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Vertical linear complementarity and discounted zero-sum stochastic games with ARAT structure

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Abstract. In this paper we consider a two-person zero-sum discounted stochastic game with ARAT structure and formulate the problem of computing a pair of pure optimal stationary strategies and the corresponding value vector of such a game as a vertical linear complementarity problem. We show that Cottle-Dantzig's algorithm (a generalization of Lemke's algorithm) can solve this problem under a mild assumption.

Key words. ARAT – Cottle-Dantzig's algorithm – VLCP

1. Introduction

In this paper we consider a two-person discounted zero-sum stochastic game in which for each state s, Player I and Player II have a finite set of actions A_s and B_s respectively. Let S be the set of states and let k be its cardinality. When the game is played in state s, Player I chooses an action $i \in A_s$ and Player II chooses an action $j \in B_s$, the payoff to Player I is $r(s, i, j)$; the payoff to Player II is $-r(s, i, j)$. The game makes a transition to state t with probability $p(t|s, i, j)$ on the next day. The stream of resulting payoffs to Player I over an infinite number of days, i.e., the time horizon of the game, is evaluated by the total discounted sum $\sum_{N=1}^{\infty} \beta^{N-1} r(s, i, j)$ assuming that on day N the game is played in state s, and the actions chosen by players are i and j respectively. The transition probability $p(t|s, i, j)$ and the reward function $r(s, i, j)$ satisfy the following additive property:

$$p(t|s, i, j) = p_1(t|s, i) + p_2(t|s, j)$$
$$r(s, i, j) = r_1(s, i) + r_2(s, j)$$

Due to this additive property assumed on the transition and reward functions, the game is called β-discounted zero-sum ARAT(Additive Reward & Additive Transition) Game. As is usual in game theory, players are allowed to choose a probability distribution over the set of actions available to them in each state and then choose an action with the probability specified by the chosen distribution. The space of probability distributions over A_s is called the space of *mixed strategies* for player I in state s. A mixed strategy that assigns probability mass 1 to a particular action is called a *pure strategy.* In a stochastic game the players are required to choose a mixed strategy each day and such a sequence

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of actions or mixed strategies chosen by a player may be called a *policy*. A policy is said to be *stationary* if the mixed strategies chosen on any day are the same whenever the game is played at a specified state, i.e., the chosen strategies depend only on the state the game is played. A stationary policy may therefore be identified with a mixed strategy at a particular state. For more details on these and related concepts see [14] and [3]. We denote the matrix $((p_1(t|s,i), t \in S,\ i \in A_s))$ as $P_1(s)$ where S is the set of states. This is a $m_1(s) \times k$ matrix where $m_1(s)$ is the cardinality of A_s and k is the cardinality of S. Similarly the matrix $P_2(s)$ of order $m_2(s) \times k$ is defined where $m_2(s)$ denotes the cardinality of the set B_s.

ARAT games have been studied in the literature earlier by Raghavan et al. [17]. See also [16] and [3]. Both the discounted and the limiting average criterion of evaluation of strategies have been considered. It is known for example, that for a β-discounted zero-sum ARAT game, the value exists and both players have stationary optimal strategies, which may also be taken as pure strategies. In [17] a finite step method to compute a pair of pure stationary optimal strategies and the value of the game has been suggested. However this approach involves solving a series (finite number) of Markov decision problems. It is interesting to ask whether one can find a *one step solution method* like solving one linear program or one LCP instead of solving a series of Markov decision problems. (Recall that a Markov decision problem can be solved as a linear program. See [17, p. 459] in this connection.) We shall show in this paper that this is indeed possible, with the following assumption on the ARAT game: Either for all s and for all $j \in B_s$, $p_2(s|s,j)$ is positive or for all s, and for each t there exists a $i \in A_s$ such that $p_1(t|s,i) > 0$ and $P_2(s)$ is not a null matrix. In otherwords, a pair of pure stationary optimal strategies and the corresponding value for a zero-sum discounted ARAT game with the above assumption, can be computed by solving a single vertical linear complementarity problem.

In Sect. 2, we define the vertical linear complementarity problem (VLCP) and supply relevant material on the VLCP. In Sect. 3, we formulate the zero-sum discounted ARAT game as a vertical linear complementarity problem. In Sect. 4, we show that the Cottle-Dantzig algorithm can process this problem under a mild assumption.

