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The mixed vertex packing problem*

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Abstract. We study a generalization of the vertex packing problem having both binary and bounded continuous variables, called the mixed vertex packing problem (MVPP). The well-known vertex packing model arises as a subproblem or relaxation of many 0-1 integer problems, whereas the mixed vertex packing model arises as a natural counterpart of vertex packing in the context of mixed 0-1 integer programming. We describe strong valid inequalities for the convex hull of solutions to the MVPP and separation algorithms for these inequalities. We give a summary of computational results with a branch-and-cut algorithm for solving the MVPP and using it to solve general mixed-integer problems.

1. Introduction

The vertex packing problem arises as a subproblem or relaxation of many 0-1 integer problems. In the context of mixed 0-1 integer problems, the *mixed vertex packing problem* (MVPP) is a natural counterpart of the vertex packing problem. MVPP arises, for example, as a column generation pricing subproblem, Lagrangian subproblem, or as a mixed-integer combinatorial relaxation of mixed 0-1 integer problems. MVPP, formulated as

$$\max\{cx + dy : (x, y) \in \text{MVP}\}, \text{ where}$$

$$\begin{aligned} \text{MVP} = \{x \in \mathbb{B}^n, y \in \mathbb{R}^m : x_i + x_j &\leq 1, & (i, j) \in E \\ a_{ik}x_i + y_k &\leq u_k, & (i, k) \in F \\ 0 \leq y_k &\leq u_k, & k \in M\} \end{aligned}$$

is a generalization of the vertex packing problem having both binary and bounded continuous variables.

We use N to denote the index set of binary variables with $n = |N|$ and M to denote the index set of continuous variables with $m = |M|$. Inequalities over $E \subseteq \{(i, j) : i, j \in N\}$ are called *binary edge inequalities*, whereas the inequalities over $F \subseteq \{(i, k) : i \in N, k \in M\}$ are called *mixed edge inequalities*. We assume

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that $u_k < \infty$ for all $k \in M$. In order to eliminate uninteresting cases, we also assume that $u_k > 0$, otherwise $y_k = 0$, and that $0 < a_{ik} \leq u_k$, otherwise either $a_{ik}x_i + y_k \leq u_k$ is redundant or $x_i = 0$ in every feasible solution. Without loss of generality, we assume that $c_i > 0$ for all $i \in N$ and $d_k > 0$ for all $k \in M$, since there is an optimal solution with $x_i = 0$ if $c_i \leq 0$ and $y_k = 0$ if $d_k \leq 0$. An arbitrary inequality $ax_i + by_k \leq h$ with positive data can be put into the form $a_{ik}x_i + y_k \leq u_k$, by writing it as $(u_k - \frac{h-a}{b})x_i + y_k \leq u_k$ after reducing u_k to h/b, if $u_k > h/b$. Similarly, $ax_i + bx_j \leq h$ can be put into the form $x_i + x_j \leq 1$ if $a + b > h$, otherwise it is redundant. Of particular interest is that a variable upper bound $y_i \leq u_i x_i$ becomes a mixed edge inequality after complementing the binary variable x_i, whereas a variable lower bound $l_i x_i \leq y_i$ becomes a mixed edge inequality after complementing the continuous variable y_i.

Since there are two variables in each constraint, MVP can be represented by a graph $G = (N \cup M, E \cup F)$ where weights on F denote the conflicts and weights on M denote the upper bounds. G is called a *mixed conflict graph* because it has two types of vertices: binary vertices for binary variables and continuous vertices for continuous variables. The following notation is used in the remainder of the paper. For $i \in N \cup M$

$$N(i) = \{j \in N : (i, j) \in E \cup F\} \text{ and } M(i) = \{k \in M : (i, k) \in F\}.$$

Thus for vertex i, $N(i)$ denotes the index set of binary vertices adjacent to i, whereas $M(i)$ denotes the index set of continuous vertices adjacent to i.

Although the $\mathcal{NP}$-hard vertex packing problem is one of the most studied problems in combinatorial optimization ([6, 8, 15–17, 19] to mention a few), the mixed vertex packing problem has apparently not been defined and studied in its own right before.

Applications

Minoux [11] describes a column generation method for optimal decomposition of a satellite traffic matrix into switching mode submatrices, where the objective is to minimize the sum of the maximum entry in each submatrix. The associated pricing subproblem is

$$\min_P \left\{ \max_{i \in P} a_i - \sum_{i \in P} c_i \right\}, \tag{1}$$

where packing P denotes a feasible switching mode submatrix, a_i the entries of the submatrix, and c_i the dual variables corresponding to the constraints of the master problem. Since MVPP can alternatively be written as

$$\max_P \sum_{i \in P} c_i + \sum_{k \in M} d_k \big(u_k - \max_{i \in P} a_{ik}\big), \tag{2}$$

where P is a packing in $G(N)$, the subgraph induced by N, and the term $\sum_{k \in M} d_k u_k$ in (2) is a constant, Minoux's pricing subproblem is a mixed vertex packing problem with a single continuous vertex. Minoux [12, 13] presents many other problems ranging from TV broadcasting to weighted edge coloring of graphs, where (1) is the column generation pricing subproblem. Minoux [13] shows that (1) can be solved in polynomial time if the vertex packing problem on $G(N)$ can be solved in polynomial time.

Another application of MVPP is noxious facility location. A mixed 0-1 integer model described by Erkut and Neuman [7] for opening p noxious facilities in n candidate locations (N) while maximizing the sum of minimum distances to m population areas (M) is

$$\max \sum_{k \in M} y_k$$

$$\text{s.t.: } y_k \leq d_{ik} + w(1 - x_i), \quad i \in N, \ k \in M \tag{3}$$

$$\sum_{i \in N} x_i = p \tag{4}$$

$$x \in \mathbb{B}^n, y \in \mathbb{R}^m,$$

with $w \geq \max_{i \in N, k \in M} d_{ik}$. Letting $u_k = \max_{i \in N} d_{ik}$, (3) can be written as $(u_k - d_{ik})x_i + y_k \leq u_k$. The Lagrangian function of this problem based on relaxing constraint (4) is a mixed vertex packing problem with independent binary variables, i.e., $E = \emptyset$, which is solvable in polynomial time as we show in Sect. 2.

Yet another application of the mixed vertex packing model is that it provides a combinatorial mixed-integer relaxation for general mixed-integer problems. In recent years valid inequalities from vertex packing relaxations have been shown to be valuable in deriving cutting planes for 0-1 integer programming, see for example Atamtürk et al. [3], Borndörfer and Weismantel [5], and Hoffman and M.W. Padberg [9]. In 0-1 integer programming, a vertex packing relaxation is obtained by considering pairwise conflicts between binary variables. We generalize this concept to mixed 0-1 integer programming by considering pairwise conflicts between continuous variables and binary variables as well. As far as we know the closest work in this context is by Johnson [10], where he strengthens variable upper bound constraints in the presence of binary edges and gives a special case of the mixed clique inequalities described here.

The following example illustrates the derivation of a mixed vertex packing relaxation of a mixed 0-1 integer program.

