Math. Program., Ser. A 88: 183–192 (2000)

Digital Object Identifier (DOI) 10.1007/s101070000161

Paul Tseng

Co-NP-completeness of some matrix classification problems*

Received: April 1999 / Accepted: March 1, 2000
Published online May 12, 2000 –

Abstract. The classes of P-, P_0-, R_0-, semimonotone, strictly semimonotone, column sufficient, and nondegenerate matrices play important roles in studying solution properties of equations and complementarity problems and convergence/complexity analysis of methods for solving these problems. It is known that the problem of deciding whether a square matrix with integer/rational entries is a P- (or nondegenerate) matrix is co-**NP**-complete. We show, through a unified analysis, that analogous decision problems for the other matrix classes are also co-**NP**-complete.

Key words. P-, P_0-, R_0-, semimonotone, strictly semimonotone, column sufficient, nondegenerate matrices – complementarity problems – 1-norm maximzation – **NP**-completeness

1. Introduction

There is a number of matrix classes, in addition to the classes of positive definite and positive semidefinite matrices, that play important roles in studying solution properties of equations and complementarity problems (CP) and convergence/complexity analysis of methods for solving these problems. For example, the two classes of P- and P_0-matrices, studied by Fiedler and Pták, play important roles in the stability analysis of solutions to CP [6, 8, 17], derivation of error bounds [14, p. 320], and the convergence/complexity analysis of algorithms, e.g., Lemke's method, interior-point methods, non-interior methods, for solving CP (see [4–6, 9, 11, 12] and references therein). In particular, a CP has certain stability property and admits reformulation as a stationary-point problem if the Jacobian of the mapping is a P_0-matrix [8, 9, 17]. And, for a linear CP (LCP), existence of central path can be shown if the matrix is a P_0-matrix, in addition to some nonempty interior and boundedness assumptions [11, Lem. 4.3]. Moreover, an LCP with a P_0-matrix is **NP**-complete [11, p. 33].

An interesting question concerns the computational complexity (in the Turing machine model of computation [10]) of deciding whether a given square matrix M with integer entries belongs to a specific matrix class.[1] For the classes of positive definite and positive semidefinite matrices, this decision problem is solvable in polynomial time (via

P. Tseng: Department of Mathematics, University of Washington, Seattle, Washington 98195, USA,
e-mail: `tseng@math.washington.edu`

Mathematics Subject Classification (2000): 15A21, 68Q25, 90C33, 90C60

* This research is supported by National Science Foundation Grant CCR-9731273

[1] The case of M with rational entries is reducible to this case by multiplying M with the lowest common denominator of its entries.

Cholesky factorization, say). The same can be shown for, say, the classes of S-, S_0-, H-matrices (via linear programming) [6,12]. For the classes of P-, nondegenerate, and copositive matrices, this problem was shown to be co-**NP**-complete by, respectively, Coxson [7], Chandrasekaran et al. [3], [12, p. 462], and Murty and Kabadi [13] (also see [1,15] for related complexity results on, respectively, submatrix and interval matrix classification). This still leaves a number of important matrix classes, described in the books [6,11,12], for which complexity of the corresponding decision problems is unknown.

In this paper, we study the complexity of decision problems for the classes of P_0-, R_0-, semimonotone, strictly semimonotone, and column sufficient matrices (see [6, §3.13] for a history of these matrix classes). In particular, we show that these problems are all co-**NP**-complete. A key part of our proof is a reduction from the **NP**-complete problem of 1-norm maximization over a parallelotope [2, Thm. 15] to the decision problems for P-, strictly semimonotone, and column sufficient matrices (see Thm. 1). This reveals an interesting relation among these problems and yields, as a byproduct, Coxson's result for P-matrices. This reduction is analogous to a reduction from the **NP**-complete knapsack problem to the decision problems for R_0- and nondegenerate matrices (see Thm. 3 and [12, p. 462]). Our arguments differ from those of Coxson and Chandrasekaran et al. in that they do not involve principal minors and, as such, can more readily be extended to other matrix classes.

