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On the rank of mixed 0,1 polyhedra*

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Abstract. For a polytope in the $[0,1]^n$ cube, Eisenbrand and Schulz showed recently that the maximum Chvátal rank is bounded above by $O(n^2 logn)$ and bounded below by $(1+\epsilon)n$ for some $\epsilon > 0$. Chvátal cuts are equivalent to Gomory fractional cuts, which are themselves dominated by Gomory mixed integer cuts. What do these upper and lower bounds become when the rank is defined relative to Gomory mixed integer cuts? An upper bound of n follows from existing results in the literature. In this note, we show that the lower bound is also equal to n. This result still holds for mixed 0,1 polyhedra with n binary variables.

Key words. mixed integer cut – disjunctive cut – split cut – rank – mixed 0,1 program

1. Introduction

Consider a mixed integer program $P_I \equiv \{(x,y) \in Z^n_+ \times R^p_+ |\ Ax + Gy \le b\}$, where A and G are given rational matrices (dimensions $m \times n$ and $m \times p$ respectively) and b is a given rational column vector (dimension m). Let $P \equiv \{(x,y) \in R^{n+p}_+ |\ Ax + Gy \le b\}$ be its standard linear relaxation. Assume w.l.o.g. that $x \ge 0$, $y \ge 0$ and $0 \le 1$ are part of the constraints $Ax + Gy \le b$ (thus any valid inequality for P is of the form $u(Ax + Gy) \le ub$ for $u \in R^m_+$). In [13], Gomory introduced a family of valid inequalities for P_I, called *mixed integer cuts*, that can be used to strengthen P. These cuts are obtained from P by considering an equivalent equality form. Let $P' = \{(x,y,s) \in R^{n+p+m}_+ |\ Ax + Gy + s = b\}$ and $P'_I = \{(x,y,s) \in Z^n_+ \times R^{p+m}_+ |\ Ax + (G,I)\binom{y}{s} = b\}$. Introduce $z = \binom{y}{s}$. For any $u \in R^m$, let $\bar{a} = uA$, $\bar{g} = u(G,I)$ and $\bar{b} = ub$. Let $\bar{a}_i = \lfloor \bar{a}_i \rfloor + f_i$ and $\bar{b} = \lfloor \bar{b} \rfloor + f_0$. Gomory showed that the following inequality is valid for P'_I:

$$\sum_{(i:f_i \le f_0)} f_i x_i + \frac{f_0}{1-f_0} \sum_{(i:f_i > f_0)} (1-f_i) x_i + \sum_{(j:\bar{g}_j \ge 0)} \bar{g}_j z_j - \frac{f_0}{1-f_0} \sum_{(j:\bar{g}_j < 0)} \bar{g}_j z_j \ge f_0. \qquad (1)$$

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Plugging $s = b - Ax - Gy$ into it, we get a valid inequality for P_I. Any such inequality $\alpha_u x + \gamma_u y \leq \beta_u$ is called a *mixed integer cut*. The convex set P^1 defined as the intersection of P with all mixed integer cuts is called the *mixed integer closure* of P. In fact P^1 is a polyhedron (see below). By recursively taking the mixed integer closure of P^{k-1}, for integers $k \geq 2$, we obtain the polyhedron P^k. Clearly $P_I \subseteq P^k \subseteq P^{k-1} \ldots \subseteq P^1 \subseteq P$. We say that P_I is a *mixed 0,1 program with n binary variables* if $P \subseteq [0, 1]^n \times R^p_+$. For mixed 0,1 programs, there is always a finite k such that $P^k = Conv(P_I)$ (see below). The smallest such k is called the *mixed integer rank* of P. In this note, we show the following.

Theorem 1. *The maximum mixed integer rank of P, taken over all mixed 0,1 programs P_I with n binary variables, is equal to n.*

In particular, the maximum mixed integer rank for a pure integer program in the $[0, 1]^n$ cube is equal to n. This is in contrast to the maximum Chvátal rank which was shown by Eisenbrand and Schulz [11] to lie in the interval $(1 + \epsilon)n$ to $O(n^2 logn)$ for some $\epsilon > 0$.

