Math. Program., Ser. A 92: 301–314 (2002)

Digital Object Identifier (DOI) 10.1007/s101070100278

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On error bounds for lower semicontinuous functions

Received: April 27, 2001 / Accepted: November 6, 2001
Published online April 12, 2002 – © Springer-Verlag 2002

Abstract. We give some sufficient conditions for proper lower semicontinuous functions on metric spaces to have error bounds (with exponents). For a proper convex function f on a normed space X the existence of a local error bound implies that of a global error bound. If in addition X is a Banach space, then error bounds can be characterized by the subdifferential of f. In a reflexive Banach space X, we further obtain several sufficient and necessary conditions for the existence of error bounds in terms of the lower Dini derivative of f.

Key words. local error bound – global error bound – subdifferential – lower Dini derivative

1. Introduction

Let (X, d) be a metric space, $f : X \to (-\infty, +\infty]$ a proper function (that is, *dom* $f :=$ $\{x \in X : f(x) < +\infty\}$ is nonempty) and $S := \{x \in X : f(x) \leq 0\}$. We say that f has a *local* (*global*) *error bound* if for some $0 < \epsilon < +\infty$ ($\epsilon = +\infty$) there exists $\mu > 0$ such that

$$d_S(x) \leq \mu f(x)_+ \quad \forall x \in X \text{ with } f(x) < \epsilon$$

where $f(x)_+ = \max\{f(x), 0\}$, and $d_S(x) := \inf\{d(x, s) : s \in S\}$ if S is nonempty and $d_S(x) = +\infty$ if S is empty.

Error bounds have important applications in sensitivity analysis of mathematical programming and in convergence analysis of some algorithms. In recent years, the study of error bounds has received a lot of attention in the mathematical programming literature. The reader is referred to the survey paper [5] for the relevant work and the references. However, most previous error bound results assume continuous or convex functions. Recently Ng and Zheng [6–8] and Wu and Ye [11] have made progress on the study of error bounds for discontinuous functions on general spaces. The purpose of this paper is to further extend several results in [6,7,11].

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Mathematics Subject Classification (2000): 49J52, 90C26, 90C31

2. Error bounds for nonconvex functions

For nonconvex functions Ng and Zheng [7] have obtained several interesting results about the existence of global error bounds. For a proper function on a metric space Ng and Zheng [7, Lemma 2.2] have presented a sufficient condition on the existence of a global error bound. Applying their result to the function f_ϵ defined by

$$f_\epsilon(x) = f(x) + \psi_{S_\epsilon}(x) \quad \forall x \in X$$

where ψ_{S_ϵ} is the indicator function of the set $S_\epsilon := \{x \in X : f(x) < \epsilon\}$, we can easily derive the following corresponding sufficient condition for error bounds.

Theorem 1. *Let (X, d) be a metric space and $f : X \to (-\infty, +\infty]$ a proper function. Suppose that the set S is nonempty. If for some constants $\mu > 0, 0 \le \rho < 1$ and $0 < \epsilon \le +\infty$ and for each $x \in f^{-1}(0, \epsilon) := \{y \in X : 0 < f(y) < \epsilon\}$ there exists $x' \in f^{-1}[0, \epsilon)$ such that*

$$d_S(x') \le \rho d_S(x) \tag{1}$$

and

$$d(x, x') \le \mu[f(x) - f(x')], \tag{2}$$

then

$$d_S(x) \le \mu f(x)_+ \quad \forall x \in f^{-1}(-\infty, \epsilon).$$

Note that the assumptions in Theorem 1 are very weak in that the space X is only required to be a metric space and f is only a proper function. If the space X is a normed space and the function f is lower semicontinuous, then one can replace condition (2) with an inequality involving the *lower Dini derivative* of f at $x \in dom\, f$ in the direction $v \in X$ given by

$$f^-(x; v) := \liminf_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}.$$

We need the following result of the mean-valued theorem given by Ng and Zheng [7, Lemma 2.1].

Lemma 1 ([7, Lemma 2.1]). *Let X be a normed space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function; let $x \in dom\ f$, $h \in X$ with $\|h\| = 1$ and $t > 0$. Assume that there exists $\delta \in R$ such that for each $\alpha \in [0, t)$, $f^-(x + \alpha h; h) \le \delta$. Then*

$$f(x + th) - f(x) \le t\delta.$$

As Ng and Zheng showed in [7, Theorem 2.4] that inequality (2) in Theorem 1 can be replaced with a condition in terms of the lower Dini derivative for the case $\epsilon = +\infty$, we use Lemma 1 to prove the corresponding result for the case $0 < \epsilon < +\infty$.

