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Series A

Convergence of a relaxed inertial proximal algorithm for maximally monotone operators

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Abstract
In a Hilbert space $\mathcal{H}$, given $A: \mathcal{H} \to 2^{\mathcal{H}}$ a maximally monotone operator, we study the convergence properties of a general class of relaxed inertial proximal algorithms. This study aims to extend to the case of the general monotone inclusion $Ax \ni 0$ the acceleration techniques initially introduced by Nesterov in the case of convex minimization. The relaxed form of the proximal algorithms plays a central role. It comes naturally with the regularization of the operator A by its Yosida approximation with a variable parameter, a technique recently introduced by Attouch–Peypouquet (Math Program Ser B, 2018. https://doi.org/10.1007/s10107-018-1252-x) for a particular class of inertial proximal algorithms. Our study provides an algorithmic version of the convergence results obtained by Attouch–Cabot (J Differ Equ 264:7138–7182, 2018) in the case of continuous dynamical systems.

Keywords Maximally monotone operators · Yosida regularization · Inertial proximal method · Large step proximal method · Lyapunov analysis · (Over)Relaxation

Mathematics Subject Classification 49M37 · 65K05 · 65K10 · 90C25

1 Introduction

Throughout this paper, $\mathcal{H}$ is a real Hilbert space endowed with the scalar product $\langle ., . \rangle$ and the corresponding norm $\|.\|$. Given $A : \mathcal{H} \to 2^{\mathcal{H}}$ a maximally monotone operator,

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we will study the convergence properties of a general class of inertial proximal based algorithms that aim to solve the inclusion $Ax \ni 0$, whose solution set is denoted by $\operatorname{zer} A$. Given initial data $x_0, x_1 \in \mathcal{H}$, we consider the Relaxed Inertial Proximal Algorithm, (RIPA) for short, defined by, for $k \geq 1$

$$\text{(RIPA)} \quad \begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} = (1-\rho_k)y_k + \rho_k J_{\mu_k A}(y_k). \end{cases}$$

In the above formula, $J_{\mu A} = (I + \mu A)^{-1}$ is the *resolvent* of A with index $\mu > 0$, where I is the identity operator. It plays a central role in the analysis of (RIPA), along with the *Yosida regularization* of A with parameter $\mu > 0$, which is defined by $A_\mu = \frac{1}{\mu}\left(I - J_{\mu A}\right)$, (see "Appendix A" for their main properties). We assume the following set of hypotheses

$$\begin{cases} \bullet\ A : \mathcal{H} \to 2^{\mathcal{H}} \text{ is a maximally monotone operator;} \\ \bullet\ (\alpha_k) \text{ is a sequence of nonnegative numbers;} \\ \bullet\ (\mu_k) \text{ and } (\rho_k) \text{ are sequences of positive numbers.} \end{cases} \tag{H}$$

If $\rho_k = 1$ for every $k \geq 1$, then algorithm (RIPA) reduces to the Inertial Proximal Algorithm

$$\text{(IPA)} \quad \begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} = J_{\mu_k A}(y_k). \end{cases}$$

On the other hand, if $\alpha_k = 0$ for every $k \geq 1$, then algorithm (RIPA) boils down to the Relaxed Proximal Algorithm

$$\text{(RPA)} \quad x_{k+1} = (1-\rho_k)x_k + \rho_k J_{\mu_k A}(x_k).$$

For classical references on relaxed proximal algorithms, see for example [10,18,19]. An inertial version of such algorithms was first studied in [1], see also [22]. Recent studies showed the importance of the case $\alpha_k \to 1$. When $A = \partial\Psi$, where $\Psi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is a proper closed convex function, this is a key property for obtaining fast convergent methods, in line with the Nesterov and FISTA methods [9,24]. The case of inertial methods for general maximally monotone operators remains largely to be explored. Inertial proximal splitting methods were recently considered in [12,17,21,23, 27]. One important application concerns the design of inertial ADMM algorithms for linearly constrained minimization problems. Recently, a new approach was delineated by Attouch and Peypouquet [7] based on the Yosida regularization of A with a varying parameter. In this paper, we will provide a unifying approach to these problems that extends [7] and opens new perspectives. Our study is the natural extension, in the algorithmic case, of the convergence results obtained by Attouch–Cabot [5] in the case of continuous dynamical systems.

1.1 Relaxed proximal algorithms

The classical proximal algorithm is obtained by the implicit discretization of the differential inclusion

$$\dot{x}(t) + A(x(t)) \ni 0. \tag{1}$$

By contrast, the relaxed proximal algorithm (RPA) comes naturally by discretizing the regularized differential equation

$$\dot{x}(t) + A_\lambda(x(t)) = 0, \tag{2}$$

where A_λ is the Yosida approximation of A of index $\lambda > 0$. Since A_λ is Lipschitz continuous, (2) is relevant owing to the Cauchy–Lipschitz theorem. Indeed, implicit time discretization of (2), with step size $h_k > 0$, gives the relaxed proximal algorithm (details of the proof are given below in the inertial case)

$$x_{k+1} = (1 - \rho_k)\, x_k + \rho_k\, J_{\mu_k A}(x_k),$$

with $\mu_k = \lambda + h_k$ and $\rho_k = \frac{h_k}{\lambda + h_k}$. System (2) has many advantages over the differential inclusion (1). Note that $\operatorname{zer} A = \operatorname{zer} A_\lambda$, so the equilibria are the same for both systems. Since A_λ is cocoercive, the trajectories of (2) converge weakly to equilibria, which is a great contrast to the semigroup generated by A, for which, in general we only have weak ergodic convergence. Thus, one can expect that the associated algorithms also benefit from these favorable properties.

These aspects are even more striking when one considers inertial dynamics. The damped inertial dynamics $\ddot{x}(t) + \gamma(t)\dot{x}(t) + A(x(t)) \ni 0$ is ill-posed, and no general convergence theory is available for this system. By contrast, the regularized dynamics

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + A_{\lambda(t)}(x(t)) = 0 \tag{3}$$

is well-posed. Some first results concerning the adjustments of the parameters $\gamma(t)$ and $\lambda(t)$, in order to have good asymptotic convergence properties have been obtained in [2,6] and [7]. A closely connected dynamical system with variable damping and step sizes has been considered in [11].

Let's proceed to the implicit temporal discretization of (3). Indeed, implicit discretizations tend to follow the continuous-time trajectories more closely than explicit discretizations. Note that, due to the Yosida regularization, the explicit discretization of (3) has the same numerical complexity as the implicit discretization (they each need one resolvent computation per iteration). Taking a time step $h_k > 0$, and setting $t_k = \sum_{i=1}^{k} h_i$, $x_k = x(t_k)$, $\lambda_k = \lambda(t_k)$, $\gamma_k = \gamma(t_k)$, an implicit finite-difference scheme for (3) with centered second-order variation gives

$$\frac{1}{h_k^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\gamma_k}{h_k}(x_k - x_{k-1}) + A_{\lambda_k}(x_{k+1}) = 0. \tag{4}$$

After expanding (4), we obtain $x_{k+1} + h_k^2 A_{\lambda_k}(x_{k+1}) = x_k + (1 - \gamma_k h_k)(x_k - x_{k-1})$. Setting $s_k = h_k^2$, we have

$$x_{k+1} = J_{s_k A_{\lambda_k}} \left(x_k + (1 - \gamma_k h_k)(x_k - x_{k-1})\right),$$

where $J_{s_k A_{\lambda_k}}$ is the resolvent of index $s_k > 0$ of the maximally monotone operator A_{λ_k}. Setting $\alpha_k = 1 - \gamma_k h_k$, this gives the following algorithm

$$\begin{cases} y_k = x_k + \alpha_k (x_k - x_{k-1}) \\ x_{k+1} = J_{s_k A_{\lambda_k}}(y_k). \end{cases} \tag{5}$$

The resolvent equation, $(A_\lambda)_s = A_{\lambda+s}$, gives $J_{sA_\lambda} = \frac{\lambda}{\lambda+s} I + \frac{s}{\lambda+s} J_{(\lambda+s)A}$. Hence, (5) is equivalent to

$$\begin{cases} y_k = x_k + \alpha_k (x_k - x_{k-1}) \\ x_{k+1} = \dfrac{\lambda_k}{\lambda_k + s_k} y_k + \dfrac{s_k}{\lambda_k + s_k} J_{(\lambda_k + s_k)A}(y_k). \end{cases}$$

That's algorithm (RIPA) with $\mu_k = \lambda_k + s_k$ and $\rho_k = \frac{s_k}{\lambda_k + s_k}$.

1.2 Geometrical aspects of (RIPA)

(RIPA) has a simple geometrical interpretation. This is illustrated in Fig. 1, where, as a distinctive feature of the proximal method, the closed affine half-space $\left\{z \in \mathcal{H} : \left\langle y_k - J_{\mu_k A}(y_k), z - J_{\mu_k A}(y_k)\right\rangle \le 0\right\}$ separates y_k from $\text{zer} A$.

In the Fig. 1, the parameter ρ_k has been taken between zero and one, so that x_{k+1} belongs to the line segment $[y_k, J_{\mu_k A}(y_k)]$. Indeed, we will see that, under certain conditions, the parameter ρ_k can vary in the interval $[0, 2]$. The case $\rho_k > 1$ is particularly interesting, since it allows to combine inertial effect with over-relaxation. These combined aspects have been little studied until now, see [20] for some recent results in the case of a fixed resolvent operator $J_{\mu A}$.

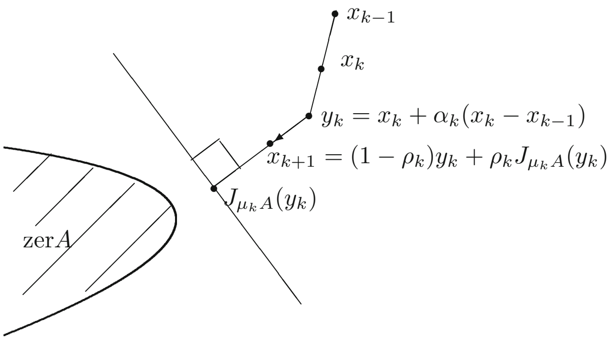

Fig. 1 Algorithm (RIPA)

For a maximally monotone operator A such that $\mathrm{zer}A \neq \emptyset$, we have $\lim_{\mu\to+\infty} J_{\mu A}(x) = \mathrm{proj}_{\mathrm{zer}A}x$, see [8, Theorem 23.47 (i)]. Thus, in the case $\mu_k \to +\infty$ (which, we will see, is important for obtaining fast methods) we have $J_{\mu_k A}(y_k) \sim \mathrm{proj}_{\mathrm{zer}A}y_k$, as shown in Fig. 1.

1.3 Presentation of the results

The case $A = 0$ already reveals some crucial notions. In this case, $J_{\mu_k A} = I$ and algorithm (RIPA) becomes $x_{k+1} = x_k + \alpha_k(x_k - x_{k-1})$. It ensues that for every $k \geq 1$,

$$x_{k+1} - x_k = \left(\prod_{j=1}^{k}\alpha_j\right)(x_1 - x_0) \text{ which gives } x_k = x_1 + \left(\sum_{l=1}^{k-1}\prod_{j=1}^{l}\alpha_j\right)(x_1 - x_0).$$

Hence, (x_k) converges if and only if $x_1 = x_0$ or if the following condition is satisfied $\sum_{l=1}^{+\infty}\prod_{j=1}^{l}\alpha_j < +\infty$. When $A = \partial\Psi$ is the subdifferential of a convex function $\Psi : \mathcal{H} \to \mathbb{R}\cup\{+\infty\}$ with a continuum of minima, this condition has been identified as a necessary condition for the convergence of the iterates of (IPA), see [16]. From now on, we assume that

$$\sum_{l=i}^{+\infty}\left(\prod_{j=i}^{l}\alpha_j\right) < +\infty \quad \text{for every } i \geq 1, \tag{K_0}$$

and we define the sequence (t_i) by

$$t_i = 1 + \sum_{l=i}^{+\infty}\left(\prod_{j=i}^{l}\alpha_j\right). \tag{6}$$

The sequence (t_i) plays a crucial role in the study of the asymptotic behavior of the iterates of (IPA). This was recently highlighted by Attouch and Cabot [4] in the potential case. However, most of the energetical arguments used in [4] are not available in the general framework of maximally monotone operators. It follows that we must develop different techniques.

As a model example of our results, let us give the following shortened version of Theorem 2.6.

Theorem 1.1 *Under (H), assume that* $\mathrm{zer}A \neq \emptyset$. *Suppose that* $\alpha_k \in [0, 1]$ *and* $\rho_k \in]0, 2]$ *for every* $k \geq 1$. *Under* (K_0), *let* (t_i) *be the sequence defined by* (6). *Assume that there exists* $\varepsilon \in]0, 1[$ *such that for* k *large enough,*

$$(1-\varepsilon)\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1}) \\ \geq \alpha_k t_{k+1}\left(1+\alpha_k+\left[\frac{2-\rho_k}{\rho_k}(1-\alpha_k)-\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})\right]_+\right). \tag{K_1}$$

Then for any sequence (x_k) *generated by* (RIPA)*, we have*

(i) $\sum_{i=1}^{+\infty} \alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty$.
(ii) $\sum_{i=1}^{+\infty} \rho_i(2-\rho_i)\, t_{i+1}\|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.
(iii) *For any* $z \in \operatorname{zer} A$, $\lim_{k\to+\infty}\|x_k - z\|$ *exists, and hence* (x_k) *is bounded.*

Assume moreover that $\limsup_{k\to+\infty}\rho_k < 2$, *and* $\liminf_{k\to+\infty}\rho_k > 0$. *Then the following holds*

(iv) $\lim_{k\to+\infty}\mu_k A_{\mu_k}(x_k) = 0$.
(v) *If* $\liminf_{k\to+\infty}\mu_k > 0$, *then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

These results are complemented in Theorem 2.14 so as to cover the case of a possibly vanishing parameter ρ_k. Then, the assumption $\liminf_{k\to+\infty}\rho_k > 0$ is removed and replaced with an alternative set of assumptions.

1.4 Organization of the paper

Our main convergence results are established in Sect. 2, see Theorem 2.6 and Theorem 2.14. Based on the behavior of the sequences (α_k), (μ_k), (ρ_k), we show the weak convergence of the sequences (x_k) generated by algorithm (RIPA). Thus, in the general context of maximally monotone operators acting on Hilbert spaces, we unify and extend most of the previously known results concerning the combination of the proximal methods with relaxation and inertia. These results are illustrated in Sect. 3 which presents applications to special classes of sequences (α_k), (μ_k), (ρ_k). In particular, we find the recent result obtained by Attouch–Peypouquet based on the accelerated method of Nesterov. Finally, in Sect. 4, we provide ergodic convergence results, extending the seminal result of Brezis–Lions. The paper is supplemented by some auxiliary technical lemmas contained in the appendix.

2 Convergence results

2.1 Equivalent forms of (RIPA)

Let us give several equivalent formulations of (RIPA). Observe that

$$(1-\rho_k)y_k + \rho_k J_{\mu_k A}(y_k) = y_k - \rho_k\mu_k A_{\mu_k}(y_k).$$

It ensues that (RIPA) can be equivalently rewritten as

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} = y_k - \rho_k\mu_k A_{\mu_k}(y_k). \end{cases} \tag{7}$$

Recalling that $\xi = A_\mu(y)$ if and only if $\xi \in A(y - \mu\xi)$, we obtain the following equivalences

$$\begin{aligned} &-\frac{1}{\rho_k\mu_k}(x_{k+1} - y_k) = A_{\mu_k}(y_k) \\ &\iff -\frac{1}{\rho_k\mu_k}(x_{k+1} - y_k) \in A\left(y_k + \frac{1}{\rho_k}(x_{k+1} - y_k)\right) \\ &\iff -\frac{1}{\rho_k\mu_k}(x_{k+1} - y_k) \in A\left(x_{k+1} + \left(\frac{1}{\rho_k} - 1\right)(x_{k+1} - y_k)\right). \end{aligned}$$

This gives rise to the equivalent formulation of (RIPA)

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} - y_k \in -\rho_k\mu_k\, A\left(x_{k+1} + \left(\frac{1}{\rho_k} - 1\right)(x_{k+1} - y_k)\right). \end{cases} \tag{8}$$

Depending on the situation, we will use one of the above mentioned equivalent formulations.