In our notation, $\Re^n$ denotes the space of n-dimensional real column vectors and ^{T} denotes transpose. For any $x \in \Re^n$, we denote by x_i the ith component of x and by $\|x\|_1$, $\|x\|_\infty$ the 1-norm and ∞-norm of x. For $x, y \in \Re^n$, we denote $x \circ y := [x_1 y_1 \; \cdots \; x_n y_n]^T$. For any $J \subseteq \{1, ..., n\}$, $|J|$ denotes the cardinality of J and, for any $n \times n$ matrix N, N_{JJ} denotes the principal submatrix obtained by removing from N all rows and columns not indexed by J.

2. P-, strictly semimonotone, column sufficient matrices

It is known that an $m \times m$ matrix M is *not* a P-matrix if and only if there exists a nonzero $u \in \Re^m$ satisfying $u \circ Mu \leq 0$ [6,11,12]. Also, by definition, M is *not* column sufficient if and only if there exists a nonzero $u \in \Re^m$ satisfying $u \circ Mu \leq 0$ and $u \circ Mu \neq 0$ [6, p. 157]. By definition, M is *not* in the class E of strictly semimonotone matrices if and only if there exists a nonzero $u \in \Re^m$ satisfying $u \geq 0$ and, for each $i \in \{1, ..., m\}$ either $u_i = 0$ or $[Mu]_i \leq 0$ [6, p. 188], [12, p. 227]. This condition can be written as $u \geq 0$ and $u \circ Mu \leq 0$. We formally state the corresponding decision problems below.

NOT-PMAT
Instance: Positive integer m and an $m \times m$ matrix M with integer entries.
Question: Does there exist nonzero $u \in \Re^m$ satisfying $u \circ Mu \leq 0$?

NOT-CSMAT
Instance: Positive integer m and an $m \times m$ matrix M with integer entries.
Question: Does there exist nonzero $u \in \Re^m$ satisfying $u \circ Mu \leq 0$ and $u \circ Mu \neq 0$?

NOT-EMAT
Instance: Positive integer m and an $m \times m$ matrix M with integer entries.
Question: Does there exist nonzero $u \in \Re^m$ satisfying $u \circ Mu \leq 0$ and $u \geq 0$?

Our reduction is from the following decision version of the problem of 1-norm maximization over a parallelotope, shown (via a reduction from NOT-ALL-EQUAL 3-SAT) by Bodlaender et al. [2, p. 213 and Thm. 15] to be **NP**-complete.

$[0,1]\mathrm{PARMAX}_1$
Instance: Positive integers n and γ; n linearly independent integer vectors $a_1, ..., a_n$ in $\Re^n$.
Question: Does there exist $y \in \sum_{i=1}^n [0, a_i]$ satisfying $\|y\|_1 \geq \gamma$?

Theorem 1. *Consider positive integers n and γ, and n linearly independent integer vectors $a_1, ..., a_n$ in $\Re^n$. Let $A := [a_1 \ \cdots \ a_n]$ and*

$$m := 3n+1, \quad M := \begin{bmatrix} I & 0 & 0 & -e \\ A & I & 0 & 0 \\ -A & 0 & I & 0 \\ 0 & 2e^T & 2e^T & 2\gamma - 1 \end{bmatrix}, \quad M' := \begin{bmatrix} I & 0 & 0 & -e \\ A & I & 0 & 0 \\ -A & 0 & I & 0 \\ 0 & -e^T & -e^T & \gamma \end{bmatrix}, \tag{1}$$

where $e := [1 \ \cdots \ 1]^T$. The following statements are equivalent:

(a) The answer to $[0,1]\mathrm{PARMAX}_1$ *with instance $n, \gamma, a_1, ..., a_n$ is yes.*
(b) The answer to NOT-PMAT *with instance m, M is yes.*
(c) The answer to NOT-CSMAT *with instance m, M is yes.*
(d) The answer to NOT-EMAT *with instance m, M' is yes.*

Proof. By using the nonsingularity of A to make the substitution $y = Ax$, we see that there exists $y \in \sum_{i=1}^n [0, a_i]$ satisfying $\|y\|_1 \geq \gamma$ if and only if there exists $x \in \Re^n$ satisfying

$$x \in [0, 1]^n, \quad \|Ax\|_1 \geq \gamma. \tag{2}$$

(a) $\Rightarrow$ (c). Since the answer to $[0,1]\mathrm{PARMAX}_1$ is yes, there exists $x \in \Re^n$ satisfying (2). Let $w_+ := -\max\{0, Ax\}$, $w_- := -\max\{0, -Ax\}$, $z := 1$. Then, for

$$u := \begin{bmatrix} x \\ w_+ \\ w_- \\ z \end{bmatrix}, \tag{3}$$

we have from (1) that

$$Mu = \begin{bmatrix} x - e \\ Ax + w_+ \\ -Ax + w_- \\ 2e^T w_+ + 2e^T w_- + 2\gamma - 1 \end{bmatrix}. \tag{4}$$

