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Short Communication

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Approximating the complexity measure of Vavasis-Ye algorithm is NP-hard

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Abstract. Given an $m \times n$ integer matrix A of full row rank, we consider the problem of computing the maximum of $\|B^{-1}A\|_2$ where B varies over all bases of A. This quantity appears in various places in the mathematical programming literature. More recently, logarithm of this number was the determining factor in the complexity bound of Vavasis and Ye's primal-dual interior-point algorithm. We prove that the problem of approximating this maximum norm, even within an exponential (in the dimension of A) factor, is NP-hard. Our proof is based on a closely related result of L. Khachiyan [1].

Key words. linear programming – computational complexity – complexity measure

1. Introduction and preliminaries

Consider the primal-dual pair of linear programming (LP) problems expressed in the following form:

$$\begin{array}{lrl} (P) \text{ minimize } & c^T x & \\ & Ax & = b, \\ & x & \geq 0, \end{array}$$

$$\begin{array}{lrl} (D) \text{ maximize } & b^T y & \\ & A^T y + s & = c, \\ & s & \geq 0, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. In this note, all vectors are column vectors. Without loss of generality, we assume $\text{rank}(A) = m$ and that $n > m \geq 2$. For a matrix $M \in \mathbb{R}^{m \times n}$, $\|M\|_p$ denotes the matrix p-norm (induced by the vector p-norms in $\mathbb{R}^m$ and $\mathbb{R}^n$) $\|M\|_p := \max\{\|Mx\|_p : \ x \in \mathbb{R}^n, \|x\|_p = 1\}$.

Vavasis and Ye [5] proposed a primal-dual interior-point algorithm for LP with the property that the number of Newton steps required by the algorithm is bounded by a function of only the coefficient matrix A. Based on the condition measure

$$\bar{\chi}(A) \ := \ \sup \left\{ \left\| A^T \left(ADA^T \right)^{-1} AD \right\|_2 : \ D \in \mathcal{D} \right\}$$

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(where $\mathcal{D}$ is the set of $n \times n$, diagonal, positive definite matrices), they established the bound of $O\left(n^{3.5}\left(\log \bar{\chi}(A) + \log n\right)\right)$ on the number of Newton steps taken by their algorithm in the worst case.

There has been a significant amount of work in mathematical programming which involves or relates to $\bar{\chi}(A)$. Many of these works include characterizations of $\bar{\chi}(A)$. Every such known characterization seems to lead only to exponential-time algorithms for computing $\bar{\chi}(A)$. In this note, we are concerned with the computational complexity of computing this number.

We will investigate the question in the context of the Turing Machine Model. Therefore, for the rest of the note, we assume that $A \in \mathbb{Z}^{m\times n}$. (The main result goes through for all A with rational entries as well.)

A related condition number of A is defined as

$$\chi(A) \ := \ \sup\left\{\left\|\left(ADA^T\right)^{-1} AD\right\|_2 : \ D \in \mathcal{D}\right\}.$$

It is not hard to show that

$$\chi(A) = \max\left\{\left\|B^{-1}\right\|_2 : \ B \in \mathcal{B}(A)\right\}, \tag{1}$$

where $\mathcal{B}(A)$ is the set of all bases ($m \times m$ non-singular sub-matrices) of A. Let poly(n) denote a polynomial function of n. Khachiyan [1] proved (in addition to many related results),

Theorem 1. *Approximating $\chi(A)$ within a factor of $2^{\mathrm{poly}(n)}$ is NP-hard.*

Khachiyan [2], and Vavasis and Ye [5] suspected that the statement of the above theorem would most likely apply to $\bar{\chi}(A)$ as well. Utilizing Khachiyan's Theorem, we prove that their suspicions were well placed. The main result of this note follows.

Theorem 2. *Approximating $\bar{\chi}(A)$ within a factor of $2^{\mathrm{poly}(n)}$ is NP-hard.*

Even though the paper [5] contains elementary ways of avoiding the accurate computation of $\bar{\chi}(A)$, and the modification of Vavasis-Ye algorithm by Megiddo-Mizuno-Tsuchiya [3] also avoids this computation, our result adds to the relevance of these techniques. Moreover, our result provides further motivation for the probabilistic approaches to the subject, as done by Todd, Tunçel and Ye [4].

