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Stability for the Timoshenko Beam System with Local Kelvin–Voigt Damping

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Abstract In this paper, we consider a vibrating beam with one segment made of viscoelastic material of a Kelvin–Voigt (shorted as K–V) type and other parts made of elastic material by means of the Timoshenko model. We have deduced mathematical equations modelling its vibration and studied the stability of the semigroup associated with the equation system. We obtain the exponential stability under certain hypotheses of the smoothness and structural condition of the coefficients of the system, and obtain the strong asymptotic stability under weaker hypotheses of the coefficients.

Keywords Timoshenko beam, Kelvin-Voigt damping, Semigroup, Stability
MR(2000) Subject Classification 35B37, 35B40

1 Introduction

In recent years, there has been much interest in the problems of exponential stability for elastic systems with locally distributed damping (the damping is distributed only in a subdomain). Most of the works were devoted to the viscous damping, i.e., damping is proportional to the velocity (see, such as [1, 2, 3]). Structures with local viscoelasticity arise from use of smart material or passive stabilization of structures. However, very little is known about exponential stability for elastic systems with local viscoelastic damping, although there is a fairly deep understanding when the damping is distributed over the entire domain, see [4] for a review. To our knowledge, the first paper in this direction was published in 1998 by Liu and Liu [5] where they obtained exponential stability for the Euler–Bernoulli beam equation with local K-V damping. They [5] also gave the surprising result that the energy of the solution to the string equation with local K-V damping and piecewise constant coefficients does not exponentially decay. This suggests that the geometric optics condition given in [6] may be insufficient for the exponential stability of elastic systems with the local viscoelastic damping. Then, the exponential decay of the energy of a vibrating string with local viscoelasticity of both K-V and Boltzmann types is established in [4] under certain hypotheses of smoothness of the coefficients. We also refer the readers to Rivera et al. [7, 8] for wave equations with local Boltzmann damping. Zero initial history data were assumed in both [7] and [8]. In this paper, we consider the Timoshenko beam with local K-V viscoelasticity.

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Consider a nonhomogeneous cantilever beam. Suppose that its motion is in the X-Z plane (no torsion). Furthermore, we assume that the cross-section being perpendicular to the centroid trace before deformation is still in a plane after deformation, and there is no strain in it during deformation [9, 10]. Thus we have

$$\sigma_y = \sigma_z = \tau_{yz} = \tau_{yx} = 0.$$

Let u, w be the longitudinal and transverse displacements of the median plane of the beam and φ be the total rotatory angle of the fiber on the centroid trace (it is positive from positive direction of x-axis to positive direction of y-axis). Under the above assumptions, all u, w and φ are functions of the spatial coordinate x only, and the displacement vector is

$$\boldsymbol{u} = (u - z\varphi, 0, w). \tag{1.1}$$

Under the linear strain-displacement assumption, the nonzero strains are

$$\varepsilon_x = u' - z\varphi', \quad \gamma_{xz} = w' - \varphi. \tag{1.2}$$

Hereafter, the prime denotes the derivative with respect to the spatial variable x. Assume that the beam is made of viscoelastic material [11] with K-V constitutive relations (damping is proportional to strain rate). Then we have

$$\sigma_x = E\varepsilon_x + E_1\dot{\varepsilon}_x, \quad \tau_{xz} = G\gamma_{xz} + G_1\dot{\gamma}_{xz}, \tag{1.3}$$

where $E = E(x) \geq E_0 > 0, G = G(x) \geq G_0 > 0, E_1 = E_1(x) \geq 0, G_1 = G_1(x) \geq 0$. E_0 and G_0 are constants. Here and after, the dot denotes the derivative with respect to the time variable t. By the principle of virtual work (see [10]), the work done by the stress through virtual strain equals the work done by the external force through the virtual displacement, i.e.,

$$\iiint \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon} dV = \iiint (\boldsymbol{F} - m\ddot{\boldsymbol{r}}) \cdot \delta\boldsymbol{r} dV + \iint \boldsymbol{F_s} \cdot \delta\boldsymbol{r} dS,$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are stress and strain tensors, m denotes density, $\boldsymbol{F}$ and $\boldsymbol{F_s}$ are volume and area distributions of external forces. In our case, $\boldsymbol{F}$=$\boldsymbol{F_s}$=0. The deformation vector $\boldsymbol{r} = (x, y, z) + \boldsymbol{u}$ and the virtual displacement $\delta\boldsymbol{r} = \delta\boldsymbol{u}$ satisfy geometric constraint conditions. Thus we have

$$\begin{aligned}
0 &= \iiint (m\ddot{\boldsymbol{r}} \cdot \delta\boldsymbol{r} + \boldsymbol{\sigma} \cdot \delta\boldsymbol{\varepsilon}) dV \\
&= \iiint \{m(\ddot{u} - z\ddot{\varphi})(\delta u - z\delta\varphi) + m\ddot{w}\delta w \\
&\quad + (E\varepsilon_x + E_1\dot{\varepsilon}_x)\delta\varepsilon_x + (G\gamma_{xz} + G_1\dot{\gamma}_{xz})\delta\gamma_{xz}\} dV \\
&= \iiint \{m(\ddot{u} - z\ddot{\varphi})(\delta u - z\delta\varphi) + m\ddot{w}\delta w \\
&\quad + [E(u' - z\varphi') + E_1(\dot{u}' - z\dot{\varphi}')](\delta u' - z\delta\varphi') \\
&\quad + [G(w' - \varphi) + G_1(\dot{w}' - \dot{\varphi})](\delta w' - \delta\varphi)\} dV \\
&= \iiint \{m\ddot{u}\delta u + mz^2\ddot{\varphi}\delta\varphi + m\ddot{w}\delta w \\
&\quad + (Eu' + E_1\dot{u}')\delta u' + (E\varphi' + E_1\dot{\varphi}')z^2\delta\varphi' \\
&\quad + [G(w' - \varphi) + G_1(\dot{w}' - \dot{\varphi})]\delta w' - [G(w' - \varphi) + G_1(\dot{w}' - \dot{\varphi})]\delta\varphi\} dV \\
&= \int_0^L \{\rho\ddot{u}\delta u + I_\rho\ddot{\varphi}\delta\varphi + \rho\ddot{w}\delta w + (pu' + D_e\dot{u}')\delta u' + (EI\varphi' + D_b\dot{\varphi}')\delta\varphi' \\
&\quad + [K(w' - \varphi) + D_s(\dot{w}' - \dot{\varphi})]\delta w' - [K(w' - \varphi) + D_s(\dot{w}' - \dot{\varphi})]\delta\varphi\} dx.
\end{aligned}$$

