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FULL LENGTH PAPER

Choquet representability of submodular functions

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Abstract Let Ω be an arbitrary set, equipped with an algebra $\mathcal{A} \subseteq 2^{\Omega}$ and let $f : B(\mathcal{A}) \to \mathbb{R}$ be a functional defined on the set $B(\mathcal{A})$ of bounded measurable functions $x : \Omega \to \mathbb{R}$. We provide necessary and sufficient conditions for a submodular functional f to be representable as a Choquet integral. From standard properties of the Choquet integral the functional f should be positively homogeneous and constant additive. Our first result shows that these two properties, together with submodularity, characterize a subadditive Choquet integral, when Ω is finite. In the general case, f is a subadditive Choquet integral if and only if it satisfies the three previous properties, together with sup-norm continuity. This result provides another characterization of subadditive Choquet integrals different from the seminal paper by Schmeidler (Proc Am Math Soc 97(2):255–261, 1986) that relies on comonotonic additivity.

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1 Introduction

Let Ω be a set, equipped with an algebra $\mathcal{A} \subseteq 2^{\Omega}$ of subsets of Ω, let $B(\mathcal{A})$ be the set of bounded measurable functions $x : \Omega \to \mathbb{R}$, and denote by $\mathbf{1}_{\Omega}$ the constant

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function on Ω equal to 1. The goal of this paper is to provide necessary and sufficient conditions for $f : B(\mathcal{A}) \to \mathbb{R}$ to be a subadditive Choquet functional, the class of functionals in which Choquet [4] was the most interested, that is, f is subadditive, i.e., $f(x + y) \leq f(x) + f(y)$ for all x, y in $B(\mathcal{A})$ such that $x + y \in B(\mathcal{A})$, and, there exists some set function $v : \mathcal{A} \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ such that for all $x \in B(\mathcal{A})$, $f(x)$ is the Choquet integral of x with respect to v. Interestingly, positive homogeneity and constant additivity (i.e., $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)$ for all $t > 0$, all $x \in B(\mathcal{A})$) are the two basic properties enjoyed by Choquet functionals. Moreover, a Choquet functional is subadditive if and only if it is submodular, that is, $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$ for all x, y in $B(\mathcal{A})$, where the lattice operations on $B(\mathcal{A})$ are defined for its point-wise order. Our first result (Theorem 2.1) shows that the properties of positive homogeneity and constant additivity, together with submodularity characterize subadditive Choquet integrals, when Ω is finite. Our second result (Theorem 2.2) shows that sup-norm continuity of the functional needs merely to be added to the three previous properties to get the characterization in the general case.

Our results provide another characterization of subadditive Choquet integrals, different from the seminal one by Schmeidler [12] that relies on comonotonic additivity,[1] a fundamental property satisfied by the Choquet integral, as proved by Dellacherie [5]. Assuming that f is monotonic, then f is a subadditive Choquet functional if and only if f is subadditive and comonotonic additive (Schmeidler [12]) and also, from our result, if and only if it is positively homogeneous, constant additivity, and submodular. The proof of our result is direct without invoking comonotonic additivity. Moreover, the monotonicity assumption is not assumed in our characterization results.

We now show the relationship with the literature dealing with an old conjecture by Choquet [4] who claimed that on certain lattice cones, every positively homogeneous and submodular functional is subadditive, a result that was known since Choquet when f is twice continuously differentiable (see Huber [7]). This conjecture has been recently proved in full generality by König [8] for the positive cone $\mathbb{R}^n_+$ endowed with the point-wise order and by Marinacci and Montrucchio [10] in the case of hyper-Archimedean Riesz spaces, thus allowing orders on the space other than the point-wise one. In [10], they also provide another result showing that continuous, constant additive and submodular functionals are also subadditive when they are defined on $\mathbb{R}^n_+$, or on $\mathbb{R}^n$, and more generally on a hyper-Archimedean Riesz space. Our main result is more specific in terms of the space $B(\mathcal{A})$ we consider, but more precise in its conclusion, namely a submodular functional $f : B(\mathcal{A}) \to \mathbb{R}$ is a subadditive Choquet functional if and only if it is additionally assumed to be positively homogeneous, constant additive, and sup-norm continuous (unless Ω is finite).

We recall some notations used throughout the paper. Let Ω be an arbitrary set, we let $\mathbb{R}^\Omega$ be the vector space of functions $x : \Omega \to \mathbb{R}$. Then $\mathbb{R}^\Omega_+$ denotes the set of non-negative functions $x \geq 0$, that is, $x(\omega) \geq 0$ for all $\omega \in \Omega$ and we define the point-wise order $x \geq x'$ by $x(\omega) \geq x'(\omega)$ for all $\omega \in \Omega$. The lattice operations $\wedge$ and $\vee$ are defined by $(x \wedge y)(\omega) := \min\{x(\omega), y(\omega)\}$, $(x \vee y)(\omega) := \max\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$. We denote by $\mathbf{1}_A$ the indicator (or characteristic) function of a subset

[1] The functional f is said to be comonotonic additive if $f(x + y) = f(x) + f(y)$ for all x, y in $B(\mathcal{A})$ such that $x + y \in B(\mathcal{A})$ and $(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \geq 0$ for all ω, ω' in Ω.

$A \subseteq \Omega$, i.e., $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbf{1}_A(\omega) = 0$ otherwise, and, by convention, $\mathbf{1}_\emptyset = 0$. We will consider successively the two following cases (i) Ω is finite,[2] and (ii) Ω is a set equipped with an algebra $\mathcal{A} \subseteq 2^\Omega$ of subsets of Ω, hence in particular a measurable space space when $\mathcal{A}$ is a σ-algebra.

