



The maximum measure of non-trivial 3-wise intersecting families

Norihide Tokushige¹

Received: 25 May 2022 / Accepted: 17 April 2023 / Published online: 9 May 2023
© Springer-Verlag GmbH Germany, part of Springer Nature and Mathematical Optimization Society 2023

Abstract

Let \mathcal{G} be a family of subsets of an n -element set. The family \mathcal{G} is called non-trivial 3-wise intersecting if the intersection of any three subsets in \mathcal{G} is non-empty, but the intersection of all subsets is empty. For a real number $p \in (0, 1)$ we define the measure of the family by the sum of $p^{|G|}(1-p)^{n-|G|}$ over all $G \in \mathcal{G}$. We determine the maximum measure of non-trivial 3-wise intersecting families. We also discuss the uniqueness and stability of the corresponding optimal structure. These results are obtained by solving linear programming problems.

Keywords Intersecting family · Product measure · Linear programming · Weak duality

Mathematics Subject Classification 05D05 · 90C05

1 Introduction

We determine the maximum measure of non-trivial 3-wise intersecting families, and discuss the stability of the optimal structure. To make the statement precise let us start with some definitions.

Let $n \geq t \geq 1$ and $r \geq 2$ be integers. For a finite set X let 2^X denote the power set of X . We say that a family of subsets $\mathcal{G} \subset 2^X$ is r -wise t -intersecting if $|G_1 \cap \dots \cap G_r| \geq t$ for all $G_1, \dots, G_r \in \mathcal{G}$. If $t = 1$ then we omit t and say an r -wise intersecting family to mean an r -wise 1-intersecting family.

The author was supported by JSPS KAKENHI 18K03399 and 23K032101.

✉ Norihide Tokushige
hide@edu.u-ryukyu.ac.jp

¹ University of the Ryukyus, 1 Senbaru, Okinawa 903-0213, Japan

Let $0 < p < 1$ be a real number and let $q = 1 - p$. For $\mathcal{G} \subset 2^X$ we define its measure (or p -measure) $\mu_p(\mathcal{G} : X)$ by

$$\mu_p(\mathcal{G} : X) := \sum_{G \in \mathcal{G}} p^{|G|} q^{|X|-|G|}.$$

We mainly consider the case $X = [n]$, where $[n] := \{1, 2, \dots, n\}$. In this case we just write $\mu_p(\mathcal{G})$ to mean $\mu_p(\mathcal{G} : [n])$.

We say that an r -wise t -intersecting family $\mathcal{G} \subset 2^{[n]}$ is non-trivial if $|\bigcap \mathcal{G}| < t$, where $\bigcap \mathcal{G} := \bigcap_{G \in \mathcal{G}} G$. Let us denote the maximum p -measure of such families by $M_r^t(n, p)$, that is,

$$M_r^t(n, p) := \max\{\mu_p(\mathcal{G}) : \mathcal{G} \subset 2^{[n]} \text{ is non-trivial } r\text{-wise } t\text{-intersecting}\}.$$

If a family $\mathcal{G} \subset 2^{[n]}$ is non-trivial r -wise t -intersecting, then so is

$$\mathcal{G}' := \mathcal{G} \sqcup \{G \sqcup \{n + 1\} : G \in \mathcal{G}\} \subset 2^{[n+1]}.$$

Since \mathcal{G} and \mathcal{G}' have the same p -measure, the function $M_r^t(n, p)$ is non-decreasing in n for fixed r, t, p , and we can define

$$M_r^t(p) := \lim_{n \rightarrow \infty} M_r^t(n, p).$$

For simplicity if $t = 1$ then we just write $M_r(n, p)$ and $M_r(p)$.

What is generally known about $M_r(n, p)$ and $M_r(p)$? For the case $r = 2$ we have the following.

$$M_2(p) = \begin{cases} p & \text{if } 0 < p \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < p < 1. \end{cases}$$

Indeed it is easy to see that $M_2(n, \frac{1}{2}) = \frac{1}{2}$, and it is known from [1] that $M_2(n, p) < p$ for $p < \frac{1}{2}$. Thus $M_2(p) \leq p$ for $p \leq \frac{1}{2}$. On the other hand we construct a non-trivial r -wise intersecting family by $\mathcal{F} := (\{F \in 2^{[n]} : 1 \in F\} \setminus \{\{1\}\}) \cup \{\{2, n\}\}$, where $[i, j] := [j] \setminus [i - 1]$. Then we have $\mu_p(\mathcal{F}) = p - pq^{n-1} + qp^{n-1} \rightarrow p$ as $n \rightarrow \infty$. Thus $M_2(p) = p$ for $p \leq \frac{1}{2}$. For the case $p > \frac{1}{2}$ we construct a non-trivial r -wise intersecting family $\mathcal{G} := \{F \in 2^{[n]} : |F| > n/2\}$. Then $\mu_p(\mathcal{G}) = \sum_{k > n/2} \binom{n}{k} p^k q^{n-k} \rightarrow 1$ as $n \rightarrow \infty$, and so $M_2(p) = 1$ for $p > \frac{1}{2}$.

The case $p = \frac{1}{2}$ (and arbitrary $r \geq 2$) is also known. Brace and Daykin [5] determined the maximum size of non-trivial r -wise intersecting families. In other words, they determined $M_r(n, \frac{1}{2})$. To state their results we define a non-trivial r -wise intersecting family $BD_r(n)$ by

$$BD_r(n) := \{F \in 2^{[n]} : |F \cap [r + 1]| \geq r\}.$$

Then $\mu_p(BD_r(n)) = (r + 1)p^r q + p^{r+1}$.

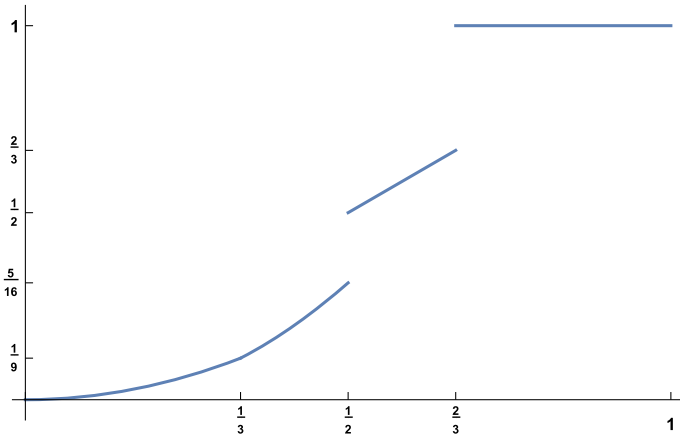


Fig. 1 The graph of $M_3(p)$

Theorem 1 (Brace and Daykin [5]) *For $r \geq 2$ we have $M_r(n, \frac{1}{2}) = \mu_{\frac{1}{2}}(\text{BD}_r(n))$. If $r \geq 3$ then $\text{BD}_r(n)$ is the only optimal family (up to isomorphism) whose measure attains $M_r(n, \frac{1}{2})$.*

Here two families $\mathcal{F}, \mathcal{G} \subset 2^{[n]}$ are isomorphic if there is a permutation τ on $[n]$ such that $\mathcal{F} = \{\{\tau(g) : g \in G\} : G \in \mathcal{G}\}$. In this case we write $\mathcal{F} \cong \mathcal{G}$.

The other thing we know is about the case p close to $\frac{1}{2}$. In this case we can extend Theorem 1 if $r \geq 8$ as follows.

Theorem 2 [20] *Let $r \geq 8$. Then there exists $\epsilon = \epsilon(r) > 0$ such that $M_r(n, p) = \mu_p(\text{BD}_r(n))$ for $|p - \frac{1}{2}| < \epsilon$, and $\text{BD}_r(n)$ is the only optimal family (up to isomorphism).*

In [20] it is conjectured the same holds for $r = 6$ and 7 as well. On the other hand there is a construction showing that if $r \leq 5$ then $M_r(n, p) > \mu_p(\text{BD}_r(n))$ for p close to $1/2$.

In this paper we focus on the case $r = 3$. We determine $M_3(p)$ for all p .

Theorem 3 *For non-trivial 3-wise intersecting families we have*

$$M_3(p) = \begin{cases} p^2 & \text{if } p \leq \frac{1}{3}, \\ 4p^3q + p^4 & \text{if } \frac{1}{3} \leq p \leq \frac{1}{2}, \\ p & \text{if } \frac{1}{2} < p \leq \frac{2}{3}, \\ 1 & \text{if } \frac{2}{3} < p < 1. \end{cases}$$

In case $M_2(p)$ there is a jump at $p = \frac{1}{2}$. In case $M_3(p)$ there are two jumps at $p = \frac{1}{2}$ and $p = \frac{2}{3}$ as in Fig. 1, and $M_3(p)$ is continuous at $p = \frac{1}{3}$ but not differentiable at this point. We also note that $\mu_p(\text{BD}_3(n)) = 4p^3q + p^4$.

The most interesting part is the case $\frac{1}{3} \leq p \leq \frac{1}{2}$. In this case we determine $M_3(n, p)$ and the corresponding optimal structure.

Theorem 4 *Let $\frac{1}{3} \leq p \leq \frac{1}{2}$. Then we have $M_3(n, p) = \mu_p(\text{BD}_3(n))$. Moreover, $\text{BD}_3(n)$ is the only optimal family (up to isomorphism), that is, if $\mathcal{F} \subset 2^{[n]}$ is a non-trivial 3-wise intersecting family with $\mu_p(\mathcal{F}) = M_3(n, p)$ then $\mathcal{F} \cong \text{BD}_3(n)$.*

We also consider the stability of the optimal family for $\frac{1}{3} \leq p \leq \frac{1}{2}$. Roughly speaking we will claim that if a non-trivial 3-wise intersecting family has measure close to $M_3(n, p)$ then the family is close to $\text{BD}_3(n)$ in structure. A similar result is known for 2-wise t -intersecting families. For comparison with our case let $\frac{1}{3} < p < \frac{2}{5}$ and $t = 2$. Note that $\text{BD}_3(n)$ is a 2-wise 2-intersecting family. If $\mathcal{F} \subset 2^{[n]}$ is a 2-wise 2-intersecting family, then it follows from the Ahlswede–Khachatrian theorem (see Theorem 11) that $\mu_p(\mathcal{F}) \leq \mu_p(\text{BD}_3(n))$. Moreover, if $\mu_p(\mathcal{F})$ is close to $\mu_p(\text{BD}_3(n))$ then \mathcal{F} is close to $\text{BD}_3(n)$. This follows from a stability result (corresponding to Theorem 11) proved by Ellis, Keller, and Lifshitz. Here we include a version due to Filmus applied to the case $\frac{1}{3} < p < \frac{2}{5}$ and $t = 2$.

Theorem 5 (Ellis–Keller–Lifshitz [6], Filmus [7]) *Let $\frac{1}{3} < p < \frac{2}{5}$. There is a constant $\epsilon_0 = \epsilon_0(p)$ such that the following holds. If $\mathcal{F} \subset 2^{[n]}$ is a 2-wise 2-intersecting family with $\mu_p(\mathcal{F}) = \mu_p(\text{BD}_3(n)) - \epsilon$, where $\epsilon < \epsilon_0$, then there is a family $\mathcal{G} \cong \text{BD}_3(n)$ such that $\mu_p(\mathcal{F} \Delta \mathcal{G}) = O(\epsilon)$, where the hidden constant depends on p only.*

We note that the condition $\mu_p(\mathcal{F} \Delta \mathcal{G}) = O(\epsilon)$ in Theorem 5 cannot be replaced with the condition $\mathcal{F} \subset \mathcal{G}$. To see this, consider a 2-wise 2-intersecting family

$$\mathcal{F} = (\text{BD}_3(n) \setminus \{\{1, 3, 4\}, \{2, 3, 4\}\}) \cup \{[n] \setminus \{3, 4\}\}. \tag{1}$$

Then $\mu_p(\mathcal{F}) = \mu_p(\text{BD}_3(n)) - 2p^3q^{n-3} + p^2q^{n-2} \rightarrow \mu_p(\text{BD}_3(n))$ as $n \rightarrow \infty$, but \mathcal{F} is not contained in $\text{BD}_3(n)$ (or any isomorphic copy of $\text{BD}_3(n)$).

Note that a non-trivial r -wise t -intersecting family is necessarily an $(r - 1)$ -wise $(t + 1)$ -intersecting family. (Otherwise there are $r - 1$ subsets whose intersection is of size exactly t , and so all subsets contain the t vertices to be r -wise t -intersecting, which contradicts the non-trivial condition.) Thus Theorem 5 also holds if we replace the assumption that \mathcal{F} is a 2-wise 2-intersecting family with the assumption that \mathcal{F} is a non-trivial 3-wise intersecting family. We also note that the family \mathcal{F} defined by (1) is 2-wise 2-intersecting, but not 3-wise intersecting. This suggests a possibility of a stronger stability for non-trivial 3-wise intersecting families than 2-wise 2-intersecting families.

Conjecture 1 *Let $\frac{1}{3} < p \leq \frac{1}{2}$. There is a constant $\epsilon_0 = \epsilon_0(p)$ such that the following holds. If $\mathcal{F} \subset 2^{[n]}$ is a non-trivial 3-wise intersecting family with $\mu_p(\mathcal{F}) = \mu_p(\text{BD}_3(n)) - \epsilon$, where $\epsilon < \epsilon_0$, then there is a family $\mathcal{G} \cong \text{BD}_3(n)$ such that $\mathcal{F} \subset \mathcal{G}$.*

We verify the conjecture for the case $\frac{2}{5} \leq p \leq \frac{1}{2}$ provided that the family is shifted. Here we say that a family $\mathcal{F} \subset 2^{[n]}$ is shifted if $F \in \mathcal{F}$ and $\{i, j\} \cap F = \{j\}$ for some $1 \leq i < j \leq n$, then $(F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$. The following is our main result in this paper.

Theorem 6 *Let $\frac{2}{5} \leq p \leq \frac{1}{2}$, and let $\mathcal{F} \subset 2^{[n]}$ be a shifted non-trivial 3-wise intersecting family. If $\mathcal{F} \not\subset \text{BD}_3(n)$ then $\mu_p(\mathcal{F}) < \mu_p(\text{BD}_3(n)) - 0.0018$.*

For the proof of Theorem 6 we divide the family into some subfamilies. These subfamilies are not only 3-wise intersecting, but also satisfy some additional intersection conditions. To capture the conditions we need some more definitions. We say that r families $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ are r -cross t -intersecting if $|F_1 \cap \dots \cap F_r| \geq t$ for all $F_1 \in \mathcal{F}_1, \dots, F_r \in \mathcal{F}_r$. If moreover $\mathcal{F}_i \neq \emptyset$ for all $1 \leq i \leq r$, then the r families are called non-empty r -cross t -intersecting. As usual we say r -cross intersecting to mean r -cross 1-intersecting. The following result is used to prove Theorem 6.