Example 1. Consider the mixed 0-1 integer set

$$\begin{aligned} \mathrm{S} = \{\, x \in \mathbb{B}^4, y \in \mathbb{R}^3_+ : 3x_1 \quad\quad\quad\quad\quad\quad\quad +6x_4 \ +y_1 \quad\quad\quad\quad\quad & \leq 9 \\ 13x_3 \quad\quad -2y_1 +2y_2 +3y_3 & \leq 6 \\ 2x_1 +5x_2 +3x_3 \quad\quad\quad\quad\quad\quad\quad\quad\quad & \leq 6 \\ y_1 \leq 9, \ y_2 \leq 10, \ y_3 & \leq 8 \,\}. \end{aligned}$$

The following logical implications, which can be found by probing [18], are valid for S:

$$\begin{aligned} &x_1 = 1 \Rightarrow x_2 = 0, y_1 \leq 6 \Rightarrow \ y_2 \leq 9, \ y_3 \leq 6, \\ &x_2 = 1 \Rightarrow x_1 = 0, \ x_3 = 0, \\ &x_3 = 1 \Rightarrow x_2 = 0, y_1 \geq \tfrac{7}{2} \Rightarrow x_4 = 0, \\ &x_4 = 1 \Rightarrow y_1 \leq 3 \Rightarrow x_3 = 0, \ y_2 \leq 6, \ y_3 \leq 4. \end{aligned}$$

Writing these implications as linear inequalities gives us the packing relaxation

$$\mathrm{MVP} = \{\, x \in \mathbb{B}^4, y \in \mathbb{R}^3_+ : \begin{array}{llllllllll} 3x_1 & & & & +y_1 & & & \leq & 9 \\ & & & 6x_4 & +y_1 & & & \leq & 9 \\ 1x_1 & & & & & +y_2 & & \leq & 10 \\ & & & 4x_4 & & +y_2 & & \leq & 10 \\ 2x_1 & & & & & & +y_3 & \leq & 8 \\ & & & 4x_4 & & & +y_3 & \leq & 8 \\ x_1 & +x_2 & & & & & & \leq & 1 \\ & x_2 & +x_3 & & & & & \leq & 1 \\ & & x_3 & +x_4 & & & & \leq & 1 \,\}. \end{array}$$

Since MVP is a relaxation of S, valid inequalities for MVP are also valid for S. This relation motivates the study of the polyhedral structure of MVP in Sect. 2. Figure 1 shows the mixed conflict graph for the packing relaxation of S. We use circles to denote the binary vertices and squares for the continuous vertices. Note that there are no edges between continuous vertices.

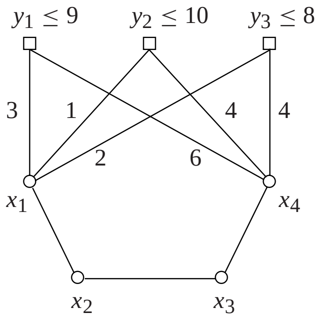

Fig. 1. Mixed conflict graph of S

The outline of this paper is as follows. In Sect. 2, we study the facial structure of the mixed vertex packing polytope. We derive several classes of valid inequalities for this polytope and give separation algorithms for these inequalities. In Sect. 3, we present computational experiments that indicate the effectiveness of the inequalities described in Sect. 2 in solving mixed vertex packing problems and general mixed-integer programs.

2. Mixed vertex packing polytope

In this section we study the facial structure of the mixed vertex packing polytope, *conv*(MVP), and derive strong valid inequalities for it. Let LMVP be the linear relaxation of MVP. Thus,

$$\text{LMVP} = \{(x, y) \in \mathbb{R}^{n+m} \text{ that satisfy (5)–(8)}\}, \text{ where}$$

$$x_i + x_j \leq 1, \qquad (i, j) \in E \tag{5}$$

$$a_{ik}x_i + y_k \leq u_k, \qquad (i, k) \in F \tag{6}$$

$$0 \leq x_i \leq 1, \qquad i \in N \tag{7}$$

$$0 \leq y_k \leq u_k, \qquad k \in M. \tag{8}$$

Below we summarize basic results on the dimension of *conv*(MVP) and the strength of inequalities (5)–(8) defining LMVP.

Proposition 1.

1. *The dimension of conv(MVP) is* $n + m$.
2. $x_i \geq 0,\ i \in N$ *and* $y_k \geq 0,\ k \in M$ *are facet-defining for conv(MVP).*
3. $x_i \leq 1,\ i \in N$ *defines a facet of conv(MVP) if and only if* $N(i) = \emptyset$ *and* $a_{ik} < u_k$ *for all* $k \in M(i)$.
4. $y_k \leq u_k,\ k \in M$ *defines a facet of conv(MVP) if and only if* $M(k) = \emptyset$.
5. $x_i + x_j \leq 1$ *defines a facet of conv(MVP) if and only if* $N(i) \cap N(j) = \emptyset$ *and* $\min\{a_{ik}, a_{jk}\} < u_k$ *for all* $k \in M(i) \cup M(j)$.
6. $a_{ik}x_i + y_k \leq u_k$ *defines a facet of conv(MVP) if and only if* $N(i) \cap N(k) = \emptyset$ *and* $a_{ik} = \max_{j \in N(k)} a_{jk}$.

The following theorem characterizes the graphs for which the linear relaxation LMVP is sufficient to describe *conv*(MVP).

Theorem 1. *Inequalities (5)–(8) of LMVP are sufficient to describe conv(MVP) if and only if G is bipartite and* $a_{ik} = a_k$*, for all* $i \in N(k)$*, for all* $k \in M$.

Proof. Suppose $a_{ik} < a_{jk}$ for some $k \in M$. In Proposition 3 we show that $(a_{jk} - a_{ik})x_j + a_{ik}x_i + y_k \leq u_k$ is valid for *conv*(MVP). This inequality dominates $a_{ik}x_i + y_k \leq u_k$. Now, suppose $a_{ik} = a_k$, for all $i \in N(k)$ for all $k \in M$ but G is not bipartite. In that case, consider the odd cycle given in Fig. 2. It is easily seen that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, u - \frac{a}{2})$ is a fractional basic feasible solution of LMVP if u is the upper bound of the continuous variable.

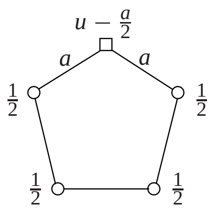

Fig. 2. Fractional basic feasible solution

Conversely, define $y'_k = (u_k - y_k)/a_k$ and rewrite inequality (6) as $x_i - y'_k \leq 0$ and inequality (8) as $0 \leq y'_k \leq u_k/a_k$. Since G is bipartite, by multiplying the binary variables associated with vertices that are not adjacent to a continuous vertex by -1, we obtain a constraint matrix with exactly one $+1$ and one -1 coefficient in each row (5) and (6) and an identity for (7) and (8), which is totally unimodular. The right-hand side of this formulation is integral, except for the upper bound constraints on $y'_k \le u_k/a_k$, $k \in M$ with $u_k > a_k$. However, if $u_k > a_k$, then in a feasible solution $y'_k \le u_k/a_k$ is tight only if $N(k) = \emptyset$. Hence an extreme point of LMVP is integral for all x variables.