2.2 The anchor sequence (h_k)

Given $z \in \mathcal{H}$ and a sequence (x_k) generated by (RIPA), let us define the sequence (h_k) by $h_k = \frac{1}{2}\|x_k - z\|^2$. The difference $h_{k+1} - h_k - \alpha_k(h_k - h_{k-1})$ plays a central role in the study of the asymptotic behavior of (x_k) as $k \to +\infty$. Let us start with a basic result that relies on algebraic manipulations of the terms h_{k-1}, h_k and h_{k+1}.

Lemma 2.1 *Let (x_k) be a sequence in $\mathcal{H}$, and let (α_k) be a sequence of real numbers. Given $z \in \mathcal{H}$, let us define the sequence (h_k) by $h_k = \frac{1}{2}\|x_k - z\|^2$. We then have*

$$\begin{aligned} h_{k+1} - h_k - \alpha_k(h_k - h_{k-1}) &= \frac{1}{2}(\alpha_k + \alpha_k^2)\|x_k - x_{k-1}\|^2 + \langle x_{k+1} - y_k, x_{k+1} - z\rangle \\ &\quad -\frac{1}{2}\|x_{k+1} - y_k\|^2, \end{aligned} \tag{9}$$

where $y_k = x_k + \alpha_k(x_k - x_{k-1})$.

Proof Observe that

$$\begin{aligned} \|y_k - z\|^2 &= \|x_k + \alpha_k(x_k - x_{k-1}) - z\|^2 \\ &= \|x_k - z\|^2 + \alpha_k^2\|x_k - x_{k-1}\|^2 + 2\alpha_k\langle x_k - z, x_k - x_{k-1}\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|x_k - z\|^2 + \alpha_k^2 \|x_k - x_{k-1}\|^2 \\
&\quad + \alpha_k \|x_k - z\|^2 + \alpha_k \|x_k - x_{k-1}\|^2 - \alpha_k \|x_{k-1} - z\|^2 \\
&= \|x_k - z\|^2 + \alpha_k (\|x_k - z\|^2 - \|x_{k-1} - z\|^2) \\
&\quad + (\alpha_k + \alpha_k^2) \|x_k - x_{k-1}\|^2 \\
&= 2[h_k + \alpha_k (h_k - h_{k-1})] + (\alpha_k + \alpha_k^2) \|x_k - x_{k-1}\|^2.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
h_{k+1} - h_k - \alpha_k (h_k - h_{k-1}) &= \frac{1}{2} \|x_{k+1} - z\|^2 - \frac{1}{2} \|y_k - z\|^2 \\
&\quad + \frac{1}{2} (\alpha_k + \alpha_k^2) \|x_k - x_{k-1}\|^2 \\
&= \langle x_{k+1} - y_k, x_{k+1} - z \rangle - \frac{1}{2} \|x_{k+1} - y_k\|^2 \\
&\quad + \frac{1}{2} (\alpha_k + \alpha_k^2) \|x_k - x_{k-1}\|^2.
\end{aligned}
$$

□

Let us particularize Lemma 2.1 to sequences generated by (RIPA). In the following statement, $\operatorname{gph} A$ stands for the graph of A, see "Appendix A".

Lemma 2.2 *Assume* (H) *and let* $(z, q) \in \operatorname{gph} A$. *Given a sequence* (x_k) *generated by* (RIPA)*, let* (h_k) *be the sequence defined by* $h_k = \frac{1}{2}\|x_k - z\|^2$. *Then we have for every* $k \geq 1$,

$$
\begin{aligned}
& h_{k+1} - h_k - \alpha_k (h_k - h_{k-1}) + \rho_k \mu_k \left\langle x_{k+1} + \left(\frac{1}{\rho_k} - 1\right)(x_{k+1} - y_k) - z, q \right\rangle \\
& \leq \frac{1}{2} (\alpha_k + \alpha_k^2) \|x_k - x_{k-1}\|^2 - \frac{2 - \rho_k}{2\rho_k} \|x_{k+1} - y_k\|^2. \qquad (10)
\end{aligned}
$$

Assume moreover that $\operatorname{zer} A \neq \emptyset$*, and let* $z \in \operatorname{zer} A$. *The following holds true for every* $k \geq 1$

$$
h_{k+1} - h_k - \alpha_k (h_k - h_{k-1}) \leq \frac{1}{2} (\alpha_k + \alpha_k^2) \|x_k - x_{k-1}\|^2 - \frac{2 - \rho_k}{2\rho_k} \|x_{k+1} - y_k\|^2. \qquad (11)
$$

Proof Iteration (RIPA) can be expressed as

$$
x_{k+1} - y_k \in -\rho_k \mu_k \, A \left(x_{k+1} + \left(\frac{1}{\rho_k} - 1\right)(x_{k+1} - y_k) \right),
$$

see (8). Since $q \in A(z)$, the monotonicity of A yields

$$
\left\langle x_{k+1} - y_k + \rho_k \mu_k \, q, x_{k+1} + \left(\frac{1}{\rho_k} - 1\right)(x_{k+1} - y_k) - z \right\rangle \leq 0.
$$

Hence

$$
\begin{aligned}
\langle x_{k+1}-y_k, x_{k+1}-z\rangle \leq &-\left(\frac{1}{\rho_k}-1\right)\|x_{k+1}-y_k\|^2\\
&-\rho_k\mu_k\left\langle q, x_{k+1}+\left(\frac{1}{\rho_k}-1\right)(x_{k+1}-y_k)-z\right\rangle.
\end{aligned}
$$

From equality (9) of Lemma 2.1, we deduce immediately that

$$
\begin{aligned}
&h_{k+1}-h_k-\alpha_k(h_k-h_{k-1})+\rho_k\mu_k\left\langle q, x_{k+1}+\left(\frac{1}{\rho_k}-1\right)(x_{k+1}-y_k)-z\right\rangle\\
&\leq \frac{1}{2}(\alpha_k+\alpha_k^2)\|x_k-x_{k-1}\|^2-\left(\frac{1}{\rho_k}-\frac{1}{2}\right)\|x_{k+1}-y_k\|^2,
\end{aligned}
$$

which is nothing but (10). Finally, if $z \in \operatorname{zer} A$, then inequality (11) is obtained by taking $q=0$ in (10). $\square$

Lemma 2.3 *Under (H), assume that $\rho_k \in]0,2]$ for every $k \geq 1$. Suppose that $\operatorname{zer} A \neq \emptyset$ and let $z \in \operatorname{zer} A$. Given a sequence (x_k) generated by (RIPA), let (h_k) be the sequence defined by $h_k=\frac{1}{2}\|x_k-z\|^2$. Then we have for every $k \geq 1$,*

$$
\begin{aligned}
&h_{k+1}-h_k-\alpha_k(h_k-h_{k-1})\\
&\quad\leq \left(\frac{1}{2}(\alpha_k+\alpha_k^2)-\frac{2-\rho_k}{2\rho_k}(1-\alpha_k)^2\right)\|x_k-x_{k-1}\|^2\\
&\quad\quad-\frac{2-\rho_k}{2\rho_k}(1-\alpha_k)(\|x_{k+1}-x_k\|^2-\|x_k-x_{k-1}\|^2).
\end{aligned}
\tag{12}
$$

Proof Let us formulate Lemma 2.2 in a recursive form. Observe that

$$
\begin{aligned}
\|x_{k+1}-y_k\|^2 &= \|x_{k+1}-x_k-\alpha_k(x_k-x_{k-1})\|^2\\
&= \|x_{k+1}-x_k-(x_k-x_{k-1})+(1-\alpha_k)(x_k-x_{k-1})\|^2\\
&= \|x_{k+1}-2x_k+x_{k-1}\|^2+(1-\alpha_k)^2\|x_k-x_{k-1}\|^2\\
&\quad+2(1-\alpha_k)\langle x_{k+1}-2x_k+x_{k-1}, x_k-x_{k-1}\rangle.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|x_{k+1}-x_k\|^2 = &\|x_{k+1}-2x_k+x_{k-1}\|^2+\|x_k-x_{k-1}\|^2\\
&+2\langle x_{k+1}-2x_k+x_{k-1}, x_k-x_{k-1}\rangle.
\end{aligned}
$$

By combining the above equalities, we obtain

$$
\begin{aligned}
\|x_{k+1}-y_k\|^2 = &\alpha_k\|x_{k+1}-2x_k+x_{k-1}\|^2+(1-\alpha_k)^2\|x_k-x_{k-1}\|^2\\
&+(1-\alpha_k)(\|x_{k+1}-x_k\|^2-\|x_k-x_{k-1}\|^2)
\end{aligned}
$$

$$\geq (1-\alpha_k)^2\|x_k - x_{k-1}\|^2 \\ + (1-\alpha_k)(\|x_{k+1}-x_k\|^2 - \|x_k - x_{k-1}\|^2).$$

Since $\rho_k \in]0,2]$ by assumption, the expected inequality follows immediately from (11). ◻

2.3 The sequences (t_i) and $(t_{i,k})$

Let us introduce the sequences (t_i) and $(t_{i,k})$ which will play a central role in the analysis of algorithm (RIPA) (the sequence (t_i) has already been briefly defined in the introduction). Throughout the paper, we use the convention $\prod_{j=i}^{i-1}\alpha_j = 1$ for $i \geq 1$. Given $i, k \geq 1$, we write $t_{i,k}$ the quantity defined by

$$t_{i,k} = \sum_{l=i-1}^{k-1}\left(\prod_{j=i}^{l}\alpha_j\right) = 1 + \sum_{l=i}^{k-1}\left(\prod_{j=i}^{l}\alpha_j\right) \quad \text{if } i \leq k, \tag{13}$$

and $t_{i,k} = 0$ if $i > k$. Observe that for every $i \geq 1$ and $k \geq i+1$,

$$1 + \alpha_i t_{i+1,k} = 1 + \alpha_i\left(\sum_{l=i}^{k-1}\left(\prod_{j=i+1}^{l}\alpha_j\right)\right) = 1 + \sum_{l=i}^{k-1}\left(\prod_{j=i}^{l}\alpha_j\right) = t_{i,k}. \tag{14}$$

From now on, we assume that

$$\sum_{l=i}^{+\infty}\left(\prod_{j=i}^{l}\alpha_j\right) < +\infty \quad \text{for every } i \geq 1. \tag{K_0}$$

We define the sequence (t_i) by

$$t_i = \sum_{l=i-1}^{+\infty}\left(\prod_{j=i}^{l}\alpha_j\right) = 1 + \sum_{l=i}^{+\infty}\left(\prod_{j=i}^{l}\alpha_j\right). \tag{15}$$

For each $i \geq 1$, the sequence $(t_{i,k})_k$ converges increasingly to t_i. By letting $k \to +\infty$ in (14), we obtain

$$1 + \alpha_i t_{i+1} = t_i,$$

for every $i \geq 1$. Let us summarize the above results.

Lemma 2.4 *Let (α_k) be a sequence of nonnegative real numbers. Then we have*

(i) *The sequence $(t_{i,k})$ defined by (13) satisfies the recursive relation: for every $i \geq 1$ and $k \geq i+1$*

$$1 + \alpha_i t_{i+1,k} = t_{i,k}$$

(ii) *Under* (K_0)*, the sequence* (t_i) *given by* (15) *is well-defined and satisfies for every* $i \geq 1$

$$1 + \alpha_i t_{i+1} = t_i.$$

2.4 Weak convergence of the iterates

Our convergence results are based on Lyapunov analysis. The weak convergence of the sequences generated by (RIPA) is based on the Opial lemma [25], which we recall in its discrete form.

Lemma 2.5 (Opial). *Let S be a nonempty subset of $\mathcal{H}$, and (x_k) a sequence of elements of $\mathcal{H}$ satisfying*

(i) *for every $z \in S$, $\lim_{k\to+\infty} \|x_k - z\|$ exists;*
(ii) *every sequential weak cluster point of (x_k), as $k \to +\infty$, belongs to S.*

Then the sequence (x_k) converges weakly as $k \to +\infty$ toward some $x_\infty \in S$.

Let us state the main result of this section.

Theorem 2.6 *Under (H), assume that* $\operatorname{zer} A \neq \emptyset$. *Suppose that $\alpha_k \in [0, 1]$ and $\rho_k \in]0, 2]$ for every $k \geq 1$. Under (K_0), let (t_i) be the sequence defined by* (15). *Assume that there exists $\varepsilon \in]0, 1[$ such that for k large enough,*

$$(1-\varepsilon)\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1}) \geq \alpha_k t_{k+1}\left(1+\alpha_k+\left[\frac{2-\rho_k}{\rho_k}(1-\alpha_k)-\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})\right]_+\right). \tag{K_1}$$

Then for any sequence (x_k) generated by (RIPA)*, we have*

(i) $\sum_{i=1}^{+\infty} \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1})\|x_i - x_{i-1}\|^2 < +\infty$, *and as a consequence* $\sum_{i=1}^{+\infty} \alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty$.
(ii) $\sum_{i=1}^{+\infty} \rho_i(2-\rho_i)\, t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 < +\infty$, *and* $\sum_{i=1}^{+\infty} \rho_i(2-\rho_i)\, t_{i+1} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.
(iii) *For any $z \in \operatorname{zer} A$, $\lim_{k\to+\infty} \|x_k - z\|$ exists, and hence (x_k) is bounded.*

Assume moreover that

$$\limsup_{k\to+\infty} \rho_k < 2 \tag{K_2}$$

$$\liminf_{k\to+\infty} \rho_k > 0. \tag{K_3}$$

Then the following holds

(iv) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(y_k) = 0$, *and* $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$.

(v) *If* $\liminf_{k\to+\infty}\mu_k > 0$, *then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

Proof (i) Let $z \in \operatorname{zer} A$, and let us set $h_k = \frac{1}{2}\|x_k - z\|^2$ for every $k \geq 1$. Setting $a_k = h_k - h_{k-1}$ and

$$w_k := \left(\frac{1}{2}(\alpha_k + \alpha_k^2) - \frac{2-\rho_k}{2\rho_k}(1-\alpha_k)^2\right)\|x_k - x_{k-1}\|^2 \\ -\frac{2-\rho_k}{2\rho_k}(1-\alpha_k)(\|x_{k+1}-x_k\|^2 - \|x_k - x_{k-1}\|^2),$$

we can rewrite inequality (12) of Lemma 2.3 in the condensed form $a_{k+1} \leq \alpha_k a_k + w_k$. By applying Lemma B.1 (i), we obtain for every $k \geq 1$

$$\begin{aligned} h_k - h_0 = \sum_{i=1}^{k} a_i &\leq t_{1,k}(h_1 - h_0) + \sum_{i=1}^{k-1} t_{i+1,k} w_i \\ &= t_{1,k}(h_1 - h_0) - \sum_{i=1}^{k-1} t_{i+1,k}\left[\left(\frac{2-\rho_i}{2\rho_i}(1-\alpha_i)^2 - \frac{1}{2}(\alpha_i + \alpha_i^2)\right)\|x_i - x_{i-1}\|^2\right. \\ &\quad \left.+\frac{2-\rho_i}{2\rho_i}(1-\alpha_i)(\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2)\right]. \end{aligned}$$

Since $t_{1,k} \leq t_1$ and $h_k \geq 0$, we deduce that

$$\sum_{i=1}^{k-1} t_{i+1,k}\left[\left(\frac{2-\rho_i}{\rho_i}(1-\alpha_i)^2 - (\alpha_i + \alpha_i^2)\right)\|x_i - x_{i-1}\|^2\right. \\ \left.+\frac{2-\rho_i}{\rho_i}(1-\alpha_i)(\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2)\right] \leq C,$$

with $C := 2h_0 + 2t_1|h_1 - h_0|$. Now observe that (we perform a discrete form of integration by parts)

$$\begin{aligned} &\sum_{i=1}^{k-1} t_{i+1,k}\frac{2-\rho_i}{\rho_i}(1-\alpha_i)(\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2) \\ &= \sum_{i=1}^{k-1}\left(t_{i,k}\frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - t_{i+1,k}\frac{2-\rho_i}{\rho_i}(1-\alpha_i)\right)\|x_i - x_{i-1}\|^2 \\ &\quad + t_{k,k}\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})\|x_k - x_{k-1}\|^2 - t_{1,k}\frac{2-\rho_0}{\rho_0}(1-\alpha_0)\|x_1 - x_0\|^2. \end{aligned}$$