Theorem 2. *Let X be a normed space and f a proper lower semicontinuous function on X; let $0 < \epsilon \le +\infty$, $0 < \mu < +\infty$ and $0 \le \rho < 1$. Suppose that S is nonempty and that for each $x \in f^{-1}(0, \epsilon) = \{y \in X : 0 < f(y) < \epsilon\}$ there exist $t_x > 0$ and $h_x \in X$ with $\|h_x\| = 1$ such that*

$$d_S(x + t_x h_x) \le \rho d_S(x) \text{ and } f^-(x + th_x; h_x) \le -\mu^{-1} \quad \forall t \in [0, t_x).$$

Then

$$d_S(x) \le \mu[f(x)]_+ \quad \forall x \in f^{-1}(-\infty, \epsilon).$$

Proof. The result for the case $\epsilon = +\infty$ has been given in [7, Theorem 2.4], we only need to consider the case $0 < \epsilon < +\infty$.

For any $n > 1$ with $n \in N$, let

$$F_n(x) = f(x) + \psi_{S_n}(x)$$

where $S_n = \{x \in X : f(x) \le (1 - \frac{1}{n})\epsilon\}$ and ψ_{S_n} is the indicator function of S_n. Then for $x \in F_n^{-1}(0, +\infty)$, by the assumption, there exist $t_x > 0$ and $h_x \in X$ with $\|h_x\| = 1$ such that

$$f^-(x + th_x; h_x) \le -\mu^{-1} \quad \forall t \in [0, t_x).$$

By Lemma 1, the above inequality implies that

$$f(x + th_x) - f(x) \le t(-\mu^{-1}) \quad \forall t \in (0, t_x),$$

that is,

$$f(x + th_x) \le f(x) - t\mu^{-1} < \left(1 - \frac{1}{n}\right)\epsilon \quad \forall t \in (0, t_x).$$

Consequently

$$F_n^-(x + th_x; h_x) = f^-(x + th_x; h_x) \quad \forall t \in [0, t_x).$$

Therefore applying the result of this theorem for the case $\epsilon = +\infty$ to the lower semicontinuous function F_n, one has

$$d_S(x) \le \mu F_n(x)_+ \quad \forall x \in F_n^{-1}(-\infty, +\infty).$$

Since $1 < n \in N$ is arbitrary, the desired result is proven.

□

Another direction for simplifying the conditions in Theorem 1 is to assume that X is a complete metric space on which the well-known Ekeland variational principle holds. Indeed using an equivalent form of the Ekeland variational principle by Hamel [4], Ng and Zheng [7, Lemma 2.3] showed that in a complete metric space X condition (1) in Theorem 1 can be omited for the case $\epsilon = +\infty$. In fact we can further show that in a complete metric space the nonemptiness of S has already been implied by the other conditions in Theorem 1, that is, the nonemptiness of S comes as a conclusion instead of an assumption thanks to the existence theorem given by Takahashi [10]. To prove the above claim, we summarize the results on the existence of minima and the equivalent statement of the Ekeland variational principle due to Takahashi and Hamel as follows.

Proposition 1. *Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty]$ a proper lower semicontinuous function bounded from below. Denote*

$$\gamma := \inf\{f(x) : x \in X\} \text{ and } Z := \{z \in X : f(z) = \gamma\}.$$

If for some $\mu > 0$ and each $x \in X$ with $\gamma < f(x)$ there exists $x' \in X$ such that

$$0 < d(x, x') \leq \mu[f(x) - f(x')],$$

then

(*i*) *the set Z is nonempty* ([10, Theorem 1]), *and*
(*ii*) $d_Z(x) \leq \mu[f(x) - f(z)] \quad \forall x \in X, z \in Z$ ([4, Theorem 2]).

Theorem 3. *Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function. Suppose that for some $\mu > 0$ and $0 < \epsilon \leq +\infty$ the set $f^{-1}(-\infty, \epsilon)$ is nonempty and for each $x \in f^{-1}(0, \epsilon)$ there exists a point $x' \in f^{-1}[0, \epsilon)$ such that*

$$0 < d(x, x') \leq \mu[f(x) - f(x')].$$

Then S is nonempty and

$$d_S(x) \leq \mu f(x)_+ \quad \forall x \in f^{-1}(-\infty, \epsilon).$$

Proof. Let $\mu > 0$ and $0 < \epsilon \leq +\infty$ satisfy the given condition. Then $f(\cdot)_+$ is a lower semicontinuous function bounded from below with $S = \{x \in X : f(x)_+ = 0\}$ and $\gamma := \inf\{f(x)_+ : x \in X\} \geq 0$.

For the case $\epsilon = +\infty$, by Proposition 1, the set $Z = \{z \in X : f(z)_+ = \gamma\}$ is nonempty. To show that S is nonempty, it suffices to prove $S = Z$, that is, $\gamma = 0$. This must be true. Otherwise if γ were greater than 0 then for any $z \in Z$ we have $f(z) > 0$. Hence by the assumption there exists $z' \in f^{-1}[0, +\infty)$ such that

$$0 < d(z, z') \leq \mu[f(z) - f(z')],$$

from which it follows that $f(z')_+ < f(z)_+ = \gamma$, contradicting the definition of γ.

