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Duality and martingales: a stochastic programming perspective on contingent claims*

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Abstract. The hedging of contingent claims in the discrete time, discrete state case is analyzed from the perspective of modeling the hedging problem as a stochastic program. Application of conjugate duality leads to the arbitrage pricing theorems of financial mathematics, namely the equivalence of absence of arbitrage and the existence of a probability measure that makes the price process into a martingale. The model easily extends to the analysis of options pricing when modeling risk management concerns and the impact of spreads and margin requirements for writers of contingent claims. However, we find that arbitrage pricing in incomplete markets fails to model incentives to buy or sell options. An extension of the model to incorporate pre-existing liabilities and endowments reveals the reasons why buyers and sellers trade in options. The model also indicates the importance of financial equilibrium analysis for the understanding of options prices in incomplete markets.

Key words. options pricing – martingales – incomplete markets – stochastic programming

1. Introduction

This paper develops a mathematical structure for contingent claims analysis from the unique perspective of managing the claim as a liability of the writer. The hedging problem is modeled as a stochastic program and analyzed using the mathematical technique of conjugate duality (cf. Rockafellar [21]). The dual problem turns out to require the existence of a valuation operator that integrates the claim's future cash flows against a martingale probability measure. In complete markets this step establishes the fundamental pricing theorems of contingent claims as in Ross [22], Harrison and Kreps [11], and Harrison and Pliska [12], since a complete market is one in which there is a unique valuation operator.

But incomplete markets can support many valuation operators, and different players can have different optimal valuation operators. Risk aversion, bid-ask spreads, transactions costs, shorting costs, margin requirements, etc, all affect the valuation operator. The goal of this paper is to indicate the ways that optimization models of the management of claims can contribute to an understanding of how valuation operators can be formed and what impact they may have on observed market prices. The advantage of the framework is that it shows how options pricing can be linked to optimal actions of investors. In particular, the development of this model shows that probably the most

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important influence on pricing options actually arises from the pre-existing liability structures or endowments of the various market players.

This paper treats the finite state theory, as surveyed recently by Naik [18], Duffie [7, Chap. 2], and Pliska [19]. It is intended for readers in optimization and mathematical programming looking for an introduction to the interesting field of finance, as well as for researchers in finance looking for a framework in which the tools of optimization can be brought to bear on key issues in pricing claims in incomplete markets. Recent work in the modeling of claims in incomplete markets using realistic investor models focusses on taxes and transactions costs as the driver of market imperfections. See Jouini and Kallal [13], Dermody and Prisman [5], a recent contribution of Wang and Poon [23], and the references cited in these papers. However, such extensions of simple arbitrage pricing frameworks do not adequately model the incentives for buyers and sellers of options. One must introduce pre-existing liability structures or endowments to complete the picture for options pricing, as in Sect. 8 of the present paper.

A recent paper of Cvitanic, Shachermayer and Wang [4], which the author received while the present paper was in process, investigates the structure of the investor's problem of Sect. 8 in the continuous semi-martingale setting. Their paper is an important step in the development of a theory of financial equilibrium in incomplete markets, essentially resolving the specification of the dual problem in a much more sophisticated mathematical setting than the one of this paper. It is hoped that the simple structures investigated in this paper may lead to useful progress in understanding options pricing in incomplete markets. The developments of the present paper will be extended to continuous time and states in a future paper (King and Korf [17]).

The outline of the paper is as follows. The first two sections show how arbitrage and pricing arise from duality considerations. The third section presents the main result of the paper, which is the connection between boundedness and feasibility of the hedging problem of the writer of the claim and arbitrage pricing theory. The next section discusses who would buy the claim offered by the writer. It concludes that one must introduce differences in writer and buyer's problems in order to model actual buying and selling. The next two sections develop models with various differences in risk aversion and transactions costs, but we show that these extensions do not in themselves lead to actual transactions in options. The next section introduces pre-existing liability positions or endowments, and analyzes the impact on models for valuation operators and market prices. In this case, simple assumptions on the desirability of the option as the price moves do lead to the possibility of a market-clearing price for the option. At this point we are in the realm of equilibrium theory and outside of the subject of the paper, so must leave the remainder of the analysis for future work.

2. Martingales and absence of arbitrage

This section develops the model of the decision space of the investor when security prices evolve on a discrete-time, discrete-state path space. The concept of arbitrage is introduced and shown to be equivalent to the impossibility of finding a solution to a certain set of equations that comprise the dual to the investor's decision space.

All random quantities in this paper will be supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms are sequences of real-valued vectors (security prices and payments) over discrete time periods $t = 0, \dots, T$. It is convenient to model a finite probability space by a *scenario tree*, in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at depth t in the tree. The root node $n = 0$ corresponds to the trivial partition $\mathcal{N}_0 = \Omega$ consisting of the entire probability space, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. (Although we do not need to use them in this finite-state probability setup, the σ-algebras $\mathcal{F}_t$ generated by the partitions $\mathcal{N}_t$ satisfy the usual properties: $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $0 \leq t < T$, and $\mathcal{F}_T = \mathcal{F}$.)

In the scenario tree, every node $n \in \mathcal{N}_t$ for $1 \leq t \leq T$ has a unique parent denoted by $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$ for $0 \leq t \leq T-1$ has a (nonempty) set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. The probability distribution P is modeled by attaching weights $p_n > 0$ to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-terminal node one has, recursively,

$$p_n = \sum_{m \in \mathcal{C}(n)} p_m \quad \forall n \in \mathcal{N}_t, \quad t = T-1, \dots, 0$$

and so each node receives a probability mass equal to the combined mass of the paths passing through it. The ratios p_m/p_n, $m \in \mathcal{C}(n)$, are the conditional probabilities that the child node m occurs given that the parent node $n = a(m)$ has occurred.

A *random variable* X is a real-valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbf{R}\}$ is either the empty-set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω. Such a random variable is said to be *measurable* (or observable, or knowable) with respect to the information contained in the nodes of $\mathcal{N}_t$. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect to $\mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$E^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

Since the node collection $\mathcal{N}_t$ is contained in the one-step-ahead collection $\mathcal{N}_{t+1}$, it follows that X_{t+1} cannot generally be lifted to $\mathcal{N}_t$. Its *conditional expectation*

$$E^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{C}(n)} \frac{p_m}{p_n} X_m$$

is a random variable taking values over the nodes $n \in \mathcal{N}_t$.

