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$L^2(\mathbb{R}^n)$ Boundedness for a Class of Multilinear Singular Integral Operators

Guo En HU
Department of Applied Mathematics, University of Information Engineering, P. O. Box 1001-747, *Zhengzhou* 450002, *P. R. China*
E-mail: huguoen@eyou.com

Abstract The $L^2(\mathbb{R}^n)$ boundedness for the multilinear singular integral operators defined by

$$T_A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}}(A(x) - A(y) - \nabla A(y)(x-y))f(y)dy$$

is considered, where Ω is homogeneous of degree zero, integrable on the unit sphere and has vanishing moment of order one, A has derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$. A sufficient condition based on the Fourier transform estimate and implying the $L^2(\mathbb{R}^n)$ boundedness for the multilinear operator T_A is given.

Keywords Multilinear singular integral operator, $\mathrm{BMO}(\mathbb{R}^n)$, Fourier transform estimate
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1 Introduction

We will work on $\mathbb{R}^n$, $n \geq 2$. For a point $x \in \mathbb{R}^n$, we denote by x_j $(1 \leq j \leq n)$ the j-th variable of x. Let Ω be homogeneous of degree zero, integrable on the unit sphere S^{n-1} and satisfy the vanishing condition

$$\int_{S^{n-1}} \Omega(x')x_j dx' = 0, \text{ for each } j \text{ with } 1 \leq j \leq n. \tag{1}$$

Let A be a function on $\mathbb{R}^n$ having derivatives of order one in $\mathrm{BMO}(\mathbb{R}^n)$. Define the multilinear singular integral operator T_A by

$$T_A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}}\big(A(x) - A(y) - \nabla A(y)(x-y)\big)f(y)dy. \tag{2}$$

A well-known result of Cohen [1] states that if $\Omega \in \mathrm{Lip}_1(S^{n-1})$, then T_A is a bounded operator on $L^p(\mathbb{R}^n)$ with bound $C\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for $1 < p < \infty$. Hu [2] proved that $\Omega \in \mathrm{Lip}_1(S^{n-1})$ is a sufficient condition such that T_A maps $L\log L(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$ boundedly. Hofmann [3] improved the result of Cohen and showed that $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$ is a sufficient condition

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such that T_A is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The purpose of this paper is to present a sufficient condition on Ω that implies the $L^2(\mathbb{R}^n)$ boundedness for T_A, and this condition is based on the Fourier transform estimate. We remark that in this paper we are very much motivated by the work of Pérez [4], some ideas are from Hofmann's papers [3], [5] and our previous work [6]. Our main result in this paper can be stated as follows:

Theorem 1 *Let Ω be homogeneous of degree zero, integrable on the unit sphere and satisfy the vanishing condition* (1)*, A have derivities of order one in* $\mathrm{BMO}(\mathbb{R}^n)$*. For $1 \le j \le n$, set*

$$k_0^j(x) = \frac{\Omega(x)x_j}{|x|^{n+1}}\chi_{\{1<|x|\le 2\}}(x),\ k_0(x) = \frac{\Omega(x)}{|x|^{n+1}}\chi_{\{1<|x|\le 2\}}(x).$$

Suppose that there exists a constant $\beta > 3$, such that

$$|\widehat{k_0}(\xi)| + \sum_{j=1}^{n}|\widehat{k_0^j}(\xi)| \le C\min\{1,\, \log^{-\beta}(2+|\xi|)\}. \tag{3}$$

Then the operator T_A defined by (2) *is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}$.*

As an application of Theorem 1, we will have:

Theorem 2 *Let Ω be homogeneous of degree zero, integrable on the unit sphere and satisfy the vanishing condition* (1)*, A have derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^n)$*. Suppose that for some $\beta > 3$,*

$$\sup_{\xi\in S^{n-1}}\int_{S^{n-1}}|\Omega(\theta)|\left(\log\frac{1}{|\theta\cdot\xi|}\right)^{\beta}d\theta < \infty. \tag{4}$$

Then the operator T_A defined by (2) *is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}$.*

Remark 1 The size condition (4) for $\beta \ge 1$ was introduced by Grafakos and Stefanov [7] in order to study the $L^p(\mathbb{R}^n)$ boundedness for the singular integral operator

$$Tf(x) = \int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}f(y)dy.$$

It has been shown in [7] that there exist integrable functions on S^{n-1} which are not in $H^1(S^{n-1})$, but satisfy (4) for all $\beta > 1$.

