\sqrt[2k]{2\pi}\right)^{2k-1}(m+M)+\sum_{i=1}^{2k-1}\alpha_i\left(\int_0^{2\pi}|x(t)|^{\frac{2ki}{2k-1}}dt\right)^{\frac{2k-1}{2k}}\right.\\
&\quad\left.+\beta_1\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}+\beta_2\left(\int_0^{2\pi}|\dot{x}(t)|^{k}dt\right)^{\frac{2k-1}{2k}}\right].
\end{aligned}$$

That is,

$$\begin{aligned}
|b|\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{(1-\frac{1}{2k})}&\le\left(\sqrt[2k]{2\pi}\right)^{2k-1}(m+M)+\sum_{i=1}^{2k-1}\alpha_i\left(\int_0^{2\pi}|x(t)|^{\frac{2ki}{2k-1}}dt\right)^{\frac{2k-1}{2k}}\\
&\quad+\beta_1\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}+\beta_2\left(\int_0^{2\pi}|\dot{x}(t)|^{k}dt\right)^{\frac{2k-1}{2k}}. \qquad (2.2)
\end{aligned}$$

Using the inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^r dt\right)^{\frac{1}{r}}\le\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^s dt\right)^{\frac{1}{s}}\quad\text{for }0\le r\le s,\ \text{ and }\ f\in C(R,R), \qquad (2.3)$$

from (2.2) it follows that

$$\begin{aligned}
(|b|-\beta_1)\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{1-\frac{1}{2k}}&\le\left(\sqrt[2k]{2\pi}\right)^{2k-1}(m+M)\\
&\quad+\sum_{i=1}^{2k-1}\alpha_i(2\pi)^{(1-\frac{1+i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{2k}}\\
&\quad+\beta_2(2\pi)^{(\frac{1}{2}-\frac{1}{4k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}. \qquad (2.4)
\end{aligned}$$

Thus

$$\begin{aligned}
&2(|b|-\beta_1)\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}\\
&\le\ \beta_2(2\pi)^{(\frac{1}{2}-\frac{1}{4k})}+\left\{\beta_2^2(2\pi)^{(1-\frac{1}{2k})}+4(|b|-\beta_1)\left[(\sqrt[2k]{2\pi})^{(2k-1)}(m+M)\right.\right.
\end{aligned}$$

$$+\sum_{i=1}^{2k-1}\alpha_i(2\pi)^{(1-\frac{1+i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{2k}}\Bigg]\Bigg\}^{\frac{1}{2}}. \tag{2.5}$$

Using the inequality

$$(a+b)^r\le a^r+b^r,\ \text{for } a\ge 0,\ b\ge 0 \quad \text{and } 0\le r\le 1, \tag{2.6}$$

it follows from (2.5) that

$$(|b|-\beta_1)\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}\le \beta_2(2\pi)^{(\frac{1}{2}-\frac{1}{4k})}+\sqrt{|b|-\beta_1}\Bigg[(\sqrt[4k]{2\pi})^{2k-1}\sqrt{m+M}$$

$$+\sum_{i=1}^{2k-1}\sqrt{\alpha_i}(2\pi)^{(\frac{1}{2}-\frac{1+i}{4k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{4k}}\Bigg]. \tag{2.7}$$

Substituting (2.7) into (2.4), then

$$(|b|-\beta_1)^2\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{1-\frac{1}{2k}}$$

$$\le A+(|b|-\beta_1)\sum_{i=1}^{2k-1}\alpha_i(2\pi)^{\left(1-\frac{1}{2k}-\frac{i}{2k}\right)}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{2k}}$$

$$+\beta_2\sqrt{|b|-\beta_1}(2\pi)^{(\frac{1}{2}-\frac{1}{4k})}\sum_{i=1}^{2k-1}\sqrt{\alpha_i}(2\pi)^{\left(\frac{1}{2}-\frac{1+i}{4k}\right)}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{4k}}, \tag{2.8}$$

where A is a positive constant.