The market consists of $J+1$ tradable securities indexed by $j = 0, \dots, J$ whose prices at each node n are denoted by the vector $S_n = (S_n^0, \dots, S_n^J)$. Following Harrison and Pliska [12], suppose one of the securities, security 0, say, always has strictly positive values. This security is chosen to be the *numeraire*. Introduce the discounts $\beta_n = (1/S_n^0)$ and let $Z_n^j = \beta_n S_n^j$ for $j = 0, \dots, J$ denote the discounted security prices *relative to the numeraire*. The price Z_n^0 of the numeraire in any state n is exactly 1.

Investors have no influence on the prices of any security and may undertake trades at every time-step based on information accumulated up to time t. The amount of security j held by the investor in state $n \in \mathcal{N}_t$ is denoted θ_n^j. The (discounted) value of the portfolio in state n is

$$Z_n \cdot \theta_n := \sum_{j=0}^{J} Z_n^j \theta_n^j$$

An *arbitrage* is a sequence of portfolio holdings that begins with zero initial value at time 0, maintains a non-negative value in each future state, and has a positive probability of achieving at least one strictly positive value in some terminal state, all through *self-financing* portfolio transactions

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{a(n)} \quad n > 0$$

which states that the funds available for investment at state n are restricted to those generated by price changes in the portfolio held at state $a(n)$.

Arbitrage is a way of making something out of nothing. This important concept can be simplified in a couple of ways without losing generality. First, non-negative portfolio values need hold only at the terminal states $n \in \mathcal{N}_T$. (Harrison and Pliska [12, page 228] demonstrate how to construct an arbitrage from a portfolio process with non-negative terminal values and one strictly negative value in a non-terminal state.) Second, since a discrete nonnegative random variable has a positive probability of a strictly positive value if and only if its expected value is strictly positive, then the arbitrage conditions are equivalent to having non-negative values in each terminal state with a positive expected value overall.

To find an arbitrage one can solve an optimization problem, called a *stochastic program* (an optimization problem in which some parameters are random variables, see Birge [2]). The following is called the *arbitrage problem*.

$$\begin{array}{lllr} \max_{(\theta)} & \sum_{n \in \mathcal{N}_T} p_n Z_n \cdot \theta_n & & \\ \text{subject to} & Z_0 \cdot \theta_0 & = 0 & : y_0 \\ & Z_n \cdot [\theta_n - \theta_{a(n)}] & = 0 \ (n \in \mathcal{N}_t,\ t \geq 1) & : y_n \\ & Z_n \cdot \theta_n & \geq 0 \ (n \in \mathcal{N}_T) & : x_n \end{array} \qquad (1)$$

A positive optimal value for this stochastic program corresponds to an arbitrage: it begins with a portfolio that has value 0, makes self-financing trades at each time step, has non-negative terminal values in every scenario, and has a positive expected value at time T. In fact one can easily verify that the problem is unbounded if an arbitrage exists. The price process is called an *arbitrage-free market price process* if no arbitrage is possible.

The basic theme of this paper is to analyze the problem of the investor through a closely related (indeed, equivalent) problem called the dual. Computing the dual in the discrete time and state setting of this paper is a basic calculation whose steps should be familiar to students of linear programming. The interesting detail comes from the fact that problem (1) is a stochastic program, and as such describes special relationships between time-indexed variables and constraints. The first step in calculating the the dual is to multiply all the constraints through by the dual variables (indicated in the right margin) to form the Lagrangian:

$$L(\theta; x, y) = \sum_{n\in\mathcal{N}_T} p_n Z_n \cdot \theta_n - \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} y_n Z_n \cdot [\, \theta_n - \theta_{a(n)} \,] - \sum_{n\in\mathcal{N}_T} x_n Z_n \cdot \theta_n \quad (x_n \le 0)$$

and rearrange to isolate terms in θ_n:

$$= \sum_{n\in\mathcal{N}_T} [\, p_n - x_n - y_n \,] Z_n \cdot \theta_n - \sum_{t=0}^{T-1} \sum_{n\in\mathcal{N}_t} [\, y_n Z_n - \sum_{m\in\mathcal{C}(n)} y_m Z_m \,] \cdot \theta_n \quad (x_n \le 0).$$

The dual problem is generated by maximizing over the unrestricted primal variables θ_n. Constraints in the dual arise from the requirement that the factors of θ_n must evaluate to zero for a feasible dual solution. Since there is no term not involving θ_n, the dual program reduces to a feasibility problem in the dual variables (x, y):

$$\begin{aligned} & x_n && \le 0 \ (n \in \mathcal{N}_T) \\ & [\, p_n - y_n - x_n \,] Z_n && = 0 \ (n \in \mathcal{N}_T) \\ & y_n Z_n - \textstyle\sum_{m\in\mathcal{C}(n)} y_m Z_m && = 0 \ (n \in \mathcal{N}_t,\ t \le T-1). \end{aligned} \tag{2}$$

The basic theorem of linear programming states that problem (1) has an optimal solution if and only if the dual (2) does too, and both optimal values are equal. Furthermore, since the problem (1) is always feasible, it follows again from the basic theory of linear programming that it has an optimal solution if and only if it is bounded.

The last equation in the dual system resembles a *martingale* condition: that the value of each coordinate of Z_n equal its conditional expected value one step ahead. Martingale properties needed for this paper are formalized in the following definition.

Definition 1. *If there exists a probability measure Q such that*

$$Z_t = E^Q [\, Z_{t+1} | \mathcal{N}_t \,] \quad (t \le T-1) \tag{3}$$

then the (vector) process $\{Z_t\}$ is called a (vector-valued) martingale *under Q and Q is called a* martingale probability measure *for the process. If one has (coordinatewise) $Z_t \ge E^Q [\, Z_{t+1} | \mathcal{N}_t \,]$ (respectively, $Z_t \le E^Q [\, Z_{t+1} | \mathcal{N}_t \,]$) the process called a (vector)* supermartingale *(respectively,* submartingale*) under Q.*

The following theorem develops the key link between arbitrage and martingales.