In the fourth equality, we have used the symmetry of the centroid trace with respect to the spatial coordinate z. Here, $\rho = \int\int_{S(x)} m dydz$ is the line density, $S(x)$ is the cross-section at x, $I_\rho = \int\int_{S(x)} mz^2 dydz$ is the rotatory inertia, $P = E|S(x)|, D_e = E_1|S(x)|$ with $|S(x)|$ being the area of $S(x)$, $I = \int\int_{S(x)} z^2 dydz$ is the moment of inertia, $D_b = E_1 I, K = G|S(x)|$ is the shear modulus, $D_s = G_1|S(x)|$. Then we have

$$\begin{cases} \displaystyle\int_0^L [\rho\ddot{u}\delta u + (pu' + D_e\dot{u}')\delta u']dx = 0, \\ u(0,t) = \delta u(0,t) = 0, \end{cases} \tag{1.4}$$

$$\begin{cases} \displaystyle\int_0^L \{\rho\ddot{w}\delta w + I_\rho\ddot{\varphi}\delta\varphi + (EI\varphi' + D_b\dot{\varphi}')\delta\varphi' \\ \qquad +[K(w' - \varphi) + D_s(\dot{w}' - \dot{\varphi})](\delta w' - \delta\varphi)\}dx = 0, \\ w(0,t) = \delta w(0,t) = \varphi(0,t) = \delta\varphi(0,t) = 0. \end{cases} \tag{1.5}$$

Integrating by parts in (1.4) and (1.5) and by the arbitrariness of $\delta u, \delta w$ and $\delta\varphi$, we have

$$\begin{cases} \rho\ddot{u} - (pu' + D_e\dot{u}')' = 0, \\ u|_{x=0} = 0, \quad (pu' + D_e\dot{u}')'|_{x=L} = 0, \end{cases} \tag{1.6}$$

$$\begin{cases} \rho\ddot{w} - [K(w' - \varphi) + D_s(\dot{w}' - \dot{\varphi})]' = 0, \\ I_\rho\ddot{\varphi} - (EI\varphi' + D_b\dot{\varphi}')' - [K(w' - \varphi) + D_s(\dot{w}' - \dot{\varphi})] = 0, \\ w|_{x=0} = 0, \quad \varphi|_{x=0} = 0, \\ [K(w' - \varphi) + D_s(\dot{w}' - \dot{\varphi})]|_{x=0} = 0, \quad (EI\varphi' + D_b\dot{\varphi}')|_{x=L} = 0. \end{cases} \tag{1.7}$$

System (1.6) describes the longitudinal vibration of the beam, and system (1.7) describes the transverse and the shearing vibrations; (1.7) is called the Timoshenko cantilever beam equation with K-V damping.

In Section 2, we consider the well-posedness and asymptotic stability of system (1.7). In Section 3, we will show that the energy of system (1.7) decays exponentially when the coefficient functions are smooth and satisfy a certain structural condition.

2 Semigroup Setting, Well-Posedness and Strong Asymptotic Stability

First, we assume that the coefficient functions of system (1.7) satisfy

(H1) $\rho, I_\rho, K, EI, D_b, D_s \in L^\infty(0,L),\ \rho, I_\rho, K, EI \geq C_0 > 0,\ D_b, D_s \geq 0.$

The energy of the system (1.7), with initial conditions

$$w(x,0) = w_0, \quad \dot{w}(x,0) = w_1, \quad \varphi(x,0) = \varphi_0, \quad \dot{\varphi}(x,0) = \varphi_1, \tag{2.1}$$

is defined by

$$E(t) = \frac{1}{2}\int_0^L (K|w' - \varphi|^2 + EI|\varphi'|^2 + \rho|\dot{w}|^2 + I_\rho|\dot{\varphi}|^2)dx. \tag{2.2}$$

Let

$$H_0(0,L) = \{w \in H^1(0,L) \mid w(0) = 0\}, \quad H = L^2(0,L) \times L^2(0,L), \quad V = H_0(0,L) \times H_0(0,L),$$

$$\|(w,\varphi)\|_H^2 = \int_0^L (\rho|w|^2 + I_\rho|\varphi|^2)dx, \quad \|(w,\varphi)\|_V^2 = \int_0^L (K|w' - \varphi|^2 + EI|\varphi'|^2 dx),$$

and

$$b((w_1,\varphi_1),(w_2,\varphi_2)) = \int_0^L [D_s(w_1' - \varphi_1)(\bar{w}_2' - \bar{\varphi}_2) + D_b\varphi_1'\bar{\varphi}_2']dx,$$

where $H^1(0,L)$ is the Sobolev space [12]. Then we have:

(C1) V, H are Hilbert spaces with norms $\|(\cdot,\cdot)\|_V$ and $\|(\cdot,\cdot)\|_H$;

(C2) $V \hookrightarrow H$ is a continuous, dense and compact embedding;

(C3) $b(\cdot,\cdot)$ is a continuous, nonnegative, symmetric sesquilinear form on V [13].