2. Disjunctive cuts

In this section, we review three results from the literature. To prove Theorem 1, we use the equivalence between mixed integer cuts and disjunctive cuts from 2-term disjunctions shown by Nemhauser and Wolsey [16]. Disjunctive cuts were introduced by Balas [1], [2]. The disjunctive cuts from 2-term disjunctions were also studied by Cook, Kannan and Schrijver [9] under the name of split cuts. We use this terminology in the remainder. Given the polyhedron $P \equiv \{(x, y) \in R^{n+p}_+ | \ Ax + Gy \leq b\}$, an inequality is called a *split cut* if it is valid for $Conv((P \cap \{x | \ \pi x \leq \pi_0\}) \cup (P \cap \{x | \ \pi x \geq \pi_0 + 1\}))$ for some $(\pi, \pi_0) \in Z^{n+1}$.

Many of the classical cutting planes can be interpreted as split cuts. For instance, in the case of pure integer programs, Chvátal cuts [5] are split cuts where at least one of the two polyhedra $P \cap \{x | \ \pi x \leq \pi_0\}$ or $P \cap \{x | \ \pi x \geq \pi_0 + 1\}$ is empty. (Indeed, if say $P \cap \{x | \ \pi x \geq \pi_0 + 1\}$ is empty, then $\pi x < \pi_0 + 1$ is valid for P, which implies that the split cut $\pi x \leq \pi_0$ is a Chvátal cut and, conversely, any Chvátal cut can be obtained this way). As another example, it is well known that the lift-and-project cuts [3] are split cuts obtained from the disjunction $x_j \leq 0$ or $x_j \geq 1$, i.e. they are valid inequalities for $Conv((P \cap \{x | \ x_j \leq 0\}) \cup (P \cap \{x | \ x_j \geq 1\}))$.

Nemhauser and Wolsey [16] showed that split cuts are equivalent to mixed integer cuts, using the concepts of MIR inequalities and superadditive inequalities as intermediate steps. In the next theorem, we give a direct proof of this equivalence. The convex set defined as the intersection of all split cuts is called the *split closure* of P.

Theorem 2. *The split closure of P is identical to the mixed integer closure of P.*

Proof. We first show that any split cut $cx + hy \leq c_0$ that is not valid for P is equal to or dominated by a mixed integer cut. From the definition of a split cut, there exists $(\pi, \pi_0) \in Z^{n+1}$ such that the inequality $cx + hy \leq c_0$ is valid for both polyhedra

$P \cap \{x | \pi x \le \pi_0\}$ and $P \cap \{x | \pi x \ge \pi_0 + 1\}$. It follows from linear programming duality that there exist scalars $\alpha,\ \beta > 0$ such that

$$cx + hy - \alpha(\pi x - \pi_0) \le c_0 \tag{2}$$

$$cx + hy + \beta(\pi x - \pi_0 - 1) \le c_0 \tag{3}$$

are both valid inequalities for P. Introduce nonnegative slack variables t_1 and t_2 in (2) and (3) respectively. Since these inequalities are valid for P, it follows that $t_1 = u^1 s$ and $t_2 = u^2 s$ for some vectors $u^1, u^2 \in R^m_+$. Let $u = u^2 - u^1$, $u_i^+ = \max\{0, u_i\}$ and $u_i^- = \max\{0, -u_i\}$. Subtract (2) with its slack from (3) with its slack. The resulting equality

$$\pi x - \frac{1}{\alpha + \beta} u^- s + \frac{1}{\alpha + \beta} u^+ s = \pi_0 + \frac{\beta}{\alpha + \beta} \tag{4}$$

is valid for the higher dimensional equality form P' of P. Now apply Gomory's formula (1) to equation (4) to obtain the following mixed integer cut:

$$\frac{\beta}{\alpha} \frac{1}{\alpha + \beta} u^- s + \frac{1}{\alpha + \beta} u^+ s \ge \frac{\beta}{\alpha + \beta}.$$