Integrating Eq. (2.1) over $[0,2\pi]$, we obtain

$$c\int_0^{2\pi}x^{2k-1}(t)dt=-\int_0^{2\pi}\left[b\dot{x}^{2k-1}(t)+g(x(t-\tau_1),\ \dot{x}(t-\tau_2))-p(t)\right]dt.$$

Thus there exists a $\xi\in(0,2\pi)$ such that

$$2\pi c x^{2k-1}(\xi)=-\int_0^{2\pi}\left[b\dot{x}^{2k-1}(t)+g(x(t-\tau_1),\ \dot{x}(t-\tau_2))-p(t)\right]dt,$$

which leads to

$$2\pi|c||x(\xi)|^{2k-1}\le\int_0^{2\pi}\Bigg[m+M+|b||\dot{x}(t)|^{2k-1}+\sum_{i=1}^{2k-1}\alpha_i|x(t-\tau_1)|^i$$

$$+\beta_1|\dot{x}(t-\tau_2)|^{2k-1}+\beta_2|\dot{x}(t-\tau_2)|^{k-\frac{1}{2}}\Bigg]dt$$

$$=2\pi(m+M)+\sum_{i=1}^{2k-1}\alpha_i\int_0^{2\pi}|x(t)|^i dt$$

$$+(\beta_1+|b|)\int_0^{2\pi}|\dot{x}(t)|^{2k-1}dt+\beta_2\int_0^{2\pi}|\dot{x}(t)|^{k-\frac{1}{2}}dt. \tag{2.9}$$

Using inequality (2.3), from (2.9) we get

$$2\pi|c||x(\xi)|^{2k-1}\le 2\pi(m+M)+(|b|+\beta_1)\sqrt[2k]{2\pi}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$

$$+ \beta_2(2\pi)^{(\frac{1}{2}+\frac{1}{4k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}$$

$$+ \sum_{i=1}^{2k-1}\alpha_i(2\pi)^{(1-\frac{i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{2k}}. \tag{2.10}$$

Substituting (2.7) and (2.8) into (2.10), then

$$\begin{aligned}
&2\pi|c|(|b|-\beta_1)^2|x(\xi)|^{2k-1}\\
&\le \mu + 2|b|(|b|-\beta_1)\sum_{i=1}^{2k-1}\alpha_i(2\pi)^{(1-\frac{i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{2k}}\\
&\quad + 2\beta_2|b|\sqrt{|b|-\beta_1}\sum_{i=1}^{2k-1}\sqrt{\alpha_i}(2\pi)^{(1-\frac{i}{4k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i}{4k}},
\end{aligned} \tag{2.11}$$

where μ is a positive constant.

Using inequality (2.6), from (2.11) it follows that, for $0\le j\le 2k-1$,

$$\begin{aligned}
&(2\pi|c|)^{\frac{j}{2k-1}}(|b|-\beta_1)^{\frac{2j}{2k-1}}|x(\xi)|^j\\
&\le \mu^{\frac{j}{2k-1}} + [2|b|(|b|-\beta_1)]^{\frac{j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{j}{2k-1}}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{ij}{2k(2k-1)}}\\
&\quad + (2\beta_2|b|)^{\frac{j}{2k-1}}(|b|-\beta_1)^{\frac{j}{2(2k-1)}}\sum_{i=1}^{2k-1}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{4k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{ij}{4k(2k-1)}}.
\end{aligned} \tag{2.12}$$

Using inequality (2.6), from (2.8) it follows that, for $0\le j\le 2k-1$,

$$\begin{aligned}
&(|b|-\beta_1)^{\frac{2(2k-1-j)}{2k-1}}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1-j}{2k}}\\
&\le A + (|b|-\beta_1)^{\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{2k-1-j}{2k-1}}(2\pi)^{\frac{2k-1-j}{2k-1}(1-\frac{1+i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i(2k-1-j)}{2k(2k-1)}}\\
&\quad + \left(\beta_2\sqrt{|b|-\beta_1}\right)^{\frac{2k-1-j}{2k-1}}(2\pi)^{(\frac{1}{2}-\frac{1}{4k})\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{2k-1-k}{2(2k-1)}}(2\pi)^{\frac{2k-1-j}{2k-1}(\frac{1}{2}-\frac{1+i}{4k})}\\
&\quad \times\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i(2k-1-j)}{4k(2k-1)}},
\end{aligned} \tag{2.13}$$

where A is a positive constant.