Theorem 7 *Let $\frac{1}{3} \leq p \leq \frac{1}{2}$. If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset 2^{[n]}$ are non-empty 3-cross intersecting families, then*

$$\mu_p(\mathcal{F}_1) + \mu_p(\mathcal{F}_2) + \mu_p(\mathcal{F}_3) \leq 3p. \tag{2}$$

Suppose, moreover, that $\frac{1}{3} < p \leq \frac{1}{2}$, all \mathcal{F}_i are shifted, and $\bigcap F = \emptyset$, where the intersection is taken over all $F \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Then,

$$\mu_p(\mathcal{F}_1) + \mu_p(\mathcal{F}_2) + \mu_p(\mathcal{F}_3) \leq 3p - \epsilon_p, \tag{3}$$

where $\epsilon_p = (2 - 3p)(3p - 1)$.

The first inequality (2) is an easy consequence of a recent result on r -cross t -intersecting families obtained by Gupta et al. [13], while the second inequality (3) is proved by solving linear programming (LP) problems. We mention that equality holds in (2) only if $|\bigcap_{F \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3} F| = 1$ unless $p = \frac{1}{3}$. This fact will not be used for the proof of Theorem 6, but it follows easily from (3) and Lemma 1.

Here we outline the proof of Theorem 6. This is done by solving LP problems as follows. First we divide \mathcal{F} into subfamilies, say, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_k$. Let $x_i = \mu_p(\mathcal{F}_i)$. These subfamilies satisfy some additional conditions, which give us (not necessarily linear) constraints on the variables x_i . Under the constraints we need to maximize $\mu_p(\mathcal{F}) = \sum_{i=1}^k x_i$. In principle this problem can be solved by the method of Lagrange multipliers, but in practice it is not so easy even if concrete p is fixed. To overcome the difficulty we first replace non-linear constraints with weaker piecewise linear constraints. In this way the problem turns into an LP problem with the parameter p , which can be solved efficiently provided p is given. Next, instead of solving this primal problem, we seek a feasible solution to the dual LP problem with the parameter. Finally the desired upper bound for $\mu_p(\mathcal{F})$ is obtained by the weak duality theorem. As to related proof technique we refer [24] for application of LP methods to some other extremal problems, and also [18, 19] for SDP methods.

In the next section we gather tools we use to prove our results. Then in Sect. 3 we deduce Theorem 3 and Theorem 4 from Theorem 6. In Sect. 4 we prove Theorem 7, whose proof is a prototype of the proof of Theorem 6. In Sect. 5 we prove our main result Theorem 6. Finally in the last section we discuss possible extensions to non-trivial r -wise intersecting families for $r \geq 4$, and a related k -uniform problem. In particular we include counterexamples to a recent conjecture posed by O’Neill and Versträete [17] (c.f. Balogh and Linz [4]).

2 Preliminaries

2.1 Shifting

For $1 \leq i < j \leq n$ we define the shifting operation $\sigma_{i,j} : 2^{[n]} \rightarrow 2^{[n]}$ by

$$\sigma_{i,j}(\mathcal{G}) := \{G_{i,j} : G \in \mathcal{G}\},$$

where

$$G_{i,j} := \begin{cases} (G \setminus \{j\}) \sqcup \{i\} & \text{if } (G \setminus \{j\}) \sqcup \{i\} \notin \mathcal{G}, \\ G & \text{otherwise.} \end{cases}$$

By definition $\mu_p(\mathcal{G}) = \mu_p(\sigma_{i,j}(\mathcal{G}))$ follows. We say that \mathcal{G} is shifted if \mathcal{G} is invariant under any shifting operations, in other words, if $G \in \mathcal{G}$ then $G_{i,j} \in \mathcal{G}$ for all $1 \leq i < j \leq n$. If \mathcal{G} is not shifted then $\sum_{G \in \mathcal{G}} \sum_{g \in G} g > \sum_{G' \in \sigma_{i,j}(\mathcal{G})} \sum_{g' \in G'} g'$ for some i, j , and so starting from \mathcal{G} we get a shifted \mathcal{G}' by applying shifting operations repeatedly finitely many times. It is not difficult to check that if \mathcal{G} is r -wise t -intersecting, then so is $\sigma_{i,j}(\mathcal{G})$. Therefore if \mathcal{G} is an r -wise t -intersecting family, then there is a shifted r -wise t -intersecting family \mathcal{G}' with $\mu_p(\mathcal{G}') = \mu_p(\mathcal{G})$. It is also true that if $\mathcal{G}_1, \dots, \mathcal{G}_r$ are r -cross t -intersecting families, then there are shifted r -cross t -intersecting families $\mathcal{G}'_1, \dots, \mathcal{G}'_r$ with $\mu_p(\mathcal{G}_i) = \mu_p(\mathcal{G}'_i)$ for all $1 \leq i \leq r$.

For the proof of Theorem 4 we use the fact that if $\sigma_{i,j}(\mathcal{G}) \cong \text{BD}_3(n)$ then $\mathcal{G} \cong \text{BD}_3(n)$. More generally the following holds.

Lemma 1 *Let n, a, b be positive integers with $a \geq 1, b \geq 0$, and $n \geq a + 2b$, and let $\mathcal{F} = \{F \subset [n] : |F \cap [a + 2b]| \geq a + b\}$. If $\mathcal{G} \subset 2^{[n]}$ satisfies $\sigma_{i,j}(\mathcal{G}) = \mathcal{F}$ then $\mathcal{G} \cong \mathcal{F}$.*

The above result is well-known, see, e.g., Lemma 6 in [16] for a proof and a history. We note that the condition $\sigma_{i,j}(\mathcal{G}) = \mathcal{F}$ can be replaced with $\sigma_{i,j}(\mathcal{G}) \cong \mathcal{F}$. Indeed if $\sigma_{i,j}(\mathcal{G}) = \mathcal{F}'$ and $\mathcal{F}' \cong \mathcal{F}$, then by Lemma 1 (and by renaming the vertices) we have $\mathcal{G} \cong \mathcal{F}'$, and so $\mathcal{G} \cong \mathcal{F}$. By choosing $a = r - 1$ and $b = 1$, we see that if $\sigma_{i,j}(\mathcal{G}) \cong \text{BD}_r(n)$ then $\mathcal{G} \cong \text{BD}_r(n)$.

For $G, H \subset [n]$ we say that G shifts to H , denoted by $G \rightsquigarrow H$, if $G = \emptyset$, or if $|G| \leq |H|$ and the i th smallest element of G is greater than or equal to that of H for each $i \leq |G|$. Note that the relation \rightsquigarrow is transitive, and this fact will be used later (Claims 16 and 17).

We say that \mathcal{G} is inclusion maximal if $G \in \mathcal{G}$ and $G \subset H$ imply $H \in \mathcal{G}$. Since we are interested in the maximum measure of non-trivial 3-wise intersecting families, we always assume that families are inclusion maximal. If \mathcal{G} is shifted and inclusion maximal, then $G \in \mathcal{G}$ and $G \rightsquigarrow H$ imply $H \in \mathcal{G}$.

2.2 Duality in linear programming

For later use we briefly record the weak duality theorem in linear programming. See e.g., chapter 6 in [12] for more details.

A primal linear programming problem (P) is formalized as follows.

maximize: $c^T x$,
subject to: $Ax \leq b$ and $x \geq 0$.

The corresponding dual programming problem (D) is as follows.

minimize: $b^T y$,
subject to: $A^T y \geq c$ and $y \geq 0$.

Theorem 8 (Weak duality) *For each feasible solution x of (P) and each feasible solution y of (D) we have $c^T x \leq b^T y$.*

2.3 Tools for the proof of Theorem 7

Let n, t, a be fixed positive integers with $t \leq a \leq n$. Define two families \mathcal{A} and \mathcal{B} by

$$\mathcal{A} = \{F \subset [n] : |F \cap [a]| \geq t\},$$

$$\mathcal{B} = \{F \subset [n] : [a] \subset F\}.$$

Then $\mu_p(\mathcal{A}) = 1 - \sum_{j=0}^{t-1} \binom{a}{j} p^j q^{a-j}$, and $\mu_p(\mathcal{B}) = p^a$. Let $\mathcal{F}_1 = \mathcal{A}, \mathcal{F}_2 = \dots = \mathcal{F}_r = \mathcal{B}$. Then $\mathcal{F}_1, \dots, \mathcal{F}_r$ are r -cross t -intersecting families with $\sum_{i=1}^r \mu_p(\mathcal{F}_i) = \mu_p(\mathcal{A}) + (r - 1)\mu_p(\mathcal{B})$. The next result is a special case of Theorem 1.4 in [13], which states that the above construction is the best choice to maximize the sum of p -measures of non-empty r -cross t -intersecting families provided $p \leq \frac{1}{2}$.

Theorem 9 (Gupta–Mogge–Piga–Schülke [13]) *Let $r \geq 2$ and $0 < p \leq \frac{1}{2}$. If $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ are non-empty r -cross t -intersecting families, then*

$$\sum_{i=1}^r \mu_p(\mathcal{F}_i) \leq \max \left\{ \left(1 - \sum_{j=0}^{t-1} \binom{a}{j} p^j q^{a-j} \right) + (r - 1)p^a : t \leq a \leq n \right\}.$$

We need the non-empty condition to exclude the case $\mathcal{F}_1 = \emptyset, \mathcal{F}_2 = \dots = \mathcal{F}_r = 2^{[n]}$.

Lemma 2 *Let $0 < p \leq \frac{1}{2}$. Suppose that $\mathcal{F}_1, \mathcal{F}_2 \subset 2^{[n]}$ are 2-cross intersecting families.*

- (i) $\mu_p(\mathcal{F}_1) + \mu_p(\mathcal{F}_2) \leq 1$.
- (ii) $\mu_p(\mathcal{F}_1)\mu_p(\mathcal{F}_2) \leq p^2$.

Proof (i) If one of the families is empty, then the inequality clearly holds. So suppose that both families are non-empty. Then, by Theorem 9, we have

$$\mu_p(\mathcal{F}_1) + \mu_p(\mathcal{F}_2) \leq \max\{(1 - q^a) + p^a : 1 \leq a \leq n\}.$$

Thus it suffices to show that $1 - q^a + p^a \leq 1$, or equivalently, $p^a \leq (1 - p)^a$ for all $a \geq 1$. Indeed this follows from the assumption $p \leq \frac{1}{2}$.

- (ii) This is proved in [21] as Theorem 2. □

Lemma 3 *Let $0 < p \leq \frac{1}{2}$, and $t \geq 2$. If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset 2^{[n]}$ are non-empty 3-cross t -intersecting families, then $\mu_p(\mathcal{F}_1) + \mu_p(\mathcal{F}_2) + \mu_p(\mathcal{F}_3) \leq 1$.*

Proof Note that if $t \geq 2$ then 3-cross t -intersecting families are 3-cross 2-intersecting families. Thus, using Theorem 9, it suffices to show that

$$f(p, a) := (1 - q^a - apq^{a-1}) + 2p^a \leq 1$$

for $a \geq t$. This inequality follows from the fact that $f(p, a)$ is increasing in p , and $f(\frac{1}{2}, a)$ is non-decreasing in a for $a = 2, 3, \dots$, and $\lim_{a \rightarrow \infty} f(\frac{1}{2}, a) = 1$. \square

2.4 Random walk

Here we extend the random walk method to deal with p -measures of r -cross t -intersecting families possibly with different p -measures. The method was originally introduced by Frankl in [9].

Let $r \geq 2$ be a positive integer. For $1 \leq i \leq r$ let p_i be a real number with $0 < p_i < 1 - \frac{1}{r}$, and let $q_i = 1 - p_i$. Let $\alpha(p_i) \in (0, 1)$ be a unique root of the equation

$$X = p_i + q_i X^r, \tag{4}$$

and let $\beta = \beta(p_1, \dots, p_r) \in (0, 1)$ be a unique root of the equation

$$X = \prod_{i=1}^r (p_i + q_i X). \tag{5}$$

Consider two types of random walks, A_i and B , in the two-dimensional grid \mathbb{Z}^2 . Both walks start at the origin, and at each step it moves from (x, y) to $(x, y + 1)$ (one step up), or from (x, y) to $(x + 1, y)$ (one step to the right). For every step the type A_i walk takes one step up with probability p_i , and one step to the right with probability q_i . On the other hand, at step j , the type B walk takes one step up with probability p_i , and one step to the right with probability q_i , where $i = j \bmod r$. Let L_j denote the line $y = (r - 1)x + j$.

Claim 1 *Let $r \geq 2$ and $t \geq 1$ be integers. Then we have*

$$\begin{aligned} \mathbb{P}(\text{the type } A_i \text{ walk hits the line } L_t) &= \alpha(p_i)^t, \\ \mathbb{P}(\text{the type } B \text{ walk hits the line } L_{rt}) &= \beta(p_1, \dots, p_r)^t. \end{aligned}$$

Proof Let $x_i(t)$ denote the probability that the walk A_i hits the line L_t . After the first step of the walk, it is at $(0, 1)$ with probability p_i , or at $(1, 0)$ with probability q_i . From $(0, 1)$ the probability for the walk hitting L_t is $x_i(t - 1)$, and from $(1, 0)$ the probability is $x_i(t - 1 + r)$. Therefore we have

$$x_i(t) = p_i x_i(t - 1) + q_i x_i(t - 1 + r). \tag{6}$$

Let a_j be the number of walks from $(0, 0)$ to $P_j := (j, (r - 1)j + t)$ which touch L_t only at P_j . (It is known that $a_j = \frac{t}{rj+t} \binom{rj+t}{j}$, but we do not need this fact.) Then we have $x_i(t) = \sum_{j \geq 0} a_j p_i^{(r-1)j+t} q_i^j$. If a walk touches the line L_{t+1} , then the walk needs to hit L_t somewhere, say, at P_j for the first time. Then the probability that the walk hit L_{t+1} starting from P_j is equal to $x_i(1)$. Thus we have

$$x_i(t + 1) = \sum_{j \geq 0} (a_j p_i^{(r-1)j+t} q_i^j) x_i(1) = x_i(t)x_i(1),$$

and so we can write $x_i(t) = z^t$, where $z := x_i(1)$. Substituting this into (6) and dividing both sides by z^{t-1} we see that z is a root of the equation (4). Let $f(X) := p_i + q_i X^r - X$. Then we have $f(0) = p_i > 0$, $f(1) = 0$, $f'(1) = q_i r - 1 > 0$, and $f''(X) = q_i r(r - 1)X^{r-2} > 0$. Thus the equation $f(X) = 0$, or equivalently, (4) has precisely two roots in $[0, 1]$, that is, $\alpha(p_i)$ and 1. We claim that $z \neq 1$. Indeed we have $\lim_{t \rightarrow \infty} x_i(t) = \lim_{t \rightarrow \infty} z^t = 0$ because a step in the type A_i walk reduces, on average, $y - (r - 1)x$ by $(r - 1) - rp_i > 0$. Consequently we have $z = \alpha(p_i)$, and so $x_i(t) = \alpha(p_i)^t$.