By (2), $x \circ (x - e) \leq 0$. Also, for any $v \in \Re$ we have $v_+(v + v_+) = 0$, where $v_+ = -\max\{0, v\}$. Thus, $w_+ \circ (Ax + w_+) = 0$ and $w_- \circ (-Ax + w_-) = 0$. Finally,

$$e^T w_+ + e^T w_- + \gamma = -e^T \max\{0, Ax\} - e^T \max\{0, -Ax\} + \gamma = -\|Ax\|_1 + \gamma \leq 0,$$

implying $2e^T w_+ + 2e^T w_- + 2\gamma - 1 < 0$. Hence, by (4), $u \circ Mu \le 0$, and, by $z = 1$, $u \circ Mu \ne 0$. Thus the answer to NOT-CSMAT is yes.

(a) $\Rightarrow$ (d). Since the answer to [0,1]PARMAX$_1$ is yes, there exists $x \in \Re^n$ satisfying (2). Let $w_+ := \max\{0, -Ax\}$, $w_- := \max\{0, Ax\}$, $z := 1$. Then, for u given by (3), we have from (1) that

$$M'u = \begin{bmatrix} x - e \\ Ax + w_+ \\ -Ax + w_- \\ -e^T w_+ - e^T w_- + \gamma \end{bmatrix}. \tag{5}$$

Similar to the proof of (a) $\Rightarrow$ (c), we obtain $x \circ (x - e) \le 0$, $w_+ \circ (Ax + w_+) = 0$ and $w_- \circ (-Ax + w_-) = 0$. Finally,

$$-e^T w_+ - e^T w_- + \gamma = -e^T \max\{0, -Ax\} - e^T \max\{0, Ax\} + \gamma = -\|Ax\|_1 + \gamma \le 0.$$

Hence, by (5), $u \circ M'u \le 0$. Also, (2) implies $x \ge 0$ and, by construction, $w_+ \ge 0$, $w_- \ge 0$, $z \ge 0$, so $u \ge 0$. Thus the answer to NOT-EMAT is yes.

(c) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (a). Since the answer to NOT-PMAT is yes, there exist $x \in \Re^n$, $w_+ \in \Re^n$, $w_- \in \Re^n$ and $z \in \Re$ such that u given by (3) is nonzero and satisfies $u \circ Mu \le 0$. Then, (1) yields

$$\begin{aligned} x \circ (x - ze) &\le 0, \\ w_+ \circ (Ax + w_+) &\le 0, \\ w_- \circ (-Ax + w_-) &\le 0, \\ z(2e^T w_+ + 2e^T w_- + (2\gamma - 1)z) &\le 0. \end{aligned} \tag{6}$$

If $z = 0$, then (6) would imply $x \circ x \le 0$ so $x = 0$ and $w_+ \circ w_+ \le 0$, $w_- \circ w_- \le 0$, so $w_+ = w_- = 0$, contradicting $u \ne 0$. Thus $z \ne 0$. Then, dividing the inequalities in (6) by z^2 and denoting $x' := x/z$, $w'_+ := w_+/z$, $w'_- := w_-/z$, we obtain

$$\begin{aligned} x' \circ (x' - e) &\le 0, \\ w'_+ \circ (Ax' + w'_+) &\le 0, \\ w'_- \circ (-Ax' + w'_-) &\le 0, \\ 2e^T w'_+ + 2e^T w'_- + 2\gamma - 1 &\le 0. \end{aligned} \tag{7}$$