Since for $t\in[0,2\pi]$,

$$|x(t)|\le|x(\xi)|+\int_0^{2\pi}|\dot{x}(t)|dt\le|x(\xi)|+(\sqrt[2k]{2\pi})^{2k-1}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{1}{2k}},$$

we have

$$(2\pi)^{\frac{1-2k}{2k}}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}\le\max_{t\in[0,2\pi]}|x(t)|^{2k-1}$$

$$\leq \left[|x(\xi)| + \left(\sqrt[2k]{2\pi}\right)^{2k-1}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{1}{2k}}\right]^{2k-1}$$

$$= \sum_{j=0}^{2k-1} C_{2k-1}^{j}(\sqrt[2k]{2\pi})^{(2k-1)(2k-1-j)}|x(\xi)|^{j}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1-j}{2k}}. \tag{2.14}$$

Substituting (2.13) and (2.12) into (2.14), then

$$(|b|-\beta_1)^2(2\pi)^{\frac{1-2k}{2k}}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$

$$\leq \sum_{j=0}^{2k-1} C_{2k-1}^{j}|c|^{\frac{-j}{2k-1}}(2\pi)^{[\frac{(2k-1)(2k-1-j)}{2k}-\frac{j}{2k-1}]}\Bigg\{\mu^{\frac{j}{2k-1}}$$

$$+[2|b|(|b|-\beta_1)]^{\frac{j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{j}{2k-1}}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{ij}{2k(2k-1)}}$$

$$+(2\beta_2|b|)^{\frac{j}{2k-1}}(|b|-\beta_1)^{\frac{j}{2(2k-1)}}\sum_{i=1}^{2k-1}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{4k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{ij}{4k(2k-1)}}\Bigg\}$$

$$\times\Bigg\{A+(|b|-\beta_1)^{\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{2k-1-j}{2k-1}}(2\pi)^{\frac{2k-1-j}{2k-1}(1-\frac{1+i}{2k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i(2k-1-j)}{2k(2k-1)}}$$

$$+\left(\beta_2\sqrt{|b|-\beta_1}\right)^{\frac{2k-1-j}{2k-1}}(2\pi)^{(\frac{1}{2}-\frac{1}{4k})\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{2k-1-j}{2(2k-1)}}$$

$$\times(2\pi)^{\frac{2k-1-j}{2k-1}(\frac{1}{2}-\frac{1+i}{4k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{i(2k-1-j)}{4k(2k-1)}}\Bigg\}. \tag{2.15}$$

Let

$$W(k,|b|,|c|,\beta_1,\alpha_{2k-1}) \triangleq \Bigg\{(|b|-\beta_1)^2(2\pi)^{\frac{1-2k}{2k}}$$

$$-2|b|(|b|-\beta_1)\left(\sum_{j=0}^{2k-1}C_{2k-1}^{j}|c|^{\frac{-j}{2k-1}}(2\pi)^{\frac{(2k-1)^2-2kj}{2k}}\right)\Bigg\}\alpha_{2k-1}.$$