Next we consider the case $0 < \epsilon < +\infty$. For each $m > 1$ with $m \in N$ such that the set

$$S_m := \left\{x \in X : f(x) \leq \left(1 - \frac{1}{m}\right)\epsilon\right\}$$

is nonempty we define $F_m : X \to (-\infty, +\infty]$ by

$$F_m(\cdot) = (f + \psi_{S_m})(\cdot)$$

where ψ_C is the indicator function of C. Then $S = \{x \in X : F_m(x) \leq 0\}$ and F_m is proper lower semicontinuous since f is lower semicontinuous and the set S_m is closed. Besides for $x \in F_m^{-1}(0, +\infty)$, that is, $x \in f^{-1}(0, (1 - \frac{1}{m})\epsilon]$, by the assumption, there exists $x' \in f^{-1}[0, \epsilon)$ such that

$$0 < d(x, x') \leq \mu[f(x) - f(x')],$$

which implies that $x' \in f^{-1}[0, (1 - \frac{1}{m})\epsilon]$. Such an x' satisfies $x' \in F_m^{-1}[0, +\infty)$ and

$$0 < d(x, x') \leq \mu[F_m(x) - F_m(x')].$$

Therefore by the conclusion for the case $\epsilon = +\infty$ the set S is nonempty and

$$\begin{aligned} d_S(x) &\leq \mu[F_m(x)_+] \quad \forall x \in F_m^{-1}(-\infty, +\infty) \\ &= \mu f(x)_+ \quad \forall x \in S_m, \end{aligned}$$

from which we obtain

$$d_S(x) \leq \mu f(x)_+ \quad \forall x \in f^{-1}(-\infty, \epsilon)$$

since $f^{-1}(-\infty, \epsilon) = \cup_{m=2}^{+\infty} S_m$.

□

As a result of Theorem 3, the conditions of Theorem 2 in a Banach space version can be greatly simplified as in [7, Theorem 2.5].

Theorem 4. *Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function. Let $0 < \epsilon \leq +\infty$ and $0 < \mu < +\infty$. Suppose that the set $f^{-1}(-\infty, \epsilon)$ is nonempty and for each $x \in f^{-1}(0, \epsilon)$ there exists $h_x \in X$ with $\|h_x\| = 1$ such that $f^-(x; h_x) \leq -\mu^{-1}$. Then S is nonempty and*

$$d_S(x) \leq \mu[f(x)]_+ \quad \forall x \in f^{-1}(-\infty, \epsilon).$$

Proof. By Theorem 3, it suffices to show that for any $\lambda > 1$ and $x \in f^{-1}(0, \epsilon)$ there exists a point $y \in f^{-1}[0, \epsilon)$ such that

$$0 < \|x - y\| \leq \lambda\mu[f(x) - f(y)].$$

Let $\lambda > 1$ be fixed. For each $x \in f^{-1}(0, \epsilon)$, by assumption, there exists $h_x \in X$ with $\|h_x\| = 1$ such that $f^-(x; h_x) \leq -\mu^{-1}$. Let $x \in f^{-1}(0, \epsilon)$ be fixed and $t_n \to 0^+$ be such that

$$\lim_{n \to +\infty} \frac{f(x + t_n h_x) - f(x)}{t_n} = f^-(x; h_x).$$

Then for sufficiently large n we have

$$\frac{f(x + t_n h_x) - f(x)}{t_n} < -\frac{1}{\lambda\mu}.$$

From this and the lower semicontinuity of f it follows that

$$0 < \frac{1}{2} f(x) < f(x + t_n h_x) < f(x) - \frac{t_n}{\lambda\mu} < \epsilon$$

for sufficiently large n. For any such n, taking $y = x + t_n h_x$, we have

$$0 < t_n = \|x - y\| \leq \lambda\mu[f(x) - f(y)].$$

This completes the proof.

□

3. Error bounds with exponents

For a proper extended-valued function f on a metric space X, we say that f has a *local* (*global*) *error bound* with exponent $\beta > 0$ if for some $0 < \epsilon < +\infty$ ($\epsilon = +\infty$) there exists $\mu > 0$ such that

$$d_S(x) \le \mu[f(x)_+]^\beta \quad \forall x \in X \text{ with } f(x) < \epsilon.$$

Simply replacing f by the function $[f_+]^\beta$ in Theorems 1 and 3, we obtain the following sufficient conditions for error bounds with exponents.