Let $\mathcal{H} = V \times H$, $\|(w,\varphi,v,\psi)\|_{\mathcal{H}}^2 = \|(w,\varphi)\|_V^2 + \|(v,\psi)\|_H^2$, $\mathcal{H}$ being a Hilbert space. We define a linear operator $\mathcal{A}$ in $\mathcal{H}$ as follows:

$$\mathcal{D}(\mathcal{A}) = \left\{ (w,\varphi,v,\psi) \in \mathcal{H} \;\middle|\; \begin{array}{l} (v,\psi) \in V,\ T := K(w'-\varphi) + D_s(v'-\psi) \in H^1(0,L), \\ R := EI\varphi' + D_b\psi' \in H^1(0,L),\ T(L) = R(L) = 0, \end{array} \right\}, \tag{2.3}$$

and

$$\mathcal{A}(w,\varphi,v,\psi) = \left(v, \psi, \frac{1}{\rho}T', \frac{1}{I_\rho}(R'+T) \right). \tag{2.4}$$

Thus (1.7) can be rewritten as an abstract evolution equation on $\mathcal{H}$:

$$\begin{cases} (\dot{w}(t), \dot{\varphi}(t), \dot{v}(t), \dot{\psi}(t)) = \mathcal{A}(w(t), \varphi(t), v(t), \psi(t)), \\ (w(0), \varphi(0), v(0), \psi(0)) = (w_0, \varphi_0, w_1, \varphi_1). \end{cases} \tag{2.5}$$

Lemma 2.1 *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$, $e^{\mathcal{A}t}$.*

Proof Since, for every $z = (w,\varphi,v,\psi) \in \mathcal{D}(\mathcal{A})$,

$$\mathrm{Re}\langle \mathcal{A}z, z\rangle_{\mathcal{H}} = -b((v,\psi),(v,\psi)) \le 0,$$

$\mathcal{A}$ is dissipative, it is easy to verify that $\ker \mathcal{A} = \{0\}$. For any $(f,\alpha,g,\beta) \in \mathcal{H}$, taking $v = f$, $\psi = \alpha$,

$$\varphi = -\int_0^x \frac{1}{EI}\left[\int_x^L \left(I_\rho\beta + \int_x^L \rho g dx \right) dx + D_b\alpha' \right] dx,$$

$$w = -\int_0^x \left[\frac{1}{K}\int_x^L \rho g dx + \frac{D_s}{K}(f'-\alpha) - \varphi \right] dx,$$

then $(w,\varphi,v,\psi) \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}(w,\varphi,v,\psi) = (f,\alpha,g,\beta)$ and $\|(w,\varphi,v,\psi)\|_{\mathcal{H}} \le M\|(f,\alpha,g,\beta)\|_{\mathcal{H}}$, for some $M > 0$ independent of (f,α,g,β). Therefore, $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$, $0 \in \rho(\mathcal{A})$, and $\mathcal{A}$ is closed. Since $\rho(\mathcal{A})$ is an open set on the complex plane, we see that $\mathcal{R}(\lambda - \mathcal{A}) = \mathcal{H}$ for sufficiently small $\lambda > 0$. By Theorem 1.4.6 in [14], $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{H}$, and we can employ the Lumer–Phillips theorem to finish our proof.

Theorem 2.1 *Let $(w_0,\varphi_0,w_1,\varphi_1) \in \mathcal{H}$, $(w(t),\varphi(t),v(t),\psi(t)) = e^{\mathcal{A}t}(w_0,\varphi_0,w_1,\varphi_1)$. Then $(w(\cdot),\varphi(\cdot)) \in C([0,\infty);V) \cap C^1([0,\infty);H)$ satisfies $(\dot{w}(\cdot),\dot{\varphi}(\cdot)) = (v(\cdot),\psi(\cdot))$ and*

$$\frac{d}{dt}\{\langle(\dot{w}(t),\dot{\varphi}(t)),(\xi,\theta)\rangle_H + b((w(t),\varphi(t)),(\xi,\theta))\} + \langle(w(t),\varphi(t)),(\xi,\theta)\rangle_V = 0,\ \ \forall(\xi,\theta) \in V,\ \ t > 0,$$

with initial conditions in (2.5)*. Moreover, if $(w_0,\varphi_0,w_1,\varphi_1) \in \mathcal{D}(\mathcal{A})$, then $(w(\cdot),\varphi(\cdot)) \in C^1([0,\infty);V) \cap C^2([0,\infty);H)$ and satisfies*

$$\langle(\ddot{w}(t),\ddot{\varphi}(t)),(\xi,\theta)\rangle_H + b((\dot{w}(t),\dot{\varphi}(t)),(\xi,\theta)) + \langle(w(t),\varphi(t)),(\xi,\theta)\rangle_V = 0,\ \forall(\xi,\theta) \in V, t > 0.$$

Proof The proof is the same as in Theorem 2.2 in [15].

For asymptotic stability, we assume that:

(H2) *ρ, EI, K, I_ρ are piecewise continuous on* $[0,L]$;

(H3) $\mathrm{int}\{x \in [0,L] | D_b > 0 \text{ or } D_s > 0\} \ne \emptyset$.

Lemma 2.2 *Suppose that hypotheses* (H1), (H2) *and* (H3) *are satisfied and $\boldsymbol{W} = (w,\varphi) \in V$. For every $\omega \in \mathbb{R}$, if*

$$\langle \boldsymbol{W}, \boldsymbol{Y}\rangle_V - \omega^2\langle \boldsymbol{W}, \boldsymbol{Y}\rangle_H = 0, \quad \forall \boldsymbol{Y} = (v,\psi) \in V \tag{2.6}$$

and

$$b(\boldsymbol{W}, \boldsymbol{W}) = 0, \tag{2.7}$$

then $\boldsymbol{W} = 0$ in V.

Proof If $\omega = 0$, then (2.6) becomes $\langle \boldsymbol{W}, \boldsymbol{Y}\rangle_V = 0$, $\forall \boldsymbol{Y} \in V$, which implies $\boldsymbol{W} = 0$ in V.