It is easy to see from (1.5) that

$$W(k,|b|,|c|,\beta_1,\alpha_{2k-1}) > 0. \tag{2.16}$$

Let $y = \left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{1}{4k(2k-1)}}$. Then from (2.15) it follows that

$$W(k,|b|,|c|,\beta_1,\alpha_{2k-1})y^{2(2k-1)^2}$$

$$\leq \sum_{j=0}^{2k-1}C_{2k-1}^{j}|c|^{\frac{-j}{2k-1}}(2\pi)^{\frac{(2k-1)^2-2kj}{2k}}\Bigg\{\mu^{\frac{j}{2k-1}}+[2|b|(|b|-\beta_1)]^{\frac{j}{2k-1}}$$

$$\cdot\sum_{i=1}^{2k-2}\alpha_i^{\frac{j}{2k-1}}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{2k})}y^{2ij}$$

$$+(2\beta_2|b|)^{\frac{j}{2k-1}}(|b|-\beta_1)^{\frac{j}{2(2k-1)}}\sum_{i=1}^{2k-1}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{4k})}y^{ij}\Bigg\}$$

$$\times\Bigg\{A+(|b|-\beta_1)^{\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-2}\alpha_i^{\frac{2k-1-j}{2k-1}}(2\pi)^{\frac{2k-1-j}{2k-1}(1-\frac{1+i}{2k})}y^{2i(2k-1-j)}$$

$$+(\beta_2\sqrt{|b|-\beta_1})^{\frac{2k-1-j}{2k-1}}(2\pi)^{(\frac{1}{2}-\frac{1}{4k})\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{2k-1-j}{2(2k-1)}}(2\pi)^{\frac{2k-1-j}{2k-1}(\frac{1}{2}-\frac{1+i}{4k})}y^{i(2k-1-j)}\Bigg\}$$

$$+\sum_{j=1}^{2k-1}C_{2k-1}^{j}|c|^{\frac{-j}{2k-1}}(2\pi)^{\frac{(2k-1)^2-2kj}{2k}}\Bigg\{\mu^{\frac{j}{2k-1}}$$

$$+[2|b|(|b|-\beta_1)]^{\frac{j}{2k-1}}\sum_{i=1}^{2k-2}\alpha_i^{\frac{j}{2k-1}}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{2k})}y^{2ij}$$

$$+(2\beta_2|b|)^{\frac{j}{2k-1}}(|b|-\beta_1)^{\frac{j}{2(2k-1)}}\sum_{i=1}^{2k-1}(2\pi)^{\frac{j}{2k-1}(1-\frac{i}{4k})}y^{ij}\Bigg\}$$

$$\times\alpha_{2k-1}^{\frac{2k-1-j}{2k-1}}y^{2(2k-1)(2k-1-j)}$$

$$+\sum_{j=0}^{2k-2}C_{2k-1}^{j}|c|^{\frac{-j}{2k-1}}(2\pi)^{\frac{(2k-1)^2-2kj}{2k}}\alpha_{2k-1}^{\frac{j}{2k-1}}(2\pi)^{\frac{j}{(2k-1)2k}}y^{2ij}$$

$$\times\Bigg\{A+(|b|-\beta_1)^{\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-2}\alpha_i^{\frac{2k-1-j}{2k-1}}(2\pi)^{\frac{2k-1-j}{2k-1}(1-\frac{i+1}{2k})}y^{2i(2k-1-j)}$$

$$+(\beta_2\sqrt{|b|-\beta_1})^{\frac{2k-1-j}{2k-1}}(2\pi)^{(\frac{1}{2}-\frac{1}{4k})\frac{2k-1-j}{2k-1}}\sum_{i=1}^{2k-1}\alpha_i^{\frac{2k-1-j}{2(2k-1)}}(2\pi)^{\frac{2k-1-j}{2k-1}(\frac{1}{2}-\frac{1+i}{4k})}y^{i(2k-1-j)}\Bigg\},$$

from which, together with (2.16) it follows that there exists a positive number ρ such that

$$y=\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{1}{4k(2k-1)}}<\rho,$$

that is,

$$\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{1}{2k}}<\rho^{2(2k-1)}, \tag{2.17}$$

from which, together with (2.8), it follows that there exists a positive constant ρ_1 such that

$$\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{1}{2k}}<\rho_1. \tag{2.18}$$