Next let $y(t)$ denote the probability that the walk B hits the line L_{rt} . After the first r steps, it is at $(x, r - x)$ for some $0 \leq x \leq r$ with probability

$$\sum_{J \in \binom{[r]}{x}} \prod_{i \in [r] \setminus J} p_i \prod_{j \in J} q_j.$$

From $(x, r - x)$ the probability for the walk hitting L_{rt} is $y(x + t - 1)$. This yields

$$y(t) = \sum_{x=0}^r y(x + t - 1) \sum_{J \in \binom{[r]}{x}} \prod_{i \in [r] \setminus J} p_i \prod_{j \in J} q_j. \tag{7}$$

Let b_s be the number of walks from $(0, 0)$ to $Q_s := (s, (r - 1)s + rt)$ which touch L_{rt} only at Q_s . Then we have

$$y(t) = \sum_{s \geq 0} b_s \sum_{J \in \binom{[r(s+t)]}{s}} \prod_{i \in [r(s+t)] \setminus J} p_i \prod_{j \in J} q_j.$$

If a walk touches the line $L_{r(t+1)}$, then the walk needs to hit L_{rt} somewhere, say, at Q_s for the first time. Then the probability that the walk hit $L_{r(t+1)}$ starting from Q_s is equal to $y(1)$. Thus we have

$$y(t + 1) = \sum_{s \geq 0} b_s \sum_{J \in \binom{[r(s+t)]}{s}} \prod_{i \in [r(s+t)] \setminus J} p_i \prod_{j \in J} q_j y(1) = y(t)y(1),$$

and so $y(t) = w^t$, where $w := y(1)$. Substituting this into (7) and dividing both sides by w^{t-1} we have

$$w = \sum_{x=0}^r w^x \sum_{J \in \binom{[r]}{x}} \prod_{i \in [r] \setminus J} p_i \prod_{j \in J} q_j = \prod_{i=1}^r (p_i + q_i w).$$

Thus w is a root of the equation (5). Let $g(X) := \prod_{i=1}^r (p_i + q_i X) - X$. Then we have $g(0) = \prod_i p_i > 0$, $g(1) = 0$, $g'(1) = \sum_i q_i > 0$, and $g''(X) > 0$. Thus the equation $g(X) = 0$, or equivalently, (5) has precisely two roots in $[0, 1]$, that is, β and 1. But we can exclude the possibility $w = 1$ in the same way as in the previous case. Thus we have $w = \beta$ and so $y(t) = \beta^t$. □

Claim 2 *Let $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ be shifted r -cross t -intersecting families. Then, for all $(F_1, \dots, F_r) \in \mathcal{F}_1 \times \dots \times \mathcal{F}_r$, there exists $j = j(F_1, \dots, F_r) \in [n]$ such that $\sum_{i=1}^r |F_i \cap [j]| \geq t + (r - 1)j$.*

This is Proposition 8.1 in [9]. We include a simple proof for convenience.

Proof Suppose the contrary. Then there exist an r -tuple of a counterexample $(F_1, \dots, F_r) \in \mathcal{F}_1 \times \dots \times \mathcal{F}_r$, which we choose $|F_1 \cap \dots \cap F_r|$ minimal. Let j be the t -th element of $F_1 \cap \dots \cap F_r$. Then we have

$$\sum_{i=1}^r |F_i \cap [j]| < t + (r - 1)j = |F_1 \cap \dots \cap F_r \cap [j]| + (r - 1)|[j]|.$$

Thus there exist some $i \in [j - 1]$ such that i is not contained in (at least) two of the r subsets, say, $i \notin F_1 \cup F_2$. By the shiftedness we have $F'_1 := (F \setminus \{j\}) \cup \{i\} \in \mathcal{F}_1$. Then $|F_1 \cap [j]| = |F'_1 \cap [j]|$ and so (F'_1, F_2, \dots, F_r) is also a counterexample. But this contradicts the minimality because $|F'_1 \cap F_2 \cap \dots \cap F_r| < |F_1 \cap F_2 \cap \dots \cap F_r|$. □

Let $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ be families of subsets. For each $(F_1, \dots, F_r) \in \mathcal{F}_1 \times \dots \times \mathcal{F}_r$ we define a vector w by

$$w = w(F_1, \dots, F_r) := (w_1^{(1)}, w_2^{(1)}, \dots, w_r^{(1)}, \dots, w_1^{(n)}, w_2^{(n)}, \dots, w_r^{(n)}) \in \{0, 1\}^{rn},$$

where

$$w_i^{(j)} = \begin{cases} 1 & \text{if } j \in F_i, \\ 0 & \text{if } j \notin F_i. \end{cases}$$

We can view w as an rn -step walk whose k -th step is up (resp. right) if the k -th entry of w is 1 (resp. 0) for $1 \leq k \leq rn$.

Claim 3 *Let $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ be shifted r -cross t -intersecting families. Then, for all $(F_1, \dots, F_r) \in \mathcal{F}_1 \times \dots \times \mathcal{F}_r$, the walk $w(F_1, \dots, F_r)$ hits the line L_{rt} .*

Proof Let $j = j(F_1, \dots, F_r)$ be from Claim 2, and let $w = w(F_1, \dots, F_r)$ be the corresponding walk. In the first rj steps of w there are at least $t + (r - 1)j$ up steps, and so at most $rj - (t + (r - 1)j) = j - t$ right steps. This means that the walk w hits the line L_{rt} within the first rj steps. \square

Theorem 10 Let p_1, \dots, p_r be positive real numbers less than $1 - \frac{1}{r}$, and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subset 2^{[n]}$ be r -cross t -intersecting families. Then we have $\prod_{i=1}^r \mu_{p_i}(\mathcal{F}_i) \leq \beta^t$, where β is the root of the equation (5).

Proof Since the shifting operation preserves r -cross t -intersecting property and p -measures, we may assume that all \mathcal{F}_i are shifted. We have

$$\begin{aligned} \prod_{i=1}^r \mu_{p_i}(\mathcal{F}_i) &= \prod_{i=1}^r \sum_{F_i \in \mathcal{F}_i} p_i^{|F_i|} q_i^{n-|F_i|} \\ &= \sum_{(F_1, \dots, F_r) \in \mathcal{F}_1 \times \dots \times \mathcal{F}_r} \prod_{i=1}^r p_i^{|F_i|} q_i^{n-|F_i|}. \end{aligned}$$

Using Claim 3 the RHS is

$$\leq \mathbb{P}(\text{type } B \text{ walk hits } L_{rt} \text{ in the first } rn \text{ steps}) \leq \mathbb{P}(\text{type } B \text{ walk hits } L_{rt}) = \beta^t,$$

where the last equality follows from Claim 1. \square

By comparing (4) and (5) it follows that if $p_1 = \dots = p_r =: p$ then $\beta(p, \dots, p) = \alpha(p)^r$. If $r = 3$ then it is not so difficult to verify that $\beta(p_1, p_2, p_3) \leq \alpha(p_1)\alpha(p_2)\alpha(p_3)$, see [15] for more details, and we have the following.

Lemma 4 Let $0 < p_1, p_2, p_3 < \frac{2}{3}$ and t be a positive integer. If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \subset 2^{[n]}$ are 3-cross t -intersecting, then

$$\mu_{p_1}(\mathcal{F}_1)\mu_{p_2}(\mathcal{F}_2)\mu_{p_3}(\mathcal{F}_3) \leq (\alpha(p_1)\alpha(p_2)\alpha(p_3))^t,$$

where

$$\alpha(p) := \frac{1}{2} \left(\sqrt{\frac{1+3p}{1-p}} - 1 \right). \tag{8}$$

2.5 Tools for the proof of Theorem 6

Let $0 < p_1 < p_2 < 1$ be fixed. Let $\mathbb{R}^{[p_1, p_2]}$ denote the set of real-valued functions defined on the interval $[p_1, p_2] := \{x \in \mathbb{R} : p_1 \leq x \leq p_2\}$. We will bound a convex function $g \in \mathbb{R}^{[p_1, p_2]}$ by a linear function connecting $(p_1, g(p_1))$ and $(p_2, g(p_2))$. To this end, define an operator $L_{p_1, p_2} : \mathbb{R}^{[p_1, p_2]} \rightarrow \mathbb{R}^{[p_1, p_2]}$ by

$$(L_{p_1, p_2}(g))(p) := \frac{g(p_2) - g(p_1)}{p_2 - p_1}(p - p_1) + g(p_1).$$

By definition we have the following.

Claim 4 Let $g \in \mathbb{R}^{[p_1, p_2]}$ be a convex function. Then $g(p) \leq (L_{p_1, p_2}(g))(p)$ for $p \in [p_1, p_2]$.

The function $\alpha = \alpha(p)$ defined by (8) is convex because $\frac{\partial^2 \alpha(p)}{\partial p^2} = \frac{6p}{(1+3p)^2 q^2} \left(\frac{1+3p}{q}\right)^{1/2} > 0$. Thus by Claim 4 we have the following.

Claim 5 For $\frac{2}{5} \leq p \leq \frac{1}{2}$ it follows that $\alpha(p) \leq \tilde{\alpha}(p)$, where

$$\begin{aligned} \tilde{\alpha}(p) &:= (L_{\frac{2}{5}, \frac{1}{2}}(\alpha))(p) = (-3 - 12\sqrt{5} + 5\sqrt{33})/6 + (30\sqrt{5} - 10\sqrt{33})p/6 \\ &\approx -0.185 + 1.60607p. \end{aligned}$$

Let $\text{AK}(n, t, p)$ denote the maximum p -measure $\mu_p(\mathcal{G})$ of 2-wise t -intersecting families $\mathcal{G} \subset 2^{[n]}$.

Theorem 11 (Ahlswede and Khachatryan [2]) Let

$$\frac{i}{t + 2i - 1} \leq p \leq \frac{i + 1}{t + 2i + 1}.$$

Then $\text{AK}(n, t, p) = \mu_p(\mathcal{A}(n, t, i))$, where

$$\mathcal{A}(n, t, i) = \{A \subset [n] : |A \cap [t + 2i]| \geq t + i\}.$$

Moreover, if $\frac{i}{t+2i-1} < p < \frac{i+1}{t+2i+1}$ (resp. $p = \frac{i}{t+2i-1}$) then $\mu_p(\mathcal{G}) = \text{AK}(n, t, p)$ and only if $\mathcal{G} \cong \mathcal{A}(n, t, i)$ (resp. $\mathcal{G} \cong \mathcal{A}(n, t, i - 1)$ or $\mathcal{G} \cong \mathcal{A}(n, t, i)$).

Let

$$f_t(p) := \limsup_{n \rightarrow \infty} \text{AK}(n, t, p).$$

By Katona’s t -intersection theorem we have $f_t(\frac{1}{2}) = \frac{1}{2}$. For $p < \frac{1}{2}$, by Theorem 11, we have $\text{AK}(n, t, p) = \max\{\mu_p(\mathcal{A}(n, t, i)) : i \leq \frac{n-t}{2}\}$, and $\text{AK}(n, t, p)$ is non-decreasing in n . In this case we have

$$f_t(p) = \lim_{n \rightarrow \infty} \text{AK}(n, t, p) = \sum_{j=t+i}^{t+2i} \binom{t+2i}{j} p^j q^{t+2i-j},$$

where $i = \left\lfloor \frac{(t-1)p}{1-2p} \right\rfloor$. The function $f_t(p)$ is left-continuous at $p = \frac{1}{2}$.

Claim 6 Let $t \geq 2$ be fixed. Then $f_t(p)$ is a convex function in p .

Proof First suppose that $\frac{i}{i+2i-1} < p < \frac{i+1}{i+2i+1}$. Then $f_t(p) = \sum_{t+i}^{t+2i} \binom{t+2i}{j} p^j q^{t+2i-j}$ $=: g(p)$, and we have

$$\frac{\partial^2}{\partial p^2} f_t(p) = \frac{(2i+t)!}{(i+t-1)!i!} p^{t-2+i} q^{i-1} (i+t-1 - (2i+t-1)p) > 0.$$

Next let $p_0 = \frac{i}{i+2i-1}$. If p is slightly larger than p_0 then we have the same $f_t(p) = g(p)$ as above, and if p is slightly smaller than p_0 then $f_t(p) = \sum_{t+i-1}^{t+2i-2} p^j q^{t+2i-2} =: h(p)$. Since $h(p) < g(p)$ for $p > p_0$ and $h(p) > g(p)$ for $p < p_0$, we see that the left derivative of $f_t(p)$ at $p = p_0$ is smaller than that of the right derivative. \square

By Claim 4 and Claim 6 we have the following.

Claim 7 For $\frac{2}{3} \leq p \leq \frac{1}{2}$ it follows that $AK(n, t, p) \leq \tilde{a}_t(p)$, where $\tilde{a}_t = L_{\frac{2}{3}, \frac{1}{2}}(f_t)$.

For convenience we record the \tilde{a}_t which will be used to prove Theorem 6.

$$\begin{aligned} \tilde{a}_2(p) &= \frac{1}{2} + (401(p - \frac{1}{2}))/125 \approx -1.104 + 3.208p, \\ \tilde{a}_3(p) &= \frac{1}{2} + (1565029(p - \frac{1}{2}))/390625 \approx -1.50324 + 4.00647p, \\ \tilde{a}_4(p) &= \frac{1}{2} + (5391614441(p - \frac{1}{2}))/1220703125 \approx -1.70841 + 4.41681p, \\ \tilde{a}_5(p) &= \frac{1}{2} + (17729648464189(p - \frac{1}{2}))/3814697265625 \approx -1.82386 + 4.64772p. \end{aligned}$$

3 Proof of Theorem 3 and Theorem 4

In this section we deduce Theorem 3 from Theorem 4, and then deduce Theorem 4 from Theorem 6 whose proof is given in the next section.

Proof of Theorem 3 Let $\mathcal{F} \subset 2^{[n]}$ be a non-trivial 3-wise intersecting family with $\mu_p(\mathcal{F}) = M_3(n, p)$. We may assume that \mathcal{F} is shifted and inclusion maximal. Since \mathcal{F} is non-trivially 3-wise intersecting, it is also 2-wise 2-intersecting, and so $M_3(n, p) \leq AK(n, 2, p)$. \square

Claim 8 If $p < \frac{1}{3}$ then $M_3(p) = p^2$.