2. Vertical linear complementarity problem

For a given square matrix $M \in R^{n\times n}$ and a vector $q \in R^n$ the linear complementarity problem (denoted by LCP(q, M)) is to find vectors $w,\ z \in R^n$ such that

$$w - Mz = q, \quad w \geq 0,\ z \geq 0 \tag{1}$$

$$w^t z = 0 \tag{2}$$

A pair (w, z) of vectors satisfying (1) and (2) is called a solution to the $LCP(q, M)$. This problem is well studied in the literature over the years. For the recent books on this topic see Cottle, Pang and Stone [2] and Murty [7]. The problem arises in some classes of stochastic game problems, for example, see [18], [10] and [11]. The algorithm presented by Lemke and Howson [6] to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke [5] to solve a $LCP(q, M)$ contributed significantly to the development of the linear complementarity theory.

Cottle and Dantzig [1] extended the problem considered above to a problem in which the matrix M is not a square matrix. The generalization of the linear complementarity problem introduced by them is given below:

We say that an $m \times k$ matrix A with the partitioned form $A = \begin{bmatrix} A^1 \\ \vdots \\ A^k \end{bmatrix}$ is a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$ if A^j is of order $m_j \times k$, $1 \le j \le k$ and $\sum_{j=1}^{k} m_j = m$.

Given a vertical block matrix $A \in R^{m \times k}$, $(m \ge k)$ of type $(m_1, \ldots, m_k)$ and $q \in R^m$ where $m = \sum_{j=1}^{k} m_j$, the generalized linear complementarity problem is to find $w \in R^m$ and $z \in R^k$ such that

$$w - Az = q, \qquad w \ge 0,\ z \ge 0 \tag{3}$$

$$z_j \prod_{i=1}^{m_j} w_i^j = 0, \qquad j = 1, 2, \ldots, k \tag{4}$$

This generalization is also known as *vertical generalization of the linear complementarity problem* [1] and it is denoted by $VLCP(q, A)$.

2.1. Algorithm for solving the vertical linear complementarity problem

Cottle and Dantzig [1] describe a procedure for solving a vertical linear complementarity problem, which is an extension of Lemke's algorithm for the ordinary linear complementary problem. For the sake of completeness we present this algorithm below. The Cottle-Dantzig algorithm for the $VLCP(q, A)$ starts with the initial solution to (3) and (4)

$$w = q + dz_0;\ z = 0$$

where z_0 is large enough so that $w > 0$ and $d \in R^m$ is a given positive vector. Let $J_1 = \{1, 2, \ldots, m_1\}$ and let $J_i = \{\sum_{j=1}^{i-1} m_j + 1, \sum_{j=1}^{i-1} m_j + 2, \ldots, \sum_{j=1}^{i} m_j\}$, $2 \le i \le k$.

Step 1: Decrease z_0 to $\bar{z_0} = \min\{z_0 \mid q + dz_0 \ge 0,\ z_0 \ge 0\}$ so that one of the variables w_i, $1 \le i \le m$, say w_p, is reduced to zero. We now have a basic feasible solution with z_0 in place of w_p. This is the initial almost proper basic feasible solution. Now let r be the unique index, $1 \le r \le k$, such that $p \in J_r$. We have exactly one pair of non-basic variables (z_r, w_p) which belong to the same set of related variables.

Step 2: At each iteration, there is exactly one pair of non-basic variables belonging to the same set of related variables. Of these, one has been eliminated from the basis in the previous iteration; the other is now selected to be included in the basis. For example, in the second iteration z_r is selected to be included in the basis.

Step 3: If the variable selected at Step 2 to enter the basis can be arbitrarily increased, then the procedure terminates in an almost proper ray, to be called a *secondary proper ray*. If a new basic feasible solution is obtained with $z_0 = 0$, or z_0 is non-basic, then we have solved (3) and (4) and have a solution for the $VLCP(q, A)$. Otherwise, we have obtained a new almost proper basic feasible solution and a new pair of nonbasic variables (x_β, y_r) belonging to the same set of related variables, say the s^{th} set, where either $(x_\beta, y_r) = (z_s, w_t)$, with $t \in J_s$ or $(x_\beta, y_r) = (w_{t_1}, w_{t_2})$, with $t_1, t_2 \in J_s$.