Remark 2 It is obvious that if $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$, then the Fourier transform estimate (3) holds for any $\beta > 1$.

2 Proof of Theorems

By the estimate of Grafakos and Stefanov [7], we see that if Ω satisfies the size condition (4) for some $\beta > 1$, then the Fourier transform estimate (3) holds for the same β. Thus Theorem 2 can be obtained from Theorem 1 directly, and so it is enough to prove Theorem 1. To do this, we begin with a preliminary lemma:

Lemma 1 [1] *Let A be a function on $\mathbb{R}^n$ with derivatives of order one in $L^q(\mathbb{R}^n)$ for some*

$q > n$. Then

$$|A(x) - A(y)| \le C|x-y|\left(\frac{1}{|I_x^y|}\int_{I_x^y} |\nabla A(z)|^q dz\right)^{1/q},$$

where I_x^y is the cube centered at x with sides parallel to the axes and having side length $2|x-y|$.

Now we choose $\phi \in C_0^\infty(\mathbb{R}^n)$ to be a radial function such that $\operatorname{supp}\phi \subset \{x : 1/4 \le |x| \le 4\}$ and

$$\sum_{l\in\mathbb{Z}} \phi(2^{-l}x) \equiv 1, \text{ for any } |x| \ne 0.$$

Set

$$K(x,\, y) = \frac{\Omega(x-y)}{|x-y|^{n+1}}\big(A(x) - A(y) - \nabla A(y)(x-y)\big),\ K^*(x,\, y) = K(y,\, x)$$

and

$$K_l(x,\, y) = K(x,\, y)\phi\big(2^{-l}(x-y)\big).$$

Let $T_{A;\,l}$ be the integral operator whose kernel is K_l. We then have the following size condition:

Lemma 2 *Let Ω be homogeneous of degree zero and integrable on S^{n-1}, $n < q < \infty$. Then there exists a positive constant $C = C_{n,\,q}$ such that for each $\lambda > 0$ and $R > 0$,*

$$\int_{|x|\le\lambda R}\int_{R\le|x-y|\le 2R} [|K^*(x,\, y)| + |K(x,\, y)|]dydx \le C\|\Omega\|_1(1+\lambda)^{n(2+1/q)}\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}R^n.$$

Proof Let B be the ball centered at the origin and having radius $(1+\lambda)R$. Set

$$\widetilde{A}(z) = A(z) - \sum_{j=1}^n m_B(\partial_j A)z_j,$$

where $\partial_j A = \partial A/(\partial x_j)$, $m_B(\partial_j A)$ is the mean value of $\partial_j A$ on the ball B. It is easy to see that, for any $x,\ y \in \mathbb{R}^n$,

$$\widetilde{A}(x) - \widetilde{A}(y) - \nabla\widetilde{A}(y)(x-y) = A(x) - A(y) - \nabla A(y)(x-y),$$

and

$$\widetilde{A}(x) - \widetilde{A}(y) - \nabla\widetilde{A}(x)(x-y) = A(x) - A(y) - \nabla A(x)(x-y).$$

Note that if $R \le |x-y| \le 2R$ and $|x| \le \lambda R$, $I_x^y \subset 10nB$. Recall that $n < q < \infty$. By Lemma 1, we know that for $x,\ y \in \mathbb{R}^n$ such that $R \le |x-y| \le 2R$,

$$\begin{aligned}|\widetilde{A}(x) - \widetilde{A}(y)| &\le C|x-y|\sum_{j=1}^n\left(\frac{1}{|x-y|^n}\int_{I_x^y}|\partial_j A(z) - m_B(\partial_j A)|^q dz\right)^{1/q}\\ &\le CR(1+\lambda)^{n/q}\sum_{j=1}^n\left(\frac{1}{|10nB|}\int_{10nB}|\partial_j A(z) - m_B(\partial_j A)|^q dz\right)^{1/q}\\ &\le C(1+\lambda)^{n/q}R.\end{aligned}$$