Proof Let $p < \frac{1}{3}$ be fixed. Then we have $\mu_p(\mathcal{F}) = M_3(n, p) \leq AK(n, 2, p) = p^2$. Moreover $\mathcal{G} \cong \mathcal{A}(n, 2, 0)$ is the only 2-wise 2-intersecting family with $\mu_p(\mathcal{G}) = p^2$. Since $\mathcal{A}(n, 2, 0)$ is not non-trivial 3-wise intersecting, we get $M_3(n, p) < p^2$.

On the other hand we can construct a non-trivial 3-wise intersecting family \mathcal{F}_1 by

$$\mathcal{F}_1 = \{F \in [n] : [2] \subset F, F \cap [3, n] \neq \emptyset\} \cup \{[n] \setminus \{1\}\} \cup \{[n] \setminus \{2\}\}.$$

Then it follows that

$$\mu_p(\mathcal{F}_1) = p^2(1 - q^{n-2}) + 2p^{n-1}q \rightarrow p^2 \text{ as } n \rightarrow \infty.$$

Thus we have $M_3(p) = p^2$ for $p < \frac{1}{3}$. \square

Claim 9 If $\frac{1}{3} \leq p \leq \frac{1}{2}$ then $M_3(p) = 4p^3q + p^4$.

Proof This is an immediate consequence of Theorem 4. □

Claim 10 If $\frac{1}{2} < p \leq \frac{2}{3}$ then $M_3(p) = p$.

Proof Let $\frac{1}{2} < p \leq \frac{2}{3}$ be fixed. It is known from [8, 10, 22] that 3-wise intersecting families \mathcal{G} have p -measure at most p for $p \leq \frac{2}{3}$, and moreover if $\mu_p(\mathcal{G}) = p$ then $|\bigcap \mathcal{G}| = 1$ for $p < \frac{2}{3}$. Thus we have $M(n, p) < p$ for $\frac{1}{2} < p < \frac{2}{3}$ and $M(n, \frac{2}{3}) \leq \frac{2}{3}$.

On the other hand, let us define a non-trivial 3-wise intersecting family \mathcal{F}_2 by

$$\mathcal{F}_2 = \{F \in [n] : 1 \in F, |F \cap [2, n]| \geq n/2\} \cup \{[2, n]\}.$$

Then it follows that, for fixed p ,

$$\mu_p(\mathcal{F}_2) = p \sum_{k \geq n/2} \binom{n-1}{k} p^k q^{n-1-k} + q p^{n-1} \rightarrow p \text{ as } n \rightarrow \infty.$$

Thus we have $M(p) = p$ for $\frac{1}{2} < p \leq \frac{2}{3}$. □

Claim 11 If $\frac{2}{3} < p < 1$ then $M_3(p) = 1$.

Proof Let $\frac{2}{3} < p < 1$ be fixed. Clearly we have $M(n, p) \leq 1$ and $M(p) \leq 1$. Let us define a non-trivial 3-wise intersecting family \mathcal{F}_3 by

$$\mathcal{F}_3 = \{F \subset [n] : |F| > \frac{2}{3}n\}.$$

Then $\mu_p(\mathcal{F}_3) = \sum_{i > \frac{2}{3}n} \binom{n}{i} p^i q^{n-i} \rightarrow 1$ as $n \rightarrow \infty$. Thus we have $M(p) = 1$ for $p > \frac{2}{3}$. □

This completes the proof of Theorem 3 assuming Theorem 4. □

Proof of Theorem 4 Let $\mathcal{F} \subset 2^{[n]}$ be a non-trivial 3-wise intersecting family.

First suppose that $\frac{1}{3} \leq p < \frac{2}{5}$. Note that \mathcal{F} is 2-wise 2-intersecting, and $\mathcal{A}(n, 2, 1) = \text{BD}_3(n)$. Thus it follows from Theorem 11 that $\mu_p(\mathcal{F}) \leq \mu_p(\text{BD}_3(n))$. Moreover equality holds if and only if $\mathcal{F} \cong \text{BD}_3(n)$ for $p > \frac{1}{3}$. If $p = \frac{1}{3}$ then $\mu_p(\mathcal{F}) = \mu_p(\text{BD}_3(n))$ if and only if $\mathcal{F} \cong \text{BD}_3(n)$ or $\mathcal{A}(n, 2, 0)$, but the latter is not non-trivial 3-wise intersecting, and so $\mathcal{F} \cong \text{BD}_3(n)$ must hold.

Next suppose that $\frac{2}{5} \leq p \leq \frac{1}{2}$. If \mathcal{F} is shifted then by Theorem 6 we have $\mu_p(\mathcal{F}) \leq \mu_p(\text{BD}_3(n))$ with equality holding if and only if $\mathcal{F} = \text{BD}_3(n)$. The same inequality holds without assuming that \mathcal{F} is shifted (see the first paragraph in Sect. 2). In this case, by Lemma 1, we have $\mu_p(\mathcal{F}) = \mu_p(\text{BD}_3(n))$ if and only if $\mathcal{F} \cong \text{BD}_3(n)$. □

4 Proof of Theorem 7

4.1 Proof of (2) of Theorem 7

Let $\frac{1}{3} \leq p \leq \frac{1}{2}$, and let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be non-empty 3-cross intersecting families. By Theorem 9 with $r = 3$ we have $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) \leq \max\{(1 - q^a) + 2p^a : a \in [n]\}$. So we need to show that $f(p, a) \geq 0$, where

$$f(p, a) := 3p - (1 - q^a) - 2p^a,$$

for all $\frac{1}{3} \leq p \leq \frac{1}{2}$ and all $1 \leq a \leq n$.

If $a = 1$ then $f(p, 1) = 3p - (1 - q) - 2p = 0$, and we are done. So we may assume that $2 \leq a \leq n$, and we show that $f(p, a) > 0$.

If $p = \frac{1}{3}$ then $f(\frac{1}{3}, a) = 1 - 1 + (\frac{2}{3})^a - 2(\frac{1}{3})^a = (\frac{1}{3})^a(2^a - 2) > 0$. We claim that $f(p, a)$ is increasing in p , which yields $f(p, a) \geq f(\frac{1}{3}, a) > 0$ (for $a \geq 2$). We have

$$\frac{\partial f}{\partial p}(p, a) = 3 - aq^{a-1} - 2ap^{a-1}. \tag{9}$$

Fix p and let $g(a)$ denote the RHS of (9). We have $g(2) = 1 - 2p > 0$ for $\frac{1}{3} \leq p < \frac{1}{2}$. Next we show that $g(a)$ is increasing in a . For this we have

$$g(a + 1) - g(a) = (ap - q)q^{a-1} + 2(aq - p)p^{a-1},$$

and we need to show that the RHS is positive. Since $aq - p \geq ap - q$ it suffices to show that $ap - q \geq 0$, or equivalently, $a \geq \frac{1-p}{p}$. Indeed $a \geq 2 \geq \frac{1-p}{p}$ because $p \geq \frac{1}{3}$. Thus $g(a)$ is increasing in a , and $g(a) \geq g(2) > 0$ as needed. \square

4.2 Proof of (3) of Theorem 7

Recall that, for $i < j$, we write $[i, j] := \{i, i + 1, \dots, j\} = [j] \setminus [i - 1]$.

We divide $\mathcal{F}_i = \{\{1\} \sqcup A : A \in \mathcal{A}_i\} \cup \mathcal{B}_i$, where

$$\begin{aligned} \mathcal{A}_i &:= \{F \setminus \{1\} : 1 \in F \in \mathcal{F}_i\} \subset 2^{[2, n]}, \\ \mathcal{B}_i &:= \{F : 1 \notin F \in \mathcal{F}_i\} \subset 2^{[2, n]}. \end{aligned}$$

Since $\mathcal{F}_i \neq \emptyset$ is shifted, we have $\mathcal{A}_i \neq \emptyset$. Let $a_i = \mu_p(\mathcal{A}_i : [2, n]) > 0$ and $b_i = \mu_p(\mathcal{B}_i : [2, n]) \geq 0$. Then $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) = \sum_{i=1}^3 (pa_i + qb_i)$. Without loss of generality we may assume that $b_1 \geq b_2 \geq b_3$. If $b_1 = 0$ then $\mathcal{B}_i = \emptyset$ for all i . In this case $1 \in \bigcap F$, where the intersection is taken over all $F \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, a contradiction. So we may assume that $b_1 \neq 0$, that is, $\mathcal{B}_1 \neq \emptyset$.

Claim 12 Let $\{i, j, k\} = [3]$.

- (1) If $\{\mathcal{A}_i, \mathcal{A}_j, \mathcal{B}_k\}$ are all non-empty, then they are 3-cross intersecting, and $a_i + a_j + b_k \leq 3p$.

Table 1 Case $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$

	a_1	a_2	a_3	b_1	
y_1		1	1	1	$3p$
y_2		1		1	1
y_3			1	1	1
y_4	1				1
y_5		1			1
y_6			1		1
y_7				1	1
	p	p	p	q	

(2) If $\{A_i, B_j, B_k\}$ are all non-empty, then they are 3-cross 2-intersecting, and $a_i + b_j + b_k \leq 1$.

Proof The item (1) follows from the assumption that $\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k$ are 3-cross intersecting, and (2) of Theorem 7.

To show (2), suppose, to the contrary, that there exist three subsets $A_i \in \mathcal{A}_i, B_j \in \mathcal{B}_j, B_k \in \mathcal{B}_k$, and $x \in [2, n]$ such that $\{x\} \supset A_i \cap B_j \cap B_k$. By definition we have $F_i := \{1\} \cup A_i \in \mathcal{F}_i$ and $F_k := B_k \in \mathcal{F}_k$. By the shiftedness we have $F_j := (B_j \setminus \{x\}) \cup \{1\} \in \mathcal{F}_j$. Then $F_i \cap F_j \cap F_k = \emptyset$, a contradiction. Thus $\{A_i, B_j, B_k\}$ are 3-wise 2-intersecting, and the inequality follows from Lemma 3. \square

Now we will show that $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) \leq 3p - \epsilon_p$, where $\epsilon_p = (2 - 3p)(3p - 1)$.

4.2.1 Case $\mathcal{B}_2 = \mathcal{B}_3 = \emptyset$

Since $\{B_1, A_2, A_3\}$ are 3-cross intersecting, any two of them are 2-cross intersecting. Then, by (2) of Theorem 7 and Lemma 2, we have the following.

- Claim 13** (1) $b_1 + a_2 + a_3 \leq 3p$,
 (2) $b_1 + a_2 \leq 1$,
 (3) $b_1 + a_3 \leq 1$.

We solve the following linear programming problem:

maximize: $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) = p(a_1 + a_2 + a_3) + qb_1$,
subject to: (1)–(3) in Claim 13, and $0 \leq a_i \leq 1$ for all $i, 0 \leq b_1 \leq 1$.

The corresponding dual problem is (Table 1)

minimize: $3py_1 + \sum_{i=2}^7 y_i$,
subject to: $y_4 \geq p, y_1 + y_2 + y_5 \geq p, y_1 + y_3 + y_6 \geq p, y_1 + y_2 + y_3 + y_7 \geq q$, and $y_i \geq 0$ for all i .

A feasible solution is given by $y_1 = 3p - 1, y_2 = y_3 = 1 - 2p, y_4 = p, y_5 = y_6 = y_7 = 0$, and the corresponding value of the objective function is

$$3p(3p - 1) + 2(1 - 2p) + p = 2 - 6p + 9p^2 = 3p - \epsilon_p.$$

Table 2 Case $\mathcal{B}_2 \neq \emptyset, \mathcal{B}_3 = \emptyset$

	a_1	a_2	a_3	b_1	b_2	
y_1		1	1	1		$3p$
y_2	1		1		1	$3p$
y_3			1	1	1	1
y_4		1		1		1
y_5	1				1	1
y_6	1					1
y_7		1				1
y_8			1			1
y_9				1		1
y_{10}					1	1
	p	p	p	q	q	

Then it follows from Theorem 8 (weak duality) that the same bound applies to the primal problem, and so $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) \leq 3p - \epsilon_p$.

4.2.2 Case $\mathcal{B}_2 \neq \emptyset, \mathcal{B}_3 = \emptyset$

In this case $\{\mathcal{A}_1, \mathcal{B}_2, \mathcal{A}_3\}$ and $\{\mathcal{B}_1, \mathcal{A}_2, \mathcal{A}_3\}$ are both 3-cross intersecting, and $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{A}_3\}$ are 3-cross 2-intersecting. Thus we have the following.

- Claim 14** (1) $b_1 + a_2 + a_3 \leq 3p$,
 (2) $a_1 + b_2 + a_3 \leq 3p$,
 (3) $b_1 + b_2 + a_3 \leq 1$,
 (4) $b_1 + a_2 \leq 1$,
 (5) $a_1 + b_2 \leq 1$.

We solve the following linear programming problem:

maximize: $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) = p(a_1 + a_2 + a_3) + q(b_1 + b_2)$,
subject to: (1)–(5) in Claim 14, and $0 \leq a_i \leq 1$ for all i , $0 \leq b_j \leq 1$ for all j .

The corresponding dual problem is (Table 2)

minimize: $3p(y_1 + y_2) + \sum_{i=3}^{10} y_i$,
subject to: $y_2 + y_5 + y_6 \geq p, y_1 + y_4 + y_7 \geq p, y_1 + y_2 + y_3 + y_8 \geq p, y_1 + y_3 + y_4 + y_9 \geq q, y_2 + y_3 + y_5 + y_{10} \geq q$, and $y_i \geq 0$ for all i .

A feasible solution is given by $y_1 = y_6 = y_7 = y_8 = y_9 = y_{10} = 0, y_2 = 3p - 1, y_3 = y_5 = 1 - 2p, y_4 = p$, and the corresponding value of the objective function is

$$3p(3p - 1) + 2(1 - 2p) + p = 3p - \epsilon_p.$$

Thus, by the weak duality, we have $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) \leq 3p - \epsilon_p$.

Table 3 Case $\mathcal{B}_2 \neq \emptyset, \mathcal{B}_3 \neq \emptyset$

	a_1	a_2	a_3	b_1	b_2	b_3
y_1		1	1	1		$3p$
y_2	1		1		1	$3p$
y_3	1	1				$3p$
y_4	1				1	1
y_5		1		1		1
y_6			1	1	1	1
y_7	1					1
y_8		1				1
y_9			1			1
y_{10}				1		1
y_{11}					1	1
y_{12}						1
	p	p	p	q	q	q

4.2.3 Case $\mathcal{B}_2 \neq \emptyset, \mathcal{B}_3 \neq \emptyset$

Let $\{i, j, k\} = [3]$. Then families $\{\mathcal{A}_i, \mathcal{A}_j, \mathcal{B}_k\}$ are 3-cross intersecting, and families $\{\mathcal{A}_i, \mathcal{B}_j, \mathcal{B}_k\}$ are 3-cross 2-intersecting. Thus we have the following.