□

2.1. Valid inequalities

There is a natural vertex packing relaxation of MVP, defined on the subgraph induced by the binary vertices. Valid inequalities for this vertex packing relaxation are valid for MVP as well.

Proposition 2. *Let MVP(N) denote the projection of MVP onto the space of binary variables. If*

$$\sum_{i \in S} b_i x_i \le r \tag{9}$$

for $S \subseteq N$ is a valid inequality for MVP(N), then it is valid for MVP as well. If (9) is facet-defining for conv(MVP(N)), then it is also facet-defining for conv(MVP) if for all $k \in M$, there exists a packing $P_k \subseteq S$ satisfying (9) at equality with $a_{ik} < u_k$ for all $i \in P_k \cap N(k)$.

Proof. The inequality is valid for MVP since $a_{ik} > 0$ for all $(i,k) \in F$. If (9) is facet-defining for $conv$(MVP(N)), then there exists n affinely independent points in MVP(N) satisfying (9) at equality. Let e_i be the ith unit vector. These n points together with $\sum_{i \in P_k} e_i + (u_k - \max_{i \in P_k} a_{ik}) e_k$, for $k \in M$, make up $n+m$ affinely independent points in $\{(x,y) \in MVP : \sum_{i \in S} b_i x_i = r\}$.

□

For a vertex k, a subgraph consisting of vertices k and $T \subseteq N(k)$ and the edges between k and T, is said to be a *star* of vertex k. Now we give the first class of new valid inequalities for MVP.

Proposition 3. *For $k \in M$, let $T = \{i_1, i_2, \ldots, i_t\}$ be a subset of $N(k)$ such that $a_{i_{j-1}k} < a_{i_j k}$ for $j = 2, 3, \ldots, t$. Then the star inequality*

$$\sum_{i \in T} \bar{a}_{ik} x_i + y_k \le u_k \tag{10}$$

where $\bar{a}_{i_1 k} = a_{i_1 k}, \bar{a}_{i_j k} = a_{i_j k} - a_{i_{j-1} k},\ j = 2, 3, \ldots, t$, is valid for MVP.

Proof. Let $(\bar{x}, \bar{y}) \in$ MVP, $S = \{i \in T : \bar{x}_i = 1\}$, and $j^* = \max_{1 \le j \le t}\{j : i_j \in S\}$.

Then $\displaystyle\sum_{i \in T} \bar{a}_{ik} \bar{x}_i + \bar{y}_k \le \sum_{i \in S} \bar{a}_{ik} + (u_k - a_{i_{j^*} k}) \le a_{i_{j^*} k} + (u_k - a_{i_{j^*} k}) = u_k.$

□

Theorem 2. *The star inequality (10) is facet-defining for conv(MVP) if* $a_{i_t k} = \max_{j \in N(k)} a_{jk}$ *and* $N(i) = \emptyset$ *for all* $i \in T$.

Proof. Suppose $N(k) = \{1, 2, \dots, l\}$ is indexed so that $a_{1k} \leq a_{2k} \leq \dots \leq a_{lk}$. Then it is easy to show that the following $n + m$ points

$$
\begin{aligned}
p_k &= u_k e_k, \\
p_i &= u_k e_k + u_i e_i, \; i \in M \setminus \{k\}, \\
q_i &= u_k e_k + e_i, \; i \in N \setminus N(k), \\
w_i &= \sum_{j \in N(k): j \leq i} e_j + (u_k - a_{ik}) e_k, i \in T, \\
z_i &= \sum_{j \in N(k): j \leq j(i), j \neq i} e_j + (u_k - a_{j(i)k}) e_k, \; i \in N(k) \setminus T,
\end{aligned}
$$

where for $i \in N(k) \setminus T$, $j(i) = \min_{1 \leq j \leq t} \{i_j \in T : a_{ik} \leq a_{i_j k}\}$ are affinely independent points of $\{(x, y) \in MVP : \sum_{i \in T} \bar{a}_{ik} x_i + y_k = u_k\}$. Note that $j(i)$ is well-defined since $a_{i_t k} = \max_{j \in N(k)} a_{jk}$.

□

Observe that the mixed edge inequalities (6) are dominated by the star inequalities (10). If the binary vertices are independent, then the star inequalities together with the upper bound and lower bound inequalities give *conv*(MVP).

Theorem 3. *If* $E = \emptyset$, *then inequalities (7), (8), and (10) are sufficient to describe conv(MVP).*

Proof. If $a_{jk} = a_k$ for $k \in M$, then the result follows from Theorem 1 since the graph is bipartite when $E = \emptyset$. So to simplify the discussion, we consider the case when a_{jk} are distinct for $k \in M$. Given an arbitrary objective function $(c, d) \neq (0, 0)$, let $(\bar{x}^l, \bar{y}^l), \; l \in \mathcal{O}$ be the optimal solutions to MVPP. We will prove the theorem by showing that there exists an inequality among (7), (8), and (10) that is satisfied at equality for *all* $l \in \mathcal{O}$. If $c_j < 0$ for some $j \in N$ then $\bar{x}^l_j = 0$ for all $l \in \mathcal{O}$; similarly, if $d_k < 0$ for some $k \in M$ then $\bar{y}^l_k = 0$ for all $l \in \mathcal{O}$. Therefore, in the following we may assume $c_j, d_k \geq 0$.

We define $S^l = \{j \in N : \bar{x}^l_j = 1\}, \; l \in \mathcal{O}$ and $S^l_r = S^l \cap N(r), \; r \in M$. There exists $t \in M$ with $d_t > 0$, since otherwise $\bar{x}^l_j = 1$ for all $l \in \mathcal{O}$ for some $j \in N$ with $c_j > 0$, which itself exists since $(c, d) \neq (0, 0)$. Then for an arbitrary $t \in M$ with $d_t > 0$, if $S^l_t = \emptyset$ for all $l \in \mathcal{O}$, we are done since $\bar{y}^l_t = u_t$ for all $l \in \mathcal{O}$; otherwise, let $T = \{j \in N(t) : j = \mathrm{argmax}_{k \in S^l_t} a_{kt}, \text{ for } l \in \mathcal{O}\}$ and $\bar{T} = T \cup \mathrm{argmax}_{k \in N(t)} a_{kt}$. We claim that the star inequality

$$\sum_{k \in \bar{T}} \bar{a}_{kt} x_k + y_t \leq u_t \tag{11}$$

is satisfied at equality for all $l \in \mathcal{O}$. To see this consider some $p \in \mathcal{O}$ and let $j = \mathrm{argmax}_{k \in S^p_t} a_{kt}$. By definition of T, it holds that $j \in T$. Notice that since (11) is a star inequality, $\sum_{k \in \bar{T}: a_{kt} \le a_{jt}} \bar{a}_{kt} = a_{jt}$ and that since $d_t > 0$, $\bar{y}^p_t = u_t - a_{jt}$. Therefore, inequality (11) has positive slack for $(\bar{x}^p, \bar{y}^p)$ if and only if there exists $i \in T$ such that $a_{it} < a_{jt}$ and $i \notin S^p_t$. Suppose there is such an index $i \in T$. By definition of T there is an optimal solution $(\bar{x}^q, \bar{y}^q)$ such that $i = \text{argmax}_{k \in S^q_t} a_{kt}$.