Let $\widetilde{\Omega}(x)=\Omega(x)\chi_{\{R\le|x|\le 2R\}}$. By the Young inequality we then have

$$\begin{aligned}&\int_{|x|\le\lambda R}\int_{R\le|x-y|\le 2R}[|K^*(x,y)|+|K(x,y)|]dydx\\&\quad\le CR^{-n}(1+\lambda)^{n/q}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\widetilde{\Omega}(x-y)|\Big(1+\sum_{j=1}^n\big|\partial_jA(y)-m_B(\partial_jA)\big|\Big)\chi_{\{|y|\le(\lambda+2)R\}}dydx\\&\qquad+C\|\Omega\|_1(1+\lambda)^{n/q}\int_{|x|\le\lambda R}\Big(1+\sum_{j=1}^n\big|\partial_jA(x)-m_B(\partial_jA)\big|\Big)dx\\&\quad\le C\|\Omega\|_1(1+\lambda)^{n(2+1/q)}R^n.\end{aligned}$$

Lemma 3 *Let Ω be integrable on S^{n-1} and satisfy the vanishing condition* (1)*, A have derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^n)$*. Then the operator T_A defined by* (2) *satisfies the weak boundedness property, that is, if η_1 and η_2 are $C_0^\infty(\mathbb{R}^n)$ functions whose supports are contained in a ball of radius r, then*

$$\Big|\int_{\mathbb{R}^n}\eta_1(x)T_A\eta_2(x)dx\Big|\le C\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\Omega\|_1r^{-n}\big(\|\eta_1\|_\infty+r\|\nabla\eta_1\|_\infty\big)\big(\|\eta_2\|_\infty+r\|\nabla\eta_2\|_\infty\big).$$

This is Lemma 4.3 in [3].

Lemma 4 *Let Ω be integrable on S^{n-1} and satisfy the vanishing condition* (3)*, A have derivatives of order one in* $\mathrm{BMO}(\mathbb{R}^n)$*. Then the operator T_A defined by* (2) *satisfies $T_A1\in$* $\mathrm{BMO}(\mathbb{R}^n)$ *and*

$$\|Q_sT_{A;\,l}1\|_\infty\le C\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\Omega\|_1(2^{-l}s)^\varepsilon,$$

for $s\le 2^l$ and some $\varepsilon>0$.

This is Lemmata 4.1 and 4.2 in [3].

Lemma 5 *Let Ω be the same as in Theorem* 1*, $\psi\in C_0^\infty(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}\psi\subset\{x:|x|\le 1\}$*, and*

$$\int_{\mathbb{R}^n}\psi(x)dx=0,\quad\int_0^\infty|\widehat{\psi}(s)|^2\frac{ds}{s}=1.$$

For each $s>0$, set $\psi_s(x)=s^{-n}\psi(s^{-1}x)$ and Q_s to be the convolution operator with kernel ψ_s. Then for each $l\in\mathbb{Z}$ and $s\le 2^l$,

$$\|Q_sT_{A;\,l}f\|_2\le C\log^{-\beta+1}(2^ls^{-1}+1)\|\Omega\|_1\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.$$

Proof By scale invariance, it suffices to consider the case $l=0$, and without loss of generality, we may assume that $\|\Omega\|_1=\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)}=1$. As in the proof of Lemma 2.2 in [3], by a standard localization argument, we may assume $\operatorname{supp}f\subset Q$ for some cube Q having side length 1. Let $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that $0\le\varphi\le 1$, φ is indentically one on $50Q$ and vanishes outside $100Q$, $\|\nabla\varphi\|_\infty\le C$. Let x_0 be a point on the boundary of $200Q$. Set $\widetilde{A}(y)=A(y)-\sum_{j=1}^n m_Q(\partial_jA)y_j$ and

$$A_\varphi(y)=\big(\widetilde{A}(y)-\widetilde{A}(x_0)\big)\varphi(y).$$

Applying Lemma 1 again, we have that, for $y \in 20Q$,

$$|A_\varphi(y)| \le |\widetilde{A}(y) - \widetilde{A}(x_0)| \le C. \tag{5}$$

For each fixed integer j with $1 \le j \le n$, set

$$k_j(x) = \phi(x)\frac{\Omega(x)x_j}{|x|^{n+1}}, \quad k(x) = \phi(x)\frac{\Omega(x)}{|x|^{n+1}}.$$