If $\omega \neq 0$, then (2.6) and (2.7) can be rewritten as

$$\int_0^L \left[K(w' - \varphi)(\bar{v}' - \bar{\psi}) + EI\varphi'\bar{\psi}'\right] dx - \omega^2 \int_0^L (\rho w\bar{v} + I_\rho \varphi\bar{\psi})dx = 0, \quad \forall \boldsymbol{Y} \in V, \tag{2.8}$$

and

$$\int_0^L (D_s|w' - \varphi|^2 + D_b|\varphi'|^2)dx = 0. \tag{2.9}$$

From (2.8) we can deduce that $\boldsymbol{W}$ satisfies

$$\begin{cases} [K(w' - \varphi)]' + \omega^2 \rho w = 0, \\ (EI\varphi')' + K(w' - \varphi) + \omega^2 I_\rho \varphi = 0, \end{cases} \quad \text{on} \quad [0, L]. \tag{2.10}$$

We denote

$$P = \begin{pmatrix} K & 0 \\ 0 & EI \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -K \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \omega^2\rho & 0 \\ 0 & \omega^2\rho - K \end{pmatrix}.$$

Then (2.10) can be written as

$$(P\boldsymbol{W}' + R\boldsymbol{W})' - (R^T\boldsymbol{W}' - Q\boldsymbol{W}) = 0, \tag{2.11}$$

where R^T denotes the transposition of R. Let $\boldsymbol{Z} = P\boldsymbol{W}' + R\boldsymbol{W}$. Then (2.11) is equivalent to

$$\begin{pmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{pmatrix}' = \begin{pmatrix} -P^{-1}R & P^{-1} \\ Q - R^TP^{-1}R & R^TP^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{pmatrix}. \tag{2.12}$$

From [16] we know that, for every $x_0 \in [0, L]$ and every $\boldsymbol{W}_0, \boldsymbol{Z}_0 \in \mathbb{R}^2$, the initial-value problem (2.12) with

$$\begin{pmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{pmatrix}\Bigg|_{x=x_0} = \begin{pmatrix} \boldsymbol{W}_0 \\ \boldsymbol{Z}_0 \end{pmatrix} \tag{2.13}$$

has a unique solution $(\boldsymbol{W}, \boldsymbol{Z})^T \in C[0, L]$.

If $\text{int}\{x \in [0, L] | D_b > 0\} \neq \emptyset$, then there exists an interval

$$(x_1, x_2) \subset \text{int}\{x \in [0, L] | D_b > 0\}.$$

From (2.9) we obtain that $\varphi = 0$ in (x_1, x_2); from the second equation of (2.10) we have $w' = 0$ in (x_1, x_2). Therefore from the first equation of (2.10) we have $w = 0$ in (x_1, x_2).

If $\text{int}\{x \in [0, L] | D_s > 0\} \neq \emptyset$, then there exists an interval

$$(x_1, x_2) \subset \text{int}\{x \in [0, L] | D_s > 0\}.$$

From (2.9) we obtain that $w' - \varphi = 0$ in (x_1, x_2); from the first equation of (2.10) we have $w = 0$ in (x_1, x_2). Therefore $\varphi = w' = 0$ in (x_1, x_2).

Thus for $\omega \neq 0$, by (H3) we conclude that there exists a subinterval (x_1, x_2) of $[0, L]$ such that $\boldsymbol{W} = 0$ in (x_1, x_2), therefore $\boldsymbol{Z} = 0$ in (x_1, x_2). By the uniqueness of solutions, (2.12), (2.13) has only zero solution $\boldsymbol{W} = 0, \boldsymbol{Z} = 0$, which completes the proof.

By Theorem 4.4 in [15] and Lemma 2.2 we can obtain:

Theorem 2.2 *Under hypotheses* (H1), (H2) *and* (H3), *the semigroup* $e^{\mathcal{A}t}$ *of system* (1.7) *is strongly asymptotically stable.*

Corollary 2.1 *Under hypotheses* (H1), (H2) *and* (H3), $\boldsymbol{i}\mathbb{R} \subset \rho(\mathcal{A})$.

Proof By Lemma 4.1 in [15] we have that all nonreal spectrum points of $\mathcal{A}$ belong to the point spectrum, but, by Theorem 3.1 in [17], a pure imaginary number can only be a continuous spectrum point, since $0 \in \rho(\mathcal{A})$, $\boldsymbol{i}\mathbb{R} \subset \rho(\mathcal{A})$.

3 Exponential Stability

For the exponential stability, we assume that:

(H1)′ $\rho, I_\rho, K, EI \in C^{0,1}[0,L]$; $D_s, D_b \in C^{1,1}[0,L]$, $\rho K, I_\rho EI \in C^{1,1}[0,L]$, $D_s, D_b \geq 0$, $\rho, I_\rho, K, EI \geq C_0 > 0$, *where* $C_0 > 0$ *is a constant*;

(H3)′ $D_s(\cdot) \not\equiv 0$ *on* $[0,L]$, $D_b(\cdot) \not\equiv 0$ *on* $[0,L]$;

(H4) $D_s(L) > 0$ or $D_s(L) = D_s'(L) = 0$; *and* $D_b(L) > 0$ or $D_b(L) = D_b'(L) = 0$;

(H5) $\exists C > 0$, s.t. $D_s \leq C D_b$ *on* $[0,L]$.

Theorem 3.1 *Under hypotheses* (H1)′, (H3)′, (H4) *and* (H5), $e^{\mathcal{A}t}$ *is exponentially stable.*

Proof By Theorem 3 in [18] and Corollary 2.1 in Section 2, we need to prove only that there exists some $\delta > 0$ such that, for every $\alpha \in \mathbb{R}$,

$$\|(\boldsymbol{i}\alpha - \mathcal{A})z\|_{\mathcal{H}} \geq \delta \|z\|_{\mathcal{H}}, \quad \forall z \in \mathcal{D}(\mathcal{A}). \tag{3.1}$$

Otherwise, there exist $\alpha_n \in \mathbb{R}$, $z_n = (w_n, \varphi_n, v_n, \psi_n) \in \mathcal{D}(\mathcal{A})$, $n = 1, 2, \ldots$, such that

$$\|z_n\|_{\mathcal{H}} = 1, \quad |\alpha_n| \to \infty \tag{3.2}$$

and

$$(\boldsymbol{i}\alpha_n - \mathcal{A})z_n =: (f_n, g_n, h_n, l_n) \to 0 \quad \text{in} \quad \mathcal{H}, \tag{3.3}$$

i.e.,

$$(\boldsymbol{i}\alpha_n w_n - v_n, \boldsymbol{i}\alpha_n \varphi_n - \psi_n) = (f_n, g_n) \to 0 \quad \text{in} \quad V, \tag{3.4}$$

$$\left(\boldsymbol{i}\alpha_n v_n - \frac{1}{\rho} T_n', \boldsymbol{i}\alpha_n \psi_n - \frac{1}{I_\rho}(R_n' + T_n)\right) = (h_n, l_n) \to 0 \quad \text{in} \quad H, \tag{3.5}$$

where

$$T_n := K(w_n' - \varphi_n) + D_s(v_n' - \psi_n), \quad R_n := EI\varphi_n' + D_b\psi_n'.$$