- Claim 15** (1) $b_1 + a_2 + a_3 \leq 3p$,
 (2) $a_1 + b_2 + a_3 \leq 3p$,
 (3) $a_1 + a_2 + b_3 \leq 3p$,
 (4) $a_1 + b_2 + b_3 \leq 1$,
 (5) $b_1 + a_2 + b_3 \leq 1$,
 (6) $b_1 + b_2 + a_3 \leq 1$.

We solve the following linear programming problem:

maximize: $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) = p(a_1 + a_2 + a_3) + q(b_1 + b_2 + b_3)$,
subject to: (1)–(6) in Claim 15, and $0 \leq a_i \leq 1$ for all i , $0 \leq b_j \leq 1$ for all j .

The corresponding dual problem is (Table 3)

minimize: $3p(y_1 + y_2 + y_3) + \sum_{i=4}^{12} y_i$,
subject to: $y_2 + y_3 + y_4 + y_7 \geq p, y_1 + y_3 + y_5 + y_8 \geq p, y_1 + y_2 + y_6 + y_9 \geq p,$
 $y_1 + y_5 + y_6 + y_{10} \geq q, y_2 + y_4 + y_6 + y_{11} \geq q, y_3 + y_4 + y_5 + y_{12} \geq q,$
 and $y_i \geq 0$ for all i .

A feasible solution is given by $y_1 = 3p - 1, y_2 = y_3 = y_7 = y_8 = y_9 = y_{10} = y_{11} = y_{12} = 0, y_4 = p, y_5 = y_6 = 1 - 2p$, and the corresponding value of the objective function is

$$3p(3p - 1) + p + 2(1 - 2p) = 3p - \epsilon_p.$$

Thus we have $\sum_{i=1}^3 \mu_p(\mathcal{F}_i) \leq 3p - \epsilon_p$.

This complete the proof of (3) of Theorem 7. □

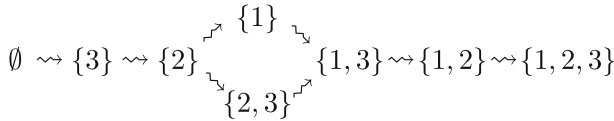


Fig. 2 Poset induced by shifting and inclusion

5 Proof of Theorem 6

Let $\frac{2}{5} \leq p \leq \frac{1}{2}$, and let $\mathcal{F} \subset 2^{[n]}$ be a non-trivial 3-wise intersecting family. Suppose that \mathcal{F} is shifted, inclusion maximal, and $\mathcal{F} \not\subset \text{BD}_3(n)$. We may also assume that \mathcal{F} is size maximal (with respect to 3-wise intersection condition), that is, for every $G \notin \mathcal{F}$, the larger family $\mathcal{F} \cup \{G\}$ is no longer 3-wise intersecting. Our goal is to show that

$$\mu_p(\mathcal{F}) < \mu_p(\text{BD}_3(n)) - 0.0018.$$

For $I \subset [3]$ define $\mathcal{F}_I \subset 2^{[n]}$ and $\mathcal{G}_I \subset 2^{[4,n]}$ by

$$\begin{aligned} \mathcal{F}_I &= \{F \in \mathcal{F} : F \cap [3] = I\}, \\ \mathcal{G}_I &= \{F \setminus [3] : F \in \mathcal{F}_I\}. \end{aligned}$$

Let $x_I = \mu_p(\mathcal{G}_I : [4, n])$. Then we have

$$\mu_p(\mathcal{F}_I) = p^{|I|}q^{3-|I|}x_I,$$

and

$$\mu_p(\mathcal{F}) = \sum_{I \subset [3]} p^{|I|}q^{3-|I|}x_I. \tag{10}$$

For simplicity we often write \mathcal{G}_I or x_I without braces and commas, e.g., we write \mathcal{G}_{12} to mean $\mathcal{G}_{\{1,2\}}$. Let \bar{I} denote $[3] \setminus I$.

Claim 16 *If $I, J \subset [3]$ satisfy $I \rightsquigarrow J$ then $\mathcal{G}_I \subset \mathcal{G}_J$ and $x_I \leq x_J$.*

Proof Suppose that $G \in \mathcal{G}_I$. Then $I \cup G \in \mathcal{F}_I$. Since \mathcal{F} is shifted, inclusion maximal, and $I \rightsquigarrow J$, we have that $J \cup G \in \mathcal{F}_J$, and so $G \in \mathcal{G}_J$. Thus $\mathcal{G}_I \subset \mathcal{G}_J$, and so $x_I \leq x_J$. □

Applying Claim 16 to the diagram in Fig. 2, we get Claim 17.

Claim 17 *We have $x_\emptyset \leq x_3 \leq x_2 \leq x_1 \leq x_{13} \leq x_{12} \leq x_{123}$, and $x_2 \leq x_{23} \leq x_{13}$.*

Let $I_1, I_2, I_3 \subset [3]$. Define a 3×3 matrix $M = M(I_1, I_2, I_3) = (m_{i,j})$ by

$$m_{i,j} = \begin{cases} 1 & \text{if } j \in I_i, \\ 0 & \text{if } j \notin I_i. \end{cases}$$

Then $I_1 \cap I_2 \cap I_3 = \emptyset$ if and only if every column of M contains (at least one) 0. In this case we say that M is acceptable, and let $\tau := 7 - s$, where s is the total sum of $m_{i,j}$.

Claim 18 *Let $M(I_1, I_2, I_3)$ be acceptable. If $\{\mathcal{G}_{I_1}, \mathcal{G}_{I_2}, \mathcal{G}_{I_3}\}$ are all non-empty, then they are 3-cross τ -intersecting, and any two of them are 2-cross τ -intersecting.*

Proof Let us start with two concrete examples.

First example is the case $I_1 = I_2 = \{1\}$, $I_3 = \{2, 3\}$, and so $s = 1 + 1 + 2 = 4$, $\tau = 7 - 4 = 3$. We show that $\{\mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_{23}\}$ are 3-cross 3-intersecting. Suppose the contrary. Then there are $G_1, G'_1 \in \mathcal{G}_1$, $G_{23} \in \mathcal{G}_{23}$ and $x, y \in [4, n]$ such that $G_1 \cap G'_1 \cap G_{23} \subset \{x, y\}$. Let $F_{12} = \{1, 2\} \cup (G_1 \setminus \{x\})$, $F_{13} = \{1, 3\} \cup (G'_1 \setminus \{y\})$, and $F_{23} = \{2, 3\} \cup G_{23} \in \mathcal{F}_{23}$. By the shiftedness we have $F_{12} \in \mathcal{F}_{12}$ and $F_{13} \in \mathcal{F}_{13}$. But $F_{12} \cap F_{13} \cap F_{23} = \emptyset$, a contradiction.

Next example is the case $I_1 = I_2 = I_3 = \emptyset$, and so $\tau = 7$. We show that $\{\mathcal{G}_\emptyset, \mathcal{G}_\emptyset, \mathcal{G}_\emptyset\}$ are 3-cross 7-intersecting, that is, \mathcal{G}_\emptyset is 3-wise 7-intersecting. Suppose the contrary. Then there are $F, F', F'' \in \mathcal{F}_\emptyset$ and $x_1, \dots, x_6 \in [4, n]$ such that $F \cap F' \cap F'' \subset \{x_1, \dots, x_6\}$. Let $F_{12} := (F \setminus \{x_1, x_2\}) \cup \{1, 2\} \in \mathcal{F}_{12}$, $F'_{13} := (F' \setminus \{x_3, x_4\}) \cup \{1, 3\} \in \mathcal{F}_{13}$, and $F''_{23} := (F'' \setminus \{x_5, x_6\}) \cup \{2, 3\} \in \mathcal{F}_{23}$. Then we have $F_{12} \cap F'_{13} \cap F''_{23} = \emptyset$, a contradiction.

The following proof for the general case is given by one of the referees. We assume by contradiction that there are sets $G_i \in \mathcal{G}_I$ such that $|G_1 \cap G_2 \cap G_3| \leq \tau - 1 = 6 - s$. The matrix M has s “taken” places out of 9. “Reserve” one empty spot in each column (these are available since the matrix is acceptable). We are left with $6 - s$ empty spots, say r_i in row i . Shift r_i elements from G_i to the empty spots on row i to construct a new set F_i , no longer belonging to G_{I_i} . By construction, F_1, F_2, F_3 have empty intersection. □

5.1 Case $\mathcal{G}_1 = \emptyset$

In this case, by Claim 17, we have $\mathcal{G}_\emptyset = \mathcal{G}_3 = \mathcal{G}_2 = \mathcal{G}_1 = \emptyset$.

First suppose that $\mathcal{G}_{23} \neq \emptyset$. If $\bigcap G \neq \emptyset$, where the intersection is taken over all $G \in \mathcal{G}_{12} \cup \mathcal{G}_{13} \cup \mathcal{G}_{23}$, then since the family is shifted, $4 \in \bigcap G$. Since $\mathcal{F} \subset \mathcal{F}_{23} \cup \mathcal{F}_{13} \cup \mathcal{F}_{12} \cup \mathcal{F}_{123}$, we have $|F \cap [4]| \geq 3$ for every $F \in \mathcal{F}$. This means that $\mathcal{F} \subset \text{BD}_3(n)$, which contradicts our assumption. Therefore we have $\bigcap G = \emptyset$. Moreover, the families $\mathcal{G}_{12}, \mathcal{G}_{13}, \mathcal{G}_{23}$ are 3-cross intersecting by Claim 18. Thus we can apply (3) of Theorem 7 with $\min\{\epsilon_p : \frac{2}{3} \leq p \leq \frac{1}{2}\} = \frac{4}{25}$ to get

$$x_{12} + x_{13} + x_{23} \leq 3p - \epsilon_p \leq 3p - 0.16.$$

Next suppose that $\mathcal{G}_{23} = \emptyset$. By Lemma 2 we have $x_{12} + x_{13} + x_{23} = x_{12} + x_{13} \leq 1 \leq 3p - 0.2$. Thus in both cases we have $x_{12} + x_{13} + x_{23} \leq 3p - 0.16$. Then it follows from (10) that

$$\begin{aligned} \mu_p(\mathcal{F}) &= p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123} \\ &\leq p^2q(3p - 0.16) + p^3 \\ &= 4p^3q + p^4 - 0.16p^2q. \end{aligned}$$

Noting that $\mu_p(\text{BD}_3(n)) = 4p^3q + p^4$ and $p^2q \geq \frac{12}{125} = 0.96$ for $\frac{2}{5} \leq p \leq \frac{1}{2}$, we have

$$\mu_p(\mathcal{F}) \leq 4p^3q + p^4 - 0.16 \cdot 0.96 < \mu_p(\text{BD}_3(n)) - 0.01,$$

as needed.

5.2 Case $\mathcal{G}_1 \neq \emptyset$ and $\mathcal{G}_2 = \emptyset$

If $\mathcal{G}_{23} = \emptyset$ then $[2, n] \notin \mathcal{F}$. This means that there are $F, F' \in \mathcal{F}$ such that $F \cap F' = \{1\}$. (Otherwise all $F, F' \in \mathcal{F}$ intersect on $[2, n]$ and we could add $[2, n]$ to \mathcal{F} , which contradicts the assumption that \mathcal{F} is size maximal.) In this case all subsets in \mathcal{F} must contain 1, which contradicts the assumption that \mathcal{F} is non-trivial. So we may assume that $\mathcal{G}_{23} \neq \emptyset$. Then both \mathcal{F}_1 and \mathcal{F}_{23} are non-empty, and so the families $\mathcal{F}_{13}, \mathcal{F}_{12}, \mathcal{F}_{123}$ are also non-empty by Claim 17.

By Claim 18 we have the following.

- Claim 19** (1) $\{\mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_{23}\}$ are 3-cross 3-intersecting, and so \mathcal{G}_1 is 2-wise 3-intersecting.
 (2) $\{\mathcal{G}_{12}, \mathcal{G}_{13}, \mathcal{G}_{23}\}$ are 3-cross intersecting, and so $\{\mathcal{G}_{12}, \mathcal{G}_{13}\}$ are 2-cross intersecting.
 (3) $\{\mathcal{G}_1, \mathcal{G}_{23}, \mathcal{G}_{123}\}$ are 3-cross intersecting, and so both $\{\mathcal{G}_1, \mathcal{G}_{123}\}$ and $\{\mathcal{G}_{23}, \mathcal{G}_{123}\}$ are 2-cross intersecting.
 (4) $\{\mathcal{G}_1, \mathcal{G}_{12}, \mathcal{G}_{23}\}$ are 3-cross 2-intersecting.
 (5) $\{\mathcal{G}_1, \mathcal{G}_{23}, \mathcal{G}_{23}\}$ are 3-cross 2-intersecting, and so \mathcal{G}_{23} is 2-wise 2-intersecting.

Claim 20 (1) $\min\{x_1, x_{23}\} \leq \tilde{\alpha}^3$ (see Claim 5 for the definition of $\tilde{\alpha}$).

- (2) $x_{12} + cx_{13} \leq p(c + 1)$, where $c = \frac{1}{2p}(1 + \sqrt{1 - 4p^2})$.
 (3) $x_1 + x_{123} \leq 1$.
 (4) $x_{23} + x_{123} \leq 1$.
 (5) $x_1 + x_{12} + x_{23} \leq 1$.
 (6) $x_{23} \leq \tilde{a}_2$ (see Claim 7 for the definition of \tilde{a}_i).
 (7) $x_1 \leq \tilde{a}_3$.

Proof Item (1): By Lemma 4 with (1) of Claim 19, we have $x_1^2x_{23} \leq \alpha^9$. Then, using $\alpha \leq \tilde{\alpha}$ from Claim 5, we get $(\min\{x_1, x_{23}\})^3 \leq x_1^2x_{23} \leq \alpha^9 \leq \tilde{\alpha}^9$.