It follows from (2.17) that there exist a $\xi\in(0,2\pi)$ and a positive constant ρ_2 such that

$$|x(\xi)|\le\rho_2. \tag{2.19}$$

Since $\forall\ t\in[0,2\pi]$,

$$|x(t)|\le|x(\xi)|+\int_0^{2\pi}|\dot{x}(t)|dt,$$

from (2.18), by using inequality (2.3), it follows that there exists a positive constant R_1 such that

$$|x|_0 < R_1. \tag{2.20}$$

Next, we will find the boundedness for $|\dot{x}(t)|$.

Since $|\dot{x}(t)| \le \int_0^{2\pi} |\ddot{x}(t)|dt$, we have from (2.1)

$$\begin{aligned} |a||\dot{x}(t)| &\le \int_0^{2\pi} |a\ddot{x}(t)|dt \\ &\le \int_0^{2\pi} \Bigg[(|b|+\beta_1)|\dot{x}(t)|^{2k-1} + |c||x(t)|^{2k-1} \\ &\quad + M + m + \beta_2|\dot{x}(t)|^{k-\frac{1}{2}} + \sum_{i=1}^{2k-1}\alpha_i|x(t)|^i\Bigg]dt, \end{aligned} \tag{2.21}$$

from which, together with (2.17) and (2.18), by using inequality (2.3), it follows that there exists a positive constant R_2 such that $|\dot{x}|_0 < R_2$ for all $t \in R$.

Let $A^* = \max\{R_1, R_2, d\}$ and take $\Omega = \{x \in X : |x|_2 < A^*\}$, where $d > 0$ is the only real root of equation $(|c| - \alpha_{2k-1})x^{2k-1} - \sum_{i=1}^{2k-2}\alpha_i x^i - M - m = 0$.

The above a priori estimates show that condition (a) in Lemma 2.1 is satisfied. When $x \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap R$, x is a constant with $|x| = A^*$. Then

$$\begin{aligned} QNx &= \frac{1}{2\pi}\int_0^{2\pi} \left[-b\dot{x}^{2k-1}(t) - cx^{2k-1}(t) - g(x(t-\tau_1), \dot{x}(t-\tau_2)) + p(t)\right] dt \\ &= -cx^{2k-1} - g(x,0) + \frac{1}{2\pi}\int_0^{2\pi} p(t)dt. \end{aligned}$$

Thus, since $A^* > d$,

$$\begin{aligned} |QNx|_0 &\ge |c|(A^*)^{2k-1} - |g(x,0)| - m \\ &\ge |c|(A^*)^{2k-1} - m - M - \sum_{i=1}^{2k-1}\alpha_i(A^*)^i \\ &= (|c| - \alpha_{2k-1})(A^*)^{2k-1} - \sum_{i=1}^{2k-2}\alpha_i(A^*)^i - m - M > 0. \end{aligned}$$

Therefore, $QNx \ne 0$ for $x \in \partial\Omega \cap R$.

Set, for $0 \le \mu \le 1, x \in R$,

$$\phi(x,\mu) = (1-\mu)cx^{2k-1} + \mu\left[cx^{2k-1} + g(x,0) - \frac{1}{2\pi}\int_0^{2\pi} p(t)dt\right].$$

When $x \in \partial\Omega \cap \mathrm{Ker}L$ and $\mu \in [0,1]$, x is constant with $|x| = A^*$. Since

$$\phi(x,\mu) = cx^{2k-1} + \mu\left[g(x,0) - \frac{1}{2\pi}\int_0^{2\pi} p(t)dt\right] \quad \text{and} \quad A^* > d,$$

we have

$$\begin{aligned} |\phi(x,\mu)| &\ge |c|(A^*)^{2k-1} - m - M - \sum_{i=1}^{2k-1}\alpha_i(A^*)^i \\ &= (|c| - \alpha_{2k-1})(A^*)^{2k-1} - \sum_{i=1}^{2k-2}\alpha_i(A^*)^i - (m+M) > 0. \end{aligned}$$