2 The main results

2.1 The finite case

When Ω is finite, a **game** is a set function $v : 2^\Omega \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. Then the **Choquet integral** (Choquet [4]) with respect to the game v is the functional $\hat{v} : \mathbb{R}^\Omega \to \mathbb{R}$ defined as follows. For $x \in \mathbb{R}^\Omega$, whose set of values, $\{x(\omega) : \omega \in \Omega\} = \{x_1, \ldots x_K\}$, is ordered decreasingly as $x_1 > \cdots > x_k > \cdots > x_K$ we define $A_k := x^{-1}(\{x_k\})$ and

$$\hat{v}(x) = \sum_{k=1}^{K-1} (x_k - x_{k+1}) v(A_1 \cup \cdots \cup A_k) + x_K v(\Omega). \tag{2.1}$$

The functional $\hat{v} : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies the following properties [direct from the definition for the first three, see for example [9] Proposition 4.11 page 64]:

[Extension] $\hat{v}(\mathbf{1}_A) = v(A)$ for all $A \subseteq \Omega$;
[Positive Homogeneity] $\hat{v}(tx) = t\hat{v}(x)$ for all $t \geq 0$, all $x \in \mathbb{R}^\Omega$;
[Constant Additivity] $\hat{v}(x + t\mathbf{1}_\Omega) = \hat{v}(x) + t\hat{v}(\mathbf{1}_\Omega)$ for all $t > 0$, all $x \in \mathbb{R}^\Omega$;
[Lipschitz] $\exists k \in \mathbb{R},\ |\hat{v}(x) - \hat{v}(y)| \leq k\|x - y\|_\infty$ for all x, y in $\mathbb{R}^\Omega$.
Moreover if v is a capacity, i.e., $v(\emptyset) = 0$ and $v(A) \leq v(B)$ for all $A \subseteq B \subseteq \Omega$, then
[Monotonicity] $\hat{v}(x) \leq \hat{v}(y)$ for all x, y in $\mathbb{R}^\Omega$ such that $x \leq y$.

The function $f : \mathbb{R}^\Omega \to \mathbb{R}$ is said to be **Choquet representable** or a **Choquet functional** if there exists a game $v : 2^\Omega \to \mathbb{R}$ such that $f(x) = \hat{v}(x)$ for all $x \in \mathbb{R}^\Omega$. Note that the game v associated with a Choquet functional f is unique since $v(A) = \hat{v}(\mathbf{1}_A) = f(A)$ for all $A \subseteq \Omega$, from the Extension property of the Choquet integral. Then $f : \mathbb{R}^\Omega \to \mathbb{R}$ is said to be a subadditive (resp. convex, submodular,…) Choquet functional if it is subadditive (resp. convex, submodular,…) as a function and if f is a Choquet functional.

From the above properties, every Choquet functional $f : \mathbb{R}^\Omega \to \mathbb{R}$ is positively homogeneous and constant additive. Moreover, a Choquet functional is subadditive

[2] When Ω is finite of cardinal n, we can identify $\mathbb{R}^\Omega$ with $\mathbb{R}^n$, thus a function $x : \Omega \to \mathbb{R}$ can also be viewed as the n-tuple $x = (x_1, \ldots, x_n)$. The previously defined order is the coordinate-wise order of $\mathbb{R}^n$, i.e., $x = (x_1, \ldots, x_n) \leq y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ means $x_i \leq y_i$ for every $i = 1, \ldots, n$. The lattice operations $\wedge$ and $\vee$ are defined by $x \wedge y := (\min\{x_1, y_1\}), \ldots, \min\{x_n, y_n\}), x \vee y := (\max\{x_1, y_1\}), \ldots, \max\{x_n, y_n\})$. With the previous identification, for $A \subseteq \{1, \ldots, n\}$, $\mathbf{1}_A$ will now be the vector in $\mathbb{R}^n$ such that $x_i = 1$ if $i \in A$ and $x_i = 0$ otherwise. Thus we denote by $\mathbf{1}_i := \mathbf{1}_{\{i\}}$ (resp. $\mathbf{1}_\Omega$) the vector with all coordinates equal to zero, but the i-th equal to 1 (resp. with all coordinates equal to 1) so that $x = (x_1, \ldots, x_n) = x_1\mathbf{1}_1 + \cdots + x_n\mathbf{1}_n$.

if and only if it is convex if and only if it is submodular. See, for example [9]. The following theorem shows that the three properties of positive homogeneity, constant additivity, and submodularity characterize the class of subadditive Choquet functionals.

Theorem 2.1 *Let Ω be a finite set, let $f : \mathbb{R}^\Omega \to \mathbb{R}$, then the two following assertions (i) and (ii) are equivalent:*

(i) *f satisfies the following three conditions:*
[Positive Homogeneity] $f(tx) = tf(x)$ for all $t \geq 0$, all $x \in \mathbb{R}^\Omega_+$,
[Constant Additivity] $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)$ for all $t > 0$, all $x \in \mathbb{R}^\Omega$,[3]
[Submodularity] $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$ for all x, y in $\mathbb{R}^\Omega_+$;[4]
(ii) *f is a Choquet functional that is subadditive on $\mathbb{R}^\Omega$, i.e.,*
[Subadditivity] $f(x + y) \leq f(x) + f(y)$ for all x, y in $\mathbb{R}^\Omega$.

If f satisfies one of the above equivalent properties, then it is Lipschitzian.

The proof of Theorem 2.1 is given in Sect. 3.1.

Subadditive Choquet functionals is a class of functionals that has been extensively studied. Connecting the previous theorem with fundamental results of this literature provides additional properties satisfied by the class of positively homogeneous, constant additive, and submodular functionals.

Corollary 2.1 *Let Ω be a finite set, let $f : \mathbb{R}^\Omega \to \mathbb{R}$, and let $v : 2^\Omega \to \mathbb{R}$ be defined by $v(A) := f(\mathbf{1}_A)$ for $A \subseteq \Omega$. Then the following assertions are equivalent:*

(i) *f is positively homogeneous, constant additive, and submodular;*
(ii) *f is a subadditive Choquet functional;*
(ii') *f is a convex Choquet functional;*
(iii) *f is a submodular Choquet functional;*
(iv) *f is a Choquet functional and v is submodular (or concave by Shapley [14])*

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B) \; for\; all\; A \subseteq \Omega, B \subseteq \Omega;$$

(v) *v is submodular and $f(x) = \sup\{x \cdot \mu : \mu \in \text{core}\,(v)\}$ for all $x \in \mathbb{R}^\Omega$,*

where $\text{core}\,(v) := \{\mu \in \mathbb{R}^\Omega \,:\, \mu(A) \leq v(A) \text{ for all } A \subseteq \Omega, \text{ and } \mu(\Omega) = v(\Omega)\}$.

Proof The equivalence $[(ii) \iff (v)]$ is due to the seminal paper by Schmeidler [12]. A general reference for the equivalence $[(ii) \iff (ii') \iff (iii) \iff (iv) \iff (v)]$ is Marinacci and Montrucchio [9] and Denneberg [6]. □

[3] This is easily proved to be equivalent to $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)$ for all $t \in \mathbb{R}$, all $x \in \mathbb{R}^\Omega$. Also, under positive homogeneity, constant additivity is equivalent to

$$f(x + \mathbf{1}_\Omega) = f(x) + f(\mathbf{1}_\Omega) \text{ for all } x \in \mathbb{R}^\Omega.$$

Indeed, $f(x + t\mathbf{1}_\Omega) = f(t(x/t + \mathbf{1}_\Omega)) = tf(x/t + \mathbf{1}_\Omega) = t(f(x/t) + f(\mathbf{1}_\Omega)) = f(x) + tf(\mathbf{1}_\Omega)$.