In order to arrive at a contradiction, we show that the objective value of $(\bar{x}, \bar{y})$ defined as $\bar{x}_k = 1, k \in S^p \cup S^q$, $\bar{x}_k = 0$ otherwise, and $\bar{y}_r = u_r - \max_{k \in (S^p_r \cup S^q_r)} a_{rk}$ is larger than of $(\bar{x}^p, \bar{y}^p)$, or equivalently, that $z(S^p \cup S^q) > z(S^p)$, where $z(S) = \sum_{k \in S} c_k - \sum_{r \in M} d_r \max_{k \in S \cap N(r)} a_{kr}$ for $S \subseteq N$. To see this, let $K = S^p \cap S^q$, $M^q = \{r \in M : \max_{k \in S^q_r} a_{kr} > \max_{k \in S^p_r} a_{kr}\}$, and $M^p = M \setminus M^q$. Then,

$$
\begin{aligned}
z(S^p \cup S^q) &= \sum_{k \in S^p} c_k - \sum_{r \in M^p} d_r \max_{k \in S^p_r} a_{kr} + \sum_{k \in S^q \setminus K} c_k - \sum_{r \in M^q} d_r \max_{k \in S^q_r \setminus K} a_{kr} \\
&= z(S^p) + \sum_{k \in S^q \setminus K} c_k - \sum_{r \in M^q} d_r \big(\max_{k \in S^q_r \setminus K} a_{kr} - \max_{k \in S^p_r} a_{kr} \big) \\
&\ge z(S^p) + \sum_{k \in S^q \setminus K} c_k - \sum_{r \in M^q} d_r \big(\max_{k \in S^q_r} a_{kr} - \max_{k \in K \cap N(r)} a_{kr} \big).
\end{aligned}
$$

However,

$$
\begin{aligned}
&\sum_{k \in S^q \setminus K} c_k - \sum_{r \in M^q} d_r \big(\max_{k \in S^q_r} a_{kr} - \max_{k \in K \cap N(r)} a_{kr} \big) > \\
&\qquad \sum_{k \in S^q \setminus K} c_k - \sum_{r \in M} d_r \big(\max_{k \in S^q_r} a_{kr} - \max_{k \in K \cap N(r)} a_{kr} \big) = z(S^q) - z(K) \ge 0.
\end{aligned}
$$

The strict inequality holds because (i) $d_r(\max_{k \in S^q_r} a_{kr} - \max_{k \in K \cap N(r)} a_{kr}) \ge 0$ for all $r \in M^p$ as $d_r \ge 0$ and $K \cap N(r) \subseteq S^q_r$ and (ii) $t \in M^p$ (since $a_{it} < a_{jt}$), $d_t > 0$, and $a_{it} = \max_{k \in S^q_r} a_{kt} > \max_{k \in K \cap N(t)} a_{kt}$ as $i \notin K$. Also, since $(\bar{x}^q, \bar{y}^q)$ is optimal, $z(S^q) \ge z(K)$ follows. Therefore it must be the case that $z(S^p \cup S^q) > z(S^p)$, which contradicts the optimality of $(\bar{x}^p, \bar{y}^p)$.

□

The next two classes of inequalities are generalizations of the clique and odd cycle inequalities [16,17] for the vertex packing problem, respectively.

Theorem 4. *If $K \subseteq N(k)$ for $k \in M$ induces a clique, then the mixed clique inequality*

$$\sum_{i \in K} a_{ik} x_i + y_k \le u_k \tag{12}$$

is valid for MVP. It is facet-defining for conv(MVP) if and only if for all $j \in N(k) \setminus K$, there exists $i \in K \setminus N(j)$ such that $a_{jk} \le a_{ik}$.

Proof. The validity of (12) is obvious since at most one of the variables in K can have value one. Suppose for some $j \in N(k) \setminus K$, $a_{jk} > a_{ik}$ holds for $i \in K \setminus N(j)$. Then $\sum_{i \in K} a_{ik} x_i + (a_{jk} - \max_{i \in K \setminus N(j)} a_{ik}) x_j + y_k \le u_k$ is valid and dominates inequality (12). Conversely, let $i(j) = \text{argmax}_{i \in K \setminus N(j)} a_{ik}$ for $j \in N(k) \setminus K$. Then

$e_i + (u_k - a_{ik})e_k,\ i \in K,\ e_{i(j)} + (u - a_{i(j)k})e_k + e_j,\ j \in N(k) \setminus K,\ e_j + u_k e_k,\ j \in N \setminus N(k)$ and $u_k e_k + u_i e_i,\ i \in M \setminus \{k\},\ u_k e_k$ are $n+m$ affinely independent points of $\{(x,y) \in \mathrm{MVP} : \sum_{i \in K} a_{ik} x_i + y_k = u_k\}$.

□

Theorem 5. *Let $C \subseteq E \cup F$ be the set of edges of an odd cycle in G, C_B be the set of binary vertices on the cycle, and C_C the set of continuous vertices on the cycle. The mixed odd cycle inequality*

$$\sum_{j \in C_B} \left(1 + \sum_{k \in M_j} \frac{a_{k_2} - a_{k_1}}{a_{k_1}}\right) x_j + \sum_{k \in C_C} \frac{y_k}{a_{k_1}} \le \left\lfloor \frac{|C_B| - |C_C|}{2} \right\rfloor + \sum_{k \in C_C} \frac{u_k}{a_{k_1}}, \quad (13)$$

where a_{k_1} and a_{k_2} are the weights of the edges incident to $k \in C_C$ in C, with $a_{k_1} \le a_{k_2}$ and $M_j = \{k \in M(j) \cap C_C : a_{k_2} = a_{jk}\}$, is valid for MVP.

Proof. For $k \in C_C$ let (k, k_1) and (k, k_2) be the edges on the cycle with weights a_{k_1} and a_{k_2}, respectively. Consider $(\bar{x}, \bar{y}) \in MVP$ and let $C_C^o = \{k \in C_C : \bar{x}_{k_1} = \bar{x}_{k_2} = 0\}$, $C_C^1 = \{k \in C_C : \bar{x}_{k_1} = 1, \bar{x}_{k_2} = 0\}$, and $C_C^2 = \{k \in C_C : \bar{x}_{k_2} = 1\}$. Then for $(\bar{x}, \bar{y})$ the left hand side of inequality (13) equals

$$\sum_{j \in C_B} \bar{x}_j + \sum_{k \in C_C} \left(\frac{\bar{y}_k}{a_{k_1}} + \left(\frac{a_{k_2} - a_{k_1}}{a_{k_1}}\right)\bar{x}_{k_2}\right) \le$$

$$\le \frac{|C| - |C_C^o| - 1}{2} + \sum_{k \in C_C^o} \frac{u_k}{a_{k_1}} + \sum_{k \in C_C^1} \frac{u_k - a_{k_1}}{a_{k_1}} + \sum_{k \in C_C^2} \left(\frac{u_k - a_{k_2}}{a_{k_1}} + \frac{a_{k_2} - a_{k_1}}{a_{k_1}}\right)$$

$$= \frac{1}{2}(|C_B| - |C_C| - 1) + \sum_{k \in C_C} \frac{u_k}{a_{k_1}},$$

which equals the rhs of inequality (13) as $|C_B| - |C_C|$ is odd for an odd cycle.