Lemma 3.1 *The sequence* $(w_n, \varphi_n, v_n, \psi_n)$ *satisfies that*

$$\lim_{n\to\infty} \int_0^L (D_s|v_n' - \psi_n|^2 + D_b|\psi_n'|^2)dx = 0, \tag{3.6}$$

$$\lim_{n\to\infty} \alpha_n^2 \int_0^L (D_s|w_n' - \varphi_n|^2 + D_b|\varphi_n'|^2)dx = 0, \tag{3.7}$$

$$\lim_{n\to\infty} \int_0^L (K|w_n' - \varphi_n|^2 + EI|\varphi_n'|^2)dx = \frac{1}{2}, \tag{3.8}$$

and

$$\int_0^L \rho D_s |\alpha_n v_n|^2 dx + \int_0^L I_\rho D_b |\alpha_n \psi_n|^2 dx \leq M, \tag{3.9}$$

where M *is a positive constant independent of* n.

Proof From (3.3) we obtain that

$$\int_0^L (D_s|v_n' - \psi_n|^2 + D_b|\psi_n'|^2)dx = \mathrm{Re}\langle(\boldsymbol{i}\alpha_n - \mathcal{A})z_n, z_n\rangle_{\mathcal{H}} \to 0.$$

This and (3.4) imply (3.7). For (3.8), we take the sum of the inner products of (3.4) with (v_n, ψ_n) and (3.5) with (w_n, φ_n) in H to get

$$\begin{aligned}\|(w_n, \varphi_n)\|_V^2 - \|(v_n, \psi_n)\|_H^2 = \mathrm{Re}\bigg(&\langle(f_n, g_n), (v_n, \psi_n)\rangle_H + \langle(h_n, l_n), (w_n, \varphi_n)\rangle_H \\ &- \int_0^L (D_s(v_n' - \psi_n)(\bar{w_n}' - \bar{\varphi_n}) + D_b\psi_n'\bar{\varphi_n})dx\bigg) \\ \to 0,&\end{aligned}$$

where we have used (3.2), (3.4)–(3.7). This and (3.2) imply (3.8).

Now we prove (3.9). First, from (3.2) and (3.6) we obtain that

$$\int_0^L |T_n|^2 dx = O(1), \quad \int_0^L |R_n|^2 dx = O(1). \tag{3.10}$$

Next, we point out that if $D_s(L) > 0$, then

$$|v_n(L)|^2 = O(1); \tag{3.11}$$

and if $D_b(L) > 0$, then

$$|\psi_n(L)|^2 = O(1). \tag{3.12}$$

Indeed, $D_s(L) > 0$ and (H5) imply that there exists $\alpha \in [0, L)$, such that $D_s(x) \geq C_1 > 0$, $D_b(x) \geq C_1 > 0$ in $[\alpha, L]$, where C_1 is a constant. Therefore, by the embedding theorem, (3.2) and (3.6), we obtain (3.11). If $D_b(L) > 0$, there exists $\alpha \in [0, L)$, such that $D_b(x) \geq C_1 > 0$ in $[\alpha, L]$, where C_1 is a constant. Therefore, by the embedding theorem, (3.2), and (3.6), we obtain (3.12).

Finally, taking the inner product of (3.5) with $(\boldsymbol{i}\alpha_n D_s v_n, \boldsymbol{i}\alpha_n D_b \psi_n)$ in H and taking the real parts, we have

$$\begin{aligned}
I :=& \int_0^L \rho D_s |\alpha_n v_n|^2 dx + \int_0^L I_\rho D_b |\alpha_n \psi_n|^2 dx \\
=& - \operatorname{Re} \int_0^L T_n' \boldsymbol{i}\alpha_n D_s \bar{v}_n dx - \operatorname{Re} \int_0^L (R_n' + T_n) \boldsymbol{i}\alpha_n D_b \bar{\psi}_n dx - \operatorname{Re} \int_0^L \rho h_n \boldsymbol{i}\alpha_n D_s \bar{v}_n dx \\
&- \operatorname{Re} \int_0^L I_\rho l_n \boldsymbol{i}\alpha_n D_b \bar{\psi}_n dx \\
=& \operatorname{Re} \int_0^L T_n \boldsymbol{i}\alpha_n D_s' \bar{v}_n dx + \operatorname{Re} \int_0^L T_n \boldsymbol{i}\alpha_n D_s \bar{v}_n' dx + \operatorname{Re} \int_0^L R_n \boldsymbol{i}\alpha_n D_b' \bar{\psi}_n dx \\
&+ \operatorname{Re} \int_0^L R_n \boldsymbol{i}\alpha_n D_b \bar{\psi}_n' dx - \operatorname{Re} \int_0^L T_n \boldsymbol{i}\alpha_n D_b \bar{\psi}_n dx - \operatorname{Re} \int_0^L \rho h_n \boldsymbol{i}\alpha_n D_s \bar{v}_n dx \\
&- \operatorname{Re} \int_0^L I_\rho l_n \boldsymbol{i}\alpha_n D_b \bar{\psi}_n dx \qquad \text{(integrating by parts)} \\
=& I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}$$