Since the second last term is nonnegative and since $t_{1,k} \leq t_1$, we deduce from the above equality that

$$\begin{aligned}
&\sum_{i=1}^{k-1} t_{i+1,k} \frac{2-\rho_i}{\rho_i}(1-\alpha_i)(\|x_{i+1}-x_i\|^2 - \|x_i - x_{i-1}\|^2) \\
&\quad \geq \sum_{i=1}^{k-1} \left(t_{i,k} \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - t_{i+1,k} \frac{2-\rho_i}{\rho_i}(1-\alpha_i) \right) \|x_i - x_{i-1}\|^2 \\
&\quad\quad - t_1 \frac{2-\rho_0}{\rho_0}(1-\alpha_0)\|x_1 - x_0\|^2.
\end{aligned}$$

Collecting the above results, we infer that

$$\sum_{i=1}^{k-1} \delta_{i,k} \|x_i - x_{i-1}\|^2 \leq C_1, \tag{16}$$

with $C_1 = 2h_0 + 2t_1|h_1 - h_0| + t_1 \frac{2-\rho_0}{\rho_0}(1-\alpha_0)\|x_1 - x_0\|^2$ and

$$\begin{aligned}
\delta_{i,k} = t_{i+1,k} &\left(\frac{2-\rho_i}{\rho_i}(1-\alpha_i)^2 - (\alpha_i + \alpha_i^2) \right) \\
&+ t_{i,k} \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - t_{i+1,k} \frac{2-\rho_i}{\rho_i}(1-\alpha_i).
\end{aligned}$$

Now recall that $t_{i,k} = 1 + \alpha_i t_{i+1,k}$ for every $i \geq 1$ and $k \geq i+1$, see Lemma 2.4 (i). It ensues that

$$\begin{aligned}
\delta_{i,k} &= \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) + t_{i+1,k} \left(\frac{2-\rho_i}{\rho_i}(1-\alpha_i)^2 - (\alpha_i + \alpha_i^2) \right. \\
&\quad \left. + \alpha_i \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - \frac{2-\rho_i}{\rho_i}(1-\alpha_i) \right) \\
&= \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) + t_{i+1,k} \left(-\alpha_i \frac{2-\rho_i}{\rho_i}(1-\alpha_i) - (\alpha_i + \alpha_i^2) \right. \\
&\quad \left. + \alpha_i \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) \right) \\
&= \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - \alpha_i t_{i+1,k} \left(1 + \alpha_i + \frac{2-\rho_i}{\rho_i}(1-\alpha_i) \right. \\
&\quad \left. - \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) \right) \\
&\geq \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - \alpha_i t_{i+1} \left(1 + \alpha_i + \left[\frac{2-\rho_i}{\rho_i}(1-\alpha_i) \right. \right. \\
&\quad \left. \left. - \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) \right]_+ \right).
\end{aligned}$$

We then infer from (16) that for every $k \geq 2$,

$$\sum_{i=1}^{k-1} \left[\frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) - \alpha_i t_{i+1} \left(1 + \alpha_i + \left[\frac{2-\rho_i}{\rho_i}(1-\alpha_i) - \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1}) \right]_+ \right) \right] \|x_i - x_{i-1}\|^2 \leq C_1.$$

By assumption, inequality (K_1) holds true for k large enough. Without loss of generality, we may assume that it is satisfied for every $k \geq 1$. In view of the above inequality, it ensues that

$$\sum_{i=1}^{k-1} \varepsilon \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1})\|x_i - x_{i-1}\|^2 \leq C_1.$$

Taking the limit as $k \to +\infty$, we find

$$\sum_{i=1}^{+\infty} \frac{2-\rho_{i-1}}{\rho_{i-1}}(1-\alpha_{i-1})\|x_i - x_{i-1}\|^2 \leq \frac{C_1}{\varepsilon} < +\infty.$$

By using again (K_1), we deduce that

$$\sum_{i=1}^{+\infty} \alpha_i t_{i+1} \|x_i - x_{i-1}\|^2 < +\infty. \tag{17}$$

(ii) Let us come back to inequality (11). Using that $\alpha_k \in [0, 1]$, we get

$$h_{k+1} - h_k - \alpha_k(h_k - h_{k-1}) \leq \alpha_k \|x_k - x_{k-1}\|^2 - \frac{2-\rho_k}{2\rho_k}\|x_{k+1} - y_k\|^2. \tag{18}$$

Since $x_{k+1} - y_k = -\rho_k \mu_k A_{\mu_k}(y_k)$, this implies that

$$h_{k+1} - h_k - \alpha_k(h_k - h_{k-1}) \leq \alpha_k \|x_k - x_{k-1}\|^2 - \frac{1}{2}\rho_k(2-\rho_k)\|\mu_k A_{\mu_k}(y_k)\|^2.$$

By invoking Lemma B.1 (i) with $a_k = h_k - h_{k-1}$ and

$$w_k = \alpha_k \|x_k - x_{k-1}\|^2 - \frac{1}{2}\rho_k(2-\rho_k)\|\mu_k A_{\mu_k}(y_k)\|^2,$$

we obtain for every $k \geq 1$,

$$h_k - h_0 = \sum_{i=1}^{k} a_i \leq t_{1,k}(h_1 - h_0)$$
$$+ \sum_{i=1}^{k-1} t_{i+1,k} \left[\alpha_i \|x_i - x_{i-1}\|^2 - \frac{1}{2}\rho_i(2-\rho_i)\|\mu_i A_{\mu_i}(y_i)\|^2 \right].$$

Since $h_k \geq 0$ and $t_{i+1,k} \leq t_{i+1}$, we deduce that

$$\frac{1}{2}\sum_{i=1}^{k-1} \rho_i(2-\rho_i)t_{i+1,k}\|\mu_i A_{\mu_i}(y_i)\|^2 \leq h_0 + t_{1,k}(h_1 - h_0)$$
$$+ \sum_{i=1}^{k-1} \alpha_i t_{i+1}\|x_i - x_{i-1}\|^2.$$

Recalling from (i) that $\sum_{i=1}^{+\infty} \alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty$, we infer that for every $k \geq 1$,

$$\sum_{i=1}^{k-1} \rho_i(2-\rho_i)\, t_{i+1,k}\|\mu_i A_{\mu_i}(y_i)\|^2 \leq C_2,$$

where we have set

$$C_2 := 2h_0 + 2t_1|h_1 - h_0| + 2\sum_{i=1}^{+\infty} \alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty.$$

Since $t_{i+1,k} = 0$ for $i \geq k$, this yields in turn

$$\sum_{i=1}^{+\infty} \rho_i(2-\rho_i)\, t_{i+1,k}\|\mu_i A_{\mu_i}(y_i)\|^2 \leq C_2.$$

Letting k tend to $+\infty$, the monotone convergence theorem then implies that

$$\sum_{i=1}^{+\infty} \rho_i(2-\rho_i)\, t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 \leq C_2 < +\infty, \tag{19}$$

which gives the first estimate of (ii). Using the $\frac{1}{\mu_i}$-Lipschitz continuity property of A_{μ_i}, we have

$$
\begin{aligned}
\|A_{\mu_i}(x_i)\|^2 &\leq 2\|A_{\mu_i}(y_i)\|^2 + 2\|A_{\mu_i}(x_i) - A_{\mu_i}(y_i)\|^2 \\
&\leq 2\|A_{\mu_i}(y_i)\|^2 + \frac{2}{\mu_i^2}\|x_i - y_i\|^2 \\
&= 2\|A_{\mu_i}(y_i)\|^2 + \frac{2\alpha_i^2}{\mu_i^2}\|x_i - x_{i-1}\|^2.
\end{aligned}
$$

It ensues that

$$
\begin{aligned}
\rho_i(2-\rho_i)\,t_{i+1}\|\mu_i A_{\mu_i}(x_i)\|^2 &\leq 2\rho_i(2-\rho_i)\,t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 \\
&\quad + 2\alpha_i^2\rho_i(2-\rho_i)\,t_{i+1}\|x_i - x_{i-1}\|^2 \\
&\leq 2\rho_i(2-\rho_i)\,t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 \\
&\quad + 2\alpha_i t_{i+1}\|x_i - x_{i-1}\|^2,
\end{aligned}
$$

where we have used $\alpha_i \leq 1$ and $\rho_i(2-\rho_i) \leq 1$ in the second inequality. The first (resp. second) term of the above right-hand side is summable by (19) [resp. (17)]. We deduce that the left-hand side is also summable. This proves the second estimate of (ii).

(iii) From (18), we derive that for every $k \geq 1$,

$$
h_{k+1} - h_k \leq \alpha_k(h_k - h_{k-1}) + \alpha_k\|x_k - x_{k-1}\|^2.
$$

Recall that, from (i), we have $\sum_{i=1}^{+\infty}\alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty$. Applying Lemma B.1 (ii) with $a_k = h_k - h_{k-1}$ and $w_k = \alpha_k\|x_k - x_{k-1}\|^2$, we infer that $\sum_{i=1}^{+\infty}(h_k - h_{k-1})_+ < +\infty$. This classically implies that $\lim_{k\to+\infty} h_k$ exists. Thus, we have obtained that $\lim_{k\to+\infty}\|x_k - z\|$ exists for every $z \in \operatorname{zer} A$, whence in particular the boundedness of the sequence (x_k).

(iv) From (K_2) and (K_3), there exist $\underline{r} > 0$ and $\overline{r} < 2$ such that $\rho_k \in [\underline{r}, \overline{r}]$ for k large enough. We deduce from the first estimate of (ii) that $\sum_{i=1}^{+\infty} t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 < +\infty$, hence $\lim_{i\to+\infty} t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 = 0$. Since $t_i \geq 1$ for every $i \geq 1$, this implies in turn that $\lim_{i\to+\infty}\|\mu_i A_{\mu_i}(y_i)\| = 0$. The proof of $\lim_{i\to+\infty}\|\mu_i A_{\mu_i}(x_i)\| = 0$ follows the same lines.

(v) To prove the weak convergence of (x_k) as $k \to +\infty$, we use the Opial lemma with $S = \operatorname{zer} A$. Item (iii) shows the first condition of the Opial lemma. For the second one, let (x_{k_n}) be a subsequence of (x_k) which converges weakly to some $\overline{x}$. By (iv), we have $\lim_{k\to+\infty}\mu_k A_{\mu_k}(x_k) = 0$ strongly in $\mathcal{H}$. Since $\liminf_{k\to+\infty}\mu_k > 0$, we also have $\lim_{k\to+\infty} A_{\mu_k}(x_k) = 0$ strongly in $\mathcal{H}$. Passing to the limit in

$$
A_{\mu_{k_n}}(x_{k_n}) \in A\left(x_{k_n} - \mu_{k_n} A_{\mu_{k_n}}(x_{k_n})\right),
$$

and invoking the graph-closedness of the maximally monotone operator A for the weak–strong topology in $\mathcal{H}\times\mathcal{H}$, we find $0 \in A(\overline{x})$. This shows that $\overline{x} \in \operatorname{zer} A$, which completes the proof. □

Remark 2.7 The main role of assumption (K_1) is to guarantee the summability condition

$$\sum_{i=1}^{+\infty} \alpha_i t_{i+1} \|x_i - x_{i-1}\|^2 < +\infty, \tag{20}$$

obtained in (i). A careful examination of the proof of Theorem 2.6 shows that conclusions (ii), (iii), (iv) and (v) hold true if we assume directly condition (20). The latter condition involves the sequence (x_k) that is a priori unknown. However, in practice it is easy to ensure it by using a suitable on-line rule.

Let us now particularize Theorem 2.6 to the case $\alpha_k = 0$ for every $k \geq 1$, corresponding to the absence of inertia in algorithm (RIPA). In this framework, assumptions (K_0) and (K_1) are automatically satisfied, and moreover $t_i = 1$ for every $i \geq 1$. We then derive from Theorem 2.6 the following result, which is a particular case of [18, Theorem 3]. Note that the latter also takes into account the presence of errors in the computation of the resolvents.

Corollary 2.8 (Bertsekas–Eckstein [18]). *Under* (H)*, assume that* $\operatorname{zer} A \neq \emptyset$*, and that* $\rho_k \in]0, 2]$ *for every* $k \geq 1$*. Then, for any sequence* (x_k) *generated by* (RPA)

$$x_{k+1} = (1 - \rho_k) x_k + \rho_k J_{\mu_k A}(x_k), \tag{RPA}$$

we have

(i) $\sum_{i=1}^{+\infty} \dfrac{2 - \rho_{i-1}}{\rho_{i-1}} \|x_i - x_{i-1}\|^2 < +\infty.$

(ii) *For any* $z \in \operatorname{zer} A$*,* $\lim_{k \to +\infty} \|x_k - z\|$ *exists, and hence* (x_k) *is bounded.*

Assume moreover that $\limsup_{k \to +\infty} \rho_k < 2$ *and* $\liminf_{k \to +\infty} \rho_k > 0$*. Then the following holds*

(iii) $\lim_{k \to +\infty} \mu_k A_{\mu_k}(x_k) = 0.$

(iv) *If* $\liminf_{k \to +\infty} \mu_k > 0$*, then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$*.*

Let us now assume that $\rho_k = 1$ for every $k \geq 1$. In such a case, the algorithm (RIPA) boils down to the inertial proximal iteration. We obtain directly the following corollary of Theorem 2.6.

Corollary 2.9 *Under* (H)*, assume that* $\operatorname{zer} A \neq \emptyset$*, and that* $\alpha_k \in [0, 1]$ *for every* $k \geq 1$*. Suppose* (K_0) *and let* (t_i) *be the sequence defined by* (15)*. Assume that there exists* $\varepsilon \in]0, 1[$ *such that for* k *large enough,*

$$(1 - \varepsilon)(1 - \alpha_{k-1}) \geq \alpha_k t_{k+1} \left(1 + \alpha_k + \left[\alpha_{k-1} - \alpha_k\right]_+\right). \tag{K_1}$$

Then for any sequence (x_k) *generated by* (IPA)

$$\text{(IPA)} \qquad \begin{cases} y_k = x_k + \alpha_k (x_k - x_{k-1}) \\ x_{k+1} = J_{\mu_k A}(y_k), \end{cases}$$

we have

(i) $\sum_{i=1}^{+\infty}(1-\alpha_{i-1})\|x_i-x_{i-1}\|^2 < +\infty$, *and as a consequence* $\sum_{i=1}^{+\infty}\alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty$.
(ii) $\sum_{i=1}^{+\infty} t_{i+1}\|\mu_i A_{\mu_i}(y_i)\|^2 < +\infty$, *and* $\sum_{i=1}^{+\infty} t_{i+1}\|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.
(iii) *For any* $z \in \operatorname{zer} A$, $\lim_{k\to+\infty}\|x_k - z\|$ *exists, and hence* (x_k) *is bounded.*
(iv) $\lim_{k\to+\infty}\mu_k A_{\mu_k}(y_k) = 0$, *and* $\lim_{k\to+\infty}\mu_k A_{\mu_k}(x_k) = 0$.
(v) *If* $\liminf_{k\to+\infty}\mu_k > 0$, *then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

Remark 2.10 Following Remark 2.7, items (ii) to (v) of Corollary 2.9 hold true if we suppose that

$$\sum_{i=1}^{+\infty}\alpha_i t_{i+1}\|x_i - x_{i-1}\|^2 < +\infty.$$

Assume moreover that there exists $\overline{\alpha} \in [0, 1[$ such that $\alpha_k \in [0, \overline{\alpha}]$ for every $k \geq 1$. Then it is easy to show that $t_k \leq 1/(1-\overline{\alpha})$ for every $k \geq 1$. Hence the above summability condition is ensured by the following

$$\sum_{i=1}^{+\infty}\alpha_i\|x_i - x_{i-1}\|^2 < +\infty. \tag{21}$$

To summarize, if $\alpha_k \in [0, \overline{\alpha}]$ for every $k \geq 1$, and if condition (21) is satisfied, then we obtain conclusions (ii) to (v) of Corollary 2.9. This is precisely the result stated in [3, Theorem 2.1].

As a consequence of Corollary 2.9, we also find the result of [3, Proposition 2.1], when $\alpha_k \leq \overline{\alpha} < \frac{1}{3}$.