We repeat Step 2.

The Cottle-Dantzig algorithm (Algorithm CD) consists of the repeated application of Steps 2 and 3. Under the standard nondegeneracy assumption (see [8]), the procedure either terminates in a *solution* to the $VLCP(q, A)$ or in a *secondary proper ray.*

In [8], Mohan et al. have shown that if the input matrix satisfies some property (i.e., if A belongs to certain classes) then the Cottle Dantzig algorithm can solve the $VLCP(q, A)$. See also [9].

Definition 1. *A is said to be a vertical block $E(d)$-matrix for some $d > 0$ if $VLCP(d, A)$ has a unique solution $w = d,\ z = 0$.*

Definition 2. *A is said to be a vertical block R_0-matrix if $VLCP(0, A)$ has a unique solution $w = 0,\ z = 0$.*

In what follows we denote the class of vertical block $E(d)$ matrices as $VBE(d)$ and the class of vertical block R_0 matrices by VBR_0. If the vertical block matrix $A \in VBE(d) \cap VBR_0$ then $VLCP(q, A)$ is processable by Cottle-Dantzig's algorithm. In the next section, we show that the vertical block matrix arising out of discounted zero-sum ARAT games belongs to $VBE(d) \cap VBR_0$ when the components $P_1(s)$ and $P_2(s)$ of the transition probability matrices satisfy a mild condition.

3. Computing optimal pure strategies of a discounted zero-sum ARAT game

We first state the following result.

Theorem 1. *(Theorem 6.4.2 in [3]) For ARAT stochastic games*

(i) Both players possess β discounted optimal stationary strategies that are pure.
(ii) These strategies are optimal for the average reward criterion as well.
(iii) The ordered field property holds for the discounted as well as the average reward criterion.

To formulate ARAT stochastic games we make use of the result that there is always an optimal stationary strategy among the pure strategies for both the players and the Shapley equations hold for this game.

The Shapley equations give us the following for state $s,\ s \in S$.

$$\text{Val}\ [r(s, i, j) + \beta \sum_t p(t|s, i, j) v_\beta(t)] = v_\beta(s)$$

This implies

$$r(s,i,j)+\beta\sum_t p(t|s,i,j)v_\beta(t)\le v_\beta(s) \text{ for all } i \text{ and for any fixed } j.$$

In particular, suppose the optimal pure strategy in state s is i_0 for Player I and j_0 for Player II. Then

$$r_1(s,i)+r_2(s,j_0)+\beta\sum_t p_1(t|s,i)v_\beta(t)+\beta\sum_t p_2(t|s,j_0)v_\beta(t)\le v_\beta(s)\ \forall\, i.$$

These inequalities yield

$$r_1(s,i)+\beta\sum_t p_1(t|s,i)v_\beta(t)\le v_\beta(s)-\eta_\beta(s)=\xi_\beta(s)\quad\forall\, i$$

where $\eta_\beta(s)=r_2(s,j_0)+\beta\sum_t p_2(t|s,j_0)v_\beta(t)$ and
$\xi_\beta(s)=r_1(s,i_0)+\beta\sum_t p_1(t|s,i_0)v_\beta(t)$ and
$\xi_\beta(s)+\eta_\beta(s)=v_\beta(s)$.

Thus the inequalities are

$$r_1(s,i)+\beta\sum p_1(t|s,i)\xi_\beta(t)-\xi_\beta(s)+\beta\sum p_1(t|s,i)\eta_\beta(t)\le 0\ \forall\, i\in A_s,\ s\in S \tag{5}$$

and similarly the inequalities for Player II are

$$r_2(s,j)+\beta\sum p_2(t|s,j)\eta_\beta(t)-\eta_\beta(s)+\beta\sum p_2(t|s,j)\xi_\beta(t)\ge 0\ \forall\, j\in B_s,\ s\in S. \tag{6}$$

Also for each s, in (5) there is an $i(s)$ such that equality holds. Similarly, for each s in (6) there is a $j(s)$ such that equality holds.