For each $i \in \{1, \dots, n\}$ with $[Ax']_i \le 0$, $[w'_+]_i([Ax']_i + [w'_+]_i) \le 0$ implies $0 \le [w'_+]_i \le -[Ax']_i$ and $[w'_-]_i(-[Ax']_i + [w'_-]_i) \le 0$ implies $[Ax']_i \le [w'_-]_i \le 0$. Thus,

$$-|[Ax']_i| = [Ax']_i \le [w'_+]_i + [w'_-]_i. \tag{8}$$

Similarly, for each $i \in \{1, \dots, n\}$ with $-[Ax']_i \le 0$, $[w'_+]_i([Ax']_i + [w'_+]_i) \le 0$ implies $-[Ax']_i \le [w'_+]_i \le 0$ and $[w'_-]_i(-[Ax']_i + [w'_-]_i) \le 0$ implies $0 \le [w'_-]_i \le [Ax']_i$. Thus,

$$-|[Ax']_i| = -[Ax']_i \le [w'_+]_i + [w'_-]_i. \tag{9}$$

Then, (8), (9) and the last inequality of (7) yield

$$-\|Ax'\|_1 = -\sum_{i=1}^{n} |[Ax']_i| \leq \sum_{i=1}^{n} [w'_+]_i + [w'_-]_i = e^T w'_+ + e^T w'_- \leq -\gamma + 1/2.$$

Thus, $\|Ax'\|_1 \geq \gamma - 1/2$, implying $\alpha := \max_{y \in [0,1]^n} \|Ay\|_1 \geq \gamma - 1/2$. Since $y \mapsto \|Ay\|_1$ is a convex function, its maximum value is attained at a vertex of $[0, 1]^n$, so α is an integer. Then it must be that $\alpha \geq \gamma$, so there exists $x \in \Re^n$ satisfying (2). Thus the answer to [0,1]PARMAX$_1$ is yes.

(d) $\Rightarrow$ (a). Since the answer to NOT-EMAT is yes, there exist $x \in \Re^n$, $w_+ \in \Re^n$, $w_- \in \Re^n$ and $z \in \Re$ such that u given by (3) is nonzero and satisfies $u \circ M'u \leq 0$ and $u \geq 0$. Then, as in the proof of (b) $\Rightarrow$ (a), we obtain that $z > 0$ and $x' := x/z$, $w'_+ := w_+/z$, $w'_- := w_-/z$ are nonnegative and satisfy

$$\begin{aligned} x' \circ (x' - e) &\leq 0, \\ w'_+ \circ (Ax' + w'_+) &\leq 0, \\ w'_- \circ (-Ax' + w'_-) &\leq 0, \\ -e^T w'_+ - e^T w'_- + \gamma &\leq 0. \end{aligned} \tag{10}$$

For each $i \in \{1, ..., n\}$ with $[Ax']_i \leq 0$, $[w'_+]_i([Ax']_i + [w'_+]_i) \leq 0$ implies $0 \leq [w'_+]_i \leq -[Ax']_i$ and $[w'_-]_i(-[Ax']_i + [w'_-]_i) \leq 0$ implies $[Ax']_i \leq [w'_-]_i \leq 0$. Since $[w'_-]_i \geq 0$, the latter implies $[w'_-]_i = 0$. Thus,

$$-|[Ax']_i| = [Ax']_i \leq -[w'_+]_i = -[w'_+]_i - [w'_-]_i. \tag{11}$$

Similarly, for each $i \in \{1, ..., n\}$ with $-[Ax']_i \leq 0$, $[w'_+]_i([Ax']_i + [w'_+]_i) \leq 0$ implies $-[Ax']_i \leq [w'_+]_i \leq 0$ and $[w'_-]_i(-[Ax']_i + [w'_-]_i) \leq 0$ implies $0 \leq [w'_-]_i \leq [Ax']_i$. Since $[w'_+]_i \geq 0$, the former implies $[w'_+]_i = 0$. Thus,

$$-|[Ax']_i| = -[Ax']_i \leq -[w'_-]_i = -[w'_+]_i - [w'_-]_i. \tag{12}$$

Then, (11), (12) and the last inequality of (10) imply

$$-\|Ax'\|_1 = -\sum_{i=1}^{n} |[Ax']_i| \leq \sum_{i=1}^{n} (-[w'_+]_i - [w'_-]_i) = -e^T w'_+ - e^T w'_- \leq -\gamma.$$

Thus, $\|Ax'\|_1 \geq \gamma$. Also, the first inequality in (10) implies $0 \leq x' \leq e$. Thus, x' satisfies (2), so the answer to [0,1]PARMAX$_1$ is yes.