This cut is equal to or dominates:

$$\frac{\beta}{\alpha} \frac{1}{\alpha + \beta} t_1 + \frac{1}{\alpha + \beta} t_2 \ge \frac{\beta}{\alpha + \beta}.$$

Replacing t_1 and t_2 by their expressions in (2) and (3) yields:

$$cx + hy \le c_0.$$

Conversely, the standard proof that mixed integer cuts are valid for P_I shows that they are split cuts. Indeed, let $\bar{a}x + \bar{g}z = \bar{b}$ be a valid equality for P'. Rewrite this equality by separating the integer and fractional parts of $\bar{a}_i$ and $\bar{b}$, and by grouping all the integer parts together. Thus

$$\sum_{(i: f_i \le f_0)} f_i x_i - \sum_{(i: f_i > f_0)} (1 - f_i) x_i + \bar{g}z = f_0 - \pi x + \pi_0 \tag{5}$$

is a valid equality for P', for some $(\pi, \pi_0) \in Z^{n+1}$. It follows that

$$\sum_{(i: f_i \le f_0)} f_i x_i - \sum_{(i: f_i > f_0)} (1 - f_i) x_i + \bar{g}z \ge f_0 \tag{6}$$

is valid for $P \cap \{x | \pi x \le \pi_0\}$ and that

$$-\sum_{(i: f_i \le f_0)} f_i x_i + \sum_{(i: f_i > f_0)} (1 - f_i) x_i - \bar{g}z \ge 1 - f_0 \tag{7}$$

is valid for $P \cap \{x | \pi x \ge \pi_0 + 1\}$. Since $x \ge 0$ and $z \ge 0$, it is easy to verify that the inequality (1) is dominated by both (6) and (7), so it is valid for $Conv((P \cap \{x | \pi x \le \pi_0\}) \cup (P \cap \{x | \pi x \ge \pi_0 + 1\}))$. Therefore it is a split cut.

□

Cook, Kannan and Schrijver [9] showed that the split closure of P is a polyhedron: it is the intersection of finitely many sets $Conv((P \cap \{x | \pi x \leq \pi_0\}) \cup (P \cap \{x | \pi x \geq \pi_0 + 1\}))$ for $(\pi, \pi_0) \in Z^{n+1}$. Therefore the mixed integer closure P^1 is also a polyhedron. By induction, P^k defined above is a polyhedron for all integers $k \geq 1$. If there exists an integer k such that $P^k = Conv(P_I)$, the smallest such k was defined above as the mixed integer rank of P. In general, mixed integer programs do not have a finite mixed integer rank, as shown by Cook, Kannan and Schrijver [9] using a simple example with two integer variables and one continuous variable.

Theorem 3. *There exist mixed integer programs P_I such that $P^k \neq Conv(P_I)$ for all finite integers k.*

Proof. Let $P_I \equiv \{(x_1, x_2, y) \in Z_+^2 \times R_+ | x_1 - y \geq 0,\ x_2 - y \geq 0,\ x_1 + x_2 + 2y \leq 2\}$. Then P is the convex hull of $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, whereas $Conv(P_I)$ is the convex hull of the first three points. The inequality $y \leq 0$ is valid for $Conv(P_I)$ but it is easy to show by induction that $y \leq 0$ is not valid for P^k for any finite integer k. Indeed, assume (induction hypothesis) that P^k contains points (x_1, x_2, y) with $y > 0$ for any (x_1, x_2) such that $x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 2$. Then, for any $(\pi, \pi_0) \in Z^3$, the set $\Pi \equiv Conv((P^k \cap \{x | \pi x \leq \pi_0\}) \cup (P^k \cap \{x | \pi x \geq \pi_0 + 1\}))$ contains a point (x_1, x_2, y) with $y > 0$ and $x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 2$. Since Π also contains the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$ and is convex, Π contains points (x_1, x_2, y) with $y > 0$ for any (x_1, x_2) such that $x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 2$. Since P^{k+1} is the intersection of finitely many sets of this form, the induction hypothesis holds for P^{k+1}.