□

The mixed odd cycle inequality for the odd cycle in Fig. 3 is $x_1 + \frac{3}{2}x_2 + 2x_3 + \frac{1}{2}y_1 + y_2 \le \frac{9}{2}$.

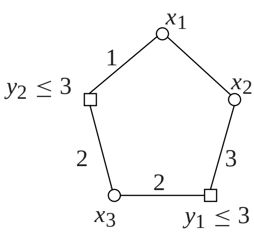

Fig. 3. Odd cycle of a mixed conflict graph

Proposition 4. *[1] The mixed odd cycle inequality (13) is facet-defining for conv(MVP) if G is a chordless odd cycle.*

Example 1 (cont.). The valid star inequalities for MVP (hence for S) are

$$\begin{array}{rrrl} 3x_1 & +3x_4 & +y_1 & \leq 9 \\ & 6x_4 & +y_1 & \leq 9 \\ x_1 & +3x_4 & +y_2 & \leq 10 \\ & 4x_4 & +y_2 & \leq 10 \\ 2x_2 & +2x_4 & +y_2 & \leq 8 \\ & 4x_4 & +y_2 & \leq 8 \end{array}$$

and the valid mixed odd cycle inequalities are

$$\begin{array}{l} x_1 + x_2 + x_3 + 2x_4 + \frac{1}{3}y_1 \leq 4 \\ x_1 + x_2 + x_3 + 4x_4 \quad + y_2 \leq 11 \\ x_1 + x_2 + x_3 + 3x_4 + \frac{1}{2}y_3 \leq 5. \end{array}$$

Although MVP is a relaxation of S, some of the extreme points of the linear relaxation of S, SL, may not be feasible for the linear relaxation of MVP, MVPL. For example $(\frac{1}{2}, 1, 0, \frac{1}{2}, 7, 10, 0)$ is a feasible point of SL but it is not feasible for MVPL. This point is cutoff by edge inequalities $x_1 + x_2 \leq 1$, $x_1 + y_2 \leq 10$, and $4x_4 + y_2 \leq 10$. To see that the valid inequalities above are potentially useful as cutting planes for S, consider the extreme point $(\frac{4}{19}, \frac{15}{19}, \frac{4}{19}, 0, \frac{159}{19}, 10, 0)$ of SL $\cap$ MVPL. This point is cutoff by the star inequality $x_1 + 3x_4 + y_2 \leq 10$ and also by the mixed odd cycle inequality $x_1 + x_2 + x_3 + 4x_4 + y_2 \leq 11$, both of which are facet-defining for $conv$(MVP).

2.2. *Separation*

Here we discuss the separation problems for the inequalities derived in Sect. 2.1. Given a point $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+m} \setminus conv(\text{MVP})$, we want to find a valid inequality violated by this point.

Theorem 6. *The separation problem for star inequalities (10) can be solved in polynomial time.*

Proof. For $k \in M$ suppose $N(k) = \{1, 2, \ldots, l\}$ is indexed so that $a_{1k} \leq a_{2k} \leq \ldots \leq a_{lk}$. We will reduce the separation problem for the star inequalities of k to a longest path problem on an acyclic directed graph with $l + 1$ layers. The graph has one layer for each variable $x_1, x_2, \ldots, x_l$ and an auxiliary layer zero. A vertex in layer i, $1 \leq i \leq l$, represents the sum of coefficients of $x_1, x_2, \ldots, x_i$ in a star inequality. Layer zero has a single vertex, representing the zero coefficient. Since the sum of the coefficients in a star inequality equals a_{lk}, layer l has a single vertex representing coefficient a_{lk}. Two arcs leave a vertex representing sum s at layer $i - 1$, both to vertices in layer i for $0 \leq i < l$. The first one is to the vertex for the same value s at layer i, representing coefficient zero for x_i in the star inequality, and the second one is to the vertex for value a_{ik}, representing coefficient $a_{ik} - s$ for x_i. There is a single arc from each vertex in layer $l - 1$ to the unique vertex in layer l representing sum a_{lk}.

With this construction, if all a_{ik} are distinct, there are $i + 1$ vertices in layer i, $0 \leq i < l$ and a single vertex in layer l, which gives a total of $l(l + 1)/2 + 1$ vertices and l^2 arcs. Furthermore, there are exactly 2^{l-1} directed paths from layer zero to layer l, each representing a particular star inequality of vertex k. If $a_{i-1k} = a_{ik}$, then the number of vertices in layers $i-1$ and i are equal; hence the number of arcs from layer $i-1$ to layer i is one less than otherwise.

Given $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$, we assign a length of $c\bar{x}_i$ to an arc representing coefficient c for variable x_i in the star inequality. Then a longest path from layer zero to layer l corresponds to an inequality with the largest left hand side value.

□

Example 2. Consider

$$S = \{ (x, y) \in \mathbb{B}^4 \times R^1_+ : 1x_1 + y \leq 10,\ 2x_2 + y \leq 10,\ 5x_3 + y \leq 10,\ 7x_4 + y \leq 10 \}.$$

The layered directed graph corresponding to S is shown in Fig. 4. In this graph each path from layer 0 to layer 4 represents one of the star inequalities below:

$$\begin{aligned}
x_1 + x_2 + 3x_3 + 2x_4 + y &\leq 10\\
2x_2 + 3x_3 + 2x_4 + y &\leq 10\\
x_1 + 4x_3 + 2x_4 + y &\leq 10\\
x_1 + x_2 + 5x_4 + y &\leq 10\\
x_1 + 6x_4 + y &\leq 10\\
2x_2 + 5x_4 + y &\leq 10\\
5x_3 + 2x_4 + y &\leq 10\\
7x_4 + y &\leq 10.
\end{aligned}$$

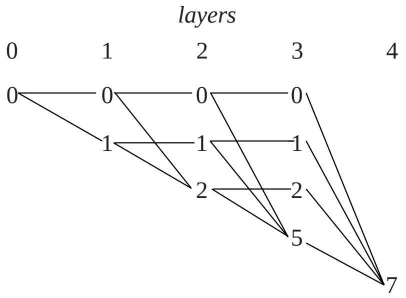

Fig. 4. Layered directed graph of S

Using the fact that arcs representing coefficient zero have zero length, we have the following simple $\Theta(l^2)$ algorithm for the separation problem of star inequalities.