Item (2): By (i) of Lemma 2 with (2) of Claim 19, we have $x_{13} + x_{12} \leq 1$. Moreover, by (ii) of Lemma 2 with $x_{13} \leq x_{12}$ from Claim 17, we have $x_{13}^2 \leq x_{13}x_{12} \leq p^2$ and $x_{13} \leq p$. By solving $x_{13}x_{12} = p^2$ and $x_{13} + x_{12} = 1$ with $x_{13} \leq x_{12}$ we get $(x_{13}, x_{12}) = (\frac{1}{2}(1 - \sqrt{1 - 4p^2}), \frac{1}{2}(1 + \sqrt{1 - 4p^2})) = (1 - cp, cp)$. Also, by solving $x_{13}x_{12} = p^2$ and $x_{13} = x_{12}$ we get $(x_{13}, x_{12}) = (p, p)$. Thus (x_{13}, x_{12}) exists

Fig. 3 Item (2) of Claim 20: (x_{13}, x_{12}) is included in the gray area

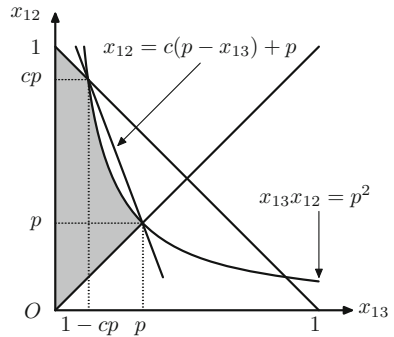


Table 4 Subcase $x_1 \leq x_{23}$

	x_1	x_{23}	x_{13}	x_{23}	x_{123}
y_0	1	-1			0
y_1	1				$\tilde{\alpha}^3$
y_2			c	1	$p(c + 1)$
y_3	1				1
y_4		1			1
y_5	1	1		1	1
y_6		1			$\tilde{\alpha}_2$
	pq^2	p^2q	p^2q	p^2q	p^3

only under the line connecting these two points, that is, $x_{12} \leq c(p - x_{13}) + p$ (See Fig. 3). □

- Items (3) and (4): These follow from (i) of Lemma 2 with (3) of Claim 19.
 - Items (5): This follows from Lemma 3 with (4) of Claim 19.
 - Items (6): This follows from Claim 7 with (5) of Claim 19.
 - Items (7): This follows from Claim 7 with (1) of Claim 19.
- Recall from (10) that $\mu_p(\mathcal{F}) = pq^2x_1 + p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123}$.

5.2.1 Subcase $x_1 \leq x_{23}$.

We solve the following linear programming problem:

maximize: $pq^2x_1 + p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123}$,
subject to: $x_1 - x_{23} \leq 0$, (1)–(6) in Claim 20, and $x_i \geq 0$ for all i .

The corresponding dual problem is (Table 4)

minimize: $\tilde{\alpha}^3y_1 + p(c + 1)y_2 + y_3 + y_4 + y_5 + \tilde{\alpha}_2y_6$,
subject to: $y_0 + y_1 + y_3 + y_5 \geq pq^2$, $-y_0 + y_4 + y_5 + y_6 \geq p^2q$, $cy_2 \geq p^2q$,
 $y_2 + y_5 \geq p^2q$, $y_3 + y_4 \geq p^3$, and $y_i \geq 0$ for all i .

A feasible solution is given by $y_0 = y_6 = 0$, $y_1 = pq^2 - p^2q(1 - \frac{2}{c}) - p^3$, $y_2 = y_4 = \frac{p^2q}{c}$, $y_3 = p^3 - \frac{p^2q}{c}$, $y_5 = p^2q(1 - \frac{1}{c})$, and the corresponding value of

the objective function is

$$p \left(p + p^2 - p^3 + \tilde{\alpha}^3(1 - 3p + p^2) \right) - \frac{1}{c} p^2 q(1 - 2\tilde{\alpha}^3 - p). \tag{11}$$

By the Taylor expansion of $\frac{1}{c}$ at $p = \frac{2}{5}$ it follows that $\frac{1}{c} > d(p)$, where $d(p) = \frac{1375p^2}{216} - \frac{325p}{108} + \frac{37}{54}$ for $\frac{2}{5} \leq p \leq \frac{1}{2}$. Since $1 - 2\tilde{\alpha}^3 - p > 0$ in this interval, the value (11) satisfies

$$\begin{aligned} &< p \left(p + p^2 - p^3 + \tilde{\alpha}^3(1 - 3p + p^2) \right) - d(p)p^2q(1 - 2\tilde{\alpha}^3 - p) =: f(p) \\ &\approx -0.00633167p + 0.490037p^2 + 3.71975p^3 - 8.76595p^4 \\ &\quad + 21.3084p^5 - 61.7755p^6 + 95.9036p^7 - 52.7438p^8. \end{aligned}$$

Let $g(p) := (4p^3q + p^4 - 0.00194) - f(p)$, and let $g^{(i)}(p)$ denote the i -th derivative. Let $\frac{2}{5} \leq p \leq \frac{1}{2}$. We have $g^{(6)}(p) \approx 1063314.937p^2 - 483354.3468p + 44478.39041$ and $g^{(6)}(p) > 0$. Thus $g^{(4)}(p)$ is convex. Since $g^{(4)}(\frac{2}{5}) < -213 < 0$ and $g^{(4)}(\frac{1}{2}) < -112 < 0$, we have $g^{(4)}(p) < 0$. Thus $g^{(3)}(p)$ is decreasing in p . Since $g^{(3)}(\frac{2}{5}) < -7 < 0$ we have $g^{(3)}(p) < 0$. Thus $g'(p)$ is concave. Since $g'(\frac{2}{5}) > 0.23 > 0$ and $g'(\frac{1}{2}) > 0.34 > 0$, we have $g'(p) > 0$ and $g(p)$ is increasing in p . Finally we have $g(\frac{2}{5}) > 3 \times 10^{-6} > 0$, and so $g(p) > 0$. This means that the value (11) is less than $4p^3q + p^4 - 0.00194$.

Then by the weak duality theorem we have $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.00194$.

5.2.2 Subcase $x_{23} \leq x_1$.

We solve the following linear programming problem:

maximize: $pq^2x_1 + p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123}$,
subject to: $x_{23} - x_1 \leq 0$, (1)–(5) and (7) in Claim 20, and $x_I \geq 0$ for all I .

The corresponding dual problem is (Table 5)

minimize: $\tilde{\alpha}^3y_1 + p(c + 1)y_2 + y_3 + y_4 + y_5 + \tilde{a}_3y_7$,
subject to: $-y_0 + y_3 + y_5 + y_7 \geq pq^2$, $y_0 + y_1 + y_4 + y_5 \geq p^2q$, $cy_2 \geq p^2q$,
 $y_2 + y_5 \geq p^2q$, $y_3 + y_4 \geq p^3$, and $y_i \geq 0$ for all i .

A feasible solution is given by $y_0 = y_4 = 0$, $y_1 = y_2 = p^2q(1 - \frac{1}{c})$, $y_3 = p^3$, $y_5 = p^2q(1 - \frac{1}{c})$, $y_7 = pq^2 - p^2q(1 - \frac{1}{c}) - p^3$. Noting that $c + \frac{1}{c} = p$, the corresponding value of the objective function is

$$p^3 \left(-\tilde{\alpha}^3 + \tilde{a}_3 - 1 \right) + p^2 \left(\tilde{\alpha}^3 - 3\tilde{a}_3 + 2 \right) + \tilde{a}_3p - \frac{1}{c} p^2 q \left(\tilde{\alpha}^3 - \tilde{a}_3 + 2p + 1 \right).$$

Table 5 Subcase $x_{23} \leq x_1$

	x_1	x_{23}	x_{13}	x_{23}	x_{123}
y_0	-1	1			0
y_1		1			$\tilde{\alpha}^3$
y_2			c	1	$p(c + 1)$
y_3	1				1
y_4		1			1
y_5	1	1		1	1
y_7	1				$\tilde{\alpha}_3$
	pq^2	p^2q	p^2q	p^2q	p^3

Then, using $\tilde{\alpha}^3 - \tilde{\alpha}_3 + 2p + 1 > 0$ and $\frac{1}{c} > d(p)$ (see the previous subsection), the above value satisfies

$$\begin{aligned}
 &< -1.50324p + 8.79901p^2 - 3.86493p^3 - 26.8212p^4 \\
 &\quad + 30.6062p^5 + 12.8608p^6 - 47.9518p^7 + 26.3719p^8 =: f(p).
 \end{aligned}$$

Let $g(p) := (4p^3q + p^4 - 0.00182) - f(p)$. Let $J_1 = \{p \in \mathbb{R} : \frac{2}{5} \leq p \leq \frac{9}{20}\}$, $J_2 = \{p \in \mathbb{R} : \frac{9}{20} \leq p \leq \frac{1}{2}\}$, and $J = J_1 \cup J_2$. We need to show that $g(p) > 0$ for $p \in J$.

First we show that $g^{(4)}(p) > 0$ for $p \in J_1$. We have $g^{(7)}(p) < 0$ for $p \in J$. Thus $g^{(5)}(p)$ is concave. Since $g^{(5)}(\frac{2}{5}) > 0$ and $g^{(5)}(\frac{9}{20}) > 0$ we have $g^{(5)}(p) > 0$ for $p \in J_1$. Thus $g^{(4)}(p)$ is increasing in $p \in J_1$. Since $g^{(4)}(\frac{9}{20}) < 0$, we have $g^{(4)}(p) < 0$ for $p \in J_1$.

Next we show that $g^{(4)}(p) > 0$ for $p \in J_2$. Since $g^{(7)}(p) < 0$, $g^{(6)}(p)$ is decreasing in p . Since $g^{(6)}(\frac{9}{20}) < 0$, $g^{(4)}(p)$ is concave for $p \in J_2$. Let $h(p) = 481.064p - 381.36$ be the tangent to $g^{(4)}(p)$ at $p = \frac{9}{20}$. Then we have $g^{(4)}(p) \leq h(p) \leq h(\frac{1}{2})$ for $p \in J_2$. Since $h(\frac{1}{2}) < 0$, we have $g^{(4)}(p) < 0$ for $p \in J_2$.

Let $p \in J$. We have shown that $g^{(4)}(p) > 0$. Then $g^{(2)}(p)$ is concave. Since $g^{(2)}(\frac{2}{5}) > 0$ and $g^{(2)}(\frac{1}{2}) > 0$, we have $g^{(2)}(p) > 0$. Thus $g'(p)$ is increasing in p . Since $g'(\frac{2}{5}) > 0$, we have $g'(p) > 0$ and $g(p)$ is increasing in p . Since $g(\frac{2}{5}) > 0$, we have $g(p) > 0$ as needed.

Thus it follows from the weak duality theorem that $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.0018$.

5.3 Case $\mathcal{G}_2 \neq \emptyset, \mathcal{G}_3 = \emptyset$

Using Claim 18 we have the following.

- Claim 21**
- (1) $\{\mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_2\}$ are 3-cross 4-intersecting, and so \mathcal{G}_1 is 2-wise 4-intersecting.
 - (2) $\{\mathcal{G}_2, \mathcal{G}_{13}, \mathcal{G}_{13}\}$ are 3-cross 2-intersecting, and so \mathcal{G}_{13} is 2-wise 2-intersecting.
 - (3) $\{\mathcal{G}_2, \mathcal{G}_{13}, \mathcal{G}_{123}\}$ are 3-cross intersecting, and so $\{\mathcal{G}_{13}, \mathcal{G}_{123}\}$ are 2-cross intersecting.
 - (4) $\{\mathcal{G}_2, \mathcal{G}_{13}, \mathcal{G}_{13}\}$ are 3-cross 2-intersecting.
 - (5) $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_{123}\}$ are 3-cross 2-intersecting.

Table 6 Case $\mathcal{G}_2 \neq \emptyset$

	x_2	x_1	x_{23}	x_{13}	x_{23}	x_{123}
y_1		1				$\tilde{\alpha}_4$
y_2				1		$\tilde{\alpha}_2$
y_3	1					$\tilde{\alpha}^4$
y_4				1		1
y_5		1	1		1	1
y_6	1			1	1	1
y_7	1	1				1
	pq^2	pq^2	p^2q	p^2q	p^2q	p^3

Claim 22 (1) $x_1 \leq \tilde{\alpha}_4$.

(2) $x_{13} \leq \tilde{\alpha}_2$.

(3) $x_2 \leq \tilde{\alpha}^4$.

(4) $x_{13} + x_{123} \leq 1$.

(5) $x_1 + x_{12} + x_{23} \leq 1$.

(6) $x_2 + x_{12} + x_{13} \leq 1$.

(7) $x_1 + x_2 + x_{123} \leq 1$.

Proof Item (3): By Lemma 4 with (1) of Claim 21, we have $x_1^2 x_2 \leq \alpha^{12}$. Then, using $x_2 \leq x_1$ from Claim 17 and Claim 5, we get $x_2^3 \leq x_1^2 x_2 \leq \alpha^{12} \leq \tilde{\alpha}^{12}$.

Item (5) is from Claim 20. Indeed all inequalities in Claim 20 are still valid in this case.

The other items follow from Claim 21, Claim 7, Lemma 2, and Lemma 3. □

We solve the following linear programming problem:

maximize: $pq^2(x_1 + x_2) + p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123}$,
subject to: (1)–(7) in Claim 22, and $x_I \geq 0$ for all I .

The corresponding dual problem is (Table 6)

minimize: $\tilde{\alpha}_4 y_1 + \tilde{\alpha}_2 y_2 + \tilde{\alpha}^4 y_3 + y_4 + y_5 + y_6 + y_7$,
subject to: $y_3 + y_6 + y_7 \geq pq^2$, $y_1 + y_5 + y_7 \geq pq^2$, $y_5 \geq p^2q$, $y_2 + y_4 + y_6 \geq p^2q$,
 $y_5 + y_6 \geq p^2q$, $y_4 + y_7 \geq p^3$, and $y_i \geq 0$ for all i .

A feasible solution is given by $y_1 = pq(1 - 2p)$, $y_2 = p^2(1 - 2p)$, $y_3 = pq^2$, $y_4 = p^3$, $y_5 = p^2q$, $y_6 = y_7 = 0$. Then the corresponding value of the objective function is

$$\begin{aligned} & \tilde{\alpha}_4 pq(1 - 2p) + \tilde{\alpha}_2 p^2(1 - 2p) + \tilde{\alpha}^4 pq^2 + p^2 \\ & \approx -1.70723p + 9.39501p^2 - 10.639p^3 - 1.74811p^4 \\ & \quad + 13.3146p^5 - 16.3729p^6 + 6.6536p^7 \\ & < 4p^3q + p^4 - 0.00436, \end{aligned}$$

and so $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.004$.

5.4 Case $\mathcal{G}_3 \neq \emptyset, \mathcal{G}_\emptyset = \emptyset$

Using Claim 18 we have the following.

Claim 23 (1) $\{\mathcal{G}_3, \mathcal{G}_{12}, \mathcal{G}_{12}\}$ are 3-cross 2-intersecting, and so \mathcal{G}_{12} is 2-wise 2-intersecting.

(2) $\{\mathcal{G}_3, \mathcal{G}_{12}, \mathcal{G}_{123}\}$ are 3-cross intersecting, and so $\{\mathcal{G}_{12}, \mathcal{G}_{123}\}$ are 2-cross intersecting.

Claim 24 (1) $x_1 \leq \tilde{a}_4$.

(2) $x_{12} \leq \tilde{a}_2$.

(3) $x_2 \leq \tilde{\alpha}^4$.

(4) $x_{12} + x_{123} \leq 1$.

(5) $x_1 + x_{12} + x_{23} \leq 1$.

(6) $x_2 + x_{12} + x_{13} \leq 1$.

(7) $x_1 + x_2 + x_{123} \leq 1$.