Theorem 1. *The discrete state stochastic vector process $\{Z_t\}$ is an arbitrage-free market price process if and only if there is at least one probability measure Q equivalent to P under which $\{Z_t\}$ is a martingale.*

Proof. If there is no arbitrage, then the problem (1) is bounded. It is always feasible, and hence has an optimal solution, as discussed above. By linear programming duality, it follows that the dual system (2) must have an optimal solution, say (x, y). Now, the value of the numeraire Z^0 in each state is exactly 1, so the nonpositivity of x_n in the first system of equalities implies

$$y_n \geq p_n, \quad n \in \mathcal{N}_T.$$

The last system of equalities implies

$$\sum_{m \in \mathcal{C}(n)} y_m = y_n.$$

It follows that y_n is a strictly positive process such that the sum of y_n over all states $n \in \mathcal{N}_t$ in each time period t is equal to y_0. Construct the numbers $q_n = y_n / y_0$, for each $n \in \mathcal{N}_T$, and let Q be the probability measure with weights q_n. Rewriting the last system of equations gives the vector equalities

$$\sum_{m \in \mathcal{C}(n)} q_m Z_m = q_n Z_n \quad (n \in \mathcal{N}_t \quad t \leq T-1)$$

from which (3) follows. Thus Q is an equivalent martingale probability measure for the process $\{Z_t\}$.

For the other direction, suppose there exists an equivalent martingale measure Q. Define $y_0 = \max\{p_n/q_n \,|\, n \in \mathcal{N}_T\}$, set $y_n = q_n y_0$ for $n \in \Omega$, and let $x_n = p_n - y_n$ for $n \in \mathcal{N}_T$. Note that $x_n = p_n - q_n y_0 \leq p_n - q_n(p_n/q_n) = 0$. The vector pair (x, y) is thus a feasible solution to (2). By weak duality it follows that (1) is bounded and hence there can be no arbitrage.

□

It is important to notice that Theorem 1 is invariant with respect to the underlying probability measure P. That is, if there exists a solution y to the dual problem (2) for a particular probability P, then there exists a solution y' for any other probability measure P' that is equivalent to P. To see this, suppose Q is the martingale measure that is constructed from y. Then Q can be transformed into a solution for the dual under P' by multiplying q_n through by a large enough multiplier y'_0 to guarantee $q_n y'_0 \geq p'_n$. This is summarized in the following.

Corollary 1. *The conclusions of Theorem 1 are invariant with respect to the actual probability; that is, if under some probability measure P the process $\{Z_t\}$ is shown either to be arbitrage-free or to have arbitrage then this same conclusion holds for all probability measures equivalent to P.*

The theorem is easily extended to cover dividend or interest payments. (The numeraire is assumed not to pay dividends.)

Corollary 2. *Suppose that each security $j = 1, \dots, J$ pays dividend payments D_n^j in state n. Then market prices are arbitrage free if and only if there exists a probability measure Q equivalent to P such that*

$$Z_t = E^Q [\, Z_{t+1} + \beta_{t+1} D_{t+1} \,|\, \mathcal{N}_t \,] \quad (t \leq T-1). \tag{4}$$

Proof. In the arbitrage linear program (1) the self-financing condition becomes

$$Z_n \cdot \theta_n - [\, Z_n + \beta_n D_n \,] \cdot \theta_{a(n)} = 0 \quad n \in \mathcal{N}_t, \quad t \geq 1.$$

The rest of the argument follows exactly as in Theorem 1.

□

At the risk of abusing the terminology, the probability measure of Corollary 2 will continue to be referred to as an equivalent *martingale* measure on the market price process. When dividends are not paid, the "martingale" condition is (3); when they are paid the condition is (4).

Corollary 2 also shows how security prices are related to future cash flows in an arbitrage-free market. The security prices today must satisfy

$$\beta_0 S_0 = E^Q [\sum_{t=1}^{T} \beta_t D_t + \beta_T S_T \,] := \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} \beta_n D_n q_n + \sum_{n \in \mathcal{N}_T} \beta_n S_n q_n .$$

This amounts to saying that feasible martingale measures on market price processes have the property that the integral of discounted dividend payouts equal current market prices.

To complete the discussion of martingale measures, define the set $\mathcal{Q}$ of martingale probability measures that are equivalent to P and also the slightly larger set $\overline{\mathcal{Q}}$ of all martingale probability measures on $\{Z_t\}$ (equivalent or not). The following proposition describes relevant properties of $\overline{\mathcal{Q}}$ and $\mathcal{Q}$.

Proposition 1. *The set $\mathcal{Q}$ of equivalent martingale probability measures is convex, and the set $\overline{\mathcal{Q}}$ of all martingale probability measures is its closure.*

Proof. An alternative representation for elements $Q \in \mathcal{Q}$ is

$$\begin{aligned} q_n \quad & > 0 \; n \in \mathcal{N}_T \\ q_n Z_n - \textstyle\sum_{m \in \mathcal{C}(n)} q_m Z_m & = 0 \; (n \in \mathcal{N}_t, \; t \leq T-1) \\ q_0 \quad & = 1 \, . \end{aligned}$$

This set is clearly convex. The representation of $\overline{\mathcal{Q}}$ is identical except that $q_n \geq 0$ for $n \in \mathcal{N}_T$. Hence $\overline{\mathcal{Q}}$ is the closure of $\mathcal{Q}$.

□

3. Financing of contingent claims

A contingent claim F with lifetime less than or equal to the horizon T has payouts F_n that are determined by the events n in the market price process. These are redundant securities that do not introduce any additional uncertainty. Currency futures and equity options are examples of traded contingent claims; but many bond agreements, collateralized obligations, etc., have embedded contingency features and are traded on liquid markets.

The great insight of Black, Scholes and Merton was that a portfolio of a bond and the underlying security can be constantly traded to risklessly generate payouts F_n through self-financing transactions. This idea can be captured in a stochastic program which determines the minimum amount needed to start a trading strategy that produces payouts F_n with no risk.

$$\begin{array}{lll} \min_{(\theta)} & Z_0 \cdot \theta_0 & \\ \text{subject to} & Z_n \cdot [\,\theta_n - \theta_{a(n)}\,] = -\beta_n F_n & (n \in \mathcal{N}_t,\ t \geq 1) \\ & Z_n \cdot \theta_n \geq 0 & (n \in \mathcal{N}_T). \end{array} \tag{5}$$

Note that any surplus remaining in the portfolio at the terminal stage is simply discarded in this formulation.