[4] Under constant additivity, it is easily proved to be equivalent to f submodular on the whole space $\mathbb{R}^\Omega$.

We now deduce that every positively homogeneous, constant additive and submodular functional is a polyhedral convex function, that is, it is the supremum of finitely many affine functions; see Rockafellar [11].

Corollary 2.2 *Let Ω be a finite set, let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be a function that is positively homogeneous, constant additive, and submodular, and let $v : 2^\Omega \to \mathbb{R}$ be defined by $v(A) := f(\mathbf{1}_A)$ for $A \subseteq \Omega$. Then*

- *f is a polyhedral convex function;*
- $\partial f(0) = \text{core}\ (v)$.

Proof We first prove that f is a polyhedral convex function. Indeed, from Assertion (v) of Theorem 2.1, one has $f(x) = \sup\{x \cdot \mu : \mu \in \text{core}\ (v)\}$ for all x. Thus f is the support function of the set core (v), which is a polyhedral convex set, i.e., a convex set defined by finitely many affine inequalities. Thus the function f is a polyhedral convex function ([11] Corollary 19.2.1, page 174).

We now prove that $\partial f(0) = \text{core}\ (v)$. Indeed, since f is the support function of core (v), which is nonempty convex compact, one gets ([11] Theorem 13.1, page 112)

$$\text{core}\ (v) = \{\mu \in \mathbb{R}^\Omega : x \cdot \mu \leq f(x) \text{ for all } x \in \mathbb{R}^\Omega\},$$

which is exactly $\partial f(0)$, since $f(0) = 0$ (from the positive homogeneity of f). □

2.2 The general case

When Ω is (possibly) infinite, more structure is needed on the space Ω and on the functions $x : \Omega \to \mathbb{R}$. The set Ω is now equipped with an algebra $\mathcal{A} \subseteq 2^\Omega$ (not assumed to be a σ-algebra), of subsets of Ω, that is, $\mathcal{A}$ contains the whole set Ω and is stable by union and complementation. Moreover every function $x : \Omega \to \mathbb{R}$ will be assumed to be measurable, that is, $x^{-1}(I) \in \mathcal{A}$ for every interval $I \subseteq \mathbb{R}$. We denote by $B(\mathcal{A})$ the set of bounded measurable functions $x : \Omega \to \mathbb{R}$, that is, x is measurable and $\|x\|_\infty := \sup\{|x(\omega)| : \omega \in \Omega\} < +\infty$ and by $B_0(\mathcal{A})$ the set of simple measurable functions, i.e., measurable functions $x : \Omega \to \mathbb{R}$ whose set of values $\{x(\omega) : \omega \in \Omega\}$ is finite. We let $B^+(\mathcal{A}) := B(\mathcal{A}) \cap \mathbb{R}^\Omega_+$ (resp. $B^+_0(\mathcal{A}) := B_0(\mathcal{A}) \cap \mathbb{R}^\Omega_+$) be the set of non-negative bounded measurable functions (resp. non-negative simple measurable functions). Clearly the finite case is a particular case of the measurable one, taking $\mathcal{A} = 2^\Omega$, and every function $x : \Omega \to \mathbb{R}$ is measurable, bounded, and simple. We recall that $B(\mathcal{A})$ is a lattice for the point-wise order but may not be a vector space, unless $\mathcal{A}$ is assumed to be a σ-algebra (but $B_0(\mathcal{A})$ is always a vector space).

A **game** is now a set function $v : \mathcal{A} \to \mathbb{R}$ such that $v(\emptyset) = 0$ (and v is not assumed to be a capacity, i.e., monotonic). Following Aumann and Shapley [2], the variation norm of a game v is defined as follows:

$$\|v\| := \sup \left\{ \sum_{k=1}^{K} |v(A_k) - v(A_{k-1})| \ : \ (A_k)_{k=0}^{K} \text{ finite chain in } \mathcal{A} \right\},$$

where, by finite chain $(A_k)_{k=0}^K$ in $\mathcal{A}$, we mean that $A_k \in \mathcal{A}$ for all $k = 1, \dots, K$ and $\emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_K = \Omega$. We denote by:

$$bv(\mathcal{A}) := \big\{v : \mathcal{A} \to \mathbb{R} \,:\, v(\emptyset) = 0 \text{ and } \|v\| < +\infty\big\},$$

the set of games having a finite variation norm. The variation norm is indeed a norm on the space $bv(\mathcal{A})$, which is a Banach space for this norm. Moreover, $bv(\mathcal{A})$ contains the class of finite games, the class of capacities, i.e., monotone games, and also the set $ba(\mathcal{A})$ of bounded charges, i.e., additive games of bounded variation. For all these properties we refer to [2].[5]

We follow Marinacci and Montrucchio [9], to define the Choquet integral with respect to the game $v \in bv(\mathcal{A})$, as follows. First when $x \in B(\mathcal{A})$, we let:

$$\int_\Omega^C x(\omega)\, dv(\omega) := \int_{-\infty}^0 [v(x > t) - v(\Omega)]\, dt + \int_0^{+\infty} v(x > t)\, dt, \text{ also denoted } \hat{v}(x)$$

where the integrals (with respect to the real variable t) are taken in the sense of Riemann. The Riemann integrals are well defined and the definition of Choquet integral coincides with the previous one for finite games. Moreover the functional $\hat{v} : B(\mathcal{A}) \to \mathbb{R}$ is Lipschitzian and this allows to extend the integral on the set:

$$\overline{B}(\mathcal{A}) := \big\{x = \lim_{n\to\infty} x_n \,:\, x \in \mathbb{R}^\Omega \text{ is bounded, for all } n,\ x_n \in B(\mathcal{A})\big\},$$

the closure of $B(\mathcal{A})$ in the Banach space of all bounded functions (endowed with the sup norm). Then $\overline{B}(\mathcal{A})$ is a Banach lattice (hence a vector space) containing $B(\mathcal{A})$, which is not in general a vector space, $\overline{B}(\mathcal{A})$ is also the closure of $B_0(\mathcal{A})$, the set of simple measurable functions, and $\overline{B}(\mathcal{A}) = B(\mathcal{A})$ whenever $\mathcal{A}$ is a σ-algebra. For this construction and the properties listed above and below, we refer to [9].