Since

$$\begin{aligned}
I_1 =& \operatorname{Re} \int_0^L K(w_n' - \varphi_n) \boldsymbol{i}\alpha_n D_s' \bar{v}_n dx + \operatorname{Re} \int_0^L D_s(v_n' - \psi_n) \boldsymbol{i}\alpha_n D_s' \bar{v}_n dx \\
=& \operatorname{Re} \int_0^L K(v_n' + f_n' - \psi_n - g_n) D_s' \bar{v}_n dx + \operatorname{Re} \int_0^L D_s(v_n' - \psi_n) \boldsymbol{i}\alpha_n D_s' \bar{v}_n dx \quad \text{(by(3.4))} \\
\leq& \operatorname{Re} \int_0^L K D_s' v_n' \bar{v}_n dx + O(1) + \frac{1}{2\varepsilon} \int_0^L \frac{1}{\rho} |D_s'|^2 D_s |v_n' - \psi_n|^2 dx \\
&+ \frac{\varepsilon}{2} \int_0^L \rho D_s |\alpha_n v_n|^2 dx, \quad \text{(by (3.2) and (3.4))} \\
=& \frac{1}{2} K D_s' |v_n|^2 \Big|_0^L - \frac{1}{2} \int_0^L (K D_s')' |v_n|^2 + O(1) + \frac{\varepsilon}{2} \int_0^L \rho D_s |\alpha_n v_n|^2 dx \\
&\text{(integrating by parts and (3.6))} \\
\leq& O(1) + \frac{\varepsilon}{2} \int_0^L \rho D_s |\alpha_n v_n|^2 dx, \quad \text{(by (3.2), (3.11) and (H4))}
\end{aligned}$$

$$
\begin{aligned}
I_2 &= \mathrm{Re}\int_0^L T_n \boldsymbol{i}\alpha_n D_s(\bar{v}_n' - \bar{\psi}_n)dx + \mathrm{Re}\int_0^L T_n\boldsymbol{i}\alpha_n D_s\bar{\psi}_n dx \\
&= \mathrm{Re}\int_0^L K(w_n' - \varphi_n)\boldsymbol{i}\alpha_n D_s(\bar{v}_n' - \bar{\psi}_n)dx + \mathrm{Re}\int_0^L T_n\boldsymbol{i}\alpha_n D_s\bar{\psi}_n dx \\
&\le \int_0^L K|\alpha_n(w_n' - \varphi_n)|D_s|v_n' - \psi_n|dx + \int_0^L |T_n|CD_b|\alpha_n\psi_n|dx, \quad \text{(by (H5))} \\
&\le \frac{1}{2}\int_0^L K^2\alpha_n^2 D_s|w_n' - \varphi_n|^2dx + \frac{1}{2}\int_0^L D_s|v_n' - \psi_n|^2dx + \frac{1}{2\varepsilon}\int_0^L \frac{1}{I_\rho}C^2D_b|T_n|^2dx \\
&\quad + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx \\
&= O(1) + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx, \quad \text{(by (3.6), (3.7) and (3.10))} \\
I_3 &= \mathrm{Re}\int_0^L EI\varphi_n'\boldsymbol{i}\alpha_n D_b'\bar{\psi}_n dx + \mathrm{Re}\int_0^L D_b\psi_n'\boldsymbol{i}\alpha_n D_b'\bar{\psi}_n dx \\
&= \mathrm{Re}\int_0^L EI(\psi_n' + g_n')D_b'\bar{\psi}_n dx + \mathrm{Re}\int_0^L D_b\psi_n'\boldsymbol{i}\alpha_n D_b'\bar{\psi}_n dx, \quad \text{(by (3.4))} \\
&\le \mathrm{Re}\int_0^L EID_b'\psi_n'\bar{\psi}_n dx + o(1) + \frac{1}{2\varepsilon}\int_0^L \frac{1}{I_\rho}|D_b'|^2D_b|\psi_n'|^2dx \\
&\quad + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx \quad \text{(by (3.2) and (3.4))} \\
&= \frac{1}{2}EID_b'|\psi_n|^2\,\Big|_0^L - \frac{1}{2}\int_0^L (EID_b')'|\psi_n|^2 + O(1) + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx \\
&\quad \text{(integrating by parts and (3.6))} \\
&\le O(1) + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx, \quad \text{(by (3.2), (3.12) and (H4))} \\
I_4 + I_5 &= \mathrm{Re}\int_0^L EI\varphi_n'\boldsymbol{i}\alpha_n D_b\bar{\psi}_n{}' dx - \mathrm{Re}\int_0^L T_n\boldsymbol{i}\alpha_n D_b\bar{\psi}_n dx \\
&\le \mathrm{Re}\int_0^L EI(\psi_n' + g_n')D_b\bar{\psi}_n{}'dx + \frac{1}{2\varepsilon}\int_0^L \frac{1}{I_\rho}D_b|T_n|^2dx + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx \\
&= O(1) + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx, \quad \text{(by (3.4), (3.6) and (3.10))} \\
I_6 + I_7 &\le \frac{1}{2\varepsilon}\int_0^L \rho|h_n|^2D_s dx + \frac{\varepsilon}{2}\int_0^L \rho D_s|\alpha_n v_n|^2dx + \frac{1}{2\varepsilon}\int_0^L I_\rho D_b|h_n|^2dx \\
&\quad + \frac{\varepsilon}{2}\int_0^L I_\rho D_b|\alpha_n\psi_n|^2dx \\
&= o(1) + \frac{\varepsilon}{2}I, \quad \text{(by (3.5))}
\end{aligned}
$$

where ε is some positive constant to be determined below, then we have

$$I \le O(1) + 2\varepsilon I,$$

which will immediately give (3.9) if we take $\varepsilon = \frac{1}{4}$.

Now we return to the proof of Theorem 3.1. By (3.4), (C1) and (3.2),

$$w_n \to 0, \quad \varphi_n \to 0 \quad \text{in} \quad L^2(0, L). \tag{3.13}$$

Taking $q \in C^2([0,L],\mathbb{R})$ with $q(0) = q(L) = 0$, it follows from the inner product of (3.5) with (qT_n, qR_n) in H that

$$\mathrm{Re}\int_0^L \{(\rho \boldsymbol{i}\alpha_n v_n - T_n')q\bar{T}_n + [I_\rho \boldsymbol{i}\alpha_n \psi_n - (R_n' + T_n)]q\bar{R}_n\}dx = o(1), \tag{3.14}$$