Notice that the matrix M differs from the matrix M' in the sign of two terms in their last row. The positive sign of the terms in M is needed to prove (b) $\Rightarrow$ (a), while the negative sign of the terms in M' is needed to prove (a) $\Rightarrow$ (d). We do not know if it is possible to use a single matrix to prove both.

Corollary 1. *The problems* NOT-PMAT, NOT-CSMAT, NOT-EMAT *are* **NP**-*complete.*

Proof. Suppose the answer to NOT-EMAT with instance m, M is yes. Then there exists nonzero $u \in \Re^m$ satisfying $u \circ Mu \leq 0$ and $u \geq 0$. Thus, there exist $l \in \{1, ..., m\}$ and $J \subseteq \{1, ..., m\}$ such that the linear system

$$u_l \geq 1, \quad \left\{ \begin{array}{c} u_i \geq 0, \\ [Mu]_i \leq 0, \end{array} \right\} \forall i \in J, \quad \left\{ \begin{array}{c} u_i = 0, \\ [Mu]_i \geq 0, \end{array} \right\} \forall i \notin J,$$

has a solution. Any vertex solution u^* has size polynomially bounded by the size of M (e.g., [16, p. 30]) and satisfies $u^* \circ Mu^* \leq 0$ and $0 \neq u^* \geq 0$, so u^* is a certificate for the yes answer. Thus NOT-EMAT is in **NP**. Similar arguments show that NOT-PMAT, and NOT-CSMAT are also in **NP**. [For NOT-PMAT, we can alternatively check that a given principal submatrix of M has nonpositive determinant, which is computable in polynomial time, e.g., [16, §3], [7].]

Since the size (number of bits in the binary representation) of m, M, M' given by (1) is a polynomial in the size of n, γ, A, it then follows from Thm. 1 and **NP**-completeness of [0,1]PARMAX$_1$ that NOT-PMAT, NOT-CSMAT, NOT-EMAT are **NP**-complete.

3. P_0- and semimonotone matrices

We formally state below the decision problems for P_0-matrices [6,11,12] and for the class E_0 of semimonotone matrices [6, p. 184], [12, p. 227]. We show these two problems are **NP**-complete by reduction from, respectively, NOT-PMAT and NOT-EMAT.

NOT-P0MAT
Instance: Positive integer m and an $m \times m$ matrix N with integer entries.
Question: Does there exist a principal submatrix of N whose determinant is negative?

NOT-E0MAT
Instance: Positive integer m and an $m \times m$ matrix N with integer entries.
Question: Does there exist nonzero $u \in \Re^m$ satisfying $u \geq 0$ and, for each $i \in \{1, ..., m\}$, either $u_i = 0$ or $[Nu]_i < 0$?

Theorem 2. *Consider positive integer m and an $m \times m$ matrix M with integer entries. Let μ be the maximum absolute value of the entries of M. Let*

$$\begin{array}{lll} N := \nu M - I, & \nu := m2^{m-1}\Delta, & \Delta := (m\mu)^m, \\ N' := \nu' M - I, & \nu' := (m(\mu+1))^m. & \end{array} \tag{13}$$

(Δ is an upper bound on the absolute value of the principal minors of M.) Then the following statements (a) and (b) are equivalent, and the following statements (c) and (d) are equivalent:

(a) The answer to NOT-PMAT *with instance m, M is no.*
(b) The answer to NOT-P0MAT *with instance m, N is no.*
(c) The answer to NOT-EMAT *with instance m, M is no.*
(d) The answer to NOT-E0MAT *with instance m, N' is no.*

Proof. (b) $\Rightarrow$ (a). Since the answer to NOT-P0MAT is no, then N is a P_0-matrix and, by (13) and its property (e.g., [6, Thm. 3.4.2]), M is a P-matrix. Thus, the answer to NOT-PMAT is no.