[Extension] $\hat{v}(\mathbf{1}_A) = v(A)$ for all $A \in \mathcal{A}$;
[Positive Homogeneity] $\hat{v}(tx) = t\hat{v}(x)$ for all $t \geq 0$, all $x \in \overline{B}(\mathcal{A})$;
[Constant Additivity] $\hat{v}(x + t\mathbf{1}_\Omega) = \hat{v}(x) + t\hat{v}(\mathbf{1}_\Omega)$ for all $t > 0$, all $x \in \overline{B}(\mathcal{A})$;
[Lipschitz] $\exists k \in \mathbb{R}_+,\ |\hat{v}(x) - \hat{v}(y)| \leq k\|x - y\|_\infty$ for all x, y in $\overline{B}(\mathcal{A})$.

We now provide an extension of the representation Theorem 2.1. The function $f : B(\mathcal{A}) \to \mathbb{R}$ (resp. $f : \overline{B}(\mathcal{A}) \to \mathbb{R}$) is said to be Choquet representable or a Choquet functional if there exists a game $v \in bv(\mathcal{A})$ such that $f(x) = \hat{v}(x)$ for all $x \in B(\mathcal{A})$ (resp. $x \in \overline{B}(\mathcal{A})$). Again, the game v associated with a Choquet functional f is unique since $v(A) = \hat{v}(\mathbf{1}_A) = f(\mathbf{1}_A)$ for all $A \in \mathcal{A}$, from the Extension property of the Choquet integral. We stress the fact that if f is a Choquet functional then its associated game v is of finite (bounded) variation, i.e., $v \in bv(\mathcal{A})$ (since the Choquet

[5] Indeed $bv(\mathcal{A})$ contains the class of all finite games (i.e., Ω is finite) since there are finitely many finite chains. Moreover, $bv(\mathcal{A})$ contains all capacities v since, using the monotonicity of v, for all finite chains (A_k), $\sum_{k=1}^K |v(A_k) - v(A_{k-1})| = \sum_{k=1}^K v(A_k) - v(A_{k-1}) = v(\Omega) - v(\emptyset) = v(\Omega)$. Consequently, $\|v\| = v(\Omega) < \infty$. An additive game μ is called a charge (or a signed charge) and we point out that the total variation norm of a charge $\|\mu\|$ is exactly the variation norm of the game μ defined above.

integral has been defined only in this case). Then, as previously, $f : \mathbb{R}^\Omega \to \mathbb{R}$ is said to be a subadditive (resp. convex, submodular,...) Choquet functional if it is subadditive (resp. convex, submodular,...) as a function and if it is a Choquet functional.

Theorem 2.2 *Let Ω be a set equipped with an algebra $\mathcal{A} \subseteq 2^\Omega$, let $f : B(\mathcal{A}) \to \mathbb{R}$. Then the following three assertions are equivalent:*

(i) *f satisfies the following four conditions:*
[Positive homogeneity] $f(tx) = tf(x)$ for all $t \geq 0$, all $x \in B_0^+(\mathcal{A})$,
Constant Additivity $f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega)$ for all $t > 0$, all $x \in B_0(\mathcal{A})$,
Submodularity $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$ for all x, y in $B_0^+(\mathcal{A})$,
[Continuity] f is sup-norm continuous on $B^+(\mathcal{A})$;
(ii) *f is a Choquet functional, and f is subadditive, i.e.*
[Subadditivity] $f(x+x') \leq f(x)+f(x')$ for all x, x' in $B(\mathcal{A})$, $x+x' \in B(\mathcal{A})$;
(ii') *f has a sup-norm continuous extension $\bar{f} : \overline{B}(\mathcal{A}) \to \mathbb{R}$ satisfying: $\bar{f}$ is a subadditive Choquet functional on $\overline{B}(\mathcal{A})$.*

Moreover, if f satisfies one of the above equivalent assertions, then f and $\bar{f}$ are Lipschitzian.

The proof of Theorem 2.2 is given in Sect. 3.2.

Remark 2.3 The above theorem contains an additional continuity assumption that was not made in Theorem 2.1 when Ω is finite. Moreover the continuity assumption cannot be dispensed with when Ω is not finite. Consider a linear functional f that is not sup-norm continuous, then f is positively homogeneous, constant additive, and submodular. But f cannot be a Choquet functional since Choquet functionals are always Lipschitzian. □

In the general case, subadditive Choquet functionals have been extensively studied, starting with the seminal paper by Schmeidler [12]. Connecting Theorem 2.2 with this literature provides additional properties satisfied by the class of positively homogeneous, constant additive, submodular, and sup-norm continuous functionals. The next result considers the case of functionals $\bar{f} : \overline{B}(\mathcal{A}) \to \mathbb{R}$ defined on the Banach lattice $\overline{B}(\mathcal{A})$.

Corollary 2.3 *Let Ω be a set equipped with an algebra $\mathcal{A} \subseteq 2^\Omega$, let $\bar{f} : \overline{B}(\mathcal{A}) \to \mathbb{R}$, and let $v : \mathcal{A} \to \mathbb{R}$ be defined by $v(A) := \bar{f}(\mathbf{1}_A)$ for $A \in \mathcal{A}$. Then the following assertions are equivalent.*

(i) *$\bar{f}$ satisfies the following four conditions:*
[Positive homogeneity] $\bar{f}(tx) = t\bar{f}(x)$ for all $t > 0$, all $x \in B_0^+(\Omega)$,
[Constant Additivity] $\bar{f}(x+t\mathbf{1}_\Omega) = \bar{f}(x)+t\bar{f}(\mathbf{1}_\Omega)$ for all $t \geq 0$, all $x \in B_0(\mathcal{A})$,
[Submodularity] $\bar{f}(x \vee y) + \bar{f}(x \wedge y) \leq \bar{f}(x) + \bar{f}(y)$ for all x, y in $B_0^+(\mathcal{A})$,
[Continuity] $\bar{f}$ is sup-norm continuous on $\overline{B}(\mathcal{A})$;
(ii) *$\bar{f}$ is a subadditive Choquet functional;*
(ii') *$\bar{f}$ is a convex Choquet functional;*
(iii) *$\bar{f}$ is a submodular Choquet functional;*

(iv) *$\bar{f}$ is a Choquet functional and v is submodular, i.e.,*

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B) \text{ for all } A \in \mathcal{A}, B \in \mathcal{A};$$

(v) *$v \in bv(\mathcal{A})$ is submodular, and*

$$\bar{f}(x) = \sup\{x \cdot \mu : \mu \in \text{core } (v)\} \text{ for all } x \in B(\mathcal{A}),$$

where core $(v) := \{\mu \in ba(\mathcal{A}) : \mu(A) \leq v(A) \text{ for all } A \in \mathcal{A}\ \mu(\Omega) = v(\Omega)\}$, *$ba(\mathcal{A})$ denotes the vector space of all charges with finite (total) variation norm.*

Moreover, if $\bar{f}$ satisfies one of the above equivalent assertions, then $\bar{f}$ is Lipschitzian.