Corollary 2.11 (Alvarez–Attouch [3]). *Under* (H), *assume that* $\operatorname{zer} A \neq \emptyset$. *Suppose that there exists* $\overline{\alpha} \in [0, 1/3[$ *such that* $\alpha_k \in [0, \overline{\alpha}]$ *for every* $k \geq 1$. *Then for any sequence* (x_k) *generated by* (IPA), *we have*

(i) $\sum_{i=1}^{+\infty}\|x_i - x_{i-1}\|^2 < +\infty$.
(ii) $\sum_{i=1}^{+\infty}\|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.
(iii) *For any* $z \in \operatorname{zer} A$, $\lim_{k\to+\infty}\|x_k - z\|$ *exists, and hence* (x_k) *is bounded.*
(iv) $\lim_{k\to+\infty}\mu_k A_{\mu_k}(x_k) = 0$.
(v) *If* $\liminf_{k\to+\infty}\mu_k > 0$, *there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

Proof Since $\alpha_k \leq \overline{\alpha} < 1$ for every $k \geq 1$, it is immediate to check that (K_0) is satisfied and that $t_k \leq \frac{1}{1-\overline{\alpha}}$ for every $k \geq 1$. On the one hand, observe that for every $k \geq 1$,

$$\alpha_k t_{k+1}\left(1 + \alpha_k + \left[\alpha_{k-1} - \alpha_k\right]_+\right) = \alpha_k t_{k+1}(1 + \max(\alpha_k, \alpha_{k-1}))$$

$$\leq \frac{\overline{\alpha}}{1-\overline{\alpha}}(1+\overline{\alpha}).$$

On the other hand $1-\alpha_{k-1} \geq 1-\overline{\alpha}$. It ensues that (K_1) is satisfied if there exists $\varepsilon \in]0,1[$ such that

$$(1-\varepsilon)(1-\overline{\alpha}) \geq \frac{\overline{\alpha}}{1-\overline{\alpha}}(1+\overline{\alpha}).$$

The latter condition is equivalent to $(1-\overline{\alpha})^2 > \overline{\alpha}(1+\overline{\alpha})$, which in turn is equivalent to $\overline{\alpha} < 1/3$. Therefore assumption (K_1) is satisfied, and it suffices to apply Corollary 2.9. □

By taking constant parameters α_k and ρ_k, we obtain the following consequence of Theorem 2.6.

Corollary 2.12 *Under (H), assume that* $\operatorname{zer} A \neq \emptyset$. *Suppose that $\alpha_k \equiv \alpha \in [0,1[$, $\rho_k \equiv \rho \in]0,2[$ for every $k \geq 1$, and that*

$$\frac{2-\rho}{\rho}(1-\alpha)^2 > \alpha(1+\alpha). \tag{22}$$

Then for any sequence (x_k) generated by (RIPA)*, we have*

(i) $\sum_{i=1}^{+\infty} \|x_i - x_{i-1}\|^2 < +\infty$.
(ii) $\sum_{i=1}^{+\infty} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.
(iii) *For any $z \in \operatorname{zer} A$, $\lim_{k\to+\infty} \|x_k - z\|$ exists, and hence (x_k) is bounded.*
(iv) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$.
(v) *If $\liminf_{k\to+\infty} \mu_k > 0$, there exists $x_\infty \in \operatorname{zer} A$ such that $x_k \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $k \to +\infty$.*

Proof Since $\alpha_k \equiv \alpha \in [0,1[$, we have for every $i \geq 1$, $t_i = \sum_{l=i-1}^{+\infty} \alpha^{l-i+1} = \frac{1}{1-\alpha} < +\infty$. Hence condition (K_0) holds true. Using that α_k and ρ_k are constant, condition (K_1) then amounts to

$$(1-\varepsilon)\frac{2-\rho}{\rho}(1-\alpha) \geq \frac{\alpha}{1-\alpha}(1+\alpha),$$

which is equivalent to (22). Therefore, all the assumptions of Theorem 2.6 are met, giving the result. □

Remark 2.13 The above result gives some indication of the balance between the inertial effect and the relaxation effect. The inequation (22) is equivalent to $\rho < \frac{2(1-\alpha)^2}{2\alpha^2-\alpha+1}$. Therefore, for given $0 < \alpha < 1$, the maximum value of the relaxation parameter is given by $\rho_m(\alpha) = \frac{2(1-\alpha)^2}{2\alpha^2-\alpha+1}$. Elementary differential calculus shows that the function $\alpha \mapsto \rho_m(\alpha)$ is decreasing on $[0,1]$. Thus, as expected, when the inertial effect increases ($\alpha \nearrow$), then the relaxation effect decreases ($\rho_m \searrow$), and vice versa, see also

[20]. When $\alpha \to 0$, the limiting value $\rho_m(\alpha)$ is 2, which is in accordance with Corollary 2.8. When $\alpha \to 1$, the limiting value of $\rho_m(\alpha)$ is zero, which is in accordance with the existing results concerning the case $\alpha_k \to 1$.

2.5 Case of a possibly vanishing parameter ρ_k

When $\alpha_k \to 1$, which is the case of the Nesterov accelerated method, we must take $\rho_k \to 0$ to satisfy the condition (K_1). Consequently, Theorem 2.6 does not make it possible to obtain the convergence of the iterates of (RIPA) in the case $\alpha_k \to 1$. The following result completes Theorem 2.6 by considering the case of a possibly vanishing parameter ρ_k. In the upcoming statement, assumption (K_3) is removed and replaced with an alternative set of assumptions, namely (K_4)–(K_5).

Theorem 2.14 *Under (H), assume that $\operatorname{zer} A \neq \emptyset$. Suppose that the sequences (α_k) and (ρ_k) satisfy $\alpha_k \in [0,1]$ and $\rho_k \in]0,2]$ for every $k \geq 1$, together with (K_0)–(K_1). Then for any sequence (x_k) generated by* (RIPA)*,*

(i) *There exists a constant $C \geq 0$ such that for every $k \geq 1$,*

$$\|x_{k+1} - x_k\| \leq C \sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j \right) \rho_i \right].$$

Assume additionally (K_2), together with

$$\sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j \right) \rho_i \right] = \mathcal{O}(\rho_k t_{k+1}), \tag{K_4}$$

$$\frac{|\mu_{k+1} - \mu_k|}{\mu_{k+1}} = \mathcal{O}(\rho_k t_{k+1}), \quad \rho_{k-1} t_k = \mathcal{O}(\rho_k t_{k+1}) \quad \text{as } k \to +\infty$$

$$\sum_{i=1}^{+\infty} \rho_i t_{i+1} = +\infty. \tag{K_5}$$

Then the following holds

(ii) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$. *If* $\liminf_{k\to+\infty} \mu_k > 0$, *then there exists $x_\infty \in \operatorname{zer} A$ such that $x_k \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $k \to +\infty$.*

Finally assume that condition (K_5) is not satisfied, i.e. $\sum_{i=1}^{+\infty} \rho_i t_{i+1} < +\infty$. Then we obtain

(iii) $\sum_{i=1}^{+\infty} \|x_i - x_{i-1}\| < +\infty$, *and hence the sequence (x_k) converges strongly toward some $x_\infty \in \mathcal{H}$.*

Proof (i) Iteration (RIPA) can be rewritten as

$$x_{k+1} - x_k = \alpha_k (x_k - x_{k-1}) - \rho_k \mu_k A_{\mu_k}(y_k),$$

see (7). Taking the norm of each member, we find

$$\|x_{k+1} - x_k\| \leq \alpha_k \|x_k - x_{k-1}\| + \rho_k \mu_k \|A_{\mu_k}(y_k)\|. \tag{23}$$

On the other hand, for $z \in \operatorname{zer} A = \operatorname{zer} A_{\mu_k}$, the $\frac{1}{\mu_k}$-Lipschitz continuity of A_{μ_k} yields

$$\|A_{\mu_k}(y_k)\| \leq \frac{1}{\mu_k}\|y_k - z\|.$$

Recall that the sequence (x_k) is bounded by Theorem 2.6 (iii). Since $\alpha_k \in [0, 1]$, the sequence (y_k) is also bounded. From the above inequality, we deduce the existence of $C_3 \geq 0$ such that $\|A_{\mu_k}(y_k)\| \leq \frac{C_3}{\mu_k}$ for every $k \geq 1$. In view of (23), we infer that

$$\|x_{k+1} - x_k\| \leq \alpha_k \|x_k - x_{k-1}\| + C_3 \rho_k. \tag{24}$$

An immediate recurrence shows that for every $k \geq 1$,

$$\|x_{k+1} - x_k\| \leq \left(\prod_{j=1}^{k} \alpha_j\right) \|x_1 - x_0\| + C_3 \sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j\right) \rho_i\right],$$

with the convention $\prod_{j=k+1}^{k} \alpha_j = 1$. Since $\alpha_k \in [0, 1]$, we have $\prod_{j=i+1}^{k} \alpha_j \geq \prod_{j=1}^{k} \alpha_j$ and hence

$$\sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j\right) \rho_i\right] \geq \left(\prod_{j=1}^{k} \alpha_j\right) \sum_{i=1}^{k} \rho_i \geq \left(\prod_{j=1}^{k} \alpha_j\right) \rho_1.$$

Setting $C_4 := \|x_1 - x_0\|/\rho_1 + C_3$, we deduce that for every $k \geq 1$,

$$\|x_{k+1} - x_k\| \leq C_4 \sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j\right) \rho_i\right]. \tag{25}$$

(ii) Recall the estimate of Theorem 2.6 (ii)

$$\sum_{i=1}^{+\infty} \rho_i (2 - \rho_i)\, t_{i+1} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty. \tag{26}$$

According to (K_2), there exists $\bar{r} \in]0, 2[$ such that $\rho_k \leq \bar{r}$ for k large enough. We deduce from (26) that

$$\sum_{i=1}^{+\infty} \rho_i\, t_{i+1} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty. \tag{27}$$

Since the operator $\mu_k A_{\mu_k}$ is 1-Lipschitz continuous, we have

$$\|\mu_k A_{\mu_k}(x_k)\| \leq \|x_k - z\| \leq C_5,$$

with $C_5 := \sup_{k\geq 1} \|x_k - z\| < +\infty$. It ensues that

$$\begin{aligned} &|\|\mu_{k+1} A_{\mu_{k+1}}(x_{k+1})\|^2 - \|\mu_k A_{\mu_k}(x_k)\|^2| \\ &\quad \leq 2C_5 \|\mu_{k+1} A_{\mu_{k+1}}(x_{k+1}) - \mu_k A_{\mu_k}(x_k)\|. \end{aligned} \tag{28}$$

By applying [7, Lemma A.4] with $\gamma = \mu_{k+1}$, $\delta = \mu_k$, $x = x_{k+1}$ and $y = x_k$, we find

$$\begin{aligned} \|\mu_{k+1} A_{\mu_{k+1}}(x_{k+1}) - \mu_k A_{\mu_k}(x_k)\| &\leq 2\|x_{k+1} - x_k\| + 2\|x_{k+1} - z\| \frac{|\mu_{k+1} - \mu_k|}{\mu_{k+1}} \\ &\leq 2\|x_{k+1} - x_k\| + 2C_5 \frac{|\mu_{k+1} - \mu_k|}{\mu_{k+1}}. \end{aligned}$$

In view of (25), we deduce that for every $k \geq 1$,

$$\begin{aligned} &\|\mu_{k+1} A_{\mu_{k+1}}(x_{k+1}) - \mu_k A_{\mu_k}(x_k)\| \\ &\quad \leq 2C_4 \sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j \right) \rho_i \right] + 2C_5 \frac{|\mu_{k+1} - \mu_k|}{\mu_{k+1}}. \end{aligned}$$

Recalling the assumption (K_4), we obtain the existence of $C_6 \geq 0$ such that for k large enough,

$$\|\mu_{k+1} A_{\mu_{k+1}}(x_{k+1}) - \mu_k A_{\mu_k}(x_k)\| \leq C_6 \rho_k t_{k+1}.$$

Using (28), we infer that

$$\left| \|\mu_{k+1} A_{\mu_{k+1}}(x_{k+1})\|^2 - \|\mu_k A_{\mu_k}(x_k)\|^2 \right| \leq 2C_5 C_6 \rho_k t_{k+1}.$$

It follows that for every $k \geq 1$,

$$\begin{aligned} &\sum_{i=1}^{k} \left| \|\mu_{i+1} A_{\mu_{i+1}}(x_{i+1})\|^4 - \|\mu_i A_{\mu_i}(x_i)\|^4 \right| \\ &\quad \leq 2C_5 C_6 \sum_{i=1}^{k} \rho_i t_{i+1} \left(\|\mu_{i+1} A_{\mu_{i+1}}(x_{i+1})\|^2 + \|\mu_i A_{\mu_i}(x_i)\|^2 \right). \end{aligned}$$

Given the estimate (27), together with the assumption $\rho_i t_{i+1} = \mathcal{O}(\rho_{i+1} t_{i+2})$ as $i \to +\infty$, we deduce that

$$\sum_{i=1}^{+\infty} \left| \|\mu_{i+1} A_{\mu_{i+1}}(x_{i+1})\|^4 - \|\mu_i A_{\mu_i}(x_i)\|^4 \right| < +\infty.$$

From a classical result, this implies that $\lim_{k\to+\infty} \|\mu_k A_{\mu_k}(x_k)\|^4$ exists, which entails in turn that $\lim_{k\to+\infty} \|\mu_k A_{\mu_k}(x_k)\|$ exists. Using again the estimate (27), together with the assumption (K_5), we immediately conclude that $\lim_{k\to+\infty} \|\mu_k A_{\mu_k}(x_k)\| = 0$. The proof of the weak convergence of the sequence (x_k) follows the same lines as in Theorem 2.6 (v).

(iii) Let us now assume that $\sum_{i=1}^{+\infty} \rho_i t_{i+1} < +\infty$. Recall from inequality (24) that

$$\|x_{k+1} - x_k\| \leq \alpha_k \|x_k - x_{k-1}\| + C_3 \rho_k.$$

By applying Lemma B.1 (ii) with $a_k = \|x_k - x_{k-1}\|$ and $w_k = C_3 \rho_k$, we obtain that $\sum_{i=1}^{+\infty} \|x_i - x_{i-1}\| < +\infty$. The last assertion is immediate. □

3 Application to particular classes of parameters α_k, μ_k and ρ_k

3.1 Some criteria for (K_0) and (K_1)

The following proposition provides a criterion for simply obtaining an asymptotic equivalent of t_k.

Proposition 3.1 *Let (α_k) be a sequence such that $\alpha_k \in [0, 1[$ for every $k \geq 1$. Assume that*[1]

$$\lim_{k\to+\infty} \left(\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k} \right) = c, \tag{29}$$

for some $c \in [0, 1[$. Then we have

(i) *The property (K_0) is satisfied, and*

$$t_{k+1} \sim \frac{1}{(1-c)(1-\alpha_k)} \quad \text{as } k \to +\infty.$$

(ii) *The equivalence $1 - \alpha_k \sim 1 - \alpha_{k+1}$ holds true as $k \to +\infty$, hence $t_{k+1} \sim t_{k+2}$ as $k \to +\infty$.*

(iii) $\sum_{k=1}^{+\infty} (1 - \alpha_k) = +\infty$.

Proof (i) This result was proved by the authors in [4, Proposition 15].
(ii) First assume that $c \in]0, 1[$. By a standard summation procedure, we infer from (29) that

$$\frac{1}{1-\alpha_k} \sim ck \quad \text{as } k \to +\infty.$$

[1] Note that in [4, Proposition 14], a closely related but different condition has been considered: the difference of the quotients is assumed to be less than or equal to c (and this guarantees (K_0)).