Let
$$w_1(s,i)=-r_1(s,i)-\beta\sum p_1(t|s,i)\eta_\beta(t)+\xi_\beta(s)-\beta\sum p_1(t|s,i)\xi_\beta(t)\ge 0,\quad i\in A_s \tag{7}$$
and
$$w_2(s,j)=r_2(s,j)-\eta_\beta(s)+\beta\sum p_2(t|s,j)\eta_\beta(t)+\beta\sum p_2(t|s,j)\xi_\beta(t)\ge 0\ \forall\, j\in B_s. \tag{8}$$

We may assume without loss of generality that $\eta_\beta(s)$, $\xi_\beta(s)$ are strictly positive. Since there is at least one inequality in (7) for each $s\in S$ that holds as an equality and one inequality in (8) for each $s\in S$ that holds as an equality, the following complementarity conditions will hold.

$$\eta_\beta(s)\prod_{i\in A_s} w_1(s,i)=0 \text{ for } 1\le s\le k \text{ and} \tag{9}$$

$$\xi_\beta(s) \prod_{j \in B_s} w_2(s, j) = 0 \text{ for } 1 \leq s \leq k. \tag{10}$$

The inequalities (7) and (8) along with the complementarity conditions (9), (10) lead to the VLCP(q, A) where the matrix A is of the form

$$A = \begin{bmatrix} -\beta P_1 & E - \beta P_1 \\ -E + \beta P_2 & \beta P_2 \end{bmatrix} \text{ and } q = \begin{bmatrix} -r_1(\cdot,\cdot) \\ r_2(\cdot,\cdot) \end{bmatrix}.$$

In the above VLCP, $P_1 = [p_1(t|s, i)]$, $P_2 = [p_2(t|s, j)]$ and

$$E = \begin{bmatrix} e^1 & 0 & \dots & 0 \\ 0 & e^2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & e^k \end{bmatrix}$$

is a vertical block identity matrix where e^j, $1 \leq j \leq k$ is a column vector of all 1's of appropriate order.

In the next section, to show the convergence of Cottle-Dantzig algorithm we show that the vertical block matrix arising from a zero-sum discounted ARAT game belongs to a processable class under a mild assumption.

4. Convergence of Cottle-Dantzig algorithm

We first observe the following property of the additive components P_1 and P_2 of the transition probability matrix P.

Lemma 1. *If $p_2(t|s, j) = 0$ for all $t \in S$ and for some $j \in B(s)$, then $P_2(s) = 0$.*

Proof. Suppose $p_2(t|s, j^0) = 0$ for all t. From the condition

$$\sum_{t=1}^{k} p_1(t|s, i) + \sum_{t=1}^{k} p_2(t|s, j^0) = 1,$$

we obtain that $\sum_{t=1}^{k} p_1(t|s, i) = 1$. Let $j \neq j^0$. Now since

$$\sum_{t=1}^{k} p_1(t|s, i) + \sum_{t=1}^{k} p_2(t|s, j) = 1,$$

it follows that $\sum_{t=1}^{k} p_2(t|s, j) = 0$ for all $j \neq j^0$. Thus the matrix $P_2(s) = 0$. This completes the proof.

□

We have the following theorem.

Theorem 2. *Consider the vertical block matrix A arising from the zero-sum ARAT game. Then $A \in VBE(e)$ where e is the vector each of whose entries is* 1.

Proof. Let $d = \begin{bmatrix} d^1 \\ d^2 \end{bmatrix}$ where $d^1 > 0$ and $d^2 > 0$. Consider the VLCP(d, A) where

$$A = \begin{bmatrix} -\beta P_1 & E - \beta P_1 \\ -E + \beta P_2 & \beta P_2 \end{bmatrix}.$$

We shall show by contradiction that $VLCP(d, A)$ has only the trivial solution $w = d,\ z = 0$, when $d = e$.

Let $\begin{bmatrix} w^1 \\ w^2 \end{bmatrix}, \begin{bmatrix} \eta_\beta \\ \xi_\beta \end{bmatrix}$ be a solution to $VLCP(d, A)$. Then $\begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = \begin{bmatrix} d^1 \\ d^2 \end{bmatrix} + A \begin{bmatrix} \eta_\beta \\ \xi_\beta \end{bmatrix}$.
Assume $\begin{bmatrix} \eta_\beta \\ \xi_\beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Let $v_\beta(s) = \xi_\beta(s) + \eta_\beta(s)$ and $v_\beta(s^*) = \max_{s \in S} v_\beta(s)$.
Now $v_\beta(s^*) = \xi_\beta(s^*) + \eta_\beta(s^*) > 0$.