The equivalence $[(i) \iff (ii)]$ is a direct consequence of Theorem 2.2, considering the restriction f of the functional $\bar{f}$ to $B(\mathcal{A})$. For the other equivalences $[(ii) \iff (ii') \iff (iii) \iff (iv) \iff (v)]$, we refer to Marinacci and Montrucchio [9].

The next corollary replaces the previous continuity assumption by the monotonicity of the functional f and the associated game v will then be a capacity.

Corollary 2.4 (Monotone) *Let Ω be a set equipped with an algebra $\mathcal{A} \subseteq 2^{\Omega}$, let $f : B(\mathcal{A}) \to \mathbb{R}$, then the following two assertions are equivalent:*

(i) *f is positively homogeneous, constant additive, submodular on $B(\mathcal{A})$,*[6] *and*
[Monotone Increasing] $f(x) \leq f(y)$ for all x, y in $B(\mathcal{A})$ such that $x \leq y$;

(ii) *v is a capacity and f is a subadditive Choquet functional.*

Proof The proof of $[(i) \implies (ii)]$ is a direct consequence of Theorem 2.2 and the fact that f is Lipschitzian whenever f is monotone increasing and constant additive. Indeed, for all x, y in $B(\mathcal{A})$ one has, $x \leq y + \|x - y\|_\infty \mathbf{1}_\Omega$. Since f is monotone increasing and constant additive, one gets:

$$f(x) \leq f\big(y + \|x - y\|_\infty \mathbf{1}_\Omega\big) = f(y) + \|x - y\|_\infty f(\mathbf{1}_\Omega).$$

Interchanging the role of x and y one gets $f(y) \leq f(x) + \|y - x\|_\infty f(\mathbf{1}_\Omega)$, hence

$$|f(x) - f(y)| \leq f(\mathbf{1}_\Omega)\|x - y\|_\infty.$$

Consequently, from Theorem 2.2, f is a subadditive Choquet functional. Since f is monotone increasing, one then checks that v is monotonic, i.e., $v(A) := f(\mathbf{1}_A) \leq f(\mathbf{1}_B) := v(B)$ for all $A \subseteq B$, $A \in \mathcal{A}$, $B \in \mathcal{A}$, that is, v is a capacity. The proof of the converse $[(ii) \implies (i)]$ relies on standard properties of the Choquet integral. □

We can now deduce the Riesz' representation theorem for linear and sup-norm continuous functions f, in the case of an algebra $\mathcal{A}$ (and the case of a σ- algebra $\mathcal{A}$ follows immediately since in this case $\overline{B}(\mathcal{A}) = B(\mathcal{A})$).

6 As in Condition (i) of Theorem 2.2.

Corollary 2.5 (Riesz) *Let* Ω *be a set equipped with an algebra* $\mathcal{A} \subseteq 2^{\Omega}$*, let* $f : \overline{B}(\mathcal{A}) \to \mathbb{R}$ *and let* $v : \mathcal{A} \to \mathbb{R}$ *be defined by* $v(A) := f(\mathbf{1}_A)$ *for* $A \in \mathcal{A}$*. Then the following two assertions are equivalent:*

(i) f *is linear and sup-norm continuous;*
(ii) $v \in ba(\mathcal{A})$ *and* $f(x) = \int_{\Omega}^{C} x(\omega) dv(\omega)$ *for all* $x \in \overline{B}(\mathcal{A})$.

Proof $[(i) \implies (ii)]$ is a direct consequence of Corollary 2.3 since a linear functional is positively homogeneous, constant additive, and modular (see Topkis [15]), hence submodular.[7] Thus f is a Choquet functional. Then one checks that v is additive (i.e., a charge) since f is linear and $v \in ba(\mathcal{A})$ since $v \in bv(\mathcal{A})$ and v is additive.
$[(ii) \implies (i)]$. If f is a Choquet functional with respect to $v \in ba(\mathcal{A})$, then it is linear from standard properties of the Choquet integral. Moreover the Choquet functional f is sup-norm continuous by Corollary 2.3. □

3 Proof of the theorems

3.1 Proof of Theorem 2.1

Hereafter, we provide the proof in a slightly more general framework than Ω finite, since it will be needed in the next section for the proof of the general case. More precisely, hereafter we assume that Ω is a set equipped with an algebra $\mathcal{A} \subseteq 2^{\Omega}$, we let $f : B_0(\mathcal{A}) \to \mathbb{R}$, where $B_0(\mathcal{A})$ denotes the set of simple measurable functions, and the definition of the Choquet integral is given by the same formula (2.1) as in the finite case. Clearly this framework extends the case of Ω finite, taking $\mathcal{A} = 2^{\Omega}$, and noticing that $B_0(\mathcal{A}) = \mathbb{R}^{\Omega}$.
$[(ii) \implies (i)]$ Assume that f is a subadditive Choquet functional. By standard properties of the Choquet integral, f is constant additive and positively homogeneous on $B_0(\mathcal{A})$. Moreover the Choquet functional f is subadditive on $B_0(\mathcal{A})$ if and only if it is submodular on $B_0(\mathcal{A})$ (see [9]). Hence f is submodular, constant additive and positively homogeneous on $B_0(\mathcal{A})$, which shows that it satisfies Condition (i). □
$[(i) \implies (ii)]$ We assume that $f : B_0(\mathcal{A}) \to \mathbb{R}$ satisfies (i), we let $x \in B_0(\mathcal{A})$, whose values $\{x_1, \ldots, x_K\}$ are ranked in decreasing order $x_1 > \cdots > x_k > \cdots > x_K$, and we recall that:

$x = x_1 \mathbf{1}_{A_1} + \cdots + x_K \mathbf{1}_{A_K}$ where $A_k := \{\omega \in \Omega \ : \ x(\omega) = x_k\} \in \mathcal{A}$,
$\{A_1, \ldots, A_K\}$ defines a measurable partition of Ω,
$x = \sum_{k=1}^{K} y_k$ with $y_k := (x_k - x_{k+1})\mathbf{1}_{A_1 \cup \cdots \cup A_k}$ $(k \le K-1)$, $y_K := x_K \mathbf{1}_{\Omega}$,
$\int_{\Omega}^{C} x \, dv := \sum_{k=1}^{K-1} (x_k - x_{k+1}) v(A_1 \cup \cdots \cup A_k) + x_K v(\Omega)$ (by Definition (2.1)).