Theorem 5. *Let (X, d) be a metric space and $f : X \to (-\infty, +\infty]$ be a proper function. Suppose that the set S is nonempty. If for some constants $\mu > 0, 0 \le \rho < 1$ and $0 < \epsilon \le +\infty$ and for each $x \in f^{-1}(0, \epsilon)$ there exists $x' \in f^{-1}[0, \epsilon)$ such that*

$$d_S(x') \le \rho d_S(x) \tag{3}$$

and

$$0 < d(x, x') \le \mu[f^\beta(x) - f^\beta(x')], \tag{4}$$

then

$$d_S(x) \le \mu[f(x)_+]^\beta \quad \forall x \in f^{-1}(-\infty, \epsilon). \tag{5}$$

With the stronger condition that X is complete and f is a proper lower semicontinuous function satisfying only (4), *then S is automatically nonempty and the conclusion* (5) *holds.*

Note that Theorem 5 extends [6, Theorem 1] from a weakly lower semicontinuous function f on a reflexive Banach space X to a proper lower semicontinuous function on a Banach space.

In order to derive the corresponding sufficient conditions for error bounds with exponents in terms of the lower Dini derivative of function f, we first give a chain rule for the lower Dini derivative of f^β.

Lemma 2. *Let X be a normed space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function; let $0 < \mu < +\infty$, $0 < \beta < +\infty$ and $0 < \epsilon \le +\infty$ be constants. Then for $x \in f^{-1}(0, \epsilon)$ and $h_x \in X$ with $\|h_x\| = 1$ the following are equivalent:*

(i) $f^-(x; h_x) \le -\mu^{-1} f^{1-\beta}(x)$.
(ii) $(f^\beta)^-(x; h_x) \le -\mu^{-1}\beta$.

Moreover if (i) *or* (ii) *holds then*

$$(f^\beta)^-(x; h_x) = \beta f^{\beta-1}(x) f^-(x; h_x).$$

Proof. Let $x \in f^{-1}(0, \epsilon)$ and $h_x \in X$ with $\|h_x\| = 1$. Suppose that inequality (i) holds. Then there exists a sequence $\{t_n\}$ such that $t_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{f(x + t_n h_x) - f(x)}{t_n} = f^-(x; h_x) \le -\mu^{-1} f^{1-\beta}(x).$$

Since $0 < f(x) < \epsilon$, for sufficiently large n, we have

$$\frac{f(x + t_n h_x) - f(x)}{t_n} < -(2\mu)^{-1} f^{1-\beta}(x),$$

which with the lower semicontinuity of f implies that the following properties hold for f:

(1) $\lim_{n\to+\infty}[f(x + t_n h_x) - f(x)] = 0$.
(2) $f(x + t_n h_x) - f(x) < 0$ for sufficiently large n.
(3) $f^\beta(x + t_n h_x) - f^\beta(x) = \beta f^{\beta-1}(x)[f(x + t_n h_x) - f(x)] + o(f(x + t_n h_x) - f(x))$ as $n \to +\infty$.

Therefore

$$\frac{f^\beta(x + t_n h_x) - f^\beta(x)}{t_n} = \frac{f(x + t_n h_x) - f(x)}{t_n} \left[\beta f^{\beta-1}(x) + \frac{o(f(x + t_n h_x) - f(x))}{f(x + t_n h_x) - f(x)}\right],$$

from which it follows that

$$(f^\beta)^-(x; h_x) \le \lim_{n\to+\infty} \frac{f^\beta(x + t_n h_x) - f^\beta(x)}{t_n} = \beta f^{\beta-1}(x) f^-(x; h_x) \le -\frac{\beta}{\mu}.$$

This proves the implication $(i) \Rightarrow (ii)$.

Conversely let inequality (ii) be true and $t_n \to 0^+$ be such that

$$\lim_{n\to+\infty} \frac{(f^\beta)(x + t_n h_x) - (f^\beta)(x)}{t_n} = (f^\beta)^-(x; h_x) \le -\frac{\beta}{\mu}.$$

From this inequality and the lower semicontinuity of f with $0 < f(x) < \epsilon$ we see again that f still satisfies the above properties (1)–(3). Hence

$$\begin{aligned} f^-(x; h_x) &\le \lim_{n\to+\infty} \frac{f(x + t_n h_x) - f(x)}{t_n} \\ &= \beta^{-1} f^{1-\beta}(x)(f^\beta)^-(x; h_x) \\ &\le -\mu^{-1} f^{1-\beta}(x), \end{aligned}$$

that is, inequality (i) follows.

Note that in the proof of implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ we have

$$(f^\beta)^-(x; h_x) \le \beta f^{\beta-1}(x) f^-(x; h_x)$$

and

$$f^-(x; h_x) \le \beta^{-1} f^{1-\beta}(x)(f^\beta)^-(x; h_x).$$

Therefore no matter whether (i) or (ii) holds we always have

$$(f^\beta)^-(x; h_x) = \beta f^{\beta-1}(x) f^-(x; h_x).$$

□

Replacing the function f in Theorems 2 and 4 with $[f_+]^\beta$ and applying Lemma 2, we obtain the following sufficient conditions for error bounds with exponents.