(8) $x_3 - x_2 \leq 0$.

(9) $x_{23} - x_{13} \leq 0$.

(10) $x_{13} - x_{12} \leq 0$.

Proof Items (1), (3), (6), and (7) are from Claim 21. Items (2) and (4) follow from Claim 23, Claim 7, and Lemma 2. Item (5) is from Claim 19. The other items are from Claim 17. □

We solve the following linear programming problem:

maximize: $pq^2(x_1 + x_2 + x_3) + p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123}$,

subject to: (1)–(10) in Claim 24, and $x_I \geq 0$ for all I .

The corresponding dual problem is (Table 7)

minimize: $\tilde{a}_4y_1 + \tilde{a}_2y_2 + \tilde{\alpha}^4y_3 + y_4 + y_5 + y_6 + y_7$,

subject to: $y_8 \geq pq^2, y_3 + y_6 + y_7 - y_8 \geq pq^2, y_1 + y_5 + y_7 \geq pq^2, y_5 + y_9 \geq p^2q, y_6 - y_9 + y_{10} \geq p^2q, y_2 + y_4 + y_5 + y_6 - y_{10} \geq p^2q, y_4 + y_7 \geq p^3$, and $y_i \geq 0$ for all i .

We distinguish the following two subcases.

5.4.1 Subcase $\frac{2}{5} \leq p \leq 0.453264$

A feasible solution is given by $y_1 = pq^2, y_2 = p^2(3 - 4p), y_3 = 2pq^2, y_4 = p^3, y_5 = y_6 = y_7 = 0, y_8 = pq^2, y_9 = p^2q, y_{10} = 2p^2q$, and the corresponding value of the objective function is

$$\begin{aligned} & p \left(\tilde{a}_4q^2 + \tilde{a}_2(3 - 4p)p + 2\tilde{\alpha}^4q^2 + p^2 \right) \\ & \approx -1.70606p + 4.43558p^2 + 5.72241p^3 - 16.7467p^4 \\ & \quad + 26.6293p^5 - 32.7457p^6 + 13.3072p^7 \\ & < 4p^3q + p^4 - 0.00404377. \end{aligned}$$

Table 7 Case $\mathcal{G}_3 \neq \emptyset$

	x_3	x_2	x_1	x_{23}	x_{13}	x_{23}	x_{123}
y_1			1				$\tilde{\alpha}_4$
y_2						1	$\tilde{\alpha}_2$
y_3		1					$\tilde{\alpha}^4$
y_4						1	1
y_5			1	1		1	1
y_6		1			1	1	1
y_7		1	1				1
y_8	1	-1					0
y_9				1	-1		0
y_{10}					1	-1	0
	pq^2	pq^2	pq^2	p^2q	p^2q	p^2q	p^3

Thus $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.004$.

5.4.2 Subcase $0.453264 \leq p \leq \frac{1}{2}$

A feasible solution is given by $y_1 = y_6 = y_9 = 0$, $y_2 = p(1 - 2p)$, $y_3 = pq$, $y_4 = p(3p - p^2 - 1)$, $y_5 = y_{10} = p^2q$, $y_7 = pq(1 - 2p)$, $y_8 = pq^2$, and the corresponding value of the objective function is

$$\begin{aligned}
 & p \left(\tilde{\alpha}_2(1 - 2p) + \tilde{\alpha}^4q + p \right) \\
 & \approx -1.10283p + 6.37415p^2 - 5.84563p^3 - 3.59536p^4 + 9.71927p^5 - 6.6536p^6 \\
 & < 4p^3q + p^4 - 0.00404377.
 \end{aligned}$$

Thus $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.004$.

5.5 Case $\mathcal{G}_\emptyset \neq \emptyset$

Using Claim 18 we have the following.

- Claim 25** (1) $\{\mathcal{G}_\emptyset, \mathcal{G}_\emptyset, \mathcal{G}_\emptyset\}$ are 3-cross 7-intersecting, that is, \mathcal{G}_\emptyset is 3-wise 7-intersecting.
 (2) $\{\mathcal{G}_1, \mathcal{G}_1, \mathcal{G}_\emptyset\}$ are 3-cross 5-intersecting, and so \mathcal{G}_1 is 2-wise 5-intersecting.
 (3) $\{\mathcal{G}_\emptyset, \mathcal{G}_{123}, \mathcal{G}_{123}\}$ are 3-cross intersecting, and so $\{\mathcal{G}_{123}, \mathcal{G}_{123}\}$ are 2-cross intersecting.

- Claim 26** (1) $x_\emptyset \leq \tilde{\alpha}^7$.
 (2) $x_2 \leq \tilde{\alpha}^4$.
 (3) $x_1 \leq \tilde{\alpha}_5$.
 (4) $x_{123} \leq p$.
 (5) $x_1 + x_{12} + x_{23} \leq 1$.
 (6) $x_3 - x_2 \leq 0$.

Table 8 Case $\mathcal{G}_\emptyset \neq \emptyset$

	x_\emptyset	x_3	x_2	x_1	x_{23}	x_{13}	x_{23}	x_{123}	
y_1	1								$\tilde{\alpha}^7$
y_2			1						$\tilde{\alpha}^4$
y_3				1					$\tilde{\alpha}^3$
y_4								1	p
y_5				1	1		1		1
y_6		1	-1						0
y_7					1	-1			0
y_8						1	-1		0
y_9							1	-1	0
	q^3	pq^2	pq^2	pq^2	p^2q	p^2q	p^2q	p^3	

(7) $x_{23} - x_{13} \leq 0.$

(8) $x_{13} - x_{12} \leq 0.$

(9) $x_{12} - x_{123} \leq 0.$

Proof Items (1), (3), and (4) follow from Claim 25, Lemma 4, Claim 5, Claim 7, and Lemma 2. Items (2) and (5) are from Claim 22 and Claim 20, respectively. The other items are from Claim 17. □

We solve the following linear programming problem:

maximize: $q^3x_\emptyset + pq^2(x_1 + x_2 + x_3) + p^2q(x_{12} + x_{13} + x_{23}) + p^3x_{123},$
subject to: (1)–(9) in Claim 26, and $x_I \geq 0$ for all $I.$

The corresponding dual problem is (Table 8)

minimize: $\tilde{\alpha}^7y_1 + \tilde{\alpha}^4y_2 + \tilde{a}_5y_3 + py_4 + y_5,$
subject to: $y_1 \geq q^3, y_6 \geq pq^2, y_2 - y_6 \geq pq^2, y_3 + y_5 \geq pq^2, y_5 + y_7 \geq p^2q,$
 $-y_7 + y_8 \geq p^2q, y_5 - y_8 + y_9 \geq p^2q, y_4 - y_9 \geq p^3,$ and $y_i \geq 0$ for all $i.$

We distinguish the following two subcases.

5.5.1 Subcase $\frac{2}{5} \leq p \leq 0.424803$

A feasible solution is given by $y_1 = q^3, y_2 = 2pq^2, y_3 = y_6 = pq^2, y_4 = p^2(3-2p), y_5 = 0, y_7 = p^2q, y_8 = 2p^2q, y_9 = 3p^2q,$ and the corresponding value of the objective function is

$$\begin{aligned} &\tilde{\alpha}^7q^3 + 2\tilde{\alpha}^4pq^2 + \tilde{a}_5pq^2 + p^3(3 - 2p) \\ &\approx -27.5644p^{10} + 104.919p^9 - 157.051p^8 + 132.065p^7 - 82.6061p^6 + 39.2544p^5 \\ &\quad - 7.70344p^4 - 6.68845p^3 + 8.19629p^2 - 1.82104p - 7.41682 \cdot 10^{-6} \\ &< 4p^3q + p^4 - 0.004322. \end{aligned}$$

Thus $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.004.$

5.5.2 Subcase $0.424803 \leq p \leq \frac{1}{2}$

A feasible solution is given by $y_1 = q^3, y_2 = 2pq^2, y_3 = pq(1 - 2p), y_4 = p^2, y_5 = y_8 = y_9 = p^2q, y_6 = pq^2, y_7 = 0$, and the corresponding value of the objective function is

$$\begin{aligned} & \tilde{\alpha}^7 q^3 + 2\tilde{\alpha}^4 pq^2 + \tilde{a}_5 pq(1 - 2p) + p^2 \\ & \approx -27.5644p^{10} + 104.919p^9 - 157.051p^8 + 132.065p^7 - 82.6061p^6 + 39.2544p^5 \\ & \quad - 1.05572p^4 - 16.16p^3 + 11.0202p^2 - 1.82104p - 7.41682 \cdot 10^{-6} \\ & < 4p^3q + p^4 - 0.004322. \end{aligned}$$

Thus $\mu_p(\mathcal{F}) < 4p^3q + p^4 - 0.004$.

This completes the proof of Theorem 6. □

6 Concluding remarks

In this section we discuss possible extensions and related problems.

6.1 Non-trivial r -wise intersecting families for $r \geq 4$

We have determined $M_2(p)$ and $M_3(p)$ for all p . Let us consider $M_r(p)$ for the general case $r \geq 4$. Some of the facts we used for the cases $r = 2, 3$ can be easily extended for the other cases as follows.

Proposition 1 *Let $r \geq 2$.*

- (1) *For $s = 0, 1, \dots, r - 1$ we have $M_r(p) \geq p^s$ for $p > \frac{r-s-1}{r-s}$.*
- (2) *$M_r(p) = p^{r-1}$ for $0 < p \leq \frac{1}{r}$.*
- (3) *$M_r(p) = p$ for $\frac{r-2}{r-1} < p \leq \frac{r-1}{r}$.*
- (4) *$M_r(p) = 1$ for $\frac{r-1}{r} < p < 1$.*

Proof Item (1): We construct a non-trivial r -wise intersecting family

$$\mathcal{F}_r(n, s) := \{[s] \cup G : G \subset [s + 1, n], |G| \geq \frac{r-s-1}{r-s}n\} \cup \{[n] \setminus \{i\} : 1 \leq i \leq s\}.$$

Then $\mu_p(\mathcal{F}_r(n, s)) \rightarrow p^s$ as $n \rightarrow \infty$, cf. [11].

Item (2): By item (1) with $s = r - 1$ we have $M_r(p) \geq p^{r-1}$. On the other hand, a non-trivial r -wise intersecting family is 2-wise $(r - 1)$ -intersecting, and by Theorem 11 the p -measure of the family is at most p^{r-1} if $p \leq \frac{1}{r}$.

Item (3): By item (1) with $s = 1$ we have $M_r(p) \geq p$. On the other hand, it is known from [8, 10, 22] that r -wise intersecting family has p -measure at most p if $p \leq \frac{r-1}{r}$.

Item (4): By item (1) with $s = 0$ we have $M_r(p) \geq 1$, and so $M_r(p) = 1$ by definition of $M_r(p)$. □

Even for the case $r = 4$ the exact value of $M_4(p)$ is not known for $\frac{1}{4} < p \leq \frac{2}{3}$. In this case Proposition 1 and Theorem 1 give us the following. For simplicity here we write $\text{bd}_r(p) = \lim_{n \rightarrow \infty} \mu_p(\text{BD}_r(n))$ and $f_r(p, s) = \lim_{n \rightarrow \infty} \mu_p(\mathcal{F}_r(n, s))$.

Fact 1 *For non-trivial 4-wise intersecting families we have the following:*

$$M_4(p) \begin{cases} = f_4(p, 3) = p^3 & \text{if } 0 < p \leq \frac{1}{4}, \\ \geq \text{bd}_4(p) = 5p^4q + p^5 & \text{if } \frac{1}{4} \leq p < \frac{1}{2}, \\ = \text{bd}_4(p) = 5p^4q + p^5 & \text{if } p = \frac{1}{2}, \\ \geq f_4(p, 2) = p^2 & \text{if } \frac{1}{2} < p \leq \frac{1+\sqrt{17}}{8}, \\ \geq \text{bd}_4(p) = 5p^4q + p^5 & \text{if } \frac{1+\sqrt{17}}{8} < p \leq \frac{2}{3}, \\ = f_4(p, 1) = p & \text{if } \frac{2}{3} < p \leq \frac{3}{4}, \\ = 1 & \text{if } \frac{3}{4} < p < 1. \end{cases}$$

Conjecture 2 *For $r \geq 2$ it holds that $M_r(p) = \text{bd}_r(p)$ for $\frac{1}{r} \leq p \leq \frac{1}{2}$.*

It is known that $M_r(p) = \text{bd}_r(p)$ if $r \geq 13$ and $\frac{1}{2} \leq p \leq \frac{1}{2} + \epsilon_r$ for some $\epsilon_r > 0$, see [11]. Note also that $M_5(p) \geq f_5(p, 3) > \text{bd}_5(p)$ for $\frac{1}{2} < p < \frac{1+\sqrt{21}}{10}$.

Problem 1 *Let $0 < p \leq \frac{r-1}{r}$, and let $\mathcal{F} \subset 2^{[n]}$ be a non-trivial r -wise intersecting family. Is it true that*

$$M_r(p) \leq \max\{\text{bd}_r(p), f_r(p, 1), \dots, f_r(p, r - 1)\}?$$

6.2 Uniform families

One can consider non-trivial r -wise intersecting k -uniform families, that is, families in $\binom{[n]}{k} := \{F \subset [n] : |F| = k\}$, and ask the maximum size. Let us construct some candidate families to address this problem. For $1 \leq s \leq r - 1$ and $r - s + 1 \leq y \leq k - s + 1$, let $j_0 := \lceil \frac{(r-s-1)y+1}{r-s} \rceil$. Note that $j_0 < y$ and j_0 is the minimum integer j satisfying $(r - s)j \geq (r - s - 1)y + 1$. Let $\mathcal{F}_r(n, k, s, y) := \mathcal{A} \cup \mathcal{B}$, where

$$\begin{aligned} \mathcal{A} &:= \{A \in \binom{[n]}{k} : [s] \subset A, |A \cap [s + 1, s + y]| \geq j_0\}, \\ \mathcal{B} &:= \{B \in \binom{[n]}{k} : |B \cap [s]| = s - 1, [s + 1, s + y] \subset B\}. \end{aligned}$$

Then the family $\{A \setminus [s] : A \in \mathcal{A}\}$ is $(r - s)$ -wise intersecting due to the choice of j_0 and y . Thus $\mathcal{F}_r(n, k, s, y)$ is a k -uniform non-trivial r -wise intersecting family. In particular,

$$\mathcal{F}_r(n, k, 1, r) = \text{BD}_r(n) \cap \binom{[n]}{k} \quad (j_0 = r - 1),$$

and $\mathcal{F}_r(n, k, r - 1, k - r + 2)$ ($j_0 = 1$) is the so-called Hilton–Milner family. Note that different parameters may give the same family, e.g., $\mathcal{F}_r(n, k, 1, r) = \mathcal{F}_r(n, k, s, r - s + 1)$ for all $1 \leq s \leq r - 1$.