Algorithm 1 Separation for star inequalities

```
π_0 ← 0
for j = 1 to l do
    π_{a_jk} ← max_{i: a_ik < a_jk} π_{a_ik} + (a_jk − a_ik) x̄_j
end for
if π_{a_lk} + ȳ_k > u_k then
    star inequality, defined by a longest path, is violated
else
    no star inequality of vertex k is violated
end if
```

Due to the polynomial equivalence of optimization and separation [8], Theorem 3 and Theorem 6 imply polynomial solvability of the mixed vertex packing problem when the binary vertices are independent.

Corollary 1. *If $E = \emptyset$, then MVPP can be solved in polynomial time.*

The separation problem for mixed clique inequalities is equivalent to solving a weighted maximum clique problem for each $k \in M$ on the subgraph induced by $N(k)$ and therefore is $\mathcal{NP}$-hard. Given $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$, a most violated mixed clique inequality can be found by solving

$$\max_{k \in M} \left\{ \max_{K \subseteq N(k)} \sum_{j \in K} a_{jk} \bar{x}_j + \bar{y}_k \right\}$$

where $G(K)$ is a clique. Solving this separation problem may be computationally feasible by enumeration for small graphs since the search for cliques is restricted to adjacent vertices of a single continuous vertex.

Theorem 7. *Suppose $a_{jk} = a_k$ for all $j \in N(k)$ and for all $k \in M$. Then the separation problem for the mixed odd cycle inequalities (13) can be solved in polynomial time.*

Proof. Consider inequality (13) when weights of all the edges incident to a continuous vertex k are the same, say a_k,

$$\sum_{j \in C_B} x_j + \sum_{k \in C_C} \frac{y_k}{a_k} \leq \frac{1}{2}(|C_B| - |C_C| - 1) + \sum_{k \in C_C} \frac{u_k}{a_k}. \qquad (14)$$

We can rewrite (14) as

$$\sum_{j \in C_B} (1 - 2x_j) + \sum_{k \in C_C} \left(\frac{2(u_k - y_k)}{a_k} - 1 \right) \geq 1. \qquad (15)$$

Then, given $(\bar{x}, \bar{y})$, finding a most violated mixed odd cycle inequality is equivalent to finding a minimum weight odd cycle on a graph with edge weights

$$w(i,k) = \begin{cases} 1 - \bar{x}_i - \bar{x}_k, & \text{if } i, k \in N, \\ \frac{u_k - \bar{y}_k}{a_k} - \bar{x}_i, & \text{if } i \in N, k \in M. \end{cases}$$

Observe that for a point $(\bar{x}, \bar{y}) \in$ LMVP, $w(i,k) \geq 0$ for all $(i,k) \in E \cup F$. Since there is a polynomial time algorithm for finding a minimum weight odd cycle on a graph with nonnegative edge weights [8], the separation problem is solvable in polynomial time. □

2.3. Strengthening star inequalities

In this section we present a procedure for strengthening star inequalities when the binary variables appearing in the inequality are not independent. A strengthened star inequality has the form $\sum_{j \in T} \tilde{a}_{jk} x_j + y_k \leq u_k$ with $\tilde{a}_{jk} \geq \bar{a}_{jk}$ for $j \in T$. The procedure begins with a star inequality (10), and then the coefficients are increased iteratively in increasing order of $a_{ik}, i \in T$.

Proposition 5. *Let $\sum_{j\in T}\tilde{a}_{jk}x_j + y_k \le u_k$ be a strengthened star inequality such that for some $i \in T$, $\tilde{a}_{jk} = \bar{a}_{jk}$ for $j \in T$ with $a_{jk} > a_{ik}$ and $\tilde{a}_{jk} \ge \bar{a}_{jk}$ for $j \in T$ with $a_{jk} \le a_{ik}$. Then the coefficient of variable x_i can be increased by*

$$(a_{ik} - \sum_{j\in S}\tilde{a}_{jk})^+ \tag{16}$$

where $S = \{j \in T \setminus N(i) : a_{jk} \le a_{ik}\}$ and $a^+ = \max\{a, 0\}$.

Proof. Let δ_i denote the increase in the coefficient of x_i. For the inequality to remain valid for MVP, we need

$$\delta_i \le u_k - \max_{(x,y)\in MVP, x_i=1}\left\{\sum_{j\in T}\tilde{a}_{jk}x_j + y_k\right\}. \tag{17}$$

Let $U \subseteq T$ be the binary variables that have value one in an optimal solution to the right hand side of (17). Then letting $\bar{a} = \max_{j\in U} a_{jk}$, we see that

$$\max_{(x,y)\in MVP, x_i=1}\left\{\sum_{j\in T}\tilde{a}_{jk}x_j + y_k\right\} \le \sum_{j\in S}\tilde{a}_{jk} + \sum_{j\in U\setminus S}\bar{a}_{jk} + u_k - \bar{a}.$$

Since $\sum_{j\in U\setminus S}\bar{a}_{jk} \le \bar{a} - a_{ik}$, the result follows. □

Example 3. Consider the mixed conflict graph given in Fig. 5. One of the star inequalities here is $x_1 + x_2 + 3x_3 + 2x_4 + y \le 10$. Increasing the coefficients of this inequality in the order 1, 2, 3, 4, we obtain $x_1 + 2x_2 + 3x_3 + 4x_4 + y \le 10$.

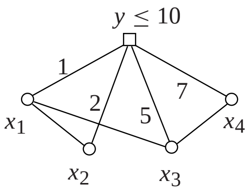

Fig. 5. Strengthening star inequalities

2.4. Sequential lifting

When the inequalities described previously are not facet-defining, we can make them stronger through lifting. We start with lifting an inequality on binary variables with a continuous variable. Let

$$\sum_{i\in S}\alpha_i x_i \le r$$

be a valid inequality for MVP(N) and consider lifting it with a continuous variable y_k. Let α_k be the coefficient of y_k in the lifted inequality. In order for the inequality to be valid, we need

$$\alpha_k \leq \min\left\{\frac{r - \sum_{i\in S}\alpha_i x_i}{y_k} : (x,y) \in MVP, y_k > 0\right\}.$$

Proposition 6. *Let $\sum_{i\in S}\alpha_i x_i \leq r$ be a valid inequality for MVP(N) . If S is a subset of $N(k)$ such that $a_{ik} = u_k$ for all $i \in S$, then*

$$\sum_{i\in S}\alpha_i x_i + \frac{r}{u_k} y_k \leq r$$

is a valid inequality for MVP.

Next, given a valid inequality of the form $\sum_{i\in S}\alpha_i x_i + y_k \leq u_k$, $S \subseteq N(k)$, we consider lifting it with binary variables in $N(k) \setminus S$. For $S \subseteq N(k)$ and $j \in N(k) \setminus S$, let $\mathcal{P}$ be the collection of packings that contain vertex j in the graph induced by the vertex set $S \cup \{j\}$. Let

$$\sum_{i\in S}\alpha_i x_i + y_k \leq u_k$$

be a valid inequality for MVP and consider lifting it with binary variable $x_j \in N(k) \setminus S$. Let α_j be the coefficient of x_j in the lifted inequality. In order for the inequality to be valid, we need

$$\alpha_j \leq u_k - \max_{(x,y)\in MVP, x_j=1}\left\{\sum_{i\in S}\alpha_i x_i + y_k\right\},$$

or equivalently,

$$\alpha_j \leq u_k - \max_{P\in\mathcal{P}}\left\{\sum_{i\in P, i\neq j}\alpha_i + \min_{i\in P}\{u_k - a_{ik}\}\right\} = \min_{P\in\mathcal{P}}\left\{\max_{i\in P} a_{ik} - \sum_{i\in P, i\neq j}\alpha_i\right\}.$$

The next proposition follows from this inequality.