Theorem 6. *Let X be a normed space and $f : X \to (-\infty, +\infty)$ be a proper lower semicontinuous function; let $0 < \mu < +\infty$, $0 \leq \rho < 1$ and $0 < \beta < +\infty$ be constants and $0 < \epsilon \leq +\infty$. Suppose that S is nonempty and for each $x \in f^{-1}(0, \epsilon)$ there exist $t_x > 0$ and $h_x \in X$ with $\|h_x\| = 1$ such that*

$$d_S(x + t_x h_x) \leq \rho d_S(x) \text{ and } \tag{6}$$

and

$$f^-(x + th_x; h_x) \leq -\mu^{-1} f^{1-\beta}(x + th_x) \quad \forall t \in [0, t_x). \tag{7}$$

Then

$$d_S(x) \leq \frac{\mu}{\beta}[f(x)_+]^\beta \quad \forall x \in f^{-1}(-\infty, \epsilon). \tag{8}$$

With the stronger condition that X is complete (i.e., X is a Banach space) and (7) is satisfied only for $t = 0$, then S is automatically nonempty and the conclusion (8) holds.

Remark 1. It is worth pointing out that Theorem 6 extends [6, Theorem 3] and [6, Corollaries 3 and 4] in which X is a reflexive Banach space and f is weakly lower semicontinuous and directionally continuous at x in the direction h_x with $f^-(x; h_x) \leq -\mu^{-1} f^{1-\beta}(x)$.

4. Error bounds for l.s.c. convex functions

A nonconvex function may have a local error bound but no global error bounds. For example, it is easy to see that the function

$$f(x) = \begin{cases} |x| & \text{if } |x| \leq 1, \\ \sqrt{|x|} & \text{if } |x| > 1 \end{cases}$$

has no global error bounds even though it has a local error bound. However, for a proper convex function, the existence of a local error bound always implies that of a global error bound.

Proposition 2. *Let X be a normed space and $f : X \to (-\infty, +\infty]$ be a proper convex function. Then, for some $\mu > 0$ and $0 < \epsilon < +\infty$, the following statements are equivalent:*

(*i*) *S is nonempty and $d_S(x) \leq \mu f(x)_+$ for each $x \in f^{-1}(0, \epsilon)$.*
(*ii*) *$d_S(x) \leq \mu f(x)_+$ for each $x \in X$.*

Proof. The inclusion $(ii) \Rightarrow (i)$ is trivial. To show $(i) \Rightarrow (ii)$ we suppose that S is nonempty and

$$d_S(x) \leq \mu f(x)_+ \qquad \forall x \in X \text{ with } f(x) < \epsilon.$$

It suffices to prove that the above estimate holds for all $x \in X$ with $\epsilon \leq f(x) < +\infty$ as well. Let $x \in X$ with $\epsilon \leq f(x) < +\infty$ be fixed. Then for any positive integer n there exists $\bar{x} \in S$ such that

$$\|x - \bar{x}\| \leq d_S(x) + \frac{1}{n}. \tag{9}$$

Let $\lambda := \dfrac{\epsilon}{2f(x)}$ and $y = (1-\lambda)\bar{x} + \lambda x$. Then (9) implies that

$$\|x - y\| + \|y - \bar{x}\| \leq d_S(x) + \frac{1}{n}$$

from which we have

$$\|y - \bar{x}\| \leq d_S(x) - \|x - y\| + \frac{1}{n} \leq d_S(y) + \frac{1}{n}. \tag{10}$$

Since f is convex and $\bar{x} \in S$,

$$f(y) \leq (1-\lambda) f(\bar{x}) + \lambda f(x) \leq \lambda f(x) = \frac{\epsilon}{2}$$

which implies $d_S(y) \leq \mu f(y)_+$. Therefore

$$\begin{aligned}
d_S(x) \leq \|x - \bar{x}\| &= \frac{1}{\lambda}\|y - \bar{x}\| = \frac{2f(x)}{\epsilon}\|y - \bar{x}\| \\
&\leq \frac{2f(x)}{\epsilon}\left(d_S(y) + \frac{1}{n}\right) \qquad \text{(by (10))} \\
&\leq \frac{2f(x)}{\epsilon}\left(\mu f(y)_+ + \frac{1}{n}\right) \\
&\leq \frac{2f(x)}{\epsilon}\left(\mu \frac{\epsilon}{2} + \frac{1}{n}\right) \\
&\leq \mu f(x) + \frac{2f(x)}{n\epsilon}.
\end{aligned}$$

The desired estimate is obtained since n is arbitrary. Hence (ii) follows.