Conjecture 3 (O’Neill and Versträete [17]) *Let $k > r \geq 2$ and $n \geq kr/(r - 1)$. Then the unique extremal non-trivial r -wise intersecting families in $\binom{[n]}{k}$ are $\mathcal{F}_r(n, k, 1, r)$ and $\mathcal{F}_r(n, k, r - 1, k - r + 2)$ (up to isomorphism).*

O’Neill and Versträete proved the conjecture if $n \geq r + e(k^2 2^k)^{2^k} (k - r)$. This bound can be reduced to $n > (1 + \frac{r}{2})(k - r + 2)$ using the Ahlswede–Khachatrian theorem for non-trivial 2-wise t -intersecting families in [3] with the fact that an r -wise intersecting family is 2-wise $(r - 1)$ -intersecting, see [4] for more details. The case $r = 2$ in the conjecture is known to be true as the Hilton–Milner theorem [14]. The case $r = 3$ is studied in [23], and a k -uniform version of Theorem 3 is obtained, from which it follows that the conjecture fails if n and k are sufficiently large and roughly $\frac{1}{2} < \frac{k}{n} \leq \frac{2}{3}$. In this case $\mathcal{F}_3(n, k, 1, k - 1)$ or $\mathcal{F}_3(n, k, 1, k)$ has size larger than $\mathcal{F}_3(n, k, 1, 3)$ and $\mathcal{F}_3(n, k, 2, k - 1)$ (see Theorem 4 in [23]). Balogh and Linz [4] verified that $\mathcal{F}_3(11, 7, 1, 7)$ is indeed a counterexample to the conjecture. They constructed the families $\mathcal{F}_r(n, k, 1, (r - 1)i + 1)$ ($i \leq \lfloor \frac{k-1}{r-1} \rfloor$), and suggested that the largest family of them could be a counterexample if $n \approx kr/(r - 1)$. Here we show that Conjecture 3 fails if $r \geq 3$, and n and k are sufficiently large and k/n is roughly in $(\frac{r-2}{r-1}, \frac{r-1}{r})$. More precisely we prove the following. Let $M_r(n, k)$ denote the maximum size of a non-trivial r -wise intersecting family in $\binom{[n]}{k}$.

Theorem 12 *Let $r \geq 3$. For every $\epsilon > 0$ and every $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all integers n and k with $n > n_0$ and $\frac{r-2}{r-1} + \epsilon < \frac{k}{n} < \frac{r-1}{r} - \epsilon$, we have*

$$(1 - \delta) \binom{n - 1}{k - 1} \leq M_r(n, k) < \binom{n - 1}{k - 1}.$$

Before proving this result, let us check that it gives counterexamples to the conjecture. To this end, suppose that $\frac{k}{n} = p$, and n and k are sufficiently large. Then we have

$$|\mathcal{F}_r(n, k, r, 1)| = (r + 1) \binom{n - r - 1}{k - r} + \binom{n - r - 1}{k - r - 1} \approx ((r + 1)p^{r-1}q + p^r) \binom{n}{k},$$

$$|\mathcal{F}_r(n, k, r - 1, k - r + 2)| = \binom{n - r + 1}{k - r + 1} - \binom{n - k - 1}{k - r + 1} + r - 1 \approx p^{r-1} \binom{n}{k}.$$

If $p > \frac{1}{r}$ then $(r + 1)p^{r-1}q + p^r > p^{r-1}$. Indeed if $k > r$ and $n \leq r(k - r + 2)$, then $|\mathcal{F}_r(n, k, r, 1)| > |\mathcal{F}_r(n, k, r - 1, k - r + 2)|$. If moreover $p = \frac{k}{n} \leq \frac{r-1}{r}$, then

$$\lim_{n, k \rightarrow \infty} |\mathcal{F}_r(n, k, r, 1)| / \binom{n - 1}{k - 1} = (r + 1)p^r q + p^r \leq \frac{8}{9},$$

where equality holds if and only if $r = 3$ and $p = \frac{2}{3}$. This implies that under the assumptions in Theorem 12 we have $\max\{|\mathcal{F}_r(n, k, 1, r)|, |\mathcal{F}_r(n, k, r - 1, k - r + 2)|\} < (1 - \delta) \binom{n - 1}{k - 1}$ for $0 < \delta < \frac{1}{9}$.

Proof of Theorem 12 The upper bound $M_r(n, k) < \binom{n - 1}{k - 1}$ was proved by Frankl in [9].

We prove the lower bound. Let r be fixed, and let $\epsilon > 0$ and $\delta > 0$ be given. Let $\frac{r-2}{r-1} < p < \frac{r-1}{r}$, and $k = pn$. Let $c > 0$ be a constant depending on r only (specified later), and let

$$J_{n,p} = \{j \in \mathbb{N} : |j - p^2n| \leq c\sqrt{n}\}.$$

For $j \in J_{n,p}$ let

$$\theta_j(n, p) = \frac{\binom{pn}{j} \binom{n-pn}{pn-j}}{\binom{n}{pn}} = \frac{\binom{k}{j} \binom{n-k}{k-j}}{\binom{n}{k}}.$$

Let $\text{erf}(z)$ denote the error function, that is,

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2) dx.$$

Then, by Lemma 5 of [23], we have

$$\lim_{n \rightarrow \infty} \sum_{j \in J_{n,p}} \theta_j(n, p) = \text{erf}\left(\frac{3c}{\sqrt{2}p}\right).$$

The RHS is a function decreasing in p (for fixed c), and we have

$$\min_{p \in [\frac{r-2}{r-1}, \frac{r-1}{r}]} \text{erf}\left(\frac{3c}{\sqrt{2}p}\right) = \text{erf}\left(\frac{3rc}{\sqrt{2}(r-1)}\right).$$

Then the RHS is a function increasing in c and approaching 1. Thus we can choose $c > 0$ so that $\text{erf}\left(\frac{3rc}{\sqrt{2}(r-1)}\right) > 1 - \frac{\delta}{3}$, and we fix c .

We will show that $|\mathcal{F}_r(n, k, 1, k)| > (1 - \delta)\binom{n-1}{k-1}$. Let $j_0 = \lceil \frac{(r-2)k+1}{r-1} \rceil$. Choose n_1 so that if $n > n_1$ then

$$\sum_{j \in J_{n,p}} \theta_j(n, p) > 1 - \frac{\delta}{2} \tag{12}$$

for all p with $\frac{r-2}{r-1} \leq p \leq \frac{r-1}{r}$. Next choose n_2 so that if $\frac{r-2}{r-1} + \epsilon < p < \frac{r-1}{r} - \epsilon$, $n > n_2$, and $k = pn$, then $j_0 < pk - c\sqrt{n}$ and $pk + c\sqrt{n} < k - 1$. Then we have $J_{n,p} \subset [j_0, k - 1]$. Finally choose n_3 so that if $n > n_3$ then $p - \frac{c}{q\sqrt{n}} > (1 - \frac{\delta}{2})p$, and let $n_0 := \max\{n_1, n_2, n_3\}$.

We have

$$|\mathcal{F}_r(n, k, 1, k)| \geq \sum_{j=j_0}^{k-1} \binom{k}{j} \binom{n-k-1}{k-j-1} > \sum_{j \in J_{n,p}} \binom{k}{j} \binom{n-k-1}{k-j-1}.$$

The summands in the RHS is $\binom{k}{j} \binom{n-k-1}{k-j-1} = \frac{k-j}{n-k} \binom{k}{j} \binom{n-k}{k-j}$. For $j \in J_{n,p}$ we have $j < p^2n + c\sqrt{n}$ and

$$\frac{k - j}{n - k} = \frac{p - \frac{j}{n}}{1 - p} > \frac{1}{q} \left(p - p^2 - \frac{c}{\sqrt{n}} \right) = p - \frac{c}{q\sqrt{n}} > \left(1 - \frac{\delta}{2} \right) p,$$

where we used $n > n_3$ in the last inequality. We then have

$$\binom{k}{j} \binom{n - k - 1}{k - j - 1} > \left(1 - \frac{\delta}{2} \right) p \binom{k}{j} \binom{n - k}{k - j}.$$

The RHS can be rewritten as $\left(1 - \frac{\delta}{2} \right) \binom{n-1}{k-1} \theta_j(n, p)$ because $\binom{k}{j} \binom{n-k}{k-j} = \theta_j(n, p) \binom{n}{k}$ and $p \binom{n}{k} = \binom{n-1}{k-1}$. Finally we have

$$\begin{aligned} M_r(n, k) &\geq |\mathcal{F}_r(n, k, 1, k)| \\ &> \sum_{j \in J_{n,p}} \binom{k}{j} \binom{n - k - 1}{k - j - 1} \\ &> \left(1 - \frac{\delta}{2} \right) \binom{n - 1}{k - 1} \sum_{j \in J_{n,p}} \theta_j(n, p) \\ &> \left(1 - \frac{\delta}{2} \right)^2 \binom{n - 1}{k - 1} \quad (\text{by (12)}) \\ &> (1 - \delta) \binom{n - 1}{k - 1}, \end{aligned}$$

and this is the lower bound we needed. □

Fact 1 suggests that the conjecture could be false even if $\frac{k}{n} < \frac{r-2}{r-1}$. For example we have $|\mathcal{F}_4(41, 26, 2, 25)| > |\mathcal{F}_4(41, 26, 1, 4)| > |\mathcal{F}_4(41, 26, 3, 24)|$. Noting that $\frac{1+\sqrt{17}}{8} \approx 0.64$ we can expect $\mathcal{F}_4(n, k, 2, k - 1)$ is larger than $\mathcal{F}_4(n, k, 1, 4)$ if $\frac{1}{2} < \frac{k}{n} < 0.64$ and n, k sufficiently large. Indeed we have

$$\begin{aligned} |\mathcal{F}_4(1000, 514, 2, 513)|/|\mathcal{F}_4(1000, 514, 1, 4)| &\approx 1.03254, \\ |\mathcal{F}_4(1000, 630, 2, 629)|/|\mathcal{F}_4(1000, 630, 1, 4)| &\approx 1.0165, \\ |\mathcal{F}_4(1000, 650, 2, 649)|/|\mathcal{F}_4(1000, 650, 1, 4)| &\approx 0.98655. \end{aligned}$$

Problem 2 Let $k > r \geq 2$ and $n \geq kr/(r - 1)$, and let $\mathcal{F} \subset \binom{[n]}{k}$ be a non-trivial r -wise intersecting family. Is it true that

$$|\mathcal{F}| \leq \max\{|\mathcal{F}_r(n, k, s, y)| : 1 \leq s \leq r - 1, r - s + 1 \leq y \leq k - s + 1\}?$$

Acknowledgements I thank both referees for their careful reading and many helpful suggestions. This research was supported by JSPS KAKENHI 18K03399 and 23K032101.

References

1. Ahlswede, K., Katona, G.O.H.: Contributions to the geometry of Hamming spaces. *Discrete Math.* **17**, 1–22 (1977)
2. Ahlswede, R., Khachatrian, L.H.: The diametric theorem in Hamming spaces-optimal anticodes. *Adv. Appl. Math.* **20**, 429–449 (1998)
3. Ahlswede, R., Khachatrian, L.H.: The complete nontrivial-intersection theorem for systems of finite sets. *J. Comb. Theory Ser. A* **76**(1), 121–138 (1996)
4. Balogh, J., Linz, W.: Short Proofs of Three Results About Intersecting Systems. [arXiv:2104.00778](https://arxiv.org/abs/2104.00778)
5. Brace, A., Daykin, D.E.: A finite set covering theorem. *Bull. Aust. Math. Soc.* **5**, 197–202 (1971)
6. Ellis, D., Keller, N., Lifshitz, N.: Stability for the Complete Intersection Theorem, and the Forbidden Intersection Problem of Erdős and Sós. [arXiv:1604.06135](https://arxiv.org/abs/1604.06135)
7. Filmus, Y.: More complete intersection theorems. *Discrete Math.* **342**, 128–142 (2019)
8. Filmus, Y., Golubev, K., Lifshitz, N.: High dimensional Hoffman bound and applications in extremal combinatorics. *Algebr. Comb.* **4**(6), 1005–1026 (2021)
9. Frankl, P.: The Shifting Technique in Extremal Set Theory. *Surveys in Combinatorics (New Cross)*, London Mathematical Society Lecture Note Series 123, pp. 81–110 (1987)
10. Frankl, P., Tokushige, N.: Weighted multiply intersecting families. *Studia Sci. Math. Hung.* **40**, 287–291 (2003)
11. Frankl, P., Tokushige, N.: Weighted non-trivial multiply intersecting families. *Combinatorica* **26**, 37–46 (2006)
12. Gärtner, B., Matoušek, J.: *Approximation Algorithms and Semidefinite Programming*. xii+251 pp. Springer, Heidelberg (2012)
13. Gupta, P., Mogge, Y., Piga, S., Schülke, B.: r -cross t -intersecting families via necessary intersection points. *Procedia Comput. Sci.* **195**, 453–458 (2021)
14. Hilton, A.J.W., Milner, E.C.: Some intersection theorems for systems of finite sets. *Q. J. Math. Oxf. Ser. (2)* **18**, 369–384 (1967)
15. Kato, M., Kosuda, M., Tokushige, N.: Extending Muirhead’s inequality. *Graphs Comb.* **37**, 1923–1941 (2011)
16. Lee, S.J., Siggers, M., Tokushige, N.: Toward extending the Ahlswede–Khachatrian theorem to cross t -intersecting families. *Discrete Appl. Math.* **216**, 627–645 (2017)
17. O’Neill, J., Versträete, J.: Non-trivial d -wise intersecting families. *J. Comb. Theory Ser. A* **178**, 105369 (2021)
18. Suda, S., Tanaka, H.: A cross-intersection theorem for vector spaces based on semidefinite programming. *Bull. Lond. Math. Soc.* **46**, 342–348 (2014)
19. Suda, S., Tanaka, H., Tokushige, N.: A semidefinite programming approach to a cross-intersection problem with measures. *Math. Program. Ser. A* **166**, 113–130 (2017)
20. Tokushige, N.: Brace–Daykin type inequalities for intersecting families. *Eur. J. Comb.* **29**, 273–285 (2008)
21. Tokushige, N.: On cross t -intersecting families of sets. *J. Comb. Theory A* **117**, 1167–1177 (2010)
22. Tokushige, N.: Application of hypergraph Hoffman’s bound to intersecting families. *Algebr. Comb.* **5**(3), 537–557 (2022)
23. Tokushige, N.: Non-trivial 3-wise intersecting uniform families. *Discrete Math.* **346**(5), Paper No. 113368 (2023)
24. Wagner, A.Z.: Refuting conjectures in extremal combinatorics via linear programming. *J. Comb. Theory A* **169**, 105130 (2020)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.