The equality $f(x) = \int_{\Omega}^{C} x \, dv$ will be proved in the following three steps and the subadditivity of f in the fourth step.
Step 1. $f(x) \ge \int_{\Omega}^{C} x \, dv$ for all $x \ge 0$.
The proof of Step 1 is a consequence of the following two claims.

[7] If f is additive, then the functional f is modular. Indeed, since $x \vee y + x \wedge y = x + y$: $f(x \vee y) + f(x \wedge y) = f(x \vee y + x \wedge y) = f(x+y) = f(x) + f(y)$ for all x, y in $B(\mathcal{A})$.

Claim 1 $f(y_k) := (x_k - x_{k+1})v(A_1 \cup \cdots \cup A_k)$ $(k \leq K-1)$, $f(y_K) = x_K v(\Omega)$.

Indeed, for $k = K$, since $x_K \geq 0$ [because $x \geq 0$] and f is positively homogeneous, we get $f(y_K) := f(x_K \mathbf{1}_\Omega) = x_K f(\mathbf{1}_\Omega) = x_K v(\Omega)$ [from the definition of v].

For $k \leq K-1$ one has:

$$\begin{aligned} f(y_k) &= f\big((x_k - x_{k+1})\mathbf{1}_{A_1\cup\cdots\cup A_k}\big) \\ &= x_k - x_{k+1}) f(\mathbf{1}_{A_1\cup\cdots\cup A_k})[\text{since } f \text{ is pos. hom. and } x_k > x_{k+1}] \\ &= (x_k - x_{k+1}) v(A_1 \cup \cdots \cup A_k)[\text{from the definition of } v]. \end{aligned}$$

□

Claim 2 $f(x) = f(\sum_{k=1}^K y_k) \geq \sum_{k=1}^K f(y_k) = \int_\Omega^C x\, dv$.

Indeed, we will prove hereafter the following inequality:

$$f(y_1 + \cdots + y_k) \geq f(y_1 + \cdots + y_{k-1}) + f(y_k) \text{ for all } k = 2, \ldots, K,$$

that immediately implies Claim 2 (by a simple induction argument) since

$$\begin{aligned} f(x) = f\left(\sum_{k=1}^K y_k\right) &\geq f(y_1 + \cdots + y_{K-1}) + f(y_K) \geq \ldots \\ &\geq f(y_1 + \cdots + y_{k-1}) + f(y_k) + \cdots + f(y_K) \geq \cdots \geq \sum_{k=1}^K f(y_k). \end{aligned}$$

We now prove the above inequality, fixing $k = 2, \ldots, K$. We let $x_{K+1} := 0$

- $$\begin{aligned} a := y_1 + \cdots + y_k &= (x_1 - x_2)\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_1\cup\cdots\cup A_k} \\ &= (x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k} \end{aligned}$$

- $b := (x_k - x_{k+1})\mathbf{1}_\Omega \geq 0$,

recalling that $x_1 > \cdots > x_K \geq 0 := x_{K+1}$ since $x \geq 0$. Moreover one gets:

- $$\begin{aligned} a \wedge b &:= (y_1 + \cdots + y_k) \wedge (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= ((x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k}) \wedge (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= (x_k - x_{k+1})\mathbf{1}_{A_1} + \ldots + (x_k - x_{k+1})\mathbf{1}_{A_k} = (x_k - x_{k+1})\mathbf{1}_{A_1\cup\cdots\cup A_k} = y_k, \end{aligned}$$

- $$\begin{aligned} a \vee b &:= \big((x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_k - x_{k+1})\mathbf{1}_{A_k}\big) \vee (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= (x_1 - x_{k+1})\mathbf{1}_{A_1} + \cdots + (x_{k-1} - x_{k+1})\mathbf{1}_{A_{k-1}} + (x_k - x_{k+1})\mathbf{1}_{A_k} \\ &\quad + (x_k - x_{k+1})\mathbf{1}_{A_{k+1}\cup\cdots\cup A_K} \\ &= (x_1 - x_k)\mathbf{1}_{A_1} + \cdots + (x_{k-1} - x_k)\mathbf{1}_{A_{k-1}} + (x_k - x_{k+1})\mathbf{1}_\Omega \\ &= y_1 + \cdots + y_{k-1} + (x_k - x_{k+1})\mathbf{1}_\Omega[\text{from the formula giving } a \text{ at } k-1] \\ &= y_1 + \cdots + y_{k-1} + b. \end{aligned}$$

Consequently, using the submodularity and constant additivity of f we get:

$$
\begin{aligned}
&f(y_1+\cdots+y_k)+f(b)=f(a)+f(b)\\
&\quad\geq f(a\wedge b)+f(a\vee b)\ \text{[since } f \text{ is submodular]}\\
&\quad= f(y_k)+f(y_1+\cdots+y_{k-1}+b)\ \text{[from above] where } b:=(x_k-x_{k+1})\mathbf{1}_\Omega,\\
&\quad= f(y_k)+f(y_1+\cdots+y_{k-1})+f(b) \text{ since } f \text{ is constant additive.}
\end{aligned}
$$

Hence, $f(y_1+\cdots+y_k)\geq f(y_1+\cdots+y_{k-1})+f(y_k)$. □

Step 2. $\int_\Omega^C x\ dv \geq f(x)$ for all $x\geq 0$.