□

Recall that the subdifferential of a proper function f at $x \in X$ in the sense of convex analysis is given by

$$\partial f(x) := \{\xi \in X^* : f(y) - f(x) \geq \langle \xi, y - x \rangle \quad \forall y \in X\}$$

where X^* is the dual space of a normed space X. For a continuous convex function f on a reflexive Banach space X, under the condition the set S be nonempty, Ng and Zheng showed that there exists $\mu > 0$ such that $d_S(x) \leq \mu f(x)_+$ for each $x \in X$ if and only if $\|\xi\| \geq \mu^{-1}$ for each $\xi \in \partial f(x)$ and each $x \in X$ with $0 < f(x) < +\infty$

(see [7, Theorem 3.3]). Note that for a proper lower semicontinuous convex function f the subdifferential $\partial f(x)$ coincides with the Clarke subdifferential $\partial^\circ f(x)$ (see [2]). This fact allows us to use [11, Theorem 3.1] to extend their result to a proper lower semicontinuous convex function on a Banach space. In fact we can use Proposition 2 to obtain additional equivalent statements about error bounds as follows.

Theorem 7. *Let X be a Banach space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then for some constant $\mu > 0$ the following are equivalent:*

(i) $\|\xi\| \geq \mu^{-1}$ *for each* $\xi \in \partial f(x)$ *and each* $x \in f^{-1}(0, +\infty)$.
(ii) *S is nonempty and there exists* $0 < \epsilon < +\infty$ *such that* $\|\xi\| \geq \mu^{-1}$ *for each* $\xi \in \partial f(x)$ *and each* $x \in f^{-1}(0, \epsilon)$.
(iii) *S is nonempty and there exists* $0 < \epsilon < +\infty$ *such that* $d_S(x) \leq \mu f(x)_+$ *for each* $x \in f^{-1}(0, \epsilon)$.
(iv) $d_S(x) \leq \mu f(x)_+$ *for each* $x \in X$.

Proof. For each $x \in X$ with $0 < f(x) < \epsilon$, by [2, Definition 2.4.10], $\xi \in \partial^\circ f(x)$ if and only if $(\xi, -1) \in N_{epi\, f}(x, f(x))$ where $epi\, f$ is the epigraph of f given by $epi\, f = \{(x, s) \in X \times R : f(x) \leq s\}$ and $N_{epi\, f}(x, f(x))$ is the Clarke normal cone to $epi\, f$ at $(x, f(x))$. Since f is lower semicontinuous and convex, $epi\, f$ is closed and convex. By [2, Proposition 2.4.4], $N_{epi\, f}(x, f(x))$ coincides with the cone of normals in the sense of convex analysis. Thus $\xi \in \partial^\circ f(x)$ if and only if $\xi \in \partial f(x)$. We will implicitly use this relation in the following proof of $(i) \Rightarrow (ii) \Rightarrow (iii)$.

$(i) \Rightarrow (ii)$: We only need to show that S is nonempty. Suppose that S were empty. Then $0 \leq \inf\{f(v), v \in X\}$. Taking $u \in X$ with $0 < f(u) < +\infty$ and $t > 1$, we have

$$f(u) \leq \inf_{v \in X} f(v) + (t\mu)^{-1}(t\mu) f(u).$$

Applying Ekeland's variational principle [3] to f with $\sigma = (t\mu)^{-1}(t\mu) f(u)$ and $\lambda = (t\mu) f(u)$, we find $x \in X$ satisfying

$$f(v) + (t\mu)^{-1}\|v - x\| \geq f(x) \quad \forall v \in X.$$

This implies that $0 < f(x) < +\infty$ and, by [2, Corollary 1 of Theorem 2.9.8],

$$0 \in \partial(f + (t\mu)^{-1}\| \cdot - x\|)(x) \subseteq \partial f(x) + \partial((t\mu)^{-1}\| \cdot - x\|)(x). \tag{11}$$

It follows that there exists $\xi \in \partial f(x)$ such that

$$\|\xi\|_* \leq (t\mu)^{-1} < \mu^{-1}$$

which contradicts statement (i).

$(ii) \Rightarrow (iii)$: If S is nonempty and there exists $0 < \epsilon < +\infty$ such that $\|\xi\| \geq \mu^{-1}$ for each $\xi \in \partial f(x)$ and each $x \in f^{-1}(0, \epsilon)$, then taking ∂° as an abstract subdifferential ∂_ω in [11, Theorem 3.1] we have

$$d_S(x) \leq \mu f(x)_+ \quad \forall x \in X \text{ with } f(x) < \frac{\epsilon}{2}.$$

By Proposition 2 this inequality holds for each $x \in X$ with $f(x) < \epsilon$.

$(iii) \Rightarrow (iv)$ follows directly from Proposition 2. It remains to prove $(iv) \Rightarrow (i)$.