where we have used the boundedness of T_n and R_n in $L^2(0,L)$. Since

$$\begin{aligned}
\mathrm{Re}\int_0^L \rho \boldsymbol{i}\alpha_n v_n q\bar{T}_n dx &= \mathrm{Re}\int_0^L \rho \boldsymbol{i}\alpha_n v_n qK(\bar{w_n}' - \bar{\varphi}_n)dx + o(1) \quad \text{(by (3.6) and (3.9))}\\
&= \mathrm{Re}\int_0^L -\rho K q\alpha_n^2 w_n \bar{w_n}' dx + \mathrm{Re}\int_0^L -\rho K q\boldsymbol{i}\alpha_n f_n \bar{w_n}' dx + \mathrm{Re}\int_0^L \rho K q\alpha_n^2 w_n \bar{\varphi}_n dx\\
&\quad + \mathrm{Re}\int_0^L \rho K q\boldsymbol{i}\alpha_n f_n \bar{\varphi}_n dx + o(1), \quad \text{(by (3.4))}\\
&= \frac{1}{2}\int_0^L (\rho K q)'|\alpha_n w_n|^2 dx + \mathrm{Re}\int_0^L (\rho q f_n)'\boldsymbol{i}\alpha_n \bar{w_n} dx + \mathrm{Re}\int_0^L \rho K q\alpha_n^2 w_n \bar{\varphi}_n dx\\
&\quad + \mathrm{Re}\int_0^L \rho K q f_n \boldsymbol{i}\alpha_n \bar{\varphi}_n dx + o(1), \quad \text{(integrating by parts)}\\
&= \frac{1}{2}\int_0^L (\rho K q)'|\alpha_n w_n|^2 + \mathrm{Re}\int_0^L \rho K q\alpha_n^2 w_n \bar{\varphi}_n dx + o(1), \quad \text{(by (3.4) and (3.2))}\\
&\geq \frac{1}{2}\int_0^L (\rho K q)'|\alpha_n w_n|^2 dx - \frac{1}{2}\int_0^L \rho K|q||\alpha_n w_n|^2 dx - \frac{1}{2}\int_0^L \rho K|q||\alpha_n \varphi_n|^2 dx + o(1),\\
\mathrm{Re}\int_0^L I_\rho \boldsymbol{i}\alpha_n \psi_n q\bar{R}_n dx &= \frac{1}{2}\int_0^L (I_\rho EIq)'|\alpha_n \varphi_n|^2 dx + \mathrm{Re}\int_0^L (I_\rho EIqg_n)'\boldsymbol{i}\alpha_n \bar{\varphi}_n dx\\
&= \frac{1}{2}\int_0^L (I_\rho EIq)'|\alpha_n \varphi_n|^2 dx + o(1), \quad \text{(by (3.4))}\\
\mathrm{Re}\int_0^L -qT_n'\bar{T}_n dx &= \frac{1}{2}\int_0^L q'|T_n|^2 dx = \frac{1}{2}\int_0^L q'K^2|w_n' - \varphi_n|^2 dx + o(1), \quad \text{(by (3.6))}\\
\mathrm{Re}\int_0^L -q(R_n' + T_n)\bar{R}_n dx &= \frac{1}{2}\int_0^L q'(EI)^2|\varphi_n'|^2 dx + o(1) - \mathrm{Re}\int_0^L qT_n\bar{R}_n dx\\
&= \frac{1}{2}\int_0^L q'(EI)^2|\varphi_n'|^2 dx - \mathrm{Re}\int_0^L qK(w_n' - \varphi_n)EI\bar{\varphi}_n' dx + o(1), \quad \text{(by (3.6) and (3.7))}\\
&\geq \frac{1}{2}\int_0^L q'(EI)^2|\varphi_n'|^2 dx - \frac{1}{2}\int_0^L |q|K^2|w_n' - \varphi_n|^2 dx - \frac{1}{2}\int_0^L |q|(EI)^2|\varphi_n'|^2 dx\\
&\quad + o(1),
\end{aligned}$$

from (3.14) we obtain

$$\begin{aligned}
&\int_0^L [(\rho K q)' - \rho K|q|]|\alpha_n w_n|^2 dx + \int_0^L [(I_\rho EIq)' - \rho K|q|]|\alpha_n \varphi_n|^2 dx\\
&\qquad + \int_0^L (q' - |q|)K^2|w_n' - \varphi_n|^2 dx + \int_0^L (q' - |q|)(EI)^2|\varphi_n'|^2 dx\\
&\quad \leq o(1).
\end{aligned} \tag{3.15}$$

Let us take the inner product of (3.5) with $([(\rho K q)' - \rho K|q|]w_n, [(I_\rho EIq)' - \rho K|q|]\varphi_n)$ in $L^2(0,L) \times L^2(0,L)$. Then we obtain

$$-\int_0^L [(\rho K q)' - \rho K|q|]|\alpha_n w_n|^2 dx - \int_0^L [(I_\rho EIq)' - \rho K|q|]|\alpha_n \varphi_n|^2 dx$$

$$+\text{Re}\int_0^L -\frac{1}{\rho}[(\rho Kq)' - \rho K|q|]T_n'\bar{w_n}dx + \text{Re}\int_0^L -\frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]R_n'\bar{\varphi_n}dx$$
$$+\text{Re}\int_0^L -\frac{1}{I_\rho}[(\rho Kq)' - \rho K|q|]T_n\bar{\varphi_n}dx = o(1). \tag{3.16}$$

We should point out that

$$\left\{\frac{1}{\rho}[(\rho Kq)' - \rho K|q|]\right\}' \in L^\infty, \quad \left\{\frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]\right\}' \in L^\infty.$$