Proposition 7.

1. *If $S \subseteq N(j)$, then the maximum lifting coefficient of x_j equals a_{jk}.*
2. *For $j \in N(k) \setminus S$, if $a_{ik} \leq a_{jk}$ for all $i \in S$, then the maximum lifting coefficient of x_j equals $a_{jk} - \max_{P\in\mathcal{P}}\sum_{i\in P, i\neq j}\alpha_i$.*

Now we give a class of *mixed odd wheel inequalities* that can be obtained by lifting a mixed odd cycle inequality. The proof of the next proposition is a simple application of the previous results on lifting.

Proposition 8. *Let $C = (C_B, C_C)$ be a mixed odd cycle. Then the mixed odd wheel inequality*

$$\sum_{j \in C_B} \left(1 + \sum_{k \in M_j} \frac{a_{k_2} - a_{k_1}}{a_{k_1}}\right) x_j + \sum_{k \in C_C} \frac{y_k}{a_{k_1}} + \alpha_w z_w \leq \left\lfloor \frac{|C_B| - |C_C|}{2} \right\rfloor + \sum_{k \in C_C} \frac{u_k}{a_{k_1}}$$

is valid for conv(MVP), where a_{k_1} and a_{k_2} are the weights of the edges incident to $k \in C_C$ in C, with $a_{k_1} \leq a_{k_2}$, $M_j = \{k \in M(j) \cap C_C : a_{k_2} = a_{jk}\}$, and

$$\alpha_w = \begin{cases} \left\lfloor \frac{|C_B| - |C_C|}{2} \right\rfloor + \sum_{k \in C_C} \frac{a_{wk}}{a_{k_1}} & \text{if } w \in N,\ C \subseteq N(w) \cup M(w), \\ \left\lfloor \frac{|C_B| - |C_C|}{2} \right\rfloor & \text{if } w \in M,\ C_B \subseteq N(w),\ a_{ji} = u_w \forall\ j \in C_B. \end{cases}$$

The lifting coefficient in the mixed wheel inequality could be computed exactly due to the special structure of an odd cycle. In general, computing lifting coefficients is hard. Therefore, we consider approximating them.

Proposition 9. *Let $\mathcal{P}$ be defined as before. Then $\left(a_{jk} - \max_{P \in \mathcal{P}} \sum_{i \in P, i \neq j} \alpha_i\right)^+$ is a lower bound on the maximum lifting coefficient.*

Proof. Decomposing the minimization problem of the lifting function we have

$$\min_{P \in \mathcal{P}} \left\{ \max_{i \in P} a_{ik} - \sum_{i \in P, i \neq j} \alpha_i \right\} \geq \left(\min_{P \in \mathcal{P}} \max_{i \in P} a_{ik} - \max_{P \in \mathcal{P}} \sum_{i \in P, i \neq j} \alpha_i \right)^+$$
$$= \left(a_{jk} - \max_{P \in \mathcal{P}} \sum_{i \in P, i \neq j} \alpha_i \right)^+.$$

Equality follows since $j \in P$ for all $P \in \mathcal{P}$ by definition of $\mathcal{P}$.

□

Proposition 9 suggests an easy way for generating valid inequalities by sequentially lifting $y_k \leq u_k$ with $x_i,\ i \in N(k)$. Let $i_1, i_2, \ldots, i_l$ be an arbitrary ordering of $N(k)$. Then

$$\sum_{j=1}^{l} \alpha_{i_j} x_{i_j} + y_k \leq u_k$$

is a valid inequality for MVP, where the coefficients α_{i_j} are calculated as follows:

$$\alpha_{i_j} = \left(a_{i_j k} - \sum_{h \in S} \alpha_h \right)^+ \tag{18}$$

with $S = \{i_1, i_2, \ldots, i_{j-1}\} \setminus N(i_j)$. We call inequalities generated this way *lifted bound inequalities*. Note that both star and mixed clique inequalities are special cases of the lifted bound inequalities. We obtain a star inequality when $N(i) = \emptyset$ for all $i \in N(k)$ and a mixed clique inequality when $S = \emptyset$. Lifted bound inequalities may be stronger than star and mixed clique inequalities when the latter are not facet-defining. Also observe that a strengthened star inequality is a lifted bound inequality as well.

By exploiting the structure of $G(S)$, one can clearly derive stronger lifted bound inequalities. For example, if $G(S)$ is a clique, then $\alpha_{i_j} = a_{i_jk} - \max_{h \in S} \alpha_h$. A simple modification to (18) allows us not only to derive stronger lifted bound inequalities, but also to generate lifted mixed clique inequalities. Let $G(K)$ be a clique of $G(S)$, then

$$\alpha_{i_j} = \left(a_{i_jk} - \sum_{h \in S \setminus K} \alpha_h - \max_{h \in K} \alpha_h \right)^+ . \tag{19}$$

Note that if K is a singleton, the inequality is a regular lifted bound inequality.

3. Computational experiments

In order to test the effectiveness of the valid inequalities derived in Sect. 2 in solving (a) mixed vertex packing problems and (b) general mixed-integer programming problems with a branch-and-cut algorithm, we performed computational experiments on two data sets. The first set consists of randomly generated mixed vertex packing problems. The second set consists of mixed-integer problems from MIPLIB [4] for which violated star inequalities are generated. Since the mixed vertex packing model forms a relaxation of general mixed-integer problems, effectiveness of the valid inequalities for the first data set is a prerequisite for successful results on general problems. The branch-and-cut algorithm is implemented using MINTO [14] (version 3.0), which is a customizable software system that solves mixed-integer linear programs by a branch-and-bound algorithm with linear programming relaxations. In the current implementation, a best bound node selection strategy is used and new valid inequalities are added only at nodes with depth less than or equal to five. All experiments are done on a SUN Ultra 10 workstation with one hour CPU time limit.