Let x be such that $0 < f(x) < +\infty$ and $d_S(x) \le \mu f(x)$. Then $d_S(x) > 0$ and for any $\xi \in \partial f(x)$ we have

$$\|\xi\| \cdot \|y - x\| \ge -\langle \xi, y - x\rangle \ge -[f(y) - f(x)] \ge f(x) \quad \forall y \in S.$$

Taking inferiors of both sides of the inequality for all y over S we obtain

$$\|\xi\| \cdot d_S(x) \ge f(x),$$

from which we have

$$\|\xi\| \ge \frac{f(x)}{d_S(x)} \ge \mu^{-1}.$$

Therefore the inequality desired follows.

□

Remark 2. Wu and Ye pointed out in [11, Theorem 3.1] that if S is nonempty and $\|\xi\| \ge \mu^{-1}$ for each $\xi \in \partial_\omega f(x)$ and each $x \in f^{-1}(0, +\infty)$ then $d_S(x) \le \mu f(x)_+$ for each $x \in X$, where $\partial_\omega f(x)$ is an abstract subdifferential of f at x defined in [11]. As in the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 7, we can prove the property that the set $f^{-1}(-\infty, \epsilon)$ is nonempty and $\|\xi\| \ge \mu^{-1}$ holds for each $\xi \in \partial_\omega f(x)$ and each $x \in f^{-1}(0, \epsilon)$ implies that S is nonempty. Hence the condition that S be nonempty in [11, Theorem 3.1] can be omitted.

We recall that if f is convex then $f^-(x; v)$ coincides with the usual directional derivative of f at x in the direction v given by

$$f'(x; v) := \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}.$$

When f is convex and each point in a neighborhood of S has a closest point in S, the sufficient conditions for the existence of error bounds given in Theorems 3 and 4 become necessary as well.

Proposition 3. *Let X be a normed space and $f : X \to (-\infty, +\infty]$ a lower semicontinuous convex function. Suppose that for some $0 < \epsilon \le +\infty$ each point $x \in f^{-1}(0, \epsilon)$ has a closest point in $S := \{x \in X : f(x) \le 0\}$. If for some $\mu > 0$ and each $x \in f^{-1}(0, \epsilon)$ there holds $d_S(x) \le \mu f(x)_+$, then for each $x \in f^{-1}(0, \epsilon)$*

(i) *there exists $y \in f^{-1}(0, \epsilon)$ such that $0 < \|x - y\| \le \mu[f(x) - f(y)]$ and*
(ii) *there exist $t_x > 0$ and $h_x \in X$ with $\|h_x\| = 1$ such that*

$$f'(x + th_x; h_x) \le -\mu^{-1} \quad \forall t \in [0, t_x).$$

Proof. Given $x \in f^{-1}(0, \epsilon)$, let $\overline{x}$ be in S such that $\|x - \overline{x}\| = d_S(x)$. Taking $t_x = \|x - \overline{x}\|$ and $h_x = t_x^{-1}(\overline{x} - x)$, then $\overline{x} = x + t_x h_x$. Obviously for each $0 < t < t_x$ we have $d_S(\overline{x} - th_x) = t$. This implies that $0 < f(\overline{x} - th_x)$. Besides, by the convexity of f,

$$f(\overline{x} - th_x) \le \left(1 - \frac{t}{t_x}\right) f(\overline{x}) + \frac{t}{t_x} f(x) < \epsilon.$$

Thus by the assumption

$$t = d_S(\overline{x} - th_x) \le \mu f(\overline{x} - th_x) \le \mu[f(\overline{x} - th_x) - f(\overline{x})] \quad \forall 0 < t < t_x.$$

Rewriting gives the following inequality

$$\frac{f(\overline{x}) - f(\overline{x} - th_x)}{t} \le -\frac{1}{\mu} \quad \forall 0 < t < t_x.$$

For each $0 < t < t_x$, by the convexity of f again,

$$\begin{aligned}\frac{f(x + th_x) - f(x)}{t} &\le \frac{f(\overline{x}) - f(x)}{t_x} = \frac{f(\overline{x}) - f(\overline{x} - t_x h_x)}{t_x} \\ &\le \frac{f(\overline{x}) - f(\overline{x} - th_x)}{t} \le -\frac{1}{\mu}.\end{aligned}$$

For any $0 < t < t_x$, taking $y = x + th_x$, we see from the above inequality that $y \in f^{-1}(0, \epsilon)$ and $0 < \|x - y\| \le \mu[f(x) - f(y)]$. This proves (i).

To prove (ii) we note that the point $\overline{x}$ is also a closest point in S to the point $x + th_x$ for each $t \in (0, t_x)$. Thus, for each $0 < s < t_x - t$ with $t \in [0, h_x)$, as in the above discussion, we have

$$\frac{f(x + th_x + sh_x) - f(x + th_x)}{s} \le -\frac{1}{\mu}$$

from which we obtain (ii).