Since

$$\begin{aligned}
&\text{Re}\int_0^L -\frac{1}{\rho}[(\rho Kq)' - \rho K|q|]T_n'\bar{w_n}dx \\
&= \text{Re}\int_0^L \frac{1}{\rho}[(\rho Kq)' - \rho K|q|]T_n\bar{w_n}'dx + \text{Re}\int_0^L \left\{\frac{1}{\rho}[(\rho Kq)' - \rho K|q|]\right\}' T_n\bar{w_n}dx \\
&= \text{Re}\int_0^L \frac{1}{\rho}[(\rho Kq)' - \rho K|q|]T_n(\bar{w_n}' - \bar{\varphi_n})dx + \text{Re}\int_0^L \frac{1}{\rho}[(\rho Kq)' - \rho K|q|]T_n\bar{\varphi_n}dx \\
&\quad +\text{Re}\int_0^L \left\{\frac{1}{\rho}[(\rho Kq)' - \rho K|q|]\right\}' T_n\bar{w_n}dx \\
&= \text{Re}\int_0^L \frac{1}{\rho}[(\rho Kq)' - \rho K|q|]K|w_n' - \varphi_n|^2 + o(1), \quad \text{(by (3.7) and (3.13))} \\
&\text{Re}\int_0^L -\frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]R_n'\bar{\varphi_n}dx \\
&= \text{Re}\int_0^L \frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]R_n\bar{\varphi_n}'dx + \text{Re}\int_0^L \left\{\frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]\right\}' R_n\bar{\varphi_n}dx \\
&= \text{Re}\int_0^L \frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]EI|\varphi_n'|^2dx + o(1), \quad \text{(by (3.7) and (3.13))} \\
&\text{Re}\int_0^L \frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]T_n\bar{\varphi_n}dx = o(1), \quad \text{(by (3.13))},
\end{aligned}$$

(3.16) can be rewritten as

$$\begin{aligned}
&-\int_0^L [(\rho Kq)' - \rho K|q|]|\alpha_n w_n|^2dx - \int_0^L [(I_\rho EIq)' - \rho K|q|]|\alpha_n\varphi_n|^2dx \\
&\quad + \int_0^L \frac{1}{\rho}[(\rho Kq)' - \rho K|q|]K|w_n' - \varphi_n|^2dx + \int_0^L \frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|]EI|\varphi_n'|dx \\
&= o(1).
\end{aligned} \tag{3.17}$$

From (3.15) and (3.17) we obtain

$$\begin{aligned}
&\int_0^L \left\{\frac{1}{\rho}[(\rho Kq)' - \rho K|q|] + (q' - |q|)K\right\} K|w_n' - \varphi_n|^2dx \\
&\quad + \int_0^L \left\{\frac{1}{I_\rho}[(I_\rho EIq)' - \rho K|q|] + (q' - |q|)EI\right\} EI|\varphi_n'|dx \\
&\le o(1),
\end{aligned}$$

i.e.,

$$\int_0^L \left\{2Kq' + \frac{(\rho K)'}{\rho}q - 2K|q|\right\} K|w_n' - \varphi_n|^2dx$$

$$+\int_0^L \left\{2EIq' + \frac{(I_\rho EI)'}{I_\rho}q - \left(\frac{\rho K}{I_\rho}\right)|q|\right\} EI|\varphi_n'|dx$$
$$\le o(1). \tag{3.18}$$

If

$$D_s \ge C_1 > 0 \text{ and } D_b \ge C_1 > 0 \text{ on } [0, L], \tag{3.19}$$

for some constant C_1, then from (3.7) we have

$$\lim_{n\to\infty}\int_0^L (K|w_n' - \varphi_n|^2 + EI|\varphi_n'|^2)dx = 0,$$

which is in contradiction to (3.8).

If (3.19) does not hold, by (H3)′ and (H5), there exists a subinterval $[x_1, x_2] \subset [0, L]$ s.t. $D_s \ge C_2 > 0$ and $D_b \ge C_2 > 0$, for some positive constant C_2. Therefore by (3.7) we have

$$\int_{x_1}^{x_2} (K|w_n' - \varphi_n|^2 + EI|\varphi_n'|^2)dx = o(1). \tag{3.20}$$

We choose $q = \gamma(e^{\eta x} - 1)$, where η is a positive constant to be determined soon and $\gamma \in C^2[0, L], 0 \le \gamma \le 1$, satisfying

$$\gamma(x) = \begin{cases} 1, & x \in [0, x_1 + \varepsilon_0], \\ 0, & x \in [x_2 - \varepsilon_0, L], \end{cases}$$

for some $\varepsilon_0 \in (0, (x_2 - x_1)/2)$; then $q \ge 0$. From (3.18) and (3.20) we have

$$\int_0^{x_1} \left\{J_1 K|w_n' - \varphi_n|^2 + J_2 EI|\varphi_n'|\right\} dx \le o(1), \tag{3.21}$$

where

$$J_1 = \left(2K\eta + \frac{(\rho K)'}{\rho} - 2K\right)e^{\eta x} - \frac{(\rho K)'}{\rho} + 2K,$$
$$J_2 = \left[2EI\eta + \frac{(I_\rho EI)'}{I_\rho} - \frac{\rho K}{I_\rho} + EI\right]e^{\eta x} - \frac{(I_\rho EI)'}{I_\rho} + \frac{\rho K}{I_\rho} - EI.$$

Now we can choose η large enough such that $J_1 \ge 1$ and $J_2 \ge 1$. Then from (3.21) we obtain that

$$\int_0^{x_1} \left\{K|w_n' - \varphi_n|^2 + EI|\varphi_n'|\right\} dx = o(1). \tag{3.22}$$

If we choose $q = \gamma(1 - e^{\eta(L-x)})$, where η is a positive constant and $\gamma \in C^2[0, L], 0 \le \gamma \le 1$, satisfying

$$\gamma(x) = \begin{cases} 0, & x \in [0, x_1 + \varepsilon_0], \\ 1, & x \in [x_2 - \varepsilon_0, L], \end{cases}$$

then $q \le 0$. Similarly, for η large enough, from (3.18) and (3.20) we can also obtain that

$$\int_{x_2}^{L} \left\{K|w_n' - \varphi_n|^2 + EI|\varphi_n'|\right\} dx = o(1). \tag{3.23}$$

(3.20), (3.22) and (3.23) are contradictory to (3.8). The proof of Theorem 3.1 is completed.

Remark 3.1 For local damping, namely, supp$D_b \subset [0, L)$ and supp$D_s \subset [0, L)$, the hypothesis (H4) holds automatically.

Remark 3.2 For a clamped beam, (H4) can be dropped. In this case, we can prove Theorem 3.1 with the same method as above.

Remark 3.3 The structural condition (H5) can be explained from the physical property of viscoelasticity.

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