□

Mixed 0,1 programs have the property that the disjunction $x_j \leq 0$ or $x_j \geq 1$ is facial, i.e. both $P \cap \{x | x_j \leq 0\}$ and $P \cap \{x | x_j \geq 1\}$ define faces of P. If follows from a result of Balas [2] on facial disjunctive programs that the mixed integer rank of a mixed 0,1 program is at most n.

Theorem 4. *For a mixed 0,1 program P_I with n binary variables, $P^n = Conv(P_I)$.*

Proof. Define $P_0 \equiv P$ and, for $k = 1, \ldots, n$, let $P_k \equiv Conv((P_{k-1} \cap \{x_k = 0\}) \cup (P_{k-1} \cap \{x_k = 1\}))$.

We claim that $P_k = Conv(P \cap S_k)$ where $S_k \equiv \{0, 1\}^k \times [0, 1]^{n-k} \times R^p$.

The claim is true for $k = 1$. Let $k \geq 2$ and assume $P_{k-1} = Conv(P \cap S_{k-1})$. Then

$$\begin{aligned} P_k &= Conv((Conv(P \cap S_{k-1}) \cap \{x_k = 0\}) \cup (Conv(P \cap S_{k-1}) \cap \{x_k = 1\})) \\ &= Conv(Conv(P \cap S_{k-1} \cap \{x_k = 0\}) \cup (Conv(P \cap S_{k-1} \cap \{x_k = 1\})) \end{aligned}$$

because, when a set S lies entirely in the closed half-space limited by a hyperplane H, $Conv(S) \cap H = Conv(S \cap H)$. Now, since $Conv(Conv(A) \cup Conv(B)) = Conv(A \cup B)$,

$$\begin{aligned} P_k &= Conv((P \cap S_{k-1} \cap \{x_k = 0\}) \cup (P \cap S_{k-1} \cap \{x_k = 1\})) \\ &= Conv(P \cap S_k). \end{aligned}$$

The claim implies that $P_n = Conv(P_I)$. Since $P^n \subseteq P_n$, the theorem follows.

□

New results in this direction were obtained recently by Balas and Perregaard [4].

3. Proof of Theorem 1

Theorem 4 shows the upper bound in Theorem 1. Next, we exhibit an example with a lower bound of n, thus completing the proof of Theorem 1.

We show that the mixed integer rank of the following well-known polytope studied by Chvátal, Cook, and Hartmann [6] is exactly n:

$$P_n \equiv \{x \in [0,1]^n | \sum_{j\in J} x_j + \sum_{j\notin J}(1 - x_j) \geq \frac{1}{2}, \text{ for all } J \subseteq \{1, 2, \cdots, n\}\}$$

just as its Chvátal rank is.

Let F_j be the set of all vectors $x \in R^n$ such that j components of x are $\frac{1}{2}$ and each of the remaining $n - j$ components are equal to 0 or 1. The polyhedron P_n is the convex hull of F_1.

Lemma 1. *If a polyhedron $P \subseteq R^n$ contains F_j, then its mixed integer closure P^1 contains F_{j+1}.*