□

Note that for an l.s.c. convex function f on a Banach space X the set $S = \{x \in X : f(x) \le 0\}$ is closed and convex. If X is reflexive and S is nonempty then each point $x \in X \backslash S$ has a closest point in S. As a result of Theorem 7, Proposition 3, Theorems 3 and 4, the equivalent statements about the existence of error bounds can be summarized as follows.

Theorem 8. *Let X be a reflexive Banach space and $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then for some $\mu > 0$ the equivalent statements $(i) - (iv)$ in Theorem 7 are all equivalent to any one of the following:*

(v) *For some $0 < \epsilon < +\infty$ the set $f^{-1}(-\infty, \epsilon)$ is nonempty and for each $x \in f^{-1}(0, \epsilon)$ there exists a point $y \in f^{-1}[0, \epsilon)$ such that $0 < \|x - y\| \le \mu[f(x) - f(y)]$.*

(vi) *For each $x \in f^{-1}(0, +\infty)$ there exists a point $y \in f^{-1}[0, +\infty)$ such that*

$$0 < \|x - y\| \le \mu[f(x) - f(y)].$$

(vii) *For some $0 < \epsilon < +\infty$ the set $f^{-1}(-\infty, \epsilon)$ is nonempty and for each $\lambda > 1$ and each $x \in f^{-1}(0, \epsilon)$ there exists a point $y \in f^{-1}[0, \epsilon)$ such that*

$$0 < \|x - y\| \le \lambda\mu[f(x) - f(y)].$$

$(viii)$ *For each $\lambda > 1$ and each $x \in f^{-1}(0, +\infty)$ there exists a point $y \in f^{-1}[0, +\infty)$ such that $0 < \|x - y\| \le \lambda\mu[f(x) - f(y)]$.*

(*ix*) *For some* $0 < \epsilon < +\infty$ *the set* $f^{-1}(-\infty, \epsilon)$ *is nonempty and for each* $x \in f^{-1}(0, \epsilon)$ *there exist* $t_x > 0$ *and* $h_x \in X$ *with* $\|h_x\| = 1$ *such that*

$$f'(x + th_x; h_x) \leq -\mu^{-1} \quad \forall t \in [0, t_x).$$

(*x*) *For each* $x \in f^{-1}(0, +\infty)$ *there exist* $t_x > 0$ *and* $h_x \in X$ *with* $\|h_x\| = 1$ *such that* $f'(x + th_x; h_x) \leq -\mu^{-1} \quad \forall t \in [0, t_x)$.

(*xi*) *For some* $0 < \epsilon < +\infty$ *the set* $f^{-1}(-\infty, \epsilon)$ *is nonempty and for each* $x \in f^{-1}(0, \epsilon)$ *there exists* $h_x \in X$ *with* $\|h_x\| = 1$ *such that* $f'(x; h_x) \leq -\mu^{-1}$.

(*xii*) *For each* $x \in f^{-1}(0, +\infty)$ *there exists* $h_x \in X$ *with* $\|h_x\| = 1$ *such that* $f'(x; h_x) \leq -\mu^{-1}$.

Proof. The implications $(iii) \Rightarrow (v) \Rightarrow (vii) \Rightarrow (iii)$ follow directly from Proposition 3 and Theorem 3 while $(iii) \Rightarrow (ix) \Rightarrow (xi) \Rightarrow (iii)$ follow from Proposition 3 and Theorem 4. Similarly by Proposition 3 and Theorems 3 and 4 we have "$(iv) \Rightarrow (vi) \Rightarrow (viii) \Rightarrow (iv)$" and "$(iv) \Rightarrow (x) \Rightarrow (xii) \Rightarrow (iv)$".

□

As an application of Theorem 7 or 8, the following example is used to illustrate that not all convex functions have error bounds. The function in this example appears in [9] and was subsequently used in [1] and [5]. By its subdifferential we prove that this function has no error bounds even though it is convex.

Example 1. Consider the closed proper convex function

$$f(x_1, x_2) = \begin{cases} \frac{x_1^2}{x_2} & \text{if } x_2 > 0 \\ 0 & \text{if } x_1 = x_2 = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Obviously $S = \{(0, 0)\}$ and for each $n \in N$ with $0 < f(x_1, n) < \infty$ the subdifferential $\partial f(x_1, n) = \{(2\frac{x_1}{n}, -\frac{x_1^2}{n^2})\}$. For fixed x_1 and any $0 < \epsilon \leq \infty$ we have $0 < f(x_1, n) < \epsilon$ when n is large enough and $(2\frac{x_1}{n}, -\frac{x_1^2}{n^2}) \to (0, 0)$ as $n \to +\infty$. Consequently, by Theorem 7 or 8, there can not exist $\mu > 0$ such that for all n,

$$d_S((x_1, n)) \leq \mu f(x_1, n).$$

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