(a) $\Rightarrow$ (b). Since the answer to NOT-PMAT is no, then M is a P-matrix, i.e., for each nonempty $J \subseteq \{1, ..., m\}$, we have $\det[M_{JJ}] > 0$. Since M_{JJ} has integer entries, this implies $\det[M_{JJ}] \geq 1$. Also, Δ is an upper bound on the absolute value of the principal minors of M [16, p. 195]. This together with (13) and (2.2.1) in [6] imply

$$\begin{aligned}
\det[N_{JJ}] &= \det[\nu M_{JJ} - I_{JJ}] \\
&= \sum_{K \subseteq J} \det[\nu M_{KK}] \det[-I_{J \setminus K J \setminus K}] \\
&= \sum_{K \subseteq J} \nu^{|K|} \det[M_{KK}](-1)^{|J|-|K|} \\
&= \nu^{|J|} \det[M_{JJ}] + \sum_{K \subset J} \nu^{|K|} \det[M_{KK}](-1)^{|J|-|K|} \\
&\geq \nu^{|J|} - \sum_{K \subset J} \nu^{|K|} \Delta \\
&= \nu^{|J|} - \sum_{k=0}^{|J|-1} \binom{|J|}{k} \nu^k \Delta \\
&\geq \nu^{|J|} - |J| \sum_{k=0}^{|J|-1} \binom{|J|-1}{k} \nu^k \Delta \\
&= \nu^{|J|} - |J|(1+\nu)^{|J|-1} \Delta \\
&> \nu^{|J|} - |J|(2\nu)^{|J|-1} \Delta \\
&= \nu^{|J|-1}(\nu - |J| 2^{|J|-1} \Delta).
\end{aligned}$$

Since $|J| \leq m$, (13) implies the right-hand side is nonnegative. Thus, all principal minors of N are nonnegative, so the answer to NOT-P0MAT is no.

(d) $\Rightarrow$ (c). Since the answer to NOT-E0MAT is no, then N' is in E_0. So, for each nonzero $u \in \Re^m$ with $u \geq 0$, there exists $k \in \{1, ..., m\}$ such that $u_k > 0$ and $[N'u]_k \geq 0$, implying from (13) that

$$[Mu]_k = ([N'u]_k + u_k)/\nu' > 0.$$

Hence M is in E. Thus, the answer to NOT-EMAT is no.

(c) $\Rightarrow$ (d). Since the answer to NOT-EMAT is no, then M is in E. So, for each nonzero $u \in \Re^m$ with $u \geq 0$, there exists $k \in \{1, ..., m\}$ such that $u_k > 0$ and $[Mu]_k > 0$, implying that the minimum value δ of

$$f(u) := \max_{i=1,...,m} \min\{u_i, [Mu]_i\},$$

subject to $u \geq 0$ and $\|u\|_\infty = 1$, is positive. For each $u \in \Re^m$, there exists $l \in \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, m\}$ such that either (i) $f(u) = u_l$ and

$$u_l \leq [Mu]_l, \quad \left\{ \begin{array}{c} u_l \geq u_j \\ [Mu]_j \geq u_j \end{array} \right\} \forall j \in J, \quad \left\{ \begin{array}{c} u_l \geq [Mu]_j \\ u_j \geq [Mu]_j \end{array} \right\} \forall j \notin J, \tag{14}$$

or (ii) $f(u) = [Mu]_l$ and

$$[Mu]_l \leq u_l, \quad \left\{ \begin{array}{c} [Mu]_l \geq u_j \\ [Mu]_j \geq u_j \end{array} \right\} \forall j \in J, \quad \left\{ \begin{array}{c} [Mu]_l \geq [Mu]_j \\ u_j \geq [Mu]_j \end{array} \right\} \forall j \notin J. \tag{15}$$

Thus,

$$\begin{aligned} \delta &= \min_{u \geq 0, \|u\|_\infty = 1} f(u) \\ &= \min_{k=1,\ldots,m} \min_{0 \leq u \leq e, u_k = 1} f(u) \\ &= \min_{\substack{k,l=1,\ldots,m \\ J \subseteq \{1,\ldots,m\}}} \min \left\{ \begin{array}{lclcl} \min & u_l & , & \min & [Mu]_l \\ \text{s.t.} & 0 \leq u \leq e & & \text{s.t.} & 0 \leq u \leq e \\ & u_k = 1, (14) & & & u_k = 1, (15) \end{array} \right\}. \end{aligned} \tag{16}$$