Recalling that $x_1>\cdots>x_K\geq 0$, we can rewrite the integral as

$$
\begin{aligned}
\int_\Omega^C x\ dv &= (x_1-x_2)f(\mathbf{1}_{A_1})+(x_2-x_3)f(\mathbf{1}_{A_1\cup A_2})+\cdots+(x_k-x_{k+1})f(\mathbf{1}_{A_1\cup\cdots\cup A_k})\\
&\quad+\cdots+(x_{K-1}-x_K)f(\mathbf{1}_{A_1\cup\cdots\cup A_{K-1}})+x_K f(\mathbf{1}_\Omega)\\
&= x_1 f(\mathbf{1}_{A_1})+x_2\big(f(\mathbf{1}_{A_1\cup A_2})-f(\mathbf{1}_{A_1})\big)\\
&\quad+\cdots+x_k\big(f(\mathbf{1}_{A_1\cup\cdots\cup A_k})-f(\mathbf{1}_{A_1\cup\cdots\cup A_{k-1}})\big)\\
&\quad+\cdots+x_K\big(f(\mathbf{1}_\Omega)-f(\mathbf{1}_{A_1\cup\cdots\cup A_{K-1}})\big)\\
&\geq f(x_1\mathbf{1}_{A_1})+f(x_1\mathbf{1}_{A_1}+x_2\mathbf{1}_{A_2})-f(x_1\mathbf{1}_{A_1})\\
&\quad+\cdots+f(x_1\mathbf{1}_{A_1}+\cdots+x_k\mathbf{1}_{A_k})-f(x_1\mathbf{1}_{A_1}+\cdots+x_{k-1}\mathbf{1}_{A_{k-1}})\\
&\quad+\cdots+f(x_1\mathbf{1}_{A_1}+\cdots+x_K\mathbf{1}_{A_K})-f(x_1\mathbf{1}_{A_1}+\cdots+x_{K-1}\mathbf{1}_{A_{K-1}})\\
&= f(x_1\mathbf{1}_{A_1}+\cdots+x_K\mathbf{1}_{A_K})=f(x).
\end{aligned}
$$

since, on the first hand, the above equalities are simply rewriting the definition of $\int_\Omega^C x\ dv$ and $f(x)$ respectively, and, on the other hand, the above inequality is a consequence of the following two assertions that for $k=1$:

$$x_1 f(\mathbf{1}_{A_1})=f(x_1\mathbf{1}_{A_1})\text{[since } f \text{ is positively homogeneous and } x_1\geq 0],$$

and and for $k=2,\ldots,K$

$$
\begin{aligned}
&x_k\big(f(\mathbf{1}_{A_1\cup\cdots\cup A_k})-f(\mathbf{1}_{A_1\cup\cdots\cup A_{k-1}})\big)\\
&\quad= x_k\big(f(\mathbf{1}_{A_1}+\cdots+\mathbf{1}_{A_k})-f(\mathbf{1}_{A_1}+\cdots+\mathbf{1}_{A_{k-1}})\big)\\
&\quad= f(x_k\mathbf{1}_{A_1}+\cdots+x_k\mathbf{1}_{A_k})-f(x_k\mathbf{1}_{A_1}+\cdots+x_k\mathbf{1}_{A_{k-1}})\text{[since } x_k\geq 0]\\
&\quad= f(a)-f(a\wedge b)\\
&\quad\geq f(a\vee b)-f(b)\text{[since } f \text{ is submodular]}\\
&\quad= f(x_1\mathbf{1}_{A_1}+\cdots+x_k\mathbf{1}_{A_k})-f(x_1\mathbf{1}_{A_1}+\cdots+x_{k-1}\mathbf{1}_{A_{k-1}}),
\end{aligned}
$$

where above we have taken:

$$a:=x_k\mathbf{1}_{A_1}+\cdots+x_k\mathbf{1}_{A_k},\, b:=x_1\mathbf{1}_{A_1}+\cdots+x_{k-1}\mathbf{1}_{A_{k-1}}$$

and, recalling that $x_1 > \cdots > x_k > \cdots > x_K \geq 0$, we clearly have:

$$a \wedge b = x_k \mathbf{1}_{A_1} + \cdots + x_k \mathbf{1}_{A_{k-1}} \text{ and } a \vee b = x_1 \mathbf{1}_{A_1} + \cdots + x_k \mathbf{1}_{A_k}. \qquad \square$$

Step 3. $f(x) = \int_\Omega^C x \, dv$ for all $x \in \mathbb{R}^\Omega$.

Indeed, there exists $t \geq 0$ such that $x + t\mathbf{1}_\Omega \geq 0$. Thus, $\int_\Omega^C (x + t\mathbf{1}_\Omega) \, dv = f(x + t\mathbf{1}_\Omega)$ from Steps 1 and 2. Consequently, from the constant additivity properties of both f and the Choquet integral, we have

$$\int_\Omega^C x \, dv + t \int_\Omega^C \mathbf{1}_\Omega \, dv = \int_\Omega^C (x + t\mathbf{1}_\Omega) \, dv = f(x + t\mathbf{1}_\Omega) = f(x) + tf(\mathbf{1}_\Omega).$$

Thus $\int_\Omega^C x \, dv = f(x)$ since $\int_\Omega^C \mathbf{1}_\Omega \, dv = v(\Omega) := f(\mathbf{1}_\Omega)$. □

Step 4. f is subadditive on $\mathbb{R}^\Omega$.

From (i), the function f is submodular on $\mathbb{R}_+^\Omega$, which together with the constant additivity of f (by (i)) implies that f is submodular on $\mathbb{R}^\Omega$. But f is a Choquet functional by Steps 1 and 2, and a Choquet functional is submodular on $\mathbb{R}^\Omega$ if and only if it is subadditive on $\mathbb{R}^\Omega$ (see [9]). Thus f is subadditive on $\mathbb{R}^\Omega$. □

3.2 Proof of Theorem 2.2

We first prove $[(i) \implies (ii)]$. Let $f : B(\mathcal{A}) \to \mathbb{R}$ satisfy Condition (i), we define the game $v : \mathcal{A} \to \mathbb{R}$ by $v(A) := f(\mathbf{1}_A)$ for $A \in \mathcal{A}$. The proof of (ii) will proceed in several steps. The first step proves that $v \in bv(\mathcal{A})$. The second step proves the equality $f(x) = \hat{v}(x)$ when $x \in B_0(\mathcal{A})$, i.e., x is a measurable simple function; this has been mainly performed in the previous section (the finite case) and we only need to recall that the two definitions of the Choquet integral used so far coincide when $x \in B_0(\mathcal{A})$. The third step uses a standard approximation argument of $x \in B(\mathcal{A})$ by a sequence of measurable simple functions (x^n); since $f(x^n) = \hat{v}(x^n)$ by Step 2, then going to the limit when $n \to \infty$, gives the desired equality $f(x) = \hat{v}(x)$, using the sup-norm continuity of $\hat{v}$ (a property of the Choquet integral) and of f (by assumption).

Step 1 $v \in bv(\mathcal{A})$.

First, v is a bounded game since f is bounded on $B_+ := \{x \in B^+(\mathcal{A}) : \|x\|_\infty \leq 1\}$ the non-negative unit ball; indeed, since f is sup-norm continuous at 0, for every $\varepsilon > 0$ there exists $\alpha > 0$ such that $\|x\| \leq \alpha$ implies $|f(x)| \leq \varepsilon$; hence, for $x \in B_+$, $\alpha|f(x)| = |f(\alpha x)| \leq \varepsilon$ since f is positively homogeneous, thus $|f(x)| \leq \varepsilon/\alpha$. Second, v is submodular since f is submodular; indeed,

$$\begin{aligned} v(A \cup B) + v(A \cap B) &= f(\mathbf{1}_{A \cup B}) + f(\mathbf{1}_{A \cap B}) \\ &= f(\mathbf{1}_A \vee \mathbf{1}_B) + f(\mathbf{1}_A \wedge \mathbf{1}_B) \leq f(\mathbf{1}_A) + f(\mathbf{1}_B) = v(A) + v(B). \end{aligned}$$

Since v is bounded and submodular (see [9] Theorem 38 Page 40), the core (v) is nonempty and moreover, for any finite chain $(A_k)_{k=0}^K$, i.e., $A_k \in \mathcal{A}$ for all k, and

$\emptyset = A_0 \subseteq \cdots \subseteq A_k \subseteq \cdots \subseteq A_K = \Omega$, there exists $\mu \in$ core (v) (and in particular $\mu \in ba(\mathcal{A})$ by definition of the core), such that $\mu(A_k) = v(A_k)$ for all k. Thus

$$\sum_{k=1}^{K} |v(A_k) - v(A_{k-1})| = \sum_{k=1}^{K} |\mu(A_k) - \mu(A_{k-1})|$$
$$= \sum_{k=1}^{K} |\mu(A_k \backslash A_{k-1})| \leq \|\mu\|,$$

where $\|\mu\|$ is the variation norm of the (additive) game $\mu \in ba(\mathcal{A})$, i.e., of the (signed) charge μ. From a standard result of measure theory (see for example [3], [1] Corollary 10.55, page 399), $\|\mu\| < +\infty$ if and only if the range of μ is bounded, i.e., there exists $M \in \mathbb{R}$, such that $|\mu(A)| \leq M$ for all $A \in \mathcal{A}$.[8] Thus to show that $v \in bv(\mathcal{A})$, i.e., $\|v\|$ is finite, we only need to check that the range of μ is bounded. Indeed, for all $A \in \mathcal{A}$, recalling that $\mu \in$ core (v), there exists $M' \in \mathbb{R}_+$ such that

$$\mu(A) \leq v(A) := f(\mathbf{1}_A) \leq M',$$
$$v(\Omega) - \mu(A) = \mu(\Omega) - \mu(A) = \mu(A^c) \leq v(A^c) := f(\mathbf{1}_{A^c}) \leq M',$$

since f is bounded on the unit ball and $\|\mathbf{1}_A\|_\infty \leq 1$, $\|\mathbf{1}_{A^c}\|_\infty \leq 1$. Consequently, $\sup_{A \in \mathcal{A}} |\mu(A)| \leq \max\{M', M' - v(\Omega)\}$, i.e., the range of μ is bounded. □

Step 2 $f(x) = \hat{v}(x)$ for all $x \in B_0(\mathcal{A})$, i.e., for all x measurable simple function.

We have proved in the previous section that, for every simple function $x : \Omega \to \mathbb{R}$ whose (finitely many) values $\{x_1, \ldots, x_K\}$ are ranked in decreasing order $x_1 > \cdots > x_k > \cdots > x_K$, we have:

$$f(x) = \sum_{k=1}^{K-1} (x_k - x_{k+1}) v(A_1 \cup \cdots \cup A_k) + x_K v(\Omega).$$

But for a simple function $x \in B_0(\mathcal{A})$, the above sum coincides with the Choquet integral w.r.t. $v \in bv(\mathcal{A})$, that is, one has ([9] Proposition 22, page 22, [6])

$$\hat{v}(x) = \sum_{k=1}^{K-1} (x_k - x_{k+1}) v(A_1 \cup \cdots \cup A_k) + x_K v(\Omega).$$

Thus $f(x) = \hat{v}(x)$. □

Step 3 $f(x) = \hat{v}(x)$ for every $x \in B(\mathcal{A})$.

Proof We recall that the set $B_0(\mathcal{A})$ of simple and measurable functions is sup-norm dense in $B(\mathcal{A})$, that is, for every $x \in B(\mathcal{A})$, there exists a sequence of measurable simple functions $x^n : \Omega \to \mathbb{R}$ such that $\|x^n - x\|_\infty \to 0$ when $n \to \infty$.

[8] Define $\|\mu\| = |\mu|(\Omega) := \mu^+(\Omega) + \mu^-(\Omega)$, where $\mu^+(A) := \sup\{\mu(B) : B \subseteq A, A \in \mathcal{A}\}$, $\mu^-(\Omega) := -\inf\{\mu(B) : B \subseteq A, A \in \mathcal{A}\}$ for $A \in \mathcal{A}$. Then $\|\mu\| = |\mu|(\Omega)$ (see for example [3] Theorem 2.2.4 page 46, Theorem 4.1.2 page 86, [1] Corollary 10.53 page 397) and the result follows.

Consequently, for every $x \in B(\mathcal{A})$, we have

$$|f(x) - \hat{v}(x)| \leq |f(x) - f(x^n)| + |f(x^n) - \hat{v}(x^n)| + |\hat{v}(x^n) - \hat{v}(x)|.$$

But $|f(x^n) - \hat{v}(x^n)| = 0$ by Step 2, and $|\hat{v}(x^n) - \hat{v}(x)| \to 0$ when $n \to +\infty$ since $\hat{v}$ is Lipschitzian on $B(\mathcal{A})$ when $v \in bv(\mathcal{A})$ [Marinacci and Montrucchio [9] Proposition 24]. Moreover, $|f(x) - f(x^n)| \to 0$ when $n \to +\infty$ since f is sup-norm continuous at x; it suffices to notice that f is continuous on $B^+(\mathcal{A})$ by assumption, which together with constant additivity, implies that f is continuous on $B(\mathcal{A})$.

Letting $n \to +\infty$ in the above inequality, at the limit we get $f(x) = \hat{v}(x)$. □

[$(ii) \implies (ii')$]. From (ii), f is a Choquet functional w.r.t. $v \in bv(\mathcal{A})$, hence it is Lipschitzian [Marinacci and Montrucchio [9] Proposition 24]. Consequently, $f : B(\mathcal{A}) \to \mathbb{R}$ has a sup-norm continuous extension $\bar{f} : \overline{B}(\mathcal