Case 1. Let $\eta_\beta(s^*) > 0$. Then there exists an $i \in A_{s^*}$ such that

$$d_i^1 - \beta \sum p_1(t|s^*, i)\xi_\beta(t) - \beta \sum p_1(t|s^*, i)\eta_\beta(t) + \xi_\beta(s^*) = 0.$$

$$\text{or, } \xi_\beta(s^*) = -d_i^1 + \beta \sum p_1(t|s^*, i) v_\beta(t). \tag{11}$$

We also have from the feasibility condition

$$d_j^2 + \beta \sum p_2(t|s^*, j) v_\beta(t) \geq \eta_\beta(s^*) \tag{12}$$

From (11) and (12), we have

$$d_j^2 - d_i^1 + \beta \sum p(t|s^*, i, j) v_\beta(t) \geq v_\beta(s^*).$$

Note that for our choice of d, $d_j^2 = d_i^1$ so that

$$\beta \sum p(t|s^*, i, j) v_\beta(t) \geq v_\beta(s^*).$$

which is a contradiction unless $v_\beta(s^*) = 0$ or $v_\beta(t) = 0$, for all t or $\xi_\beta(t) = \eta_\beta(t) = 0$, for all t.

Case 2. Let $\xi_\beta(s^*) > 0$. Then by complementarity there exists a $j \in B_{s^*}$ such that

$$d_j^2 - \eta_\beta(s^*) + \beta \sum p_2(t|s^*, j) v_\beta(t) = 0$$

$$\text{or, } d_j^2 + \beta \sum p_2(t|s^*, j) v_\beta(t) = \eta_\beta(s^*)$$

Since $d_j^2 > 0$, it follows that $\eta_\beta(s^*) > 0$. Hence the theorem follows.

□

Theorem 3. *Consider the vertical block matrix A arising from zero-sum ARAT game. Then $A \in VBR_0$ if either the condition (a) or the set of conditions (b) stated below is satisfied.*

(a) For each s and each $j \in B_s$, $p_2(s|s, j) > 0$.
(b) (i) For each s, the matrix $P_1(s)$ does not contain any zero column and
(ii) the matrix $P_2(s)$ is not a null matrix.

Proof. Consider the VLCP(0, A) where

$$A = \begin{bmatrix} -\beta P_1 & E - \beta P_1 \\ -E + \beta P_2 & \beta P_2 \end{bmatrix}.$$

We shall show by contradiction that VLCP(0, A) has only the trivial solution $w = 0,\ z = 0$.

Let $\begin{bmatrix} w^1 \\ w^2 \end{bmatrix}, \begin{bmatrix} \eta_\beta \\ \xi_\beta \end{bmatrix}$ be a solution to VLCP(0, A). Then $\begin{bmatrix} w^1 \\ w^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + A \begin{bmatrix} \eta_\beta \\ \xi_\beta \end{bmatrix}$.

Suppose $\begin{bmatrix} \eta_\beta \\ \xi_\beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Let $v_\beta(s) = \xi_\beta(s) + \eta_\beta(s)$ and let $v_\beta(s^*) = \max_{s \in S} v_\beta(s)$.

Now $v_\beta(s^*) = \xi_\beta(s^*) + \eta_\beta(s^*) > 0$.

Case 1. Let $\eta_\beta(s^*) > 0$. Then by complementarity there exists an $i \in A_{s^*}$ such that

$$-\beta \sum p_1(t|s^*, i)\xi_\beta(t) - \beta \sum p_1(t|s^*, i)\eta_\beta(t) + \xi_\beta(s^*) = 0.$$

$$\text{This implies } \beta \sum p_1(t|s^*, i)v_\beta(t) = \xi_\beta(s^*). \tag{13}$$

We also have from the feasibility condition

$$\beta \sum p_2(t|s^*, j)v_\beta(t) \geq \eta_\beta(s^*) \tag{14}$$

From (13) and (14), we have

$$\beta \sum p(t|s^*, i, j)v_\beta(t) \geq v_\beta(s^*).$$

which is a contradiction unless $v_\beta(s^*) = 0$ or $v_\beta(t) = 0$, for all t or $\xi_\beta(t) = \eta_\beta(t) = 0$, for all t.