Denote by S_j (resp. S) the convolution operator whose kernel is k_j (resp. k). The Fourier transform estimate (3) together with the trivial estimate $|\widehat{\psi}(s\xi)| \le C\min\{1, |s\xi|\}$ says that

$$\sum_{j=1}^{n} |\widehat{\psi}(s\xi)\widehat{k}_j(\xi)| + |\widehat{\psi}(s\xi)\widehat{k}(\xi)| \le C\log^{-\beta}(s^{-1}+1).$$

This, via the Plancherel theorem, tells us that

$$\|Q_sSh\|_2 + \sum_{j=1}^{n}\|Q_sS_jh\|_2 \le C\log^{-\beta}(s^{-1}+1)\|h\|_2. \tag{6}$$

Note that, for $y \in \mathbb{R}^n$,

$$T_{A;\,0}f(y) = T_{A_\varphi;\,0}f(y).$$

Write

$$\begin{aligned} Q_sT_{A_\varphi;\,0}f(x) &= Q_s(A_\varphi Sf)(x) - Q_sS(A_\varphi f)(x) - \sum_{j=1}^{n} Q_sS_j(f\partial_jA_\varphi)(x) \\ &= \mathrm{I}(x) + \mathrm{II}(x) + \mathrm{III}(x). \end{aligned}$$

Combining estimates (5) and (6) leads to

$$\|\mathrm{II}\|_2 \le C\log^{-\beta}(s^{-1}+1)\|A_\varphi f\|_2 \le C\log^{-\beta}(s^{-1}+1)\|f\|_2.$$

Decompose the term I as

$$\mathrm{I}(x) = A_\varphi(x)Q_sSf(x) - \int_{\mathbb{R}^n}\big(A_\varphi(x) - A_\varphi(y)\big)\psi_s(x-y)(Sf)(y)dy = \mathrm{I}_1(x) + \mathrm{I}_2(x).$$

Obviously,

$$\|\mathrm{I}_1\|_2 \le C\log^{-\beta}(s^{-1}+1)\|f\|_2.$$

A familar argument involving Lemma 1 shows that, for $z \in 40Q$,

$$\begin{aligned} |\nabla A_\varphi(z)| \le |\nabla\widetilde{A}(z)| + C|\widetilde{A}(z) - \widetilde{A}(x_0)| &\le \sum_{j=1}^{n} |\partial_jA(z) - m_Q(\partial_jA)| \qquad (7) \\ &\quad + C\sum_{j=1}^{n}|z-x_0|\bigg(\frac{1}{|I_z^{x_0}|}\int_{I_z^{x_0}}|\partial_jA(w) - m_Q(\partial_jA)|^q dw\bigg)^{1/q} \\ &\le C\bigg(\sum_{j=1}^{n}|\partial_jA(z) - m_Q(\partial_jA)| + 1\bigg). \end{aligned}$$

If $y \in 10Q$ and $|x-y| \le s < 1$, then $x \in 12Q$ and $I_x^y \subset 40Q$. Choosing $q > n$, by Lemma 1 we have that, for $y \in 10Q$ and $|x-y| \le s$,

$$
\begin{aligned}
|A_\varphi(x)-A_\varphi(y)| &\le C|x-y|\bigg(\frac{1}{|x-y|^n}\int_{I_y^x}|\nabla A_\varphi(z)|^q dz\bigg)^{1/q}\\
&\le C|x-y|^{1-n/q}\sum_{j=1}^n\bigg(\int_{40Q}|\partial_j A(z)-m_Q(\partial_j A)|^q dz\bigg)^{1/q}\\
&\le C|x-y|^{1-n/q}.
\end{aligned}
$$