It ensues that $1-\alpha_k \sim \frac{1}{ck}$ as $k \to +\infty$, and hence clearly $1-\alpha_k \sim 1-\alpha_{k+1}$ as $k \to +\infty$. Now assume that $c=0$. Multiplying (29) by $1-\alpha_k$, we find

$$\frac{1-\alpha_k}{1-\alpha_{k+1}} = 1 + o(1-\alpha_k) \to 1 \quad \text{as } k \to +\infty,$$

because $\alpha_k \in [0,1[$. This completes the proof of the equivalence $1-\alpha_k \sim 1-\alpha_{k+1}$ as $k \to +\infty$. The last assertion then follows immediately from (i).
(iii) Fix $\varepsilon > 0$. In view of (29), there exists $k_0 \geq 1$ such that for every $k \geq k_0$,

$$\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k} \leq c + \varepsilon.$$

By summing the above inequality, we obtain $\frac{1}{1-\alpha_k} \leq \frac{1}{1-\alpha_{k_0}} + (c+\varepsilon)(k-k_0)$ for every $k \geq k_0$. Setting $d = 1/(1-\alpha_{k_0})$, we deduce immediately that $1-\alpha_k \geq 1/(d+(c+\varepsilon)(k-k_0))$, thus implying that $\sum_{k=1}^{+\infty}(1-\alpha_k) = +\infty$. □

Let us now analyze the condition (K_1)

$$\begin{aligned}(1-\varepsilon)&\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})\\ &\geq \alpha_k t_{k+1}\left(1+\alpha_k+\left[\frac{2-\rho_k}{\rho_k}(1-\alpha_k) - \frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})\right]_+\right). \qquad (K_1)\end{aligned}$$

Following an argument parallel to the continuous case, see [5, Proposition 3.2], let us introduce the following condition:

$$\begin{aligned}&\text{There exists } \ c' \in]-1,+1[\ \text{ such that}\\ &\lim_{k\to+\infty} \frac{\frac{2-\rho_k}{\rho_k}(1-\alpha_k) - \frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})}{\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})^2} = c'.\end{aligned} \qquad (30)$$

Proposition 3.2 *Let's make assumptions* (29) *and* (30)*, with $|c'| < 1-c$. Then (K_1) is satisfied if*

$$\liminf_{k\to+\infty} \frac{2-\rho_k}{\rho_k}(1-\alpha_k)^2 > \limsup_{k\to+\infty} \frac{\alpha_k(1+\alpha_k)}{1-c-|c'|}. \qquad (31)$$

Proof Setting $\theta_k = \frac{2-\rho_k}{\rho_k}(1-\alpha_k)$, let us rewrite (K_1) as a discrete differential inequality, as follows

$$(1-\varepsilon)\theta_{k-1} \geq \alpha_k t_{k+1}\left(1+\alpha_k+\left[\theta_k - \theta_{k-1}\right]_+\right). \qquad (32)$$

According to Proposition 3.1 we have $t_{k+1} \sim t_k \sim \frac{1}{(1-c)(1-\alpha_{k-1})}$ as $k \to +\infty$. Consequently, (32) can be equivalently formulated as

$$(1-\varepsilon)(1-c)\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})^2 \geq (1+o(1))\,\alpha_k\left(1+\alpha_k+\left[\theta_k-\theta_{k-1}\right]_+\right).$$

On the other hand, condition (30) gives

$$|\theta_k-\theta_{k-1}| = |c'|\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})^2 + o\left(\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})^2\right).$$

Setting $R_k := \frac{2-\rho_k}{\rho_k}(1-\alpha_k)^2$, we deduce that (K_1) is implied by the following condition

$$(1-\varepsilon)(1-c)R_{k-1} \geq (1+o(1))\alpha_k\left(1+\alpha_k+|c'|R_{k-1}+o(R_{k-1})\right).$$

Rearranging the terms we obtain

$$\left[(1-\varepsilon)(1-c)-\alpha_k(|c'|+o(1))\right]R_{k-1} \geq (1+o(1))\alpha_k\,(1+\alpha_k)\,.$$

Since $\alpha_k \leq 1$, the above inequality will be satisfied if

$$\left[(1-\varepsilon)(1-c)-(|c'|+o(1))\right]R_{k-1} \geq (1+o(1))\alpha_k\,(1+\alpha_k)\,.$$

This will be fulfilled if

$$\liminf_{k\to+\infty}\frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1})^2 > \limsup_{k\to+\infty}\frac{\alpha_k(1+\alpha_k)}{(1-\varepsilon)(1-c)-|c'|}. \tag{33}$$

The right member of (33) is a continuous increasing function of ε. Consequently, it is equivalent to assume that the above strict inequality is satisfied for $\varepsilon = 0$, which gives the claim. □

The next proposition brings to light a set of conditions which guarantee that condition (K_1) is satisfied.

Proposition 3.3 *Suppose that* $\alpha_k \in [0,1[$ *and* $\rho_k \in]0,2[$ *for every* $k \geq 1$. *Let us assume that there exist* $\overline{\rho} \in [0,2[$, $c \in [0,1[$ *and* $c'' \in \mathbb{R}$, *with* $-(1-\overline{\rho}/2) < c'' \leq -(1-\overline{\rho}/2)c$ *such that*

$$\lim_{k\to+\infty}\rho_k = \overline{\rho}; \tag{34}$$

$$\lim_{k\to+\infty}\left(\frac{1}{1-\alpha_{k+1}}-\frac{1}{1-\alpha_k}\right) = c; \tag{35}$$

$$\lim_{k\to+\infty}\frac{\rho_{k+1}-\rho_k}{\rho_{k+1}(1-\alpha_k)} = c''; \tag{36}$$

$$\liminf_{k\to+\infty}\frac{(1-\alpha_k)^2}{\rho_k} > \limsup_{k\to+\infty}\frac{\alpha_k(1+\alpha_k)}{2-\overline{\rho}+2c''}. \tag{37}$$

Then condition (K_1) *is satisfied.*

Proof Let us check that the conditions (30) and (31) of Proposition 3.2 are satisfied. First observe that

$$\begin{aligned}
&\frac{2-\rho_k}{\rho_k}(1-\alpha_k) - \frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1}) \\
&= \frac{2-\rho_{k-1}}{\rho_{k-1}}((1-\alpha_k)-(1-\alpha_{k-1})) \\
&\quad + \left(\frac{2-\rho_k}{\rho_k} - \frac{2-\rho_{k-1}}{\rho_{k-1}}\right)(1-\alpha_k) \\
&= \frac{2-\rho_{k-1}}{\rho_{k-1}}((1-\alpha_k)-(1-\alpha_{k-1})) - 2\,\frac{\rho_k-\rho_{k-1}}{\rho_k\rho_{k-1}}(1-\alpha_k). \qquad (38)
\end{aligned}$$

In view of assumption (35), we have

$$\begin{aligned}
(1-\alpha_k)-(1-\alpha_{k-1}) &= -c(1-\alpha_{k-1})(1-\alpha_k) + o((1-\alpha_{k-1})(1-\alpha_k)) \\
&= -c(1-\alpha_{k-1})^2 + o(1-\alpha_{k-1})^2,
\end{aligned}$$

since $1-\alpha_{k-1} \sim 1-\alpha_k$ as $k \to +\infty$, see Proposition 3.1 (ii). Setting $R_k := \frac{2-\rho_k}{\rho_k}(1-\alpha_k)^2$, this leads to

$$\frac{2-\rho_{k-1}}{\rho_{k-1}}((1-\alpha_k)-(1-\alpha_{k-1})) = -cR_{k-1} + o(R_{k-1}) \quad \text{as } k \to +\infty. \qquad (39)$$

On the other hand, assumption (36) yields

$$\begin{aligned}
\frac{\rho_k-\rho_{k-1}}{\rho_k\rho_{k-1}}(1-\alpha_k) &= \frac{c''}{\rho_{k-1}}(1-\alpha_{k-1})(1-\alpha_k) + o\left(\frac{1}{\rho_{k-1}}(1-\alpha_{k-1})(1-\alpha_k)\right) \\
&= \frac{c''}{\rho_{k-1}}(1-\alpha_{k-1})^2 + o\left(\frac{1}{\rho_{k-1}}(1-\alpha_{k-1})^2\right) \\
&= \frac{c''}{2-\overline{\rho}}R_{k-1} + o(R_{k-1}), \qquad (40)
\end{aligned}$$

where we used assumption (34) in the last equality. By combining (38), (39) and (40), we obtain

$$\begin{aligned}
&\frac{2-\rho_k}{\rho_k}(1-\alpha_k) - \frac{2-\rho_{k-1}}{\rho_{k-1}}(1-\alpha_{k-1}) \\
&= -\left(c + \frac{2c''}{2-\overline{\rho}}\right)R_{k-1} + o(R_{k-1}) \quad \text{as } k \to +\infty.
\end{aligned}$$

It ensues that condition (30) is satisfied with $c' = -\left(c + \frac{c''}{1-\overline{\rho}/2}\right)$. Since $c'' \le -(1-\overline{\rho}/2)c$ by assumption, we have $c' \ge 0$. This implies that

$$1 - c - |c'| = 1 - c - c' = 1 + \frac{c''}{1-\overline{\rho}/2}. \qquad (41)$$

Using that $-(1-\overline{\rho}/2) < c''$ by assumption, we deduce that the above quantity is positive, hence $|c'| < 1-c$.
Let us finally check condition (31). Recalling that $\lim_{k\to+\infty} \rho_k = \overline{\rho}$, condition (31) is equivalent to

$$(2-\overline{\rho}) \liminf_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} > \limsup_{k\to+\infty} \frac{\alpha_k(1+\alpha_k)}{1-c-|c'|}.$$

In view of (41), the latter condition is in turn equivalent to

$$\liminf_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} > \limsup_{k\to+\infty} \frac{\alpha_k(1+\alpha_k)}{2-\overline{\rho}+2c''},$$

which holds true by (37). Then just use Proposition 3.2. □

3.2 Application of the main results

Combining Theorem 2.6 with Proposition 3.3, we obtain the following result.

Theorem 3.4 *Under* (H)*, assume that* $\operatorname{zer} A \neq \emptyset$*. Suppose that* $\alpha_k \in [0,1[$ *and* $\rho_k \in]0,2[$ *for every* $k \geq 1$*. Let us assume that there exist* $\overline{\rho} \in [0,2[$, $c \in [0,1[$ *and* $c'' \in \mathbb{R}$*, with* $-(1-\overline{\rho}/2) < c'' \leq -(1-\overline{\rho}/2)c$ *such that*

$$\lim_{k\to+\infty} \rho_k = \overline{\rho};$$
$$\lim_{k\to+\infty} \left(\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k}\right) = c;$$
$$\lim_{k\to+\infty} \frac{\rho_{k+1}-\rho_k}{\rho_{k+1}(1-\alpha_k)} = c'';$$
$$\liminf_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} > \limsup_{k\to+\infty} \frac{\alpha_k(1+\alpha_k)}{2-\overline{\rho}+2c''}.$$

Then for any sequence (x_k) *generated by* (RIPA)*, we have*

(i) $\sum_{i=1}^{+\infty} \frac{1-\alpha_{i-1}}{\rho_{i-1}} \|x_i - x_{i-1}\|^2 < +\infty$.
(ii) $\sum_{i=1}^{+\infty} \frac{\rho_i}{1-\alpha_i} \|\mu_i A_{\mu_i}(y_i)\|^2 < +\infty$, *and* $\sum_{i=1}^{+\infty} \frac{\rho_i}{1-\alpha_i} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.
(iii) *For any* $z \in \operatorname{zer} A$, $\lim_{k\to+\infty} \|x_k - z\|$ *exists, and hence* (x_k) *is bounded.*

Assume moreover that $\overline{\rho} > 0$*. Then the following holds*

(iv) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(y_k) = 0$, *and* $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$.
(v) *If* $\liminf_{k\to+\infty} \mu_k > 0$, *then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

Proof Proposition 3.1 shows that (K_0) is satisfied and that $t_{k+1} \sim \frac{1}{(1-c)(1-\alpha_k)}$ as $k \to +\infty$. On the other hand, condition (K_1) is fulfilled in view of Proposition 3.3. Items (i) to (v) then follow immediately from Theorem 2.6. □

To apply Theorem 2.14, we must find suitable conditions that ensure that condition (K_4) is satisfied. The following result gives an equivalent, when $k \to +\infty$, of the first expression appearing in condition (K_4).

Proposition 3.5 *Let (α_k) and (ρ_k) be sequences such that $\alpha_k \in [0, 1[$ and $\rho_k \in]0, 2]$ for every $k \geq 1$. Let us assume that there exist $\overline{\alpha} \in [0, 1]$, $c \in [0, 1[$ and $c'' \in \mathbb{R}$, with $1 + c + c''\overline{\alpha} > 0$ such that*

$$\lim_{k\to+\infty} \alpha_k = \overline{\alpha}; \tag{42}$$

$$\lim_{k\to+\infty} \left(\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k} \right) = c; \tag{43}$$

$$\lim_{k\to+\infty} \frac{\rho_{k+1} - \rho_k}{\rho_{k+1}(1-\alpha_k)} = c''. \tag{44}$$

Then the following equivalence holds true

$$\sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j \right) \rho_i \right] \sim \frac{1}{(1 + c + c''\overline{\alpha})} \frac{\rho_k}{1-\alpha_k} \quad \textit{as } k \to +\infty.$$

Proof Observe that for every $i \leq k$,

$$\begin{aligned}
&\frac{\rho_i}{1-\alpha_i} \prod_{j=i+1}^{k} \alpha_j - \frac{\rho_{i-1}}{1-\alpha_{i-1}} \prod_{j=i}^{k} \alpha_j \\
&= \left(\prod_{j=i+1}^{k} \alpha_j \right) \left[\frac{\rho_i}{1-\alpha_i} - \frac{\rho_{i-1}\alpha_i}{1-\alpha_{i-1}} \right] \\
&= \left(\prod_{j=i+1}^{k} \alpha_j \right) \left[\frac{\rho_i}{1-\alpha_i} - \frac{\rho_i\alpha_i}{1-\alpha_{i-1}} + \frac{(\rho_i - \rho_{i-1})\alpha_i}{1-\alpha_{i-1}} \right] \\
&= \left(\prod_{j=i+1}^{k} \alpha_j \right) \rho_i \left[\frac{1}{1-\alpha_i} - \frac{1}{1-\alpha_{i-1}} + \frac{1-\alpha_i}{1-\alpha_{i-1}} + \frac{(\rho_i - \rho_{i-1})\alpha_i}{\rho_i(1-\alpha_{i-1})} \right]
\end{aligned} \tag{45}$$

In view of assumption (43), we have $\lim_{i\to+\infty}(1-\alpha_i)/(1-\alpha_{i-1}) = 1$, see Proposition 3.1 (ii). By using assumptions (42), (43) and (44), we then obtain that

$$\lim_{i\to+\infty} \left[\frac{1}{1-\alpha_i} - \frac{1}{1-\alpha_{i-1}} + \frac{1-\alpha_i}{1-\alpha_{i-1}} + \frac{(\rho_i - \rho_{i-1})\alpha_i}{\rho_i(1-\alpha_{i-1})} \right] = 1 + c + c''\overline{\alpha}. \tag{46}$$

Recalling that $1+c+c''\overline{\alpha}>0$ by assumption, let us fix $\varepsilon \in \left]0,\ 1+c+c''\overline{\alpha}\right[$. We infer from (46) that there exists $i_0 \geq 1$ such that for every $i \geq i_0$,

$$\begin{aligned} 1+c+c''\overline{\alpha}-\varepsilon &\leq \left[\frac{1}{1-\alpha_i}-\frac{1}{1-\alpha_{i-1}}+\frac{1-\alpha_i}{1-\alpha_{i-1}}+\frac{(\rho_i-\rho_{i-1})\alpha_i}{\rho_i(1-\alpha_{i-1})}\right] \\ &\leq 1+c+c''\overline{\alpha}+\varepsilon. \end{aligned}$$

In view of (45), this implies that for every $i \geq i_0$ and $k \geq i$,

$$\begin{aligned} &(1+c+c''\overline{\alpha}-\varepsilon)\left(\prod_{j=i+1}^{k}\alpha_j\right)\rho_i \leq \frac{\rho_i}{1-\alpha_i}\prod_{j=i+1}^{k}\alpha_j-\frac{\rho_{i-1}}{1-\alpha_{i-1}}\prod_{j=i}^{k}\alpha_j \\ &\leq (1+c+c''\overline{\alpha}+\varepsilon)\left(\prod_{j=i+1}^{k}\alpha_j\right)\rho_i. \end{aligned} \tag{47}$$

Let us sum the above inequalities from $i=i_0$ to k. We find

$$\begin{aligned} &(1+c+c''\overline{\alpha}-\varepsilon)\sum_{i=i_0}^{k}\left[\left(\prod_{j=i+1}^{k}\alpha_j\right)\rho_i\right] \leq \frac{\rho_k}{1-\alpha_k}-\frac{\rho_{i_0-1}}{1-\alpha_{i_0-1}}\prod_{j=i_0}^{k}\alpha_j \\ &\leq (1+c+c''\overline{\alpha}+\varepsilon)\sum_{i=i_0}^{k}\left[\left(\prod_{j=i+1}^{k}\alpha_j\right)\rho_i\right]. \end{aligned}$$

It ensues that

$$\begin{aligned} (1+c+c''\overline{\alpha})\sum_{i=i_0}^{k}\left[\left(\prod_{j=i+1}^{k}\alpha_j\right)\rho_i\right] &\sim \frac{\rho_k}{1-\alpha_k} \\ -\frac{\rho_{i_0-1}}{1-\alpha_{i_0-1}}\prod_{j=i_0}^{k}\alpha_j \quad &\text{as } k \to +\infty. \end{aligned}$$