Case 2. Next suppose $\eta_\beta(s^*) = 0$. This implies $\xi_\beta(s^*) > 0$. Therefore, by the vertical block complementarity condition there exists a $j \in B_{s^*}$ such that

$$\beta \sum p_2(t|s^*, j)v_\beta(t) = \eta_\beta(s^*).$$

Suppose now condition (a) holds. Note that by this condition, $p_2(s^*|s^*, j) > 0$ and $v_\beta(s^*) > 0$.

Since $\eta_\beta(s^*) = \beta \sum p_2(t|s^*, j)v_\beta(t)$ and both $p_2(s^*|s^*, j)$ and $v_\beta(s^*)$ are positive, it follows that $\eta_\beta(s^*) > 0$. Hence Case 2 does not arise if condition (a) holds.

Now suppose the set of conditions (b) holds. Since for each s, $P_1(s)$ does not have a 0 column, we have by the feasibility condition

$$\xi_\beta(s)e^s s - \beta P_1(s)(\eta_\beta + \xi_\beta) \geq 0$$

where e^s denotes the vector of order $|A_s|$ of $1's$. From here it follows that $\xi_\beta(s)$ is positive for each s. It follows from here that for $s = s^*$ we have

$$\eta_\beta(s^*) = \beta \sum p_2(t|s^*, j)v_\beta(t) > 0.$$

Thus again Case 2 does not arise if the set of conditions (b) holds. This completes the proof.

□

The following example shows that if neither (a) nor (b) holds then, Theorem 3 may not hold. In otherwords if both (a) and (b) are violated then $VLCP(0, A)$ may have a nontrivial solution.

Example 1. Consider a two player zero-sum discounted ARAT game with $s = 2$ states. In each state each of the two players has 2 actions. The transition probabilities are given by

$p_1(1\,|\,1, 1) = \frac{1}{2},\ p_1(2\,|\,1, 1) = 0,$
$p_1(1\,|\,1, 2) = \frac{1}{2},\ p_1(2\,|\,1, 2) = 0,$
$p_1(1\,|\,2, 1) = 0,\ p_1(2\,|\,2, 1) = \frac{1}{2},$
$p_1(1\,|\,2, 2) = 0,\ p_1(2\,|\,2, 2) = \frac{1}{2},$
$p_2(1\,|\,1, 1) = \frac{1}{2},\ p_2(2\,|\,1, 1) = 0,$
$p_2(1\,|\,1, 2) = 0,\ p_2(2\,|\,1, 2) = \frac{1}{2},$
$p_2(1\,|\,2, 1) = 0,\ p_2(2\,|\,2, 1) = \frac{1}{2},$
$p_2(1\,|\,2, 2) = \frac{1}{2}$ and $p_2(2\,|\,2, 2) = 0.$

Note that $p(t|s, i, j) = p_1(t|s, i) + p_2(t|s, j)$.

Let the discount factor $\beta = \frac{1}{2}$. The matrix A is given by

$$A = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{3}{4} \\ 0 & -\frac{1}{4} & 0 & \frac{3}{4} \\ -\frac{3}{4} & 0 & \frac{1}{4} & 0 \\ -1 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & -\frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & -1 & \frac{1}{4} & 0 \end{bmatrix}$$

where A is a vertical block matrix of type $(2, 2, 2, 2)$.

Now it is easy to verify that for this matrix, neither condition (a) nor the set of conditions (b) holds. Also it is easy to verify that $\eta_1 = \eta_2 = \xi_1 = 0, \xi_2 = 1$ is a nontrivial solution to $VLCP(0, A)$. Thus A is not a vertical block R_0 matrix.

Even though we have shown the convergence of Cottle and Dantzig's algorithm under the conditions (a) or (b) of Theorem 3, in practical implementation, Cottle-Dantzig's algorithm seems to succeed in computing a solution even when the assumption is not satisfied.

To see this consider the following example.

Example 2. To Example 1 we add the following rewards to complete the description of an ARAT game.