2. Review of the ingredients

$\bar{\chi}(A)$ also has a characterization in terms of the bases of A (see, for instance, [4]).

$$\bar{\chi}(A) = \max\left\{\left\|B^{-1}A\right\|_2 : \ B \in \mathcal{B}(A)\right\}. \tag{2}$$

We use some elementary and very well-known facts from the complexity analyses of LP problems (Propositions 1 and 2). All logarithms in this note are of base 2.

Given $z \in \mathbb{Z}$, size$(z) := \lceil \log(|z|+1)\rceil + 1$. Then size$(A) := \sum_{i=1}^{m}\sum_{j=1}^{n} \mathrm{size}(a_{ij})$. We denote size$(A)$ by L. dim(M) denotes the number of entries of M.

Proposition 1. *(a) Let $d \in \mathbb{Z}^n$. Then* $\|d\|_1 \le 2^{\text{size}(d)-\dim(d)}$.
(b) Let C be a square sub-matrix of A. Then $|det(C)| \le 2^{\text{size}(C)-\dim(C)} \le 2^{L-mn}$.

Proof. Proof of (a) is straightforward. Proof of (b) can be easily obtained by an induction.

□

Proposition 2. *Let C be an $r \times r$ non-singular sub-matrix of A. Let d be an r-vector whose entries are chosen from A. Then*

(a) $\left\|C^{-1}d\right\|_\infty \le 2^{L-mn}, \quad \left\|C^{-1}d\right\|_2 \le 2^{L-n}$,
(b) $\|C\|_2 \le 2^L, \quad \left\|C^{-1}\right\|_2 \ge 2^{-L}$.

Proof. (a) By Cramer's Rule, Proposition 1 (b), and the fact that all entries of C, d are integers, we have $\|C^{-1}d\|_\infty \le 2^{L-mn}$. The next inequality follows from the relationship of the vector norms.

(b) Using Proposition 1 (a), and the characterization of operator infinity-norm, we have $\|C\|_\infty \le 2^{L-n}$. Using the relationship of the operator infinity and 2-norms, we arrive at $\|C\|_2 \le 2^L$. Recall that the reciprocal of the largest singular value of C is the smallest singular value of C^{-1}. We conclude, $\|C^{-1}\|_2 \ge 2^{-L}$.

□

3. The main result

Let $\hat{B}$ denote a basis of A attaining $\chi(A) = \|\hat{B}^{-1}\|_2$. Note that for any square, non-singular sub-matrix C of A, there exists a basis $B \in \mathcal{B}(A)$ containing C as a sub-matrix. For every such B, we have the interlacing property of the singular values of B and C. In particular, $\|B\|_2 \ge \|C\|_2$ and $\|B^{-1}\|_2 \ge \|C^{-1}\|_2$. Thus,

$$\left\|C^{-1}\right\|_2 \le \left\|\hat{B}^{-1}\right\|_2. \tag{3}$$

Our main idea is to exploit the characterizations (1) and (2) of χ and $\bar{\chi}$ in the following way. We consider the $\bar{\chi}$ value of the augmented matrix $[A \,|\, \alpha I]$. We have

$$\bar{\chi}(A \,|\, \alpha I) \le \left\|B^{-1}A\right\|_2 + \alpha \left\|B^{-1}\right\|_2,$$

where $B \in \mathcal{B}(A \,|\, \alpha I)$ attains the maximum. We observe that if we choose α very large then the second term above might dominate, and we may be forced to choose B very close to $\hat{B}$. Indeed, this is a very rough idea and we have to consider various issues and verify a few bounds. But in essence, in what follows, we prove that choosing $\alpha := 2^{5L}$ works. Many of the constants in the analysis below can be improved (including $5L$); however, the conclusion of the main theorem stays the same. Therefore, the estimations below are very generous for the ease of presentation.

Lemma 1. *Let $\alpha := 2^{5L}$, and let B be a basis of $[A \,|\, \alpha I]$. Then*

(a) $\left\|B^{-1}\right\|_2 \le \left(1 + 2^{-2L}\right) \left\|\hat{B}^{-1}\right\|_2$,

(b) $\left\| B^{-1}A \right\|_2 \leq 2^L + 2^{-L}$,
(c) $\bar{\chi}(A \,|\, \alpha I) \geq 2^{4L}$.