In the next theorem, the dual to this problem will be shown to be equivalent to

$$\max_{Q \in \overline{\mathcal{Q}}} E^Q [\sum_{t=1}^{T} \beta_t F_t\,]\,. \tag{6}$$

This equation computes the maximum expected value of the discounted payouts over all possible martingale measures.

Proposition 2. *Let F_n be a contingent claim on an arbitrage-free market price process $\{Z_t\}$. The claim is attainable if and only if its price F_0 satisfies*

$$\beta_0 F_0 \geq \max_{Q \in \overline{\mathcal{Q}}} E^Q [\sum_{t=1}^{T} \beta_t F_t\,] \tag{7}$$

where $\overline{\mathcal{Q}}$ is the set of martingale probability measures on $\{Z_t\}$.

Proof. With the labeling of the dual variables as in (1), one arrives at the following Lagrangian

$$\sum_{n \in \mathcal{N}_T} [\,-x_n - y_n\,] \, Z_n \cdot \theta_n - \sum_{t=1}^{T-1} \sum_{n \in \mathcal{N}_t} [\, y_n Z_n - \sum_{m \in \mathcal{C}(n)} y_m Z_m\,] \cdot \theta_n + \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} y_n \beta_n F_n$$
$$- [\, Z_0 - \sum_{m \in \mathcal{C}(0)} y_m Z_m\,] \cdot \theta_0 \quad (x_n \leq 0).$$

Minimizing out in the θ variables, and observing again that the numeraire value Z_n^0 is always equal to one, leads to the dual problem (6). If F_0 is the price of the option then its value $\beta_0 F_0$ in terms of the numeraire must equal the optimal value of (6) in order to be feasible.

□

4. The writer's problem

The previous section computed the minimum initial investment F_0 required to hedge the claim F with no risk of falling short. Now consider the position of the "writer" of the contingent claim who has received F_0 in return for a promise to pay F_n in the future and who will then invest this money to try to generate a profit.

If the writer is confident that P is the true probability distribution and wishes to invest in such a way as to maximize expected value while hedging the claim F, then the *writer's problem* may be modeled as the stochastic program

$$\begin{array}{lll} \max_{(\theta)} & \sum_{n\in\mathcal{N}_T} p_n Z_n \cdot \theta_n & \\ \text{subject to} & Z_0 \cdot \theta_0 = \beta_0 F_0 & \\ & Z_n \cdot [\,\theta_n - \theta_{a(n)}\,] = -\beta_n F_n & (n \in \mathcal{N}_t,\ t \geq 1) \\ & Z_n \cdot \theta_n \geq 0 & (n \in \mathcal{N}_T). \end{array} \tag{8}$$

The connection of the writer's problem to the developments of the previous two sections is direct. By linear programming theory, the writer's problem has an optimal solution if and only if it is both bounded and feasible. Theorem 1 analyzes what is required to bound the writer's problem. Proposition 2 discusses what is required to have a feasible solution to the writer's problem. The conclusions one can draw from these relationships are summarized in the following theorem.

Theorem 2. *The writer's problem has an optimum if and only if*

1. *The collection $\mathcal{Q}$ of equivalent martingale probability measures on the market price process $\{Z_t\}$ is nonempty, and*
2. *The price F_0 charged by the writer to generate the payouts F_n satisfies*

$$\beta_0 F_0 \geq \max_{Q\in\overline{\mathcal{Q}}} E^Q [\sum_{t=1}^{T} \beta_t F_t] \tag{9}$$

where the maximum is taken over the collection of all martingale probability measures $\overline{\mathcal{Q}}$ on $\{Z_t\}$. This price is invariant under changes of the original probability measure P.

At a price F_0 satisfying (9) *the writer earns the expected profit*

$$W^* = y_0^* \left[\, \beta_0 F_0 - E^{Q^*} [\sum_{t=1}^{T} \beta_t F_t] \,\right] \geq 0 \tag{10}$$

where Q^* is the equivalent martingale probability measure generated by the dual solution y^* and y_0^* equals the maximum of the ratios $\{p_n/q_n^* \mid n \in \mathcal{N}_T\}$.

Proof. The development preceding Theorem 1 and the analysis in its proof may be applied to show that the problem dual to (8) is equivalent to

$$\begin{array}{lll} \min_{(y)} & \beta_0 F_0 y_0 - \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} \beta_n F_n y_n & \\ \text{subject to} & y_n \geq p_n & (n \in \mathcal{N}_T) \\ & y_n Z_n - \sum_{m\in\mathcal{C}(n)} y_m Z_m = 0 & (n \in \mathcal{N}_t,\ t \leq T-1). \end{array} \tag{11}$$

By linear programming theory, the writer's problem has an optimal solution if and both primal and dual are feasible. Feasibility of the dual was shown in Theorem 1 to correspond to a nonempty set of equivalent martingale probability measures $\mathcal{Q}$. The feasibility of the primal was shown in Proposition 2 to be equivalent to (9).

The final statement is proved by applying the analysis of Corollary 1 to the current setting. Since the process is arbitrage-free, the writers problem has an optimal value. So suppose that y^* is an optimal solution for the dual. Construct the equivalent martingale measure Q^* by $q_n^* = y_n^*/y_0^*$ as in the proof of Theorem 1. Fixing this Q^*, the dual problem can be written as a minimization in y_0

$$\min_{y_0} \quad y_0\beta_0\Big[F_0 - \sum_{n>0} \beta_n F_n q_n^*\Big]$$

subject to the constraint that $y_0\, q_n^* \geq p_n$ for $n \in \mathcal{N}_T$. This minimum is attained at the maximum of the ratios p_n/q_n^* for $n \in \mathcal{N}_T$, since $\beta_0 F_0 - \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} \beta_n F_n q_n^* > 0$.

□

One can show that the profit maximizing portfolio for the writer's problem (8) takes the form of a lottery. It selects the event that contains the maximum defining y_0^* in (10) and bets all its surplus wealth there, so the payoff is spectacularly positive in one event and zero elsewhere. The following example illustrates this behavior.

Example 1. A trivial one-period example of the writer's surplus can be developed as follows. Let there be two securities, a stock and a bond. The stock, currently valued at 1, takes values (0.90, 1.05, 1.25) in the next period; the interest rate for the bond over the period is 5 percent. Normalizing with respect to the bond, Z takes values (0.86, 1.0, 1.19) (approximately). Let P be equally weighted, and consider a call option struck at 1.1 (units of the numeraire). The payoff of the claim F is $(0, 0, 0.09)$. Solving the linear program (5) gives the arbitrage price of the claim as (approximately) $F_0^* = 0.0382$. Solving the writer's problem (8) with the arbitrage price F_0^* leads to the following conclusions. The writer takes this money, purchases approximately 0.273 units of stock and borrows approximately 0.235 units of the bond. In the next period, the writer pays out the claim F, and the terminal distribution of his portfolio values equals approximately $(0.0, 0.0382, 0.0)$. In the event that the true probability distribution is P, the writer makes a positive expected profit $W^* = 0.0126$.

5. The buyer's problem and the arbitrage interval

The buyer's problem is the reverse of the writer's: one pays F_0 in return for a promise of payments F_n in each state $n > 0$. Assuming the buyer has the same capability to trade in the market and wishes to maximize expected value at the end of the horizon, then the problem of the buyer is:

$$\begin{array}{lll}
\max_{(\theta)} & \sum_{n\in\mathcal{N}_T} p_n Z_n \cdot \theta_n & \\
\text{subject to} & Z_0 \cdot \theta_0 = -\beta_0 F_0 & \\
& Z_n \cdot [\,\theta_n - \theta_{a(n)}\,] = \beta_n F_n & (n \in \mathcal{N}_t,\ t \geq 1) \\
& Z_n \cdot \theta_n \geq 0 & (n \in \mathcal{N}_T).
\end{array}$$

The analysis of the writer's problem does not depend on the sign of F. In the setting developed so far in this paper, the buyer's acceptable price F_0 might be computed just by reversing signs in (9) and so will satisfy

$$\beta_0 F_0 \leq \min_{Q \in \overline{\mathcal{Q}}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]$$

where the minimum is taken over the collection of all martingale probability measures $\overline{\mathcal{Q}}$ on $\{Z_t\}$.

Compare the writer's minimum offering price,

$$F_0^w := \beta_0^{-1} \max_{Q \in \overline{\mathcal{Q}}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]$$

to the buyer's maximum acceptable price

$$F_0^b := \beta_0^{-1} \min_{Q \in \overline{\mathcal{Q}}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right].$$

One has $F_0^b \leq F_0^w$. The interval $[F_0^b, F_0^w]$ is called the *arbitrage interval*, in the sense that prices in this interval will not induce either writer or buyer to wish to sell or buy infinite amounts of F. They may be thought of as bounds on the price of F, as in Ritchken and Kuo [20]. But it is not clear what further use one can make of this arbitrage interval. Note that the price F_0^w is the *minimum* acceptable price for the seller and F_0^b is the *maximum* acceptable price for the buyer. When these are not equal, as is generally the case in incomplete markets, then there will be no trading activity under the assumptions of the modeling so far.

6. Risk aversion

It is often believed that the solution to pricing in incomplete markets is to model risk aversion in the hedging problem simply through the introduction of a utility function. But risk aversion alone does not introduce any fundamentally new incentives for buyers and sellers to trade options.

To fix ideas, consider the utility function in the model for the writer's problem (8). One may write it as

$$u_w(v) = v - \delta_{v \geq 0}(v)$$

where the function $\delta_{v \geq 0}(v)$ (the indicator function of convex analysis) equals 0 if $v \geq 0$ and is $+\infty$ if $v < 0$. The writer's utility $u_w(\cdot)$ is $-\infty$ for values below zero, has an infinite vertical leap at the point 0 and then continues as a linear function of unit slope on the positive reals.

The mathematical content of Theorem 2 is that the boundedness of the arbitrage problem (1) and the feasibility of the financing problem (5) turn out to be the boundedness and feasibility conditions, respectively, for the writer's problem (8). The analysis of the next theorem reveals that these relationships depend only on two large-scale features of the utility, namely that it is concave and increasing and that its domain is the nonnegative reals.

Theorem 3. *If the writer invests to maximize expected utility of terminal value*

$$\begin{array}{lll} \max_{(\theta)} & \sum_{n\in\mathcal{N}_T} p_n u(Z_n \cdot \theta_n) & \\ \textit{subject to} & Z_0 \cdot \theta_0 & = \beta_0 F_0 \\ & Z_n \cdot [\,\theta_n - \theta_{a(n)}\,] & = -\beta_n F_n \;\; (n \in \mathcal{N}_t,\; t \geq 1) \end{array} \tag{12}$$

for any strictly increasing closed concave utility function u that has domain equal to the nonnegative real numbers, then (12) *has an optimum if and only if*

1. *The collection $\mathcal{Q}$ of equivalent martingale probability measures on the market price process $\{Z_t\}$ is nonempty, and*
2. *The option price F_0 satisfies*

$$\beta_0 F_0 \geq \max_{Q\in\overline{\mathcal{Q}}} E^Q\,[\sum_{t=1}^{T} \beta_t F_t\,]$$

where the maximum is taken over the collection of all martingale probability measures $\overline{\mathcal{Q}}$ on $\{Z_t\}$.

Proof. Primal feasibility is governed by the domain of u. Since u and u_w have the same domain, it follows that the writer's problem is feasible if and only if F_0 satisfies (9). Dual feasibility is governed by the domain of the dual problem

$$\begin{array}{ll} \min_{(y)} & \beta_0 F_0 y_0 - \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} \beta_n F_n y_n - \sum_{n\in\mathcal{N}_T} p_n u^*(y_n/p_n) \\ \text{subject to} & y_n Z_n - \sum_{m\in\mathcal{C}(n)} y_m Z_m = 0 \quad (n \in \mathcal{N}_t,\; t \leq T-1) \end{array} \tag{13}$$

where u^* is the concave conjugate of u

$$u^*(y) = \inf_v [yv - u(v)].$$

Since u is closed, concave, and the domain of the primal problem is polyhedral and non-empty, Fenchel's Duality Theorem [21, Theorem 31.1] guarantees the attainment of the optimal solutions and the equality of their optimal values whenever the dual problem is feasible. Since u is strictly increasing, it follows that the subgradient sets $\partial u(\cdot)$ are bounded below by a positive constant, say $c > 0$. The domain of u^* is thus strictly greater than 0, so the domain of the dual problem (13) is equivalent to the domain of the dual writer's problem (11).

□

Thus risk aversion modeled in this way makes no difference to prices of contingent claims. There are only two issues that would alter these conclusions. One is that the utility u is merely non-decreasing and not strictly increasing. In this case it is easy to see that the martingale condition be relaxed to allow any probability measure.

Corollary 3. *If u is only non-decreasing, then criteria (1) may be weakened to the existence of a probability measure Q (not necessarily equivalent to P) under which $\{Z_t\}$ is a martingale.*

The other issue arises when the domain of the utility u is bounded away from zero, say if $u(v)$ in Theorem 3 is shifted by an amount $\alpha > 0$ to $u(v-\alpha)$. This would guarantee that the portfolio value would not fall below α in the terminal stage. Computing the conjugate throws up an additional term $-\alpha \sum_{n \in \mathcal{N}_T} y_n$ in the dual problem (13), and consequently, boundedness of the dual will require lifting the price by exactly that amount. A related issue arises when the domain of the utility does not include the point α (like the logarithm or power utilities).

Corollary 4. *If the domain of u equals $[\alpha, +\infty)$ where $\alpha > 0$, then criteria (2) must be strengthened to*

$$\beta_0 F_0 \geq \max_{Q \in \overline{\mathcal{Q}}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right] + \alpha. \tag{14}$$

Furthermore, if the domain of u equals $(\alpha, +\infty)$ then the inequality is strengthened to a strict inequality.

In conclusion, prices of contingent claims in these models are unaffected by risk aversion behavior or by the investor's assumptions concerning the original probability P. (Of course, the actual portfolios selected by the investor are affected both by the risk aversion and the probability estimates.) Differences between acceptable loss levels α for buyer and seller might cause a transaction to take place. But this begs the question: why would either the buyer or the seller be willing to accept the risk of loss? We reserve discussing this issue until Sect. 8.

7. Access to markets: spreads and margins

A further key difference between market players concerns the relative expense of trading. Transaction costs induce a spread between the bid and ask price of a security, and furthermore, selling a security short typically yields something less than the bid price to the short seller because of the risk that the short seller may go bankrupt. Differences between investor's access to markets can be modeled by such spreads, see Dermody and Rockafellar [6].

Let S^s denote the vector of shorting prices (the money a short seller would receive), S^b denote the selling (bid) prices, and S^a denote the buying (ask) prices. One has

$$S^s \leq S^b \leq S^a.$$

For simplicity, assume that the bid price and the ask price (but not the shorting price) of the numeraire security are equal, and as before, let $\beta_t = 1/S_t^{0,a} = 1/S_t^{0,b}$ denote the discount process for this numeraire. Define the normalized prices $Z_t^{[s,a,b]} = \beta_t S_t^{[s,a,b]}$, and note as before that the normalized price for buying and selling (but not shorting) the numeraire is exactly 1 in all states.

The formulation of the investor's problem is made more complex by the requirement to keep track of the long and short portfolios and the buying, selling and shorting transactions. Let $\theta_n^+ \geq 0$ and $\theta_n^- \geq 0$ denote the long and short portfolio holdings, respectively. A portfolio transaction in the long portfolio will be indicated by the variables $\delta_n^a \geq 0$, buying at the ask price, and $\delta_n^b \geq 0$, selling at the bid price:

$$\theta_n^+ - \theta_{a(n)}^+ = \delta_n^a - \delta_n^b \quad n > 0 \tag{15a}$$

$$\theta_0^+ = \delta_0^a - \delta_0^b \tag{15b}$$

and transactions in the short portfolio

$$\theta_n^- - \theta_{a(n)}^- = \epsilon_n^s - \epsilon_n^a \quad n > 0 \tag{15c}$$

$$\theta_0^- = \epsilon_0^s - \epsilon_0^a \tag{15d}$$

where an increase in the short portfolio corresponds to a short sale $\epsilon_n^s \geq 0$ and a decrease to a purchase $\epsilon_n^a \geq 0$.

Suppose that the writer has a utility function as in Theorem 3, then the writer's problem with spreads may be written as a maximization in the nonnegative variables $\theta^{[+,-]}$, $\delta^{[a,b]}$ and $\epsilon^{[s,a]}$:

$$\begin{array}{lll} \max & \sum_{n\in\mathcal{N}_T} p_n u\big(Z_n^b \cdot \theta_n^+ - Z_n^a \cdot \theta_n^-\big) & \\ \text{subject to} & Z_0^a \cdot \big[\delta_0^a - \delta_0^b\big] - Z_0^s \cdot \big[\epsilon_0^s - \epsilon_0^a\big] & = \beta_0 F_0 \\ & Z_n^a \cdot [\,\delta_n^a + \epsilon_n^a\,] - Z_n^b \cdot \delta_n^b - Z_n^s \cdot \epsilon_n^s & = -\beta_n F_n \ (n \in \mathcal{N}_t,\ t \geq 1). \end{array} \tag{16}$$

Some presentations of this problem allow inequalities in the self-financing conditions. In the model presented here, bid-ask prices for the numeraire are equal so any excess cash can be placed in the numeraire without cost. For an analysis of hedging with spreads without the assumption of costless transactions in the numeraire, see Edirisinghe, Naik and Uppal [9].

Equations (15) are assumed to be part of the writer's linear program. Assign the dual variables U^+ and U^-, respectively, to these (vector) equations, and let the dual assignments for the other equations be as in (8). The dual to the investor's problem with spreads may be written as follows.

$$\begin{array}{lll} \min_{(y,U^+,U^-)} & \beta_0 F_0 y_0 - \sum_{t=1}^T \sum_{n\in\mathcal{N}_t} \beta_n F_n y_n - \sum_{n\in\mathcal{N}_T} p_n u^*(y_n/p_n) & \\ \text{subject to} & U_n^+ - \sum_{m\in\mathcal{C}(n)} U_m^+ \geq 0 & (n \in \mathcal{N}_t,\ t \leq T-1) \\ & U_n^- - \sum_{m\in\mathcal{C}(n)} U_m^- \leq 0 & (n \in \mathcal{N}_t,\ t \leq T-1) \\ & y_n Z_n^a \geq U_n^+ \geq y_n Z_n^b & \forall n \\ & y_n Z_n^a \geq U_n^- \geq y_n Z_n^s & \forall n. \end{array} \tag{17}$$

The numeraire's buying and selling cost are equal to one, hence it follows that $U_n^{0,+} = y_n$ and one has

$$y_0 \geq \sum_{n\in\mathcal{N}_1} y_n \geq \cdots \geq \sum_{n\in\mathcal{N}_T} y_n.$$

But the shorting cost $Z_n^{0,s}$ is less than one, so one only has

$$Z_0^{0,s} y_0 \leq \sum_{n \in \mathcal{N}_T} y_n.$$

The y_n are positive because the utility function is strictly increasing, but since the sum of the y_n over each period may not be constant, it cannot be interpreted as a measure. Define q_n as the terminal probabilities generated by the y_n:

$$q_n := y_n / \sum_{n \in \mathcal{N}_T} y_n, \quad n \in \mathcal{N}_T$$

and extend q_n to the intermediate nodes in the usual way. Introduce the (variable) parameters

$$\gamma_n := \frac{y_n}{q_n y_0}, \quad \forall n.$$

By construction $\gamma_0 = 1$. The values γ_n multiplies all the state flows and prices of the system, so it may be thought of as a variable that multiplies the discount factor. This leads to the following. The proof follows the pattern of Theorem 3.

Theorem 4. *Under the assumptions on the utility function in Theorem 3, the writer's problem with spreads* (16) *has a solution if and only if there exist a probability measure Q and a discount multiplier process $\{\gamma_t\}$ with $\gamma_0 = 1$ such that*

$$\text{There is a (vector) } Q\text{-supermartingale} \quad U_t^+ \in \left[\gamma_t Z_t^a, \gamma_t Z_t^b\right] \tag{18a}$$

$$\text{There is a (vector) } Q\text{-submartingale} \quad U_t^- \in \left[\gamma_t Z_t^a, \gamma_t Z_t^s\right] \tag{18b}$$

and the price charged by the writer satisfies

$$\beta_0 F_0 \geq \sup_{Q,\gamma} E^Q \left[\sum_{t=1}^{T} \gamma_t \beta_t F_t \right] \tag{19}$$

where the supremum is taken over all probability measures Q and discount multiplier processes $\{\gamma_t\}$ satisfying (18) *and $\gamma_0 = 1$.*

Since the bid-ask spread is 0 for the numeraire, it follows from (18) that $\{\gamma_t\}$ is a supermartingale. Furthermore one can also establish that $\gamma_n = \sum_{n \in \mathcal{N}_T} y_n / y_0 \in [Z_0^{0,s}, 1]$ for $n \in \mathcal{N}_T$ so that $0 \geq \gamma_n \geq Z_0^{0,s}$ for all n. I am indebted to M.A.H. Dempster for the observation that the discount multiplier $\{\gamma_t\}$ acts to steepen the discount relative to the numeraire, due to the requirement on the investor to borrow funds at a spread above the numeraire in this model.

Investors often are required to limit their short positions to some proportion of their long position; this is called a margin requirement. Margin requirements are easily modeled in the setting of this section. Consider a margin requirement of the form

$$Z_n^b \theta_n^+ \geq M Z_n^a \theta_n^-, \quad (n \in \mathcal{N}_t, \ t \leq T-1). \tag{20}$$

This says that the investor's long position must be a multiple M of the short position in every state except the terminal states (in which all positions are closed). Adding this requirement to the problem (16), and denoting the dual multipliers for (20) as w_n, the following corollary is obtained.

Corollary 5. *If the investor of problem* (16) *has margin requirements of the form* (20), *then the dual martingale conditions for U_t^+ and U_t^- are refined as follows*

$$\gamma_t U_t^+ - w_t \gamma_t Z_t^b \geq E^Q \left[\gamma_{t+1} U_{t+1}^+ \,\middle|\, \mathcal{N}_t \right]$$
$$\gamma_t U_t^- - M w_t \gamma_t Z_t^a \leq E^Q \left[\gamma_{t+1} U_{t+1}^- \,\middle|\, \mathcal{N}_t \right].$$

Furthermore, the supremum in (19) *must be performed over the set of feasible w as well.*

Let us now examine the arbitrage interval from the perspective of Theorem 4. The writer's minimum price satisfies

$$F_0^w = \beta_0^{-1} \sup_{Q,\gamma} E^Q \left[\sum_{t=1}^{T} \gamma_t \beta_t F_t \right]$$

and the buyer's maximum price satisfies

$$F_0^b = \beta_0^{-1} \inf_{Q,\gamma} E^Q \left[\sum_{t=1}^{T} \gamma_t \beta_t F_t \right].$$

Since the dual system with spreads or margins is *less constrained* than the system (13) it follows that the gap between buyer and seller is *wider* if both buyer and seller face spreads or margins! Even if buyer and seller face different spreads or margins (as opposed to different prices) this gap will never be smaller than the situation with no spreads or margins. It follows that buyer and seller will not trade even when transactions costs are introduced.

8. Liability structures and endowments

Neither differential attitudes to risk nor differential transactions costs will induce trading between buyers and sellers of contingent claims in the framework so far developed. It seems reasonable to consider extending the framework to try to see what will. In this section we introduce existing liability structures or endowments for investors and explore the consequences for options pricing.

An existing liability structure or endowment of an investor can be modeled as if it were a contingent claim: in each state n there is a payout (positive or negative) denoted L_n. The liability may not itself be a tradeable security (perhaps for policy reasons internal to the investor) but its flows are correlated with the market. As is common in microeconomic modeling, the utility functions of the investors are assumed to be nondecreasing and to have domain equal to or contained in the nonnegative reals. The development will be in the setting of Theorem 3 (a similar development applies to the situation covered by Theorem 4). The investor optimizes the problem (12) replacing the contingent flows F_n with the liability flows L_n. The value L_0 then has the interpretation of the net capital of the investor and the optimal dual solution y^* gives rise to the investor's own "valuation operator".

Now consider the investor's attitude to a contingent claim F with payouts F_n offered on the market for the price F_0. The investor would buy the claim if

$$\beta_0 F_0 y_0^* < \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} \beta_n F_n y_n^*$$

and would sell the claim if

$$\beta_0 F_0 y_0^* > \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} \beta_n F_n y_n^*.$$

Now that the claim has been introduced into the universe of possible investments, the following problem determines how many shares of F the investor wishes to purchase:

$$\begin{array}{lll} \max_{(\theta,\epsilon_0)} & \sum_{n\in\mathcal{N}_t} p_n u(Z_n \cdot \theta_n) & \\ \text{subject to} & Z_0\theta_0 & = \beta_0 L_0 - \epsilon_0 \beta_0 F_0 \\ & Z_n \cdot [\,\theta_n - \theta_{a(n)}\,] & = -\beta_n L_n + \epsilon_0 \beta_n F_n \ (n \in \mathcal{N}_t,\ t \geq 1). \end{array} \tag{22}$$

In keeping with the theme of this paper, let us examine the dual

$$\begin{array}{ll} \min_{(y)} & \beta_0 L_0 y_0 - \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} \beta_n L_n y_n - \sum_{n\in\mathcal{N}_T} p_n u^*(y_n/p_n) \\ \text{subject to} & y_n Z_n - \sum_{m\in\mathcal{C}(n)} y_m Z_m = 0 \quad (n \in \mathcal{N}_t,\ t \leq T-1) \\ & y_0 \beta_0 F_0 = \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} y_n \beta_n F_n . \end{array} \tag{23}$$

The last constraint in the dual system shows that the introduction of the security F alters the investor's optimal dual solution. For example, to be a buyer of the option one has to have had

$$\beta_0 F_0 < E^{Q_b^*}[\sum_{t=1}^{T} \beta_t F_t\,] \tag{24}$$

but after incorporation of the claim into the optimal portfolio, the buyer's martingale measure must in fact satisfy a "risk-neutral" valuation of F — namely,

$$\beta_0 F_0 = E^{Q_b^*}[\sum_{t=1}^{T} \beta_t F_t\,]\,. \tag{25}$$

A similar statement holds for sellers. Every investor will change their valuation operator so that the current price equals the expectation of the cash-flows.

The quantity that is actually transacted is $\min(\epsilon_0^b, \epsilon_0^s)$, and the price F_0 may shift in response to the supply-demand imbalance. A partial equilibrium argument for the price of the claim F could go as follows. Suppose that some investors will desire to buy F if the price is sufficiently low (prices are permitted to be negative) and some desire to sell if the price is sufficiently high. Then as the price moves from such a sufficiently low value to higher values there exists a price F_0^* at which the excess demand function $F_0 \mapsto (\epsilon_0^b - \epsilon_0^s)$ crosses zero. We have not actually shown that investors will desire to trade precisely the quantities needed to clear the market at the price F_0^*. The question comes down to the existence of a market clearing price.

Definition 2. *Suppose that there is a finite population of investors, $k = 1, \ldots, K$ each solving problem* (22)*. We say that a market clearing price F_0^* for the contingent claim F exists when each investor's optimal choice ϵ_0^k of the claim satisfies $\sum_{k=1}^K \epsilon_0^k = 0$ and there is at least one investor with $\epsilon_0^k > 0$.*

An examination of the dual problem (23) shows that each investor's dual problem is more tightly constrained in general, so that each investor's optimal value is *the same or higher* after the introduction of the claim F. If there exists a market clearing price, then there is at least one investor whose optimal dual solution satisfied equation (24) without F and satisfies (25) after the introduction of F. Since (25) was infeasible for this investor before the introduction of F it follows that the optimal value of their investment problem must be *strictly higher* after the introduction of the claim (because the dual (23) is more tightly constrained). We have finally proved a theorem that indicates unequivocally why there is buyer and seller interest in contingent claims.

Theorem 5. *Suppose that a contingent claim F is introduced into a marketplace and that there exists a market clearing price F_0^* for the claim. Then every investor's optimal valuation operator values the claim at the market clearing price. After the introduction of the claim, every investor's optimal utility will be the same or higher and at least two investors will have a strictly higher optimal utility.*

The question of existence of market clearing prices involves technical conditions that at least guarantee no infinite jumps in the excess demand function. In addition one must introduce a dynamic "tatonnement" process by which market clearing prices are found. Of course, in general, F will not be the only security available to investors to hedge the risks of their structural liabilities or endowments and so the simple outlines of the partial equilibrium argument may not suffice. The study of general financial equilibrium issues in incomplete markets is an active research area that is outside the scope of this paper. For more on this topic, see for instance, Allen and Gale [1] or Grossman and Hart [10].

9. Conclusions and discussion

The connections drawn in this paper between arbitrage, contingent claims pricing and martingales arise from quite elementary analyses of duality relationships. Nevertheless, they provide a rich mathematical structure. Extensions of the models are quite natural, as the sections on risk aversion, liability structures, and spreads and margins demonstrate.

This framework extends naturally to those of asset-liability management problems with side constraints, such as those of regulated insurance pools or pension funds (cf. Cariño, et al [3]), or those of financial intermediaries with reserve capital constraints, or even problems having real production processes — as in the emerging energy markets of our time (cf. King, Birge, Takriti and Wu [16] and King and Ahmed [15]).

It is interesting to speculate on the interpretation of the optimal dual variable y_0. In Theorem 2 it appears as a multiplier of the difference between the price charged by the writer and the writer's valuation of the claim. In the interpretation of the investor's problem (22) it appears to be the optimal value to the investor of increasing their capital L_0 by one additional unit. One suspects that the investors of Sect. 8 who participate in a market for contingent claims ought to have y_0^* equal to the expected return of one share of equity in a financial intermediary firm.

The investors of Sect. 8 do not in general know the appropriate probability distribution — they must infer it from observed prices of series of options. This problem is treated in the framework of this paper by King, Streltchenko and Yesha [14].

There are powerful computational approaches to the solution of stochastic programming problems like the ones developed in this paper. Decomposition and sampling are natural for this class of problems, and very large problems can now be solved. For a survey of recent progress in stochastic programming, see Birge [2]. A recent paper of Edirisinghe [8] investigates techniques for establishing bounds deriving from the observation that the dual problem is a version of the well-known "moments problem". The specialized structure of the models of this paper may in particular be well-suited for stochastic programming methods adapted to the stochastic structures of market processes.

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