For each k, l, J, each of the two minimizations inside the braces is a linear program with constraint matrix entries of maximum absolute value $\mu + 1$. Let u^* be an optimal basic solution of either linear program. Then, by Cramer's rule, each entry of u^* is of the form p/q, where p is an integer, and q is the determinant of a nonsingular submatrix of the constraint matrix of the linear program. Since ν' is an upper bound on the determinant of any $k \times k$ ($k \leq m$) submatrix with integer entries of maximum absolute value $\mu + 1$ [16, p. 195], then $q \leq \nu'$, implying $u_l^* \geq 1/\nu'$ and $[Mu^*]_l \geq 1/\nu'$. By (16), $\delta \geq 1/\nu'$. Thus, for each nonzero $u \in \Re^m$ with $u \geq 0$, there exists $l \in \{1, \ldots, m\}$ such that $u_l \geq \|u\|_\infty/\nu'$ and $[Mu]_l \geq \|u\|_\infty/\nu'$, implying from (13) that

$$[N'u]_l = \nu'[Mu]_l - u_l \geq \|u\|_\infty - u_l \geq 0.$$

Hence N' is in E_0. Thus, the answer to NOT-E0MAT is no.

Corollary 2. *The problems* NOT-P0MAT *and* NOT-E0MAT *are* **NP**-*complete.*

Proof. By similar arguments as in the proof of Cor. 1, we have that NOT-P0MAT and NOT-E0MAT are in **NP**. Also, the size of N, N' given by (13) is a polynomial in the size of m, M. It then follows from Thm. 2 and Cor. 1 that NOT-P0MAT and NOT-E0MAT are **NP**-complete.

4. R_0- and nondegenerate matrices

By definition, an $m \times m$ real matrix M is degenerate if and only if there exists nonempty $J \subseteq \{1, \ldots, m\}$ such that M_{JJ} is singular [6,12]. Since M_{JJ} is singular if and only if there exists nonzero $u \in \Re^m$ satisfying $M_{JJ}u_J = 0$ and $u_i = 0$ for all $i \notin J$, this is equivalent to the existence of a nonzero $u \in \Re^m$ satisfying $u \circ Mu = 0$. Also, by definition, M is *not* an R_0-matrix if and only if there exists a nonzero $u \in \Re^m$ satisfying $u \geq 0$, $Mu \geq 0$, and $u \circ Mu = 0$ [6, p. 180], [12, p. 229]. We formally state the corresponding decision problems below.

NOT-R0MAT
Instance: Positive integer m and an $m \times m$ matrix M with integer entries.
Question: Does there exist nonzero $u \in \Re^m$ satisfying $u \circ Mu = 0$, $u \geq 0$, $Mu \geq 0$?

DEGMAT
Instance: Positive integer m and an $m \times m$ matrix M with integer entries.
Question: Does there exist nonzero $u \in \Re^m$ satisfying $u \circ Mu = 0$?

The reduction, similar to one used by Chandrasekaran et al., is from the following integer knapsack problem, known to be **NP**-complete [10, p. 247].

KNAPSACK
Instance: Positive integers n and b; an integer vector a in $\Re^n$.
Question: Does there exist $x \in \{0, 1\}^n$ satisfying $a^T x = b$?

Theorem 3. *Consider positive integers n and b, and integer vector a in $\Re^n$. Let*

$$m := n + 1, \qquad M := \begin{bmatrix} -I & e \\ -a^T & b \end{bmatrix}, \tag{17}$$

where $e := [1 \ \cdots \ 1]^T$. The following statements are equivalent:

(a) The answer to KNAPSACK *with instance n, b, a is yes.*
(b) The answer to NOT-R0MAT *with instance m, M is yes.*
(c) The answer to DEGMAT *with instance m, M is yes.*

Proof. (a) $\Rightarrow$ (b). Since the answer to KNAPSACK is yes, there exists $x \in \{0, 1\}^n$ satisfying $a^T x = b$. Then, $x \circ (x - e) = 0$, so $u := \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ is nonzero and satisfies $u \circ Mu = 0$. Also, by construction, $u \geq 0$ and $Mu = \left[\begin{smallmatrix} e-x \\ 0 \end{smallmatrix}\right] \geq 0$. Thus the answer to NOT-R0MAT is yes.