Proof. If B does not contain any column of αI, then the inequality in (a) clearly holds, the inequality in (b) also holds (as can be checked, using Proposition 2 (a)). So, for proving (a) and (b), we assume, without loss of generality, that B contains the first k columns of αI. Then we write B as

$$B = \left[\begin{array}{c|c} \alpha I & \bar{B}_1 \\ \hline 0 & \bar{B}_2 \end{array} \right]; \quad \text{thus,} \quad B^{-1} = \left[\begin{array}{c|c} \frac{1}{\alpha} I & -\frac{1}{\alpha}\bar{B}_1\bar{B}_2^{-1} \\ \hline 0 & \bar{B}_2^{-1} \end{array} \right].$$

Now, we prove (a):

$$\begin{aligned} \left\| B^{-1} \right\|_2 &\leq \frac{1}{\alpha} \left\| \left[I \,|\, -\bar{B}_1\bar{B}_2^{-1} \right] \right\|_2 + \left\| \bar{B}_2^{-1} \right\|_2 \\ &\leq \frac{1}{\alpha} \left(1 + \left\| \bar{B}_1\bar{B}_2^{-1} \right\|_2 \right) + \left\| \hat{B}^{-1} \right\|_2 \\ &\leq \frac{1}{\alpha} \left(1 + 2^L \right) + \left\| \hat{B}^{-1} \right\|_2 \\ &\leq \left(1 - 2^{-2L} \right) \left\| \hat{B}^{-1} \right\|_2 \end{aligned}$$

The second inequality above uses (3). Third inequality uses Proposition 2 (a). The last inequality follows from Proposition 2 (b).

Proof of (b): Write

$$A = \left[\begin{array}{c} A_1 \\ \hline A_2 \end{array} \right]$$

according to the row partition of B. Then

$$B^{-1}A = \left[\begin{array}{c} \frac{1}{\alpha}\left(A_1 - \bar{B}_1\bar{B}_2^{-1}A_2 \right) \\ \hline \bar{B}_2^{-1}A_2 \end{array} \right].$$

Therefore,

$$\left\| B^{-1}A \right\|_2 \leq \frac{1}{\alpha} m \left(2^L + 2^{2L} \right) + 2^L \leq 2^L + 2^{-L}.$$

Proof of (c):

$$\bar{\chi}(A \,|\, \alpha I) \geq \left\| \left[\hat{B}^{-1}A \middle| \alpha\hat{B}^{-1} \right] \right\|_2 \geq \alpha \left\| \hat{B}^{-1} \right\|_2 \geq 2^{4L}.$$

We used Proposition 2 (b).

□

Proof of Theorem 2. Let B be the basis attaining $\bar{\chi}(A\,|\,\alpha I)$. Then

$$\begin{aligned}\bar{\chi}(A\,|\,\alpha I) &\leq \left\| \left[B^{-1}A \middle| \alpha B^{-1} \right] \right\|_2 \\ &\leq \left\| B^{-1}A \right\|_2 + \alpha \left\| B^{-1} \right\|_2 \\ &\leq 2^{-L} + 2^{L} + \alpha \left(1 - 2^{-2L}\right) \chi(A).\end{aligned}$$

We used Lemma 1 (a) and (b). Since $\alpha\chi(A) \leq \bar{\chi}(A\,|\,\alpha I)$, we obtain

$$\frac{\frac{1}{\alpha}\bar{\chi}(A\,|\,\alpha I) - 2^{-6L} - 2^{-4L}}{1 + 2^{-2L}} \leq \chi(A) \leq \frac{1}{\alpha}\bar{\chi}(A\,|\,\alpha I).$$

Therefore, $\frac{1}{\alpha}\bar{\chi}(A\,|\,\alpha I)$ approximates $\chi(A)$ within a factor of

$$\frac{1 + 2^{-2L}}{1 - 2^{-3L} - 2^{-5L}},$$

we used the fact that $\frac{1}{\alpha}\bar{\chi}(A\,|\,\alpha I) \geq 2^{-L}$ (by Lemma 1 (c)). Since $n > m \geq 2$ ($L \geq 6$), this ratio is very close to 1 (bounded above by $(1 + 2^{-12})/(1 - 2^{-18} - 2^{-30})$). Clearly, if there were a polynomial time algorithm which approximated $\bar{\chi}(A)$ within a factor of $2^{\text{poly}(n)}$, we could use it on $[A\,|\,\alpha I]$ whose size is bounded by a polynomial function of size L of A (then divide the result by α) to get a polynomial time algorithm guaranteeing an approximation factor of e.g., $2^{\text{poly}(n)+1}$ for $\chi(A)$. Therefore, the problem of approximating $\bar{\chi}(A)$ within a factor of $2^{\text{poly}(n)}$ is NP-hard.

□

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