Recall that $\operatorname{supp} f \subset Q$. Thus $\operatorname{supp} Sf \subset 10Q$ and so

$$|\mathrm{I}_2(x)| \le Cs^{1-n/q}\int_{\mathbb{R}^n}|\psi_s(x-y)||Sf(y)|dy.$$

This in turn implies that

$$\|\mathrm{I}_2\|_2 \le Cs^{1-n/q}\|f\|_2.$$

To estimate the term III, we will use a basic estimate for $(Q_sS_j)^*$, the adjoint operator of Q_sS_j, that is,

$$(Q_sS_j)^*h(x) = \int_{\mathbb{R}^n}\psi_s * k_j(y-x)h(y)dy.$$

We claim that there exists a positive constant C_n such that, for $\operatorname{supp} h \subset 10Q$ and $b \in \mathrm{BMO}\mathbb{R}^n$,

$$\int_Q |b(x)-m_Q(b)|^2|(Q_sS_j)^*h(x)|^2dx \le C\log^{2(-\beta+1)}(s^{-1}+1)\|b\|^2_{\mathrm{BMO}(\mathbb{R}^n)}\|h\|_2^2. \tag{8}$$

In fact, without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|h\|_2 = 1$. Note that $\Phi(t) = t\log^2(2+t)$ is a Young function and its complementary Young function is $\Psi(t) \approx \exp t^{1/2}$. By the general Hölder inequality, it follows that

$$\int_Q |b(x)-m_Q(b)|^2|(Q_sS_j)^*h(x)|^2dx \le C\||b-m_Q(b)|^2\|_{(\exp L)^{1/2},Q}\|((Q_sS_j)^*h)^2\|_{L(\log L)^2,Q},$$

where

$$\||b-m_Q(b)|^2\|_{(\exp L)^{1/2},Q} = \inf\bigg\{\lambda>0 : \int_Q \exp\bigg(\frac{|b(y)-m_Q(b)|}{\lambda^{1/2}}\bigg)dy \le 2\bigg\}$$

and

$$\|((Q_sS_j)^*h)^2\|_{L(\log L)^2,Q} = \inf\bigg\{\lambda>0 : \int_Q \frac{|(Q_sS_j)^*h(y)|^2}{\lambda}\log^2\bigg(2+\frac{|(Q_sS_j)^*h(y)|^2}{\lambda}\bigg)dy \le 1\bigg\},$$

see [4, p. 168]. The well-known John-Nirenberg inequality now states that

$$\||b-m_Q(b)|^2\|_{(\exp L)^{1/2},Q} \le C.$$

On the other hand, note that $\operatorname{supp} h \subset 10Q$ and by the Young inequality,

$$\|(Q_sS_j)^*h\|_\infty \le C\|\Omega\|_1\|\psi_s\|_\infty\|h\|_1 \le C_ns^{-n}.$$

Set $\lambda_0 = C_n\log^{2(-\beta+1)}(s^{-1}+1)$. A straightforward computation gives us that

$$\int_Q |(Q_sS_j)^*h(y)|^2\log^2\bigg(2+\frac{|(Q_sS_j)^*h(y)|^2}{\lambda_0}\bigg)dy \le C\log^2\bigg(2+\frac{C_ns^{-2n}}{\lambda_0}\bigg)\|(Q_sS_j)^*h\|_2^2$$

$$\leq C\log^{-2\beta}(s^{-1}+1)\log^2(2+s^{-n+1})$$
$$\leq C\log^{2(-\beta+1)}(s^{-1}+1),$$

which means that

$$\|((Q_sS_j)^*h)^2\|_{L(\log L)^2,\,Q} \leq C\log^{2(-\beta+1)}(s^{-1}+1),$$

and then we establish inequality (8).

The estimate for III follows from inequality (8) directly. In fact, for each $1 \leq j \leq n$, by the standard duality argument, we have

$$\begin{aligned}\|Q_sS_j(\partial_jA_\varphi f)\|_2 &= \sup_{\operatorname{supp} h\subset 20Q,\, \|h\|_2\leq 1}\left|\int_{20Q} Q_sS_j(\partial_jA_\varphi f)(x)h(x)dx\right|\\ &= \sup_{\operatorname{supp} h\subset 20Q,\, \|h\|_2\leq 1}\left|\int_{20Q} f(y)\partial_jA_\varphi(y)(Q_sS_j)^*h(y)dy\right|\\ &\leq \sup_{\operatorname{supp} h\subset 20Q,\, \|h\|_2\leq 1}\|f\|_2\|((Q_sS_j)^*h)\partial_jA_\varphi\|_2\\ &\leq C\log^{-\beta+1}(s^{-1}+1)\|f\|_2,\end{aligned}$$

where in the last inequality, we have invoked the fact that $\operatorname{supp}(Q_sS_j)^*h \subset 50Q$ and estimate (7). Summing over all $1 \leq j \leq n$ gives the desired estimate for III, and then concludes the proof of Lemma 5.