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a). Since the answer to DEGMAT is yes, there exist $x \in \Re^n$ and $z \in \Re$ such that $u := \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$ is nonzero and satisfies $u \circ Mu = 0$. Using (17), this can be rewritten as

$$x \circ (ze - x) = 0, \quad z(zb - a^T x) = 0. \tag{18}$$

If $z = 0$, then (18) would imply $x \circ x = 0$, so $x = 0$, contradicting $u \neq 0$. Thus $z \neq 0$. Then, dividing the inequalities in (18) by z^2 and letting $x' := x/z$, we obtain

$$x' \circ (e - x') = 0, \quad b - a^T x' = 0.$$

The first equation implies $x' \in \{0, 1\}^n$. Thus, the answer to KNAPSACK is yes.

Corollary 3. *The problems* NOT-R0MAT, DEGMAT *are* **NP***-complete.*

Proof. By a similar argument as in the proof of Cor. 1, we have that NOT-R0MAT and DEGMAT are in **NP**.

Since the size of m, M given by (17) is a polynomial in the size of n, b, a, it then follows from Thm. 3 and the **NP**-completeness of KNAPSACK that NOT-R0MAT and DEGMAT are **NP**-complete.

5. Further questions

There remain a number of matrix classes, described in [6, 11, 12], for which complexity of the corresponding decision problem is unknown. Two good examples are the classes of Q- and Q_0-matrices.

Acknowledgements. The author thanks Victor Klee for suggesting the reference [2]. He also thanks Walter Morris, Katta Murty, Richard Stone, and an anonymous referee for their helpful comments on an earlier draft.

References

1. Bartholdi III, J.J. (1982): A good submatrix is hard to find. Oper. Res. Lett. **1**, 190–193
2. Bodlaender, H.L., Gritzmann, P., Klee, V., Van Leeuwen, J. (1990): Computational complexity of norm-maximization. Combinatorica **10**, 203–225
3. Chandrasekaran, R., Kabadi, S.N., Murty, K.G. (1982): Some NP-complete problems in linear programming. Oper. Res. Lett. **1**, 101–104
4. Chen, B., Chen, X.-J., Kanzow, C. (2000): A penalized Fischer-Burmeister NCP-function. Math. Program., DOI 10.1007/s101070000159
5. Chen, X.-J., Ye, Y. (1998): On smoothing methods for the P_0 matrix linear complementarity problem. School of Mathematics, The University of New South Wales, Sydney; to appear in SIAM J. Optim.
6. Cottle, R.W., Pang, J.-S., Stone, R.E. (1992): The Linear Complementarity Problem. Academic Press, New York
7. Coxson, G.E. (1994): The P-matrix problem is co-NP-complete. Math. Program. **64**, 173–178
8. Facchinei, F., Kanzow, C. (1999): Beyond monotonicity in regularization methods for nonlinear complementarity problems. SIAM J. Control Optim. **37**, 1150–1161
9. Facchinei, F., Soares, J. (1997): A new merit function for nonlinear complementarity problems and a related algorithm. SIAM J. Optim. **7**, 225–247
10. Garey, M.R., Johnson, D.S. (1979): Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, San Francisco
11. Kojima, M., Megiddo, N., Noma, T., Yoshise, A. (1991): A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems. Lecture Notes in Computer Science 538, Springer-Verlag, Berlin
12. Murty, K.G. (1988): Linear Complementarity, Linear and Nonlinear Programming. Heldermann Verlag, Berlin
13. Murty, K.G., Kabadi, S.N. (1987): Some NP-complete problems in quadratic and nonlinear programming, Math. Program. **39**, 117–129
14. Pang, J.-S. (1997): Error bounds in mathematical programming, Math. Program. **79**, 299–332
15. Rohn, J. (1998): Checking properties of interval matrices. In: V., Lakeyev, A., Rohn, J., Kahl, P., eds., Computational Complexity and Feasibility of Data Processing and Interval Computations. Kreinovich, Kluwer, Dordrecht
16. Schrijver, A. (1986): Theory of Linear and Integer Program. John Wiley & Sons, New York
17. Sznajder, R., Gowda, M.S. (1999): On the limiting behavior of the trajectory of regularized solutions of a P_0-complementarity problem. In: Fukushima, M., Qi, L., eds., Reformulation: nonsmooth, piecewise smooth, semismooth and smoothing methods, pp. 371–379. Kluwer, Dordrecht