Computational experiments on the mixed vertex packing problems are summarized in Table 1. Clique inequalities on binary variables are valid for MVP, and MINTO generates them automatically. To see the effect of the new inequalities, we compare the performance of a branch-and-cut algorithm with clique, star and lifted bound inequalities against one with only clique inequalities on randomly generated graphs with varying edge density and fraction of continuous vertices. We use the algorithm given in Sect. 2.2 to find violated star inequalities. A star inequality with the largest left hand side value is strengthened as explain in Sect. 2.3 and then checked for violation. Given a fractional solution $(\bar{x}, \bar{y})$, a lifted bound inequality is generated for each continuous variable y_k by lifting its adjacent binary variables x_i in nonincreasing order of $a_{ik}\bar{x}_i$ using equation (19). If the resulting inequality is violated by the fractional solution, it is added to the formulation. Note that even though a strengthened star inequality is a special case of lifted bound inequalities, the existence of an efficient separation algorithm may allow us to generate many violated star inequalities that may have been missed by the lifting order used to generate a lifted bound inequality. Therefore, we generate these classes of inequalities separately. We have not generated mixed odd cycle inequalities in the branch-and-cut algorithm as they are less likely to be facet-defining. In Table 1, for each

Table 1. Performance statistics for mixed vertex packing problems

vert	dns	cont	LPgap	endgap	clqs	nodes	time	LPgap	endgap	clqs	stars	lft bnds	nodes	time
100	0.1	0.2	9.60	0.00	72	147	5	5.32	0.00	59	117	8	19	3
	0.1	0.4	9.59	0.00	47	255	7	2.28	0.00	28	155	15	7	2
	0.2	0.2	27.44	0.00	814	1367	159	16.91	0.00	334	563	19	204	98
	0.2	0.4	11.67	0.00	82	74	9	0.23	0.00	37	179	40	2	2
	0.4	0.2	41.80	0.00	857	604	249	21.38	0.00	569	613	20	179	182
	0.4	0.4	5.10	0.00	56	29	7	0.00	0.00	23	41	42	1	1
150	0.1	0.2	22.22	0.00	2627	8913	1648	15.26	0.00	480	641	23	1269	938
	0.1	0.4	11.22	0.00	195	849	110	2.63	0.00	60	402	38	23	16
	0.2	0.2	46.45	5.46	7416	5764	3600	31.48	3.92	1952	1148	29	1216	3600
	0.2	0.4	9.87	0.00	95	65	27	0.00	0.00	47	206	62	1	3
	0.4	0.2	58.19	6.13	3809	2221	3600	30.08	3.84	1896	1239	27	518	3600
	0.4	0.4	5.11	0.00	97	51	44	0.00	0.00	24	56	62	1	4

case we give the average duality gap (LPgap = $100 \times \frac{\text{zroot}-\text{zopt}}{\text{zopt}}$) at the root node after all cuts are added, the percentage gap between the best upper bound and the best lower bound (endgap = $100 \times \frac{\text{zub}-\text{zlb}}{\text{zlb}}$) at termination, the number of inequalities generated, the number of nodes explored, and the total CPU time elapsed in seconds of five instances with 100 and 150 vertices. Observe that as the fraction of continuous variables increases, MVPPs become easier to solve. Problems with 20% continuous vertices could not be solved to optimality for densities 0.2 and 0.4 and 150 vertices by either algorithm. However, the duality gap is reduced considerably with the addition of the new inequalities. For these problems, since optimal solutions are unknown, we use the best feasible solution instead of an optimal one to report the duality gap at the root node. We remark that in both cases better feasible solutions are found when star and lifted bound inequalities are added. In summary, the star inequalities and the lifted bound inequalities are very effective in strengthening the LP relaxations and in reducing the number of nodes explored and the overall solution times.

Table 2. Performance statistics for MIPLIB problems

problem	LPgap	endgap	clqs	nodes	time	LPgap	endgap	clqs	stars	nodes	time
`bell3a`	1.40	0.00	0	54532	291	1.39	0.00	0	4	54157	213
`blend2`	8.99	0.00	318	2590	183	8.99	0.00	176	26	2035	94
`dcmulti`	1.46	0.00	21	817	25	1.46	0.00	19	15	802	24
`egout`	1.06	0.00	9	3	1	0.58	0.00	3	34	3	1
`fixnet4`	13.79	0.00	1	755	21	13.68	0.00	1	6	486	15
`fixnet6`	19.85	0.00	1	715	20	19.73	0.00	1	5	549	18
`gen`	0.05	0.00	14	253	10	0.05	0.00	19	22	217	10
`gesa2`	1.02	0.00	3	65783	1451	1.01	0.00	3	3	65003	1472
`gesa2_o`	1.01	0.00	2	92701	1933	1.01	0.00	2	2	93831	1865
`gesa3`	0.52	0.00	21	837	91	0.52	0.00	7	8	757	69
`gesa3_o`	0.52	0.00	17	1771	167	0.52	0.00	9	4	1675	154
`khb05250`	10.31	0.00	0	1935	53	0.18	0.00	0	98	13	3
`mod011`	17.71	6.55	0	8376	3600	8.45	1.61	0	1531	2004	3600
`qnet1`	10.95	0.00	0	153	26	10.95	0.00	0	4	149	24
`qnet1_o`	19.48	0.00	0	281	33	15.69	0.00	0	5	203	29
`rgn`	40.63	0.00	0	3701	24	38.81	0.00	0	20	3339	21
`rout`	12.83	6.12	0	111101	3600	9.74	2.63	0	3	119712	3600
`set1ch`	31.57	20.90	4	193135	3600	22.14	12.52	4	128	177648	3600
`vpm2`	19.72	5.07	1	173582	3600	16.58	0.87	1	8	161115	3600

A similar comparison is made for the second data set in Table 2. Here we report the LPgap, endgap, the number of cuts generated, the number of nodes explored in the search tree, and the total CPU time elapsed in seconds. We remark that no violated lifted bound inequality is found for any of the MIPLIB problems. The addition of the star cuts reduces the number of nodes explored and the overall solution times for almost all problems in this set, even if there is none or only a modest reduction in the duality gap at the root node. Four problems could not be solved within an hour of CPU time; however for all of these unsolved problems LPgap and endgap reduced significantly with the addition of star cuts. We use the already known optimal value to report the LPgap and the endgap of `set1ch` because no feasible solution was found by either of the algorithms within one hour of CPU time for this problem. From these computational results we conclude that inequalities derived from mixed vertex packing relaxations may be valuable in supplementing the ones from vertex packing relaxations for mixed 0-1 integer problems.

MINTO can also generate other classes of system cuts, such as lifted knapsack cover inequalities and lifted flow cover inequalities for mixed-integer problems. When star cuts are generated in addition to all system cuts, their effect is less pronounced, especially for the easily solved problems in Table 2. Nevertheless, addition of the star cuts does improve the lower bounds for harder problems, where knapsack and flow cover cuts are less effective. For example, the best lower bound at termination improves from -55781349.08 to -55244373.09 for `mod011` and from 47510.79 to 47651.13 for `set1ch` after 1313 and 122 star cuts are generated, respectively, in addition to all system cuts available in MINTO. This translates to reductions in endgap from 2.24% to 1.26% for `mod011` and from 12.88% to 12.63% for `set1ch`.

With the insights gained from studying MVP, we are investigating how to use the mixed vertex packing relaxation together with a single mixed-integer knapsack inequality for obtaining stronger relaxations of general 0-1 MIP problems. Since a variable lower/upper bound constraint is a special case of a mixed edge inequality, we can derive generalizations of the flow cover and flow pack inequalities [2, 20] in this manner.

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