Proof of Theorem 1 Also, we may assume that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. By the Littlewood-Paley theory, it suffices to prove that, for $f,\, g \in C_0^\infty(\mathbb{R}^n)$,

$$\left|\int_0^\infty\int_0^t\int_{\mathbb{R}^n} Q_s^2T_AQ_t^2f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\right| \leq C\|f\|_2\|g\|_2 \tag{9}$$

and

$$\left|\int_0^\infty\int_t^\infty\int_{\mathbb{R}^n} Q_s^2T_AQ_t^2f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\right| \leq C\|f\|_2\|g\|_2. \tag{10}$$

By Lemma 2–Lemma 5, the same argument as in the proof of Theorem 3.1 in [3] shows that the estimate (9) holds. Thus the proof of Theorem 1 can be reduced to proving (10). Note that by a standard duality argument, the inequality (10) is equivalent to the estimate

$$\left|\int_0^\infty\int_0^t\int_{\mathbb{R}^n} Q_s^2T_A^*Q_t^2f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\right| \leq C\|f\|_2\|g\|_2. \tag{11}$$

Define the operator $\widetilde{T_A}$ by

$$\widetilde{T_A}f(x) = \int_{\mathbb{R}^n}\frac{\Omega(y-x)}{|x-y|^{n+1}}\big(A(x)-A(y)-\nabla A(y)(x-y)\big)f(y)dy$$

and the operator W_j by

$$W_jf(x) = \int_{\mathbb{R}^n}[\partial_jA(x)-\partial_jA(y)]\frac{\Omega(x-y)(x_j-y_j)}{|x-y|^{n+1}}f(y)dy,$$

for integer j with $1 \leq j \leq n$. Write

$$T_A^*f(x) = \widetilde{T_A}f(x) + \sum_{j=1}^n W_jf(x).$$

Note that $\widetilde{\Omega}(x) = \Omega(-x)$ is also integrable on S^{n-1} and enjoys the same Fourier transform estimate (3). So by the same argument as that was used in the proof of the inequality, we have

$$\left|\int_0^\infty \int_0^t \int_{\mathbb{R}^n} Q_s^2 \widetilde{T_A} Q_t^2 f(x) g(x) dx \frac{ds}{s}\frac{dt}{t}\right| \le C\|f\|_2\|g\|_2.$$

The observation of Han and Sawyer [8] says that the kernel of the convolution operator

$$P_s = \int_s^\infty Q_t^2 \frac{dt}{t}$$

is a radial bounded function with bound Cs^{-n}, supported on a ball of radius Cs and having integral zero. It follows from the Littlewood-Paley theory that

$$\int_0^\infty \|P_s h\|_2^2 \frac{ds}{s} \le C\|h\|_2^2.$$

On the other hand, note that, for $1 \le j \le n$, $\Omega(x)x_j/|x|^{n+1}$ is homogeneous of degree zero and has mean value zero. Theorem 1 of [6] tells us that for Ω satisfing the Fourier transform estimate (3) for $\beta > 2$, then the commutator W_j is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|\partial_j A\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Appling the Schwarz inequality twice, we finally obtain that, for each j with $1 \le j \le n$,

$$\begin{aligned}\left|\int_0^\infty \int_0^t \int_{\mathbb{R}^n} Q_s^2 W_j Q_t^2 f(x) g(x) dx \frac{ds}{s}\frac{dt}{t}\right| &= \left|\int_0^\infty \int_s^\infty \int_{\mathbb{R}^n} W_j^* Q_s^2 g(x) Q_t^2 f(x) dx \frac{dt}{t}\frac{ds}{s}\right| \\ &\le \left(\int_0^\infty \|W_j^* Q_s^2 g\|_2^2 \frac{ds}{s}\right)^{1/2}\left(\int_0^\infty \|P_s f\|_2^2 \frac{ds}{s}\right)^{1/2} \le C\|\partial_j A\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2\|g\|_2,\end{aligned}$$

where W_j^* is the adjoint operator of W_j. This shows that inequality (10) is a easy consequence of inequality (9) and then concludes the proof of Theorem 1.

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