Thus $\phi(x,\mu)\neq 0$. Consequently

$$\deg\Big(QN,\Omega\bigcap \mathrm{Ker}L,0\Big)=\deg\left(-cx^{2k-1}-g(x,0)+\frac{1}{2\pi}\int_0^{2\pi}p(t)dt\right)$$
$$=\deg\big(-cx^{2k-1},\Omega\cap \mathrm{Ker}L,0\big)\neq 0.$$

Up to now, all conditions in Lemma 2.1 are satisfied, and hence Eq. (1.3) has at least one solution in Ω. The proof of Theorem 1.1 is completed.

Next we present the proof for Theorem 1.2.

In order to use Lemma 2.1 for Eq. (1.3), we take $X, Z, |x|_0$, and $|x|_2$ as above. Set

$$(Lx)(t)=a\ddot{x}(t),\quad x\in X,\ t\in R,$$

$$(Nx)(t)=-b\dot{x}^{2k-1}(t)-cx^{2k-1}(t)-g(x(t-\tau_1),\dot{x}(t-\tau_2))+p(t),\ x\in X,\ t\in R,$$
$$(Px)(t)=\frac{1}{2\pi}\int_0^{2\pi}x(t)dt,\ x\in X,\ t\in R,\quad (Qz)(t)=\frac{1}{2\pi}\int_0^{2\pi}z(t)dt,\ z\in Z,\ t\in R.$$

Corresponding to the operator equation $Lx=\lambda Nx$ with $\lambda\in(0,1)$, we have

$$a\ddot{x}(t)+\lambda b\dot{x}^{2k-1}(t)+\lambda cx^{2k-1}(t)+\lambda g(x(t-\tau_1),\dot{x}(t-\tau_2))=\lambda p(t). \tag{2.22}$$

Let $x\in X$ be a solution of Eq. (2.22) for some $\lambda\in(0,1)$. Integrating Eq. (2.22) over $[0,2\pi]$, we obtain

$$|c|\int_0^{2\pi}|x(t)|^{2k-1}dt\leq\int_0^{2\pi}\Bigg[(|b|+\gamma_{2k-1})|\dot{x}(t)|^{2k-1}+\sum_{i=1}^{2k-2}\gamma_i|\dot{x}(t)|^i$$
$$+\beta_3|x(t)|^{2k-1}+\beta_4|x(t)|^{k-\frac{1}{2}}+m+M^*\Bigg]dt,$$

where m is defined as above.

Thus there exists a $\xi\in(0,2\pi)$ such that

$$2\pi|c||x(\xi)|^{2k-1}\leq\int_0^{2\pi}\Bigg[(|b|+\gamma_{2k-1})|\dot{x}(t)|^{2k-1}+\sum_{i=1}^{2k-2}\gamma_i|\dot{x}(t)|^i$$
$$+\beta_3|x(t)|^{2k-1}+\beta_4|x(t)|^{k-\frac{1}{2}}+m+M^*\Bigg]dt. \tag{2.23}$$

Using inequality (2.3), from (2.23) it follows that

$$2\pi|c||x(\xi)|^{2k-1}\leq(|b|+\gamma_{2k-1})\sqrt[2k]{2\pi}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$
$$+\sum_{i=1}^{2k-2}\gamma_i(2\pi)^{(1-\frac{i}{2k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{i}{2k}}$$
$$+\sqrt[2k]{2\pi}\beta_3\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}+\beta_4(2\pi)^{(\frac{1}{2}+\frac{1}{4k})}$$
$$\times\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}+2\pi(m+M^*). \tag{2.24}$$