It remains now to prove that $\prod_{j=i_0}^{k}\alpha_j = o\left(\frac{\rho_k}{1-\alpha_k}\right)$ as $k \to +\infty$. If there exists $k_0 \geq i_0$ such that $\alpha_{k_0}=0$, then the sequence $\left(\prod_{j=i_0}^{k}\alpha_j\right)$ is stationary and equal to 0 for $k \geq k_0$. Without loss of generality, we can assume that $\alpha_k>0$ for every $k \geq i_0$. Let us come back to the left inequality of (47), and divide each member by $\prod_{j=i_0}^{k}\alpha_j$. We find

$$(1+c+c''\overline{\alpha}-\varepsilon)\frac{\rho_i}{\prod_{j=i_0}^{i}\alpha_j} \leq \frac{\rho_i}{1-\alpha_i}\frac{1}{\prod_{j=i_0}^{i}\alpha_j}-\frac{\rho_{i-1}}{1-\alpha_{i-1}}\frac{1}{\prod_{j=i_0}^{i-1}\alpha_j}. \tag{48}$$

Since $1+c+c''\overline{\alpha}>\varepsilon$, we infer that the sequence $\left(\frac{\rho_i}{1-\alpha_i}\frac{1}{\prod_{j=i_0}^{i}\alpha_j}\right)$ is increasing. This implies that for every $i \geq i_0$,

$$\frac{\rho_i}{1-\alpha_i}\frac{1}{\prod_{j=i_0}^{i}\alpha_j} \geq \frac{\rho_{i_0-1}}{1-\alpha_{i_0-1}}.$$

In view of (48), we deduce that

$$(1+c+c''\overline{\alpha}-\varepsilon)\frac{\rho_{i_0-1}}{1-\alpha_{i_0-1}}(1-\alpha_i) \leq \frac{\rho_i}{1-\alpha_i}\frac{1}{\prod_{j=i_0}^{i}\alpha_j} - \frac{\rho_{i-1}}{1-\alpha_{i-1}}\frac{1}{\prod_{j=i_0}^{i-1}\alpha_j}.$$

By summing the above inequality from $i = i_0$ to k, we obtain

$$(1+c+c''\overline{\alpha}-\varepsilon)\frac{\rho_{i_0-1}}{1-\alpha_{i_0-1}}\sum_{i=i_0}^{k}(1-\alpha_i) \leq \frac{\rho_k}{1-\alpha_k}\frac{1}{\prod_{j=i_0}^{k}\alpha_j} - \frac{\rho_{i_0-1}}{1-\alpha_{i_0-1}}.$$

Using that $\sum_{i=1}^{+\infty}(1-\alpha_i) = +\infty$ by Proposition 3.1 (iii), this entails that

$$\lim_{k\to+\infty}\frac{\rho_k}{1-\alpha_k}\frac{1}{\prod_{j=i_0}^{k}\alpha_j} = +\infty.$$

This shows that $\prod_{j=i_0}^{k}\alpha_j = o\left(\frac{\rho_k}{1-\alpha_k}\right)$ as $k \to +\infty$, which completes the proof. □

Combining Theorem 2.14 with Proposition 3.5, we obtain the following result.

Theorem 3.6 *Under (H), assume that $\operatorname{zer} A \neq \emptyset$. Suppose that the sequences (α_k) and (ρ_k) satisfy $\alpha_k \in [0, 1[$ and $\rho_k \in]0, 2[$ for every $k \geq 1$. Let us assume that there exist $\overline{\alpha} \in [0, 1]$, $\overline{\rho} \in [0, 2[$, $c \in [0, 1[$ and $c'' \in \mathbb{R}$, with $-(1-\overline{\rho}/2) < c'' \leq -(1-\overline{\rho}/2)c$ such that*

$$\lim_{k\to+\infty}\alpha_k = \overline{\alpha}; \tag{49}$$

$$\lim_{k\to+\infty}\rho_k = \overline{\rho}; \tag{50}$$

$$\lim_{k\to+\infty}\left(\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k}\right) = c; \tag{51}$$

$$\lim_{k\to+\infty}\frac{\rho_{k+1}-\rho_k}{\rho_{k+1}(1-\alpha_k)} = c''; \tag{52}$$

$$\liminf_{k\to+\infty}\frac{(1-\alpha_k)^2}{\rho_k} > \frac{\overline{\alpha}(1+\overline{\alpha})}{2-\overline{\rho}+2c''}. \tag{53}$$

Then for any sequence (x_k) generated by (RIPA), we have

(i) $\|x_{k+1}-x_k\| = \mathcal{O}\left(\frac{\rho_k}{1-\alpha_k}\right)$ *as $k \to +\infty$.*

Assume additionally that $\frac{|\mu_{k+1}-\mu_k|}{\mu_{k+1}} = \mathcal{O}\left(\frac{\rho_k}{1-\alpha_k}\right)$ as $k \to +\infty$, together with $\sum_{i=1}^{+\infty}\frac{\rho_i}{1-\alpha_i} = +\infty$.
Then the following holds

(ii) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$. *If* $\liminf_{k\to+\infty} \mu_k > 0$, *then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

Finally assume that $\sum_{i=1}^{+\infty} \dfrac{\rho_i}{1-\alpha_i} < +\infty$. *Then we obtain*

(iii) $\sum_{i=1}^{+\infty} \|x_i - x_{i-1}\| < +\infty$, *and hence the sequence* (x_k) *converges strongly toward some* $x_\infty \in \mathcal{H}$.

Proof Let us check that the assumptions of Theorem 2.14 are satisfied. Condition (K_0) is fulfilled owing to assumption (51) and Proposition 3.1 (i). Assumptions (49)–(50)–(51)–(52)–(53) ensure that condition (K_1) is satisfied, see Proposition 3.3. Since $\overline{\rho} \in [0, 2[$, condition (K_2) holds true in view of assumption (50).
(i) Observe that

$$\begin{aligned} 1 + c + c''\overline{\alpha} &\geq 1 + c + c'' \quad \text{since } \overline{\alpha} \leq 1 \text{ and } c'' \leq 0, \\ &> c + \overline{\rho}/2 \qquad \text{because } c'' > -(1 - \overline{\rho}/2), \end{aligned}$$

hence $1 + c + c''\overline{\alpha} > 0$. Proposition 3.5 then shows that

$$\sum_{i=1}^{k} \left[\left(\prod_{j=i+1}^{k} \alpha_j \right) \rho_i \right] \sim \frac{1}{(1 + c + c''\overline{\alpha})} \frac{\rho_k}{1 - \alpha_k} \quad \text{as } k \to +\infty. \tag{54}$$

By combining this equivalence with Theorem 2.14 (i), we obtain that $\|x_{k+1} - x_k\| = \mathcal{O}\left(\frac{\rho_k}{1-\alpha_k}\right)$ as $k \to +\infty$.
(ii)–(iii) In view of (54) and the equivalence $t_{k+1} \sim \frac{1}{1-c}\frac{1}{1-\alpha_k}$ as $k \to +\infty$, we immediately see that the first condition of (K_4) is satisfied. The second condition of (K_4) is guaranteed by the assumption $\frac{|\mu_{k+1}-\mu_k|}{\mu_{k+1}} = \mathcal{O}\left(\frac{\rho_k}{1-\alpha_k}\right)$ as $k \to +\infty$. From assumption (52) and $\alpha_k \in [0, 1[$, we get

$$\rho_{k+1} - \rho_k = \mathcal{O}(\rho_{k+1}) \text{ as } k \to +\infty.$$

It ensues that $\rho_k = \mathcal{O}(\rho_{k+1})$ as $k \to +\infty$. Recalling from Proposition 3.1 (ii) that $t_{k+1} \sim t_{k+2}$ as $k \to +\infty$, we deduce immediately that the third condition of (K_4) is satisfied. Finally, condition (K_5) is fulfilled owing to the assumption $\sum_{i=1}^{+\infty} \frac{\rho_i}{1-\alpha_i} = +\infty$. Points (ii) and (iii) then follow from the corresponding points of Theorem 2.14. □

3.3 Some particular cases

Let us now particularize our results to the case $\alpha_k = 1 - \alpha/k^q$ and $\rho_k = \beta/k^r$, for some $\alpha, \beta > 0$, $q \in]0, 1[$ and $r > 0$.

Corollary 3.7 *Under* (H), *assume that* $\operatorname{zer} A \neq \emptyset$. *Suppose that* $(q, r) \in]0, 1[\times \mathbb{R}_+^*$ *is such that* $r \geq 2q$, *and that* $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *satisfies* $\alpha^2/\beta > 1$ *if* $r = 2q$ *(no* *condition if $r > 2q$). Assume that $\alpha_k = 1 - \alpha/k^q$ and $\rho_k = \beta/k^r$ for every $k \geq 1$. Then for any sequence (x_k) generated by* (RIPA), *we have*

(i) $\sum_{i=1}^{+\infty} i^{r-q} \|x_i - x_{i-1}\|^2 < +\infty$.

(ii) $\sum_{i=1}^{+\infty} \frac{1}{i^{r-q}} \|\mu_i A_{\mu_i}(y_i)\|^2 < +\infty$, *and* $\sum_{i=1}^{+\infty} \frac{1}{i^{r-q}} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.

(iii) *For any $z \in \operatorname{zer} A$, $\lim_{k\to+\infty} \|x_k - z\|$ exists, and hence (x_k) is bounded.*

(iv) $\|x_{k+1} - x_k\| = \mathcal{O}\left(\frac{1}{k^{r-q}}\right)$ *as $k \to +\infty$.*

Assume additionally that $r \leq q + 1$ and that $\frac{|\mu_{k+1}-\mu_k|}{\mu_{k+1}} = \mathcal{O}\left(\frac{1}{k^{r-q}}\right)$ as $k \to +\infty$. Then the following holds

(v) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$.

(vi) *If $\liminf_{k\to+\infty} \mu_k > 0$, then there exists $x_\infty \in \operatorname{zer} A$ such that $x_k \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $k \to +\infty$.*

Finally assume that $r > q + 1$. Then we obtain

(vii) $\sum_{i=1}^{+\infty} \|x_i - x_{i-1}\| < +\infty$, *and hence the sequence (x_k) converges strongly toward some $x_\infty \in \mathcal{H}$.*

Proof We first check that the assumptions (49), (50), (51), (52) and (53) are fulfilled. Assumptions (49)–(50) are clearly satisfied, with $\overline{\alpha} = 1$ and $\overline{\rho} = 0$ respectively. Now observe that

$$\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k} = \frac{1}{\alpha}((k+1)^q - k^q) \sim \frac{q}{\alpha} k^{q-1} \to 0 \quad \text{as } k \to +\infty,$$

where we have used $q \in]0, 1[$. Hence assumption (51) is verified with $c = 0$. On the other hand, we have

$$\frac{\rho_{k+1} - \rho_k}{\rho_{k+1}(1-\alpha_k)} = \left(\frac{1}{(k+1)^r} - \frac{1}{k^r}\right)(k+1)^r \frac{k^q}{\alpha} \sim -\frac{r}{\alpha} k^{q-1} \to 0 \quad \text{as } k \to +\infty.$$

This shows that assumption (52) is fulfilled with $c'' = 0$. Finally, hypothesis (53) amounts to

$$\liminf_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} > 1.$$

We have $(1-\alpha_k)^2/\rho_k = (\alpha^2/k^{2q})(k^r/\beta) = \frac{\alpha^2}{\beta} k^{r-2q}$, hence

$$\lim_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} = \begin{cases} +\infty & \text{if } r > 2q \\ \alpha^2/\beta & \text{if } r = 2q. \end{cases}$$

It ensues that assumption (53) is automatically satisfied if $r > 2q$, while it is equivalent to $\alpha^2/\beta > 1$ if $r = 2q$. Therefore the assumptions of Theorem 3.6 are satisfied, which implies that the hypotheses of Theorem 3.4 are also fulfilled. Points (i), (ii) and (iii) follow immediately from Theorem 3.4. Item (iv) is a consequence of Theorem 3.6 (i). Condition $\sum_{i=1}^{+\infty} \frac{\rho_i}{1-\alpha_i} = +\infty$ amounts to $r \leq q + 1$. Points (v), (vi) and (vii) can be immediately derived from the corresponding points of Theorem 3.6. □

Consider finally the case $q = 1$, thus leading to a sequence (α_k) of the form $\alpha_k = 1 - \alpha/k$. This case was recently studied by Attouch and Peypouquet [7] in connection with Nesterov's accelerated methods.

Corollary 3.8 *Under* (H), *assume that* $\operatorname{zer} A \neq \emptyset$. *Let* $r \geq 2$, $\alpha > r$ *and* $\beta > 0$ *be such that* $\beta < \alpha(\alpha - 2)$ *if* $r = 2$ *(no condition on* β *if* $r > 2$*). Assume that* $\alpha_k = 1 - \alpha/k$ *and* $\rho_k = \beta/k^r$ *for every* $k \geq 1$. *Then for any sequence* (x_k) *generated by* (RIPA), *we have*

(i) $\sum_{i=1}^{+\infty} i^{r-1} \|x_i - x_{i-1}\|^2 < +\infty$.

(ii) $\sum_{i=1}^{+\infty} \frac{1}{i^{r-1}} \|\mu_i A_{\mu_i}(y_i)\|^2 < +\infty$, *and* $\sum_{i=1}^{+\infty} \frac{1}{i^{r-1}} \|\mu_i A_{\mu_i}(x_i)\|^2 < +\infty$.

(iii) *For any* $z \in \operatorname{zer} A$, $\lim_{k\to+\infty} \|x_k - z\|$ *exists, and hence* (x_k) *is bounded.*

(iv) $\|x_{k+1} - x_k\| = \mathcal{O}\left(\frac{1}{k^{r-1}}\right)$ *as* $k \to +\infty$.

Assume additionally that $r = 2$ *and that* $\frac{|\mu_{k+1}-\mu_k|}{\mu_{k+1}} = \mathcal{O}\left(\frac{1}{k}\right)$ *as* $k \to +\infty$. *Then the following holds*

(v) $\lim_{k\to+\infty} \mu_k A_{\mu_k}(x_k) = 0$.

(vi) *If* $\liminf_{k\to+\infty} \mu_k > 0$, *then there exists* $x_\infty \in \operatorname{zer} A$ *such that* $x_k \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

Finally assume that $r > 2$. *Then we obtain*

(vii) $\sum_{i=1}^{+\infty} \|x_i - x_{i-1}\| < +\infty$, *and hence the sequence* (x_k) *converges strongly toward some* $x_\infty \in \mathcal{H}$.

Proof Assumptions (49)–(50) are clearly satisfied, with $\overline{\alpha} = 1$ and $\overline{\rho} = 0$ respectively. Now observe that

$$\frac{1}{1-\alpha_{k+1}} - \frac{1}{1-\alpha_k} = \frac{1}{\alpha}(k+1) - \frac{1}{\alpha}k = \frac{1}{\alpha},$$

hence assumption (51) is verified with $c = \frac{1}{\alpha}$. On the other hand, we have

$$\frac{\rho_{k+1} - \rho_k}{\rho_{k+1}(1-\alpha_k)} = \left(\frac{1}{(k+1)^r} - \frac{1}{k^r}\right)(k+1)^r \frac{k}{\alpha} \to -\frac{r}{\alpha} \quad \text{as } k \to +\infty.$$

This shows that assumption (52) is fulfilled with $c'' = -\frac{r}{\alpha}$. The hypothesis $-(1 - \overline{\rho}/2) < c'' \leq -(1 - \overline{\rho}/2)c$ amounts to $-1 < -\frac{r}{\alpha} \leq -\frac{1}{\alpha}$, which is in turn equivalent to $1 \leq r < \alpha$. This holds true in view of the assumptions of Corollary 3.8. Finally, hypothesis (53) can be rewritten as

$$\liminf_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} > \frac{1}{1 - r/\alpha} = \frac{\alpha}{\alpha - r}.$$

We have $(1-\alpha_k)^2/\rho_k = (\alpha^2/k^2)(k^r/\beta) = \frac{\alpha^2}{\beta} k^{r-2}$, hence

$$\lim_{k\to+\infty} \frac{(1-\alpha_k)^2}{\rho_k} = \begin{cases} +\infty & \text{if } r > 2 \\ \alpha^2/\beta & \text{if } r = 2. \end{cases}$$

It ensues that assumption (53) is automatically satisfied if $r > 2$, while it is equivalent to $\alpha(\alpha-2) > \beta$ if $r = 2$. Points (i), (ii) and (iii) follow immediately from Theorem 3.4. Item (iv) is a consequence of Theorem 3.6 (i). Condition $\sum_{i=1}^{+\infty} \frac{\rho_i}{1-\alpha_i} = +\infty$ amounts to $r \leq 2$, which boils down to $r = 2$. Points (v), (vi) and (vii) can be immediately derived from the corresponding points of Theorem 3.6. □

The case $r = 2$ corresponds to the situation studied by Attouch and Peypouquet [7]. More precisely, they considered the case

$$\alpha_k = 1 - \frac{\alpha}{k}, \quad \rho_k = \frac{s}{\lambda_k + s} \quad \text{and} \quad \mu_k = \lambda_k + s,$$

where $\alpha, s > 0$ and $\lambda_k = (1+\varepsilon)\frac{s}{\alpha^2}k^2$, for some $\varepsilon > 0$. Let us recall their result, that can be obtained as a direct consequence of Theorems 3.4 and 3.6. The details are left to the reader.