$r_1(1, 1) = 4, r_1(1, 2) = 5, r_1(2, 1) = 3$ and $r_1(2, 2) = 4$.

$r_2(1, 1) = 3, r_2(1, 2) = 6, r_2(2, 1) = 6$ and $r_2(2, 2) = 2$. $r(s, i, j) = r_1(s, i) + r_2(s, j)$.

For this game our formulation leads to the $VLCP(q, A)$ where the vertical block matrix A is as in Example 1 and

$$q = \begin{bmatrix} -4 \\ -5 \\ -3 \\ -4 \\ 3 \\ 6 \\ 6 \\ 2 \end{bmatrix}.$$

Although the vertical block matrix A is not a vertical block R_0 matrix, Cottle-Dantzig algorithm processes this matrix with the covering vector as e and produces the following solution. $\eta_\beta(1) = 7, \eta_\beta(2) = 6, \xi_\beta(1) = 9$ and $\xi_\beta(2) = 7.33$. $w(1) = 1.0, w(2) = 0$, $w(3) = 1.0, w(4) = 0, w(5) = 0, w(6) = 2.33, w(7) = 3.33$ and $w(8) = 0$.

Therefore an optimal pure strategy for the players in the various states are as follows: Player I chooses action 2 in states 1 and 2. Player II chooses action 2 in states 1 and 2.

Remark 1. It is relevant to note here that a given $VLCP(q, A)$ can be equivalently formulated as a $LCP(q, M)$ as in [8]. This requires constructing the square matrix M from the given vertical block matrix A by copying its j^{th} column as many times as the j^{th} block size. We say that the matrix A is a vertical block $E(0)$ matrix if the equivalent square matrix M satisfies the following condition: $(\bar{w}, \bar{z}),\ \bar{z} \neq 0$ is a solution to the $LCP(0, M) \Rightarrow$ there exists a $x \geq 0,\ x \neq 0,\ x \in R^n$ such that $y = -M^t x \geq 0,\ x \leq \bar{z},\ y \leq \bar{w}$. It is known that Cottle-Dantzig algorithm processes $VLCP(q, A)$ if $VLCP(d, A)$ has the unique solution $w = d, z = 0$ and A is a vertical block $E(0)$ matrix. It is interesting to note that the vertical block matrix A in the example above is also not a vertical block $E(0)$ matrix.

Remark 2. The method of Raghavan et al. [17] requires solving a finite number of Markov decision problems. Each Markov decision problem can be solved as a linear program. But on the other hand, the Cottle-Dantzig algorithm can solve the VLCP formulation of the game problem if one of the conditions (a) or (b) stated in Theorem 3 holds. Then for such games a pair of pure stationary optimal strategies and the value can be computed by solving a single VLCP. However we are not sure of the computational superiority of the Cottle-Dantzig procedure over the procedure that solves a sequence of Markov decision problems as linear programs. The question of solving the $VLCP(q, A)$ for these stochastic games when the vertical block matrix A does not satisfy one of the conditions (a) or (b) by using Cottle-Dantzig algorithm still remains open. However the $VLCP(q, A)$ arising from such a game may also be solved by other methods such as the the enumerative algorithm (finite step) of Garcia and Lemke [4] for computation of pure strategies and the value vector of this game. See also [19].

Remark 3. We can enumerate various special cases where one of (a) or (b) holds. Notice that it is also easy to verify the conditions in general by examining the entries of the matrices P_1 and P_2. In particular, when both P_1 and P_2 are positive both (a) and (b) hold. If P_2 is positive condition (a) holds. When P_1 is positive and $P_2(s)$ is not a null matrix for each s condition (b) holds.

5. Conclusion

In this paper we considered the zerosum discounted stochastic game with ARAT structure and showed that a pair of stationary optimal pure strategies for both the players (such optimal strategies are known to exist) and the corresponding value can be obtained as a solution to a vertical linear complementarity problem. To show that the resulting VLCP can be solved by Cottle-Dantzig algorithm, we had to impose certain conditions on the VLCP, which restricted the scope of this approach to some extent. The possibility of solving the VLCP arising from a general zerosum discounted stochastic game with ARAT structure by Cottle-Dantzig algorithm is still open.

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