Hence

$$(2\pi)^{\frac{1-2k}{2k}}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}} \le \max_{t\in(0,2\pi]}|x(t)|^{2k-1}$$
$$\le \left[|x(\xi)| + \left(\sqrt[2k]{2\pi}\right)^{2k-1}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{1}{2k}}\right]^{2k-1}$$
$$= \sum_{j=0}^{2k-1} C^j_{2k-1}|x(\xi)|^j\left(\sqrt[2k]{2\pi}\right)^{(2k-1)(2k-1-j)}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1-j}{2k}}. \quad (2.25)$$

Using the inequality

$$ab < \frac{a^k}{k} + \frac{b^{k'}}{k'} \quad \text{for} \quad k>1, \quad \frac{1}{k}+\frac{1}{k'}=1, \quad a>0,\ b>0,$$

we get, from (2.25),

$$(2\pi)^{\frac{1-2k}{2k}}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$
$$\le \sum_{j=1}^{2k-2} C^j_{2k-1}(\sqrt[2k]{2\pi})^{(2k-1)(2k-1-j)}\left[\frac{|x(\xi)|^{2k-1}}{\frac{2k-1}{j}} + \frac{\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}}{\frac{2k-1}{2k-1-j}}\right]$$
$$+ |x(\xi)|^{2k-1} + (\sqrt[2k]{2\pi})^{(2k-1)^2}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$
$$= f_1(k)|x(\xi)|^{2k-1} + f_2(k)\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}. \quad (2.26)$$

Substituting (2.24) into (2.26), then

$$\sqrt[2k]{2\pi}(|c|-\beta_3 f_1(k))\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$
$$\le \left[f_2(k) + f_1(k)\sqrt[2k]{2\pi}(|b|+\gamma_{2k-1})\right]\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}$$
$$+ f_1(k)\sum_{j=0}^{2k-2}\gamma_i(2\pi)^{(1-\frac{i}{2k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{i}{2k}}$$
$$+ \beta_4(2\pi)^{(\frac{1}{2}+\frac{1}{4k})}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}, \quad (2.27)$$

from which, by using inequality (2.6), it follows that

$$(|c|-\beta_3 f_1(k))\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{(\frac{1}{2}-\frac{1}{4k})}$$
$$\le \beta_4(2\pi)^{(\frac{1}{2}+\frac{1}{4k})} + \sqrt{|c|-\beta_3 f_1(k)}\Bigg\{\sqrt[4k]{2\pi}[f_2(k)(|b|+\gamma_{2k-1})]^{\frac{1}{2}}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{4k}}$$

$$+\sum_{i=1}^{2k-2}\sqrt{r_i}(2\pi)^{(\frac{1}{2}-\frac{i}{4k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{i}{4k}}\Bigg\}. \tag{2.28}$$

Substituting (2.28) into (2.27), then

$$\begin{aligned}&\sqrt[2k]{2\pi}(|c|-\beta_3 f_1(k))^2\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}\\
&\le \beta_4^2(2\pi)^{(1+\frac{1}{2k})}+\beta_4(2\pi)^{(\frac{1}{2}+\frac{1}{4k})}\sqrt{|c|-\beta_3 f_1(k)}\\
&\quad\times\Bigg\{\sqrt{\sqrt[4k]{2\pi}f_2(k)(|b|+\gamma_{2k-1})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{4k}}\\
&\quad+\sum_{i=1}^{2k-2}\sqrt{r_i}(2\pi)^{(\frac{1}{2}-\frac{i}{4k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{i}{4k}}\Bigg\}\\
&\quad+(|c|-\beta_3 f_1(k))\Bigg\{\left[f_2(k)+\sqrt[2k]{2\pi}f_1(k)(|b|+\gamma_{2k-1})\right]\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}\\
&\quad+\sum_{i=1}^{2k-2}\gamma_i(2\pi)^{(1-\frac{i}{2k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{i}{2k}}\Bigg\}.\end{aligned} \tag{2.29}$$