Theorem 3.9 (Attouch–Peypouquet [7]) *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator such that $\operatorname{zer} A \neq \emptyset$. Let (x_k) be a sequence generated by the Regularized Inertial Proximal Algorithm*

$$(\mathrm{RIPA})_{\alpha,s} \qquad \begin{cases} y_k &= x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} &= \frac{\lambda_k}{\lambda_k + s} y_k + \frac{s}{\lambda_k + s} J_{(\lambda_k + s)A}(y_k). \end{cases}$$

Suppose that $\alpha > 2$, $s > 0$, $\varepsilon > \frac{2}{\alpha-2}$, and $\lambda_k = (1+\varepsilon)\frac{s}{\alpha^2}k^2$ for all $k \geq 1$. Then,

(i) $\|x_{k+1} - x_k\| = \mathcal{O}(\frac{1}{k})$ *as $k \to +\infty$, and* $\sum_{k=1}^{+\infty} k\|x_k - x_{k-1}\|^2 < +\infty$.
(ii) *There exists $x_\infty \in \operatorname{zer} A$ such that $x_k \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $k \to +\infty$.*
(iii) *The sequence (y_k) converges weakly in $\mathcal{H}$ to x_∞, as $k \to +\infty$.*

The following table gives a synthetic view of some of the situations studied previously (the large number of cases does not allow to enter all of them). Each column gives the joint tuning of the parameters α_k, ρ_k, and μ_k, which provides the convergence of the iterates generated by (RIPA). For ease of reading, we recall the definition of (RIPA)

$$(\mathrm{RIPA}) \qquad \begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}) \\ x_{k+1} = (1-\rho_k)y_k + \rho_k J_{\mu_k A}(y_k). \end{cases}$$

From left to right, the table is ordered according to the decreasing values of α_k.
As noticed before, we can observe the balance between the inertial effect and the relaxation effect. As α_k gets closer to one, the relaxation parameter ρ_k gets closer to zero.

α_k	$\alpha_k = 1 - \frac{\alpha}{k}$ $\alpha > 2$	$\alpha_k = 1 - \frac{\alpha}{k^q}$ $q \in]0,1[,\ \alpha > 0$	$\alpha_k = 1 - \frac{\alpha}{k^q}$ $q \in]0,1[,\ \alpha > 0$	$\alpha_k \equiv \alpha \in [0, 1[$
ρ_k	$\rho_k = \frac{\beta}{k^2}$, $\beta < \alpha(\alpha - 2)$	$\rho_k = \frac{\beta}{k^{2q}},\ \beta < \alpha^2$	$\rho_k = \frac{\beta}{k^r}$, $2q < r \le q + 1,\ \beta > 0$	$\rho_k \equiv \rho < \frac{2(1-\alpha)^2}{2\alpha^2-\alpha+1}$
μ_k	$\frac{\lvert\mu_{k+1}-\mu_k\rvert}{\mu_{k+1}} = \mathcal{O}\left(\frac{1}{k}\right)$ $\liminf \mu_k > 0$	$\frac{\lvert\mu_{k+1}-\mu_k\rvert}{\mu_{k+1}} = \mathcal{O}\left(\frac{1}{k^q}\right)$ $\liminf \mu_k > 0$	$\frac{\lvert\mu_{k+1}-\mu_k\rvert}{\mu_{k+1}} = \mathcal{O}\left(\frac{1}{k^{r-q}}\right)$ $\liminf \mu_k > 0$	$\liminf \mu_k > 0$

4 Ergodic convergence results

4.1 Ergodic variant of the Opial lemma

An ergodic version of the Opial lemma was derived by Passty [26] in the case of the averaging process defined by

$$\widehat{x}_k = \frac{1}{\sum_{i=1}^k s_i} \sum_{i=1}^k s_i x_i,$$

where (s_k) is a sequence of positive steps. In order to deal with a more general averaging process, let us consider a double sequence $(\tau_{i,k})_{i,k\geq 1}$ of nonnegative numbers satisfying the following assumptions

$$\sum_{i=1}^{+\infty} \tau_{i,k} = 1 \quad \text{for every } k \geq 1 \tag{55}$$

$$\lim_{k\to+\infty} \tau_{i,k} = 0 \quad \text{for every } i \geq 1. \tag{56}$$

To each bounded sequence (x_k) of $\mathcal{H}$, we associate the averaged sequence $(\widehat{x}_k)$ by

$$\widehat{x}_k = \sum_{i=1}^{+\infty} \tau_{i,k} x_i. \tag{57}$$

Lemma B.2 in the appendix shows that the sequence $(\widehat{x}_k)$ is well-defined, bounded and that convergence of (x_k) implies convergence of $(\widehat{x}_k)$ as $k \to +\infty$ toward the same limit (Cesaro property). The extension of Opial lemma to a general averaging process satisfying (55) and (56) is given hereafter. This result can be obtained as a consequence of the generalized Opial lemma established by Brézis–Browder, see [14, Lemma 1]. For the sake of the reader, we give an independent and self-contained proof.

Proposition 4.1 *Let S be a nonempty subset of $\mathcal{H}$ and let (x_k) be a bounded sequence of $(\mathcal{H})$. Let $(\tau_{i,k})$ be a double sequence of nonnegative numbers satisfying (55) and (56), and let $(\widehat{x}_k)$ be the averaged sequence defined by (57). Assume that*

(i) *For every $z \in S$, $\lim_{k\to+\infty} \|x_k - z\|$ exists;*

(ii) *every weak limit point of the sequence* $(\widehat{x}_k)$ *belongs to* S.

Then the sequence $(\widehat{x}_k)$ *converges weakly as* $k \to +\infty$ *toward some* $x_\infty \in S$.

Proof From Lemma B.2 (i), the sequence $(\widehat{x}_k)$ is bounded, therefore it is enough to establish the uniqueness of weak limit points. Let $(\widehat{x}_{k_n})$ and $(\widehat{x}_{k_m})$ be two weakly converging subsequences satisfying respectively $\widehat{x}_{k_n} \rightharpoonup \overline{x}_1$ as $n \to +\infty$ and $\widehat{x}_{k_m} \rightharpoonup \overline{x}_2$ as $m \to +\infty$. From (ii), the weak limit points $\overline{x}_1$ and $\overline{x}_2$ belong to S. From (i), we deduce that $\lim_{k\to+\infty} \|x_k - \overline{x}_1\|^2$ and $\lim_{k\to+\infty} \|x_k - \overline{x}_2\|^2$ exist. Writing that

$$\|x_k - \overline{x}_1\|^2 - \|x_k - \overline{x}_2\|^2 = 2\left\langle x_k - \frac{\overline{x}_1 + \overline{x}_2}{2}, \overline{x}_2 - \overline{x}_1 \right\rangle,$$

we infer that $\lim_{k\to+\infty} \langle x_k, \overline{x}_2 - \overline{x}_1\rangle$ exists. Observe that

$$\begin{aligned}\langle \widehat{x}_k, \overline{x}_2 - \overline{x}_1\rangle &= \left\langle \sum_{i=1}^{+\infty} \tau_{i,k} x_i, \overline{x}_2 - \overline{x}_1 \right\rangle \\ &= \sum_{i=1}^{+\infty} \tau_{i,k} \langle x_i, \overline{x}_2 - \overline{x}_1\rangle .\end{aligned}$$

By applying Lemma B.2 (ii) to the real sequence $\left(\langle x_k, \overline{x}_2 - \overline{x}_1\rangle\right)$, we deduce that $\lim_{k\to+\infty} \langle \widehat{x}_k, \overline{x}_2 - \overline{x}_1\rangle$ exists. This implies that

$$\lim_{n\to+\infty} \langle \widehat{x}_{k_n}, \overline{x}_2 - \overline{x}_1\rangle = \lim_{m\to+\infty} \langle \widehat{x}_{k_m}, \overline{x}_2 - \overline{x}_1\rangle,$$

which entails that $\langle \overline{x}_1, \overline{x}_2 - \overline{x}_1\rangle = \langle \overline{x}_2, \overline{x}_2 - \overline{x}_1\rangle$. Therefore $\|\overline{x}_2 - \overline{x}_1\|^2 = 0$, which ends the proof. □

Remark 4.2 By taking $(\tau_{i,k})$ defined by

$$\tau_{i,k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

conditions (55) and (56) are trivially satisfied and we find $\widehat{x}_k = x_k$ for every $k \geq 1$. It ensues that the Opial lemma appears as a particular case of Proposition 4.1.

4.2 Ergodic convergence of the iterates

To each sequence (x_k) generated by (RIPA), we associate a suitable averaged sequence as in (57). The weight coefficients are judiciously chosen and depend on α_k, μ_k and ρ_k. Under conditions (K_0)–(K_1)–(K_2)–(K_3), we show that the averaged sequence converges weakly toward some zero of the operator A.

Theorem 4.3 *Under* (H), *assume that* $\operatorname{zer} A \neq \emptyset$. *Suppose that* $\alpha_k \in [0, 1]$ *and* $\rho_k \in]0, 2]$ *for every* $k \geq 1$. *Under* (K_0), *let* $(t_{i,k})$ *and* (t_i) *be the sequences respectively defined by* (13) *and* (15). *Assume that conditions* (K_1)–(K_2)–(K_3) *hold, together with*

$$\sum_{i=1}^{+\infty} t_i \rho_{i-1} \mu_{i-1} = +\infty. \tag{58}$$

Let us define the sequence $(\tau_{i,k})$ *by*

$$\tau_{i,k} = \frac{t_{i,k} \rho_{i-1} \mu_{i-1}}{\sum_{i=1}^{k} t_{i,k} \rho_{i-1} \mu_{i-1}}. \tag{59}$$

Then for any sequence (x_k) *generated by* (RIPA)*, there exists* $x_\infty \in \operatorname{zer} A$ *such that*

$$\widehat{x}_k = \sum_{i=1}^{k} \tau_{i,k} x_i \rightharpoonup x_\infty \quad \textit{weakly in } \mathcal{H} \textit{ as } k \to +\infty.$$

Proof The proof relies on Proposition 4.1 applied with $S = \operatorname{zer} A$. Let us first check that conditions (55) and (56) are satisfied for the sequence $(\tau_{i,k})$ given by (59). Property (55) follows immediately from the definition of $(\tau_{i,k})$ (recall that $t_{i,k} = 0$ for $i > k$). On the other hand, observe that for every $i, k \geq 1$,

$$\tau_{i,k} \leq \frac{t_i \rho_{i-1} \mu_{i-1}}{\sum_{i=1}^{k} t_{i,k} \rho_{i-1} \mu_{i-1}}. \tag{60}$$

The quantity $t_i \rho_{i-1} \mu_{i-1}$ is finite and independent of k. Since $t_{i,k}$ tends increasingly toward t_i as $k \to +\infty$, the monotone convergence theorem implies that

$$\lim_{k \to +\infty} \sum_{i=1}^{k} t_{i,k} \rho_{i-1} \mu_{i-1} = \lim_{k \to +\infty} \sum_{i=1}^{+\infty} t_{i,k} \rho_{i-1} \mu_{i-1} = \sum_{i=1}^{+\infty} t_i \rho_{i-1} \mu_{i-1} = +\infty, \tag{61}$$

where we have used the assumption (58). We then deduce from the inequality (60) that $\lim_{k \to +\infty} \tau_{i,k} = 0$, which establishes (56).

We now have to prove that the conditions (i) and (ii) of Proposition 4.1 are fulfilled. Condition (i) is realized in view of Theorem 2.6 (iii). Let us now assume that there exist $x_\infty \in \mathcal{H}$ and a sequence (k_n) such that $k_n \to +\infty$ and $\widehat{x}_{k_n} \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $n \to +\infty$. Let us fix $(z, q) \in \operatorname{gph} A$ and define the sequence (h_k) by $h_k = \frac{1}{2}\|x_k - z\|^2$. From inequality (10) of Lemma 2.2, we have

$$\begin{aligned} & h_{k+1} - h_k - \alpha_k (h_k - h_{k-1}) + \rho_k \mu_k \left\langle x_{k+1} + \left(\frac{1}{\rho_k} - 1 \right) (x_{k+1} - y_k) - z, q \right\rangle \\ & \leq \alpha_k \|x_k - x_{k-1}\|^2, \end{aligned}$$

because the assumptions $\alpha_k \in [0, 1]$ and $\rho_k \in]0, 2]$ imply respectively $\frac{1}{2}(\alpha_k + \alpha_k^2) \leq \alpha_k$ and $\frac{2-\rho_k}{2\rho_k} \geq 0$. Since $x_{k+1} = y_k - \rho_k \mu_k A_{\mu_k}(y_k)$, the above inequality can be rewritten as

$$h_{k+1} - h_k - \alpha_k(h_k - h_{k-1}) + \rho_k \mu_k \left\langle x_{k+1} - z - (1-\rho_k)\mu_k A_{\mu_k}(y_k), q \right\rangle \\ \leq \alpha_k \|x_k - x_{k-1}\|^2. \tag{62}$$

Setting $a_k = h_k - h_{k-1}$ and

$$w_k = \alpha_k \|x_k - x_{k-1}\|^2 - \rho_k \mu_k \left\langle x_{k+1} - z - (1-\rho_k)\mu_k A_{\mu_k}(y_k), q \right\rangle,$$

inequality (62) amounts to $a_{k+1} \leq \alpha_k a_k + w_k$. By applying Lemma B.1 (i), we obtain for every $k \geq 1$,

$$\begin{aligned} h_k - h_0 = \sum_{i=1}^{k} a_i &\leq t_{1,k}(h_1 - h_0) + \sum_{i=1}^{k-1} t_{i+1,k} w_i \\ &= t_{1,k}(h_1 - h_0) + \sum_{i=1}^{k-1} t_{i+1,k} \Big[\alpha_i \|x_i - x_{i-1}\|^2 \\ &\quad -\rho_i \mu_i \Big\langle x_{i+1} - z - (1-\rho_i)\mu_i A_{\mu_i}(y_i), q \Big\rangle \Big]. \end{aligned}$$

Since $h_k \geq 0$ and $t_{i+1,k} \leq t_{i+1}$, we deduce that

$$\begin{aligned} \sum_{i=1}^{k-1} t_{i+1,k} \rho_i \mu_i \langle x_{i+1} - z, q \rangle \leq h_0 + t_{1,k}(h_1 - h_0) &+ \sum_{i=1}^{k-1} t_{i+1} \alpha_i \|x_i - x_{i-1}\|^2 \\ &+ \sum_{i=1}^{k-1} t_{i+1,k} \rho_i \mu_i \left\langle (1-\rho_i)\mu_i A_{\mu_i}(y_i), q \right\rangle. \end{aligned}$$

Recalling from Theorem 2.6 (i) that $\sum_{i=1}^{+\infty} t_{i+1} \alpha_i \|x_i - x_{i-1}\|^2 < +\infty$, we infer that for every $k \geq 1$,

$$\sum_{i=1}^{k-1} t_{i+1,k} \rho_i \mu_i \langle x_{i+1} - z, q \rangle \leq C + \sum_{i=1}^{k-1} t_{i+1,k} \rho_i \mu_i \left\langle (1-\rho_i)\mu_i A_{\mu_i}(y_i), q \right\rangle,$$

where we have set $C := h_0 + t_1|h_1 - h_0| + \sum_{i=1}^{+\infty} t_{i+1} \alpha_i \|x_i - x_{i-1}\|^2 < +\infty$. Since $\rho_k \in]0, 2]$, according to the Cauchy–Schwarz inequality we have that

$$|\left\langle (1-\rho_i)\mu_i A_{\mu_i}(y_i), q \right\rangle| \leq \|\mu_i A_{\mu_i}(y_i)\| \|q\|.$$