Proof. It suffices to show that, for every $(\pi, \pi_0) \in Z^{n+1}$, the polyhedron $\Pi = Conv((P \cap \{x| \pi x \leq \pi_0\}) \cup (P \cap \{x| \pi x \geq \pi_0 + 1\}))$ contains F_{j+1}. Let $v \in F_{j+1}$ and assume w.l.o.g. that the first $j + 1$ elements of v are equal to $\frac{1}{2}$. If $\pi v \in Z$, then clearly $v \in \Pi$. If $\pi v \notin Z$, then at least one of the first $j+1$ components of π is nonzero. Assume w.l.o.g. that $\pi_1 > 0$. Let $v_1, v_2 \in F_j$ be equal to v except for the first component which is 0 and 1 respectively. Notice that $v = \frac{v_1+v_2}{2}$. Clearly, each of the intervals $[\pi v_1, \pi v]$ and $[\pi v, \pi v_2]$ contains an integer. Since πx is a continuous function, there are points $\tilde{v}_1$ on the line segment $Conv(v, v_1)$ and $\tilde{v}_2$ on the line segment $Conv(v, v_2)$ with $\pi \tilde{v}_1 \in Z$ and $\pi \tilde{v}_2 \in Z$. This means that $\tilde{v}_1$ and $\tilde{v}_2$ are in Π. Since $v \in Conv(\tilde{v}_1, \tilde{v}_2)$, this implies $v \in \Pi$.

□

Starting from $P = P_n$ and applying the lemma recursively, it follows that the $(n-1)$st mixed integer closure P_n^{n-1} contains F_n, which is nonempty. Since $Conv((P_n)_I)$ is empty, the mixed integer rank of P_n is at least n. This completes the proof of Theorem 1.

4. Concluding remarks

In this note, we considered Gomory's mixed integer procedure applied to polytopes P in the n-dimensional 0, 1-cube. Lovász and Schrijver [14] introduced a different procedure, based on a semi-definite relaxation of P_I for strengthening a polytope P in the n-dimensional 0, 1-cube. Recently, Cook and Dash [8] and Goemans and Tuncel [12] established that the semi-definite rank of polytopes in the n-dimensional 0, 1-cube is equal to n, in the worst case, by showing that the semi-definite rank of P_n (as defined in Sect. 3) is equal to n. Although the mixed integer and semi-definite closures are incomparable (neither contains the other in general), both are contained in the lift-and-project closure as introduced by Balas, Ceria and Cornuéjols [3]. Since the lift-and-project rank is at most n [3] and the semi-definite and mixed integer ranks of P_n equal

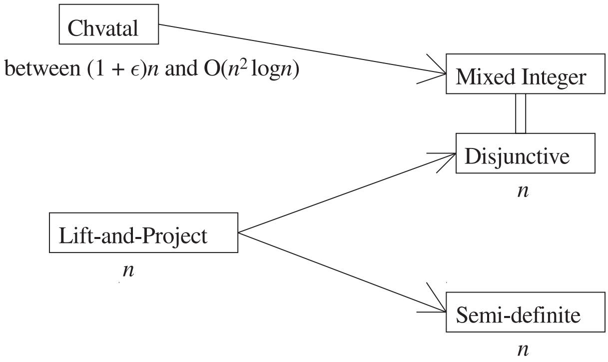

Fig. 1. Maximum rank of polytopes in the $[0, 1]^n$ cube

n, it follows that, in the worst case, all three procedures have rank n. We summarize this in Fig. 1 where $A \to B$ means that the corresponding elementary closures satisfy $P_A \supseteq P_B$ and the inclusion is strict for some instances, and A not related to B in the figure means that for some instances $P_A \not\subseteq P_B$ and for other instances $P_B \not\subseteq P_A$. A figure comparing elementary closures derived from several other cuts can be found in [10].

Cook and Dash [8] also considered the intersection of the Chvátal closure and the semi-definite closure. They showed that, even for this Chvátal + semi-definite closure, it is still the case that the rank of P_n equals n. In a similar way, we can define the disjunctive + semi-definite closure of a mixed 0,1 program P_I as the intersection of the disjunctive closure and the semi-definite closure of P. Using the approach of Cook and Dash and Sect. 3 above, it is easy to show that the mixed integer + semi-definite rank of P_n is equal to n.

Theorem 5. *The mixed integer + semi-definite rank of P_n is exactly n.*

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