Multiplying Eq. (2.22) by $\dot{x}$ and integrating over $[0,2\pi]$, we obtain

$$\begin{aligned}|b|\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\le\int_0^{2\pi}|\dot{x}(t)|\Bigg[M^*+m+\sum_{i=1}^{2k-1}\gamma_i|\dot{x}(t-\tau_2)|^i\\
+\beta_3|x(t-\tau_1)|^{2k-1}+\beta_4|x(t-\tau_1)|^{k-\frac{1}{2}}\Bigg]dt.\end{aligned} \tag{2.30}$$

Using inequality (2.3) and the Hölder inequality, from (2.30) it follows that

$$\begin{aligned}&(|b|-\gamma_{2k-1})\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{1-\frac{1}{2k}}\\
&\le\left(\sqrt[2k]{2\pi}\right)^{2k-1}(M^*+m)+\sum_{i=1}^{2k-2}\gamma_i\left(\int_0^{2\pi}|\dot{x}(t)|^{\frac{2ki}{2k-1}}dt\right)^{\frac{2k-1}{2k}}\\
&\quad+\beta_3\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}+\beta_4\left(\int_0^{2\pi}|x(t)|^{k}dt\right)^{\frac{2k-1}{2k}}\\
&\le\left(\sqrt[2k]{2\pi}\right)^{2k-1}(M^*+m)+\sum_{i=1}^{2k-2}\gamma_i(2\pi)^{(1-\frac{1+i}{2k})}\left(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt\right)^{\frac{i}{2k}}\\
&\quad+\beta_3\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{2k}}+\beta_4(2\pi)^{\frac{2k-1}{4k}}\left(\int_0^{2\pi}|x(t)|^{2k}dt\right)^{\frac{2k-1}{4k}}.\end{aligned} \tag{2.31}$$

Substituting (2.28) and (2.29) into (2.31), we obtain an inequality with respect to $(\int_0^{2\pi}|\dot{x}(t)|^{2k}dt)^{\frac{1}{4k}}$. The rest of the proof is similar to that of Theorem 1.1 and is omitted.

Finally, we give a specific example to illustrate our result.

Example Consider the following equation:

$$\ddot{x}(t)+\dot{x}^5(t)+x^5(t)+\frac{\beta_3 x^5(t-\tau)+\sqrt{x^5(t-\tau)\operatorname{sgn}x(t-\tau)}+1}{1+x^2(t-\tau)}$$
$$+\frac{\dot{x}(t-\tau)+\dot{x}^2(t-\tau)+\dot{x}^3(t-\tau)+\dot{x}^4(t-\tau)+\gamma_5\dot{x}^5(t-\tau)}{1+x^2(t-\tau)}=p(t), \qquad (2.32)$$

where β_3, γ_5, τ are constants with $|\beta_3|<1$, and p is continuous and 2π-periodic.

In this example,

$$g(x_1,x_2)=\frac{1+x_2+x_2^2+x_2^3+x_2^4+\gamma_5x_2^5+\beta_3x_1^5+\sqrt{x_1^5\operatorname{sgn}x_1}}{1+x_1^2}, \text{ and hence}$$

$$|g(x_1,x_2)|\le 1+\sum_{i=1}^{4}|x_2|^i+|\gamma_5||x_2|^5+|\beta_3||x_1|^5+|x_1|^{\frac{5}{2}}.$$

If we take β_3 and γ_5 such that

$$1>|\beta_3|\left[2\sum_{j=0}^{5}\frac{j}{5}C_5^j\left(\sqrt[6]{2\pi}\right)^{5(5-j)}+\sum_{j=0}^{5}\frac{5-j}{5}C_5^j\left(\sqrt[6]{2\pi}\right)^{5(5-j)}\right]+|\gamma_5|,$$

say, we let β_3 and γ_5 be sufficiently small, then all conditions in Theorem 1.2 are satisfied. Therefore, by Theorem 1.2, Eq. (1.3) has at least one 2π-periodic solution.

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