It ensues that

$$\sum_{i=1}^{k-1} t_{i+1,k} \rho_i \mu_i \langle x_{i+1} - z, q \rangle \leq C + \|q\| \sum_{i=1}^{k-1} t_{i+1,k} \rho_i \mu_i \|\mu_i A_{\mu_i}(y_i)\|.$$

By shifting the index of summation, we deduce from the above inequality that

$$\begin{aligned}\sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}\langle x_i - z, q\rangle &\le C + \|q\| \sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}\|\mu_{i-1}A_{\mu_{i-1}}(y_{i-1})\| \\ &\quad + t_{1,k}\rho_0\mu_0\langle x_1 - z, q\rangle - \|q\|\, t_{1,k}\rho_0\mu_0\|\mu_0 A_{\mu_0}(y_0)\| \\ &\le C' + \|q\| \sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}\|\mu_{i-1}A_{\mu_{i-1}}(y_{i-1})\|,\end{aligned}$$

where we have set $C' := C + t_1\rho_0\mu_0|\langle x_1 - z, q\rangle|$. This can be rewritten as

$$\left\langle \sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}(x_i - z), q \right\rangle \le C' + \|q\| \sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}\|\mu_{i-1}A_{\mu_{i-1}}(y_{i-1})\|.$$

Dividing by $\sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}$, we find

$$\begin{aligned}\langle \widehat{x}_k - z, q\rangle \le\ & \frac{C'}{\sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}} \\ &+ \frac{\|q\|}{\sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}} \sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}\|\mu_{i-1}A_{\mu_{i-1}}(y_{i-1})\|. \quad (63)\end{aligned}$$

By Theorem 2.6 (iv) we have $\lim_{k\to+\infty}\|\mu_k A_{\mu_k}(y_k)\| = 0$. From the Cesaro property, we infer that

$$\frac{1}{\sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}} \sum_{i=1}^{k} t_{i,k}\rho_{i-1}\mu_{i-1}\|\mu_{i-1}A_{\mu_{i-1}}(y_{i-1})\| \longrightarrow 0 \quad \text{as } k \to +\infty,$$

see Lemma B.2. Using (61) and taking the upper limit as $k \to +\infty$ in inequality (63), we then obtain

$$\limsup_{k\to+\infty} \langle \widehat{x}_k - z, q\rangle \le 0.$$

Since $\widehat{x}_{k_n} \rightharpoonup x_\infty$ weakly in $\mathcal{H}$ as $n \to +\infty$, we have $\langle \widehat{x}_{k_n} - z, q\rangle \to \langle x_\infty - z, q\rangle$ as $n \to +\infty$. From what precedes, we deduce that $\langle x_\infty - z, q\rangle \le 0$. Since this is true for every $(z, q) \in \operatorname{gph} A$, and since the operator A is maximally monotone, we infer that $0 \in A(x_\infty)$. We have proved that $x_\infty \in \operatorname{zer} A$, which shows that condition (ii) of Proposition 4.1 is satisfied. The proof is complete. □

Let us now apply Theorem 4.3 to the case $\alpha_k = 0$ for every $k \ge 1$. In this case, assumptions (K_0) and (K_1) are trivially satisfied, and moreover $t_i = t_{i,k} = 1$ for every $i \ge 1$ and $k \ge i$. We then obtain the following corollary of Theorem 4.3.

Corollary 4.4 *Under* (H), *assume that* $\operatorname{zer} A \neq \emptyset$. *Suppose moreover that* $\limsup_{k\to+\infty} \rho_k < 2$ *and* $\liminf_{k\to+\infty} \rho_k > 0$, *together with* $\sum_{i=0}^{+\infty} \rho_i \mu_i = +\infty$. *Then for any sequence* (x_k) *generated by* (RPA)

$$x_{k+1} = (1-\rho_k)x_k + \rho_k J_{\mu_k A}(x_k), \tag{RPA}$$

there exists $x_\infty \in \operatorname{zer} A$ *such that*

$$\frac{1}{\sum_{i=0}^{k} \rho_i \mu_i} \sum_{i=0}^{k} \rho_i \mu_i x_i \rightharpoonup x_\infty \quad \textit{weakly in } \mathcal{H} \textit{ as } k \to +\infty. \tag{64}$$

Proof From Theorem 4.3, we obtain that

$$\widehat{x}_k = \frac{1}{\sum_{i=1}^{k} \rho_{i-1}\mu_{i-1}} \sum_{i=1}^{k} \rho_{i-1}\mu_{i-1} x_i \rightharpoonup x_\infty \quad \text{weakly in } \mathcal{H} \text{ as } k \to +\infty.$$

We deduce immediately that

$$\frac{1}{\sum_{i=0}^{k} \rho_i \mu_i} \sum_{i=0}^{k} \rho_i \mu_i x_{i+1} \rightharpoonup x_\infty \quad \text{weakly in } \mathcal{H} \text{ as } k \to +\infty. \tag{65}$$

Recall from Corollary 2.8 (i) that $\sum_{i=1}^{+\infty} \frac{2-\rho_i}{\rho_i} \|x_{i+1} - x_i\|^2 < +\infty$. Since $\limsup_{k\to+\infty} \rho_k < 2$, this implies that $\sum_{i=1}^{+\infty} \|x_{i+1} - x_i\|^2 < +\infty$, which entails in turn that $\lim_{k\to+\infty} \|x_{k+1} - x_k\| = 0$. From the Cesaro property, we infer that

$$\frac{1}{\sum_{i=0}^{k} \rho_i \mu_i} \sum_{i=0}^{k} \rho_i \mu_i (x_{i+1} - x_i) \longrightarrow 0 \quad \text{strongly in } \mathcal{H} \text{ as } k \to +\infty. \tag{66}$$

By putting together (65) and (66), we immediately obtain (64). □

If we assume moreover that $\rho_k = 1$ for every $k \geq 1$, we recover a classical result of ergodic convergence for the proximal point algorithm, see the seminal paper of Brezis and Lions, see [15, Remarque 10].

Corollary 4.5 *Under* (H), *assume that* $\operatorname{zer} A \neq \emptyset$ *and that* $\sum_{i=0}^{+\infty} \mu_i = +\infty$. *Then for any sequence* (x_k) *generated by the algorithm*

$$x_{k+1} = J_{\mu_k A}(x_k), \tag{PA}$$

there exists $x_\infty \in \operatorname{zer} A$ *such that* $\frac{1}{\sum_{i=0}^{k} \mu_i} \sum_{i=0}^{k} \mu_i x_i \rightharpoonup x_\infty$ *weakly in* $\mathcal{H}$ *as* $k \to +\infty$.

5 Conclusion, perspective

The introduction of inertial features into proximal-based algorithms to solve general monotone inclusions is a long-standing difficult problem. (RIPA) algorithm, which addresses these issues, involves three basic parameters, α_k, μ_k, ρ_k, which depend on the iteration index k, and which take into account respectively the inertia, the proximal step size, and the relaxation. (RIPA) provides a general framework for understanding the subtle tuning of these different parameters to achieve the weak convergence of the iterates. In particular, we obtained convergence results based on the Nesterov acceleration method, in the context of maximally monotone operators, which extend the recent result of Attouch–Peypouquet [7]. Several basic splitting algorithms in optimization, naturally rely on the maximally monotone approach, such as ADMM, primal-dual methods, Douglas-Rachford. Our results provide a general way to understand the acceleration of these algorithms via inertia. Several important questions remain to be studied, such as obtaining splitting methods in this context, and studying the convergence rate of these methods. In this respect, it would be important to study the case $A = \partial\Psi$ where Ψ is a closed convex function, thus recovering the rate of convergence of the values for Nesterov methods.

Appendix A. Yosida regularization

A set-valued mapping A from $\mathcal{H}$ to $\mathcal{H}$ assigns to each $x \in \mathcal{H}$ a set $A(x) \subset \mathcal{H}$, hence it is a mapping from $\mathcal{H}$ to $2^{\mathcal{H}}$. Every set-valued mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$ can be identified with its graph defined by

$$\operatorname{gph} A = \{(x, u) \in \mathcal{H} \times \mathcal{H} : \, u \in A(x)\}.$$

The set $\{x \in \mathcal{H} : \; 0 \in A(x)\}$ of the zeros of A is denoted by $\operatorname{zer} A$. An operator $A : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be monotone if for any (x, u), $(y, v) \in \operatorname{gph} A$, one has $\langle y - x, v - u\rangle \geq 0$. It is maximally monotone if there exists no monotone operator whose graph strictly contains $\operatorname{gph} A$. If a single-valued operator $A : \mathcal{H} \to \mathcal{H}$ is continuous and monotone, then it is maximally monotone, *cf.* [13, Proposition 2.4].

Given a maximally monotone operator A and $\lambda > 0$, the *resolvent* of A with index λ and the *Yosida regularization* of A with parameter λ are defined by

$$J_{\lambda A} = (I + \lambda A)^{-1} \qquad \text{and} \qquad A_\lambda = \frac{1}{\lambda}\left(I - J_{\lambda A}\right),$$

respectively. The operator $J_{\lambda A} : \mathcal{H} \to \mathcal{H}$ is nonexpansive and everywhere defined (indeed it is firmly non-expansive). Moreover, A_λ is λ-cocoercive: for all $x, y \in \mathcal{H}$ we have

$$\langle A_\lambda y - A_\lambda x, y - x\rangle \geq \lambda\|A_\lambda y - A_\lambda x\|^2.$$

This property immediately implies that $A_\lambda : \mathcal{H} \to \mathcal{H}$ is $\frac{1}{\lambda}$-Lipschitz continuous. Another property that proves useful is the *resolvent equation* (see, for example, [13, Proposition 2.6] or [8, Proposition 23.6])

$$(A_\lambda)_\mu = A_{(\lambda+\mu)},$$

which is valid for any $\lambda, \mu > 0$. This property allows to compute simply the resolvent of A_λ by

$$J_{\mu A_\lambda} = \frac{\lambda}{\lambda+\mu} I + \frac{\mu}{\lambda+\mu} J_{(\lambda+\mu)A},$$

for any $\lambda, \mu > 0$. Also note that for any $x \in \mathcal{H}$, and any $\lambda > 0$ $A_\lambda(x) \in A(J_{\lambda A}x) = A(x - \lambda A_\lambda(x))$. Finally, for any $\lambda > 0$, A_λ and A have the same solution set, $\operatorname{zer} A_\lambda = \operatorname{zer} A$. For a detailed presentation of the maximally monotone operators and the Yosida approximation, the reader can consult [8] or [13].

Appendix B. Some auxiliary results

In this section, we present some auxiliary lemmas that are used throughout the paper.

Lemma B.1 *Let (a_k), (α_k) and (w_k) be sequences of real numbers satisfying*

$$a_{i+1} \leq \alpha_i a_i + w_i \quad \textit{for every } i \geq 1. \tag{67}$$

Assume that $\alpha_i \geq 0$ for every $i \geq 1$.

(i) *For every $k \geq 1$, we have*

$$\sum_{i=1}^{k} a_i \leq t_{1,k} a_1 + \sum_{i=1}^{k-1} t_{i+1,k} w_i, \tag{68}$$

where the double sequence $(t_{i,k})$ is defined by (13).

(ii) *Under (K_0), assume that the sequence (t_i) defined by (15) satisfies $\sum_{i=1}^{+\infty} t_{i+1}(w_i)_+ < +\infty$. Then the series $\sum_{i\geq 1}(a_i)_+$ is convergent, and*

$$\sum_{i=1}^{+\infty}(a_i)_+ \leq t_1(a_1)_+ + \sum_{i=1}^{+\infty} t_{i+1}(w_i)_+.$$

Proof (i) Recall from Lemma 2.4 (i) that $\alpha_i t_{i+1,k} = t_{i,k} - 1$ for every $i \geq 1$ and $k \geq i+1$. Multiplying inequality (67) by $t_{i+1,k}$ gives

$$t_{i+1,k} a_{i+1} \leq (t_{i,k} - 1) a_i + t_{i+1,k} w_i,$$

or equivalently

$$a_i \leq (t_{i,k} a_i - t_{i+1,k} a_{i+1}) + t_{i+1,k} w_i .$$

By summing from $i = 1$ to $k - 1$, we deduce that

$$\sum_{i=1}^{k-1} a_i \leq t_{1,k} a_1 - t_{k,k} a_k + \sum_{i=1}^{k-1} t_{i+1,k} w_i .$$

Since $t_{k,k} = 1$, inequality (68) follows immediately.
(ii) Taking the positive part of each member of (67), we find

$$(a_{i+1})_+ \leq \alpha_i (a_i)_+ + (w_i)_+ .$$

By applying (i) with $(a_i)_+$ (resp. $(w_i)_+$) in place of a_i (resp. w_i), we obtain for every $k \geq 1$

$$\sum_{i=1}^{k} (a_i)_+ \leq t_{1,k} (a_1)_+ + \sum_{i=1}^{k-1} t_{i+1,k} (w_i)_+ \leq t_1 (a_1)_+ + \sum_{i=1}^{+\infty} t_{i+1} (w_i)_+ < +\infty,$$

because $t_{i+1,k} \leq t_{i+1}$, and $\sum_{i=1}^{+\infty} t_{i+1} (w_i)_+ < +\infty$ by assumption. Then just let k tend to $+\infty$. □

Given a bounded sequence (x_k) of a Banach space $(\mathcal{X}, \|.\|)$, the next lemma gives basic properties of the averaged sequence $(\widehat{x}_k)$ defined by (57).

Lemma B.2 *Let $(\mathcal{X}, \|.\|)$ be a Banach space and let (x_k) be a bounded sequence of $\mathcal{X}$. Given a sequence $(\tau_{i,k})_{i,k \geq 1}$ of nonnegative numbers satisfying (55)–(56), let $(\widehat{x}_k)$ be the averaged sequence defined by $\widehat{x}_k = \sum_{i=1}^{+\infty} \tau_{i,k} x_i$. Then we have*

- (i) *The sequence $(\widehat{x}_k)$ is well-defined, bounded and $\sup_{k \geq 1} \|\widehat{x}_k\| \leq \sup_{k \geq 1} \|x_k\|$.*
- (ii) *If (x_k) converges toward $\overline{x} \in \mathcal{X}$, then the sequence $(\widehat{x}_k)$ is also convergent and $\lim_{k \to +\infty} \widehat{x}_k = \overline{x}$.*

Proof (i) Set $M = \sup_{k \geq 1} \|x_k\| < +\infty$. In view of (55), observe that for every $k \geq 1$,

$$\sum_{i=1}^{+\infty} \tau_{i,k} \|x_i\| \leq M \sum_{i=1}^{+\infty} \tau_{i,k} = M. \tag{69}$$

Since the space $\mathcal{X}$ is complete, we classically deduce that the series $\sum_{i \geq 1} \tau_{i,k} x_i$ is convergent. From the definition of $\widehat{x}_k$, we then have $\|\widehat{x}_k\| \leq \sum_{i=1}^{+\infty} \tau_{i,k} \|x_i\|$, and hence $\|\widehat{x}_k\| \leq M$ in view of (69).

(ii) Assume that (x_k) converges toward $\overline{x} \in \mathcal{X}$. By using (55), we have for every $k \geq 1$,

$$\|\widehat{x}_k - \overline{x}\| = \left\| \sum_{i=1}^{+\infty} \tau_{i,k}(x_i - \overline{x}) \right\| \leq \sum_{i=1}^{+\infty} \tau_{i,k} \|x_i - \overline{x}\|.$$

Fix $\varepsilon > 0$, and let $K \geq 1$ such that $\|x_i - \overline{x}\| \leq \varepsilon$ for every $i \geq K$. From the above inequality, we obtain

$$\|\widehat{x}_k - \overline{x}\| \leq \left(\sup_{i \in \{1,\dots,K\}} \|x_i - \overline{x}\| \right) \left(\sum_{i=1}^{K} \tau_{i,k} \right) + \varepsilon \sum_{i=K+1}^{+\infty} \tau_{i,k} \leq M \sum_{i=1}^{K} \tau_{i,k} + \varepsilon,$$

with $M = \sup_{i \geq 1} \|x_i - \overline{x}\| < +\infty$. Taking the upper limit as $k \to +\infty$, we deduce from (56) that

$$\limsup_{k \to +\infty} \|\widehat{x}_k - \overline{x}\| \leq \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we conclude that $\lim_{k \to +\infty} \|\widehat{x}_k - \overline{x}\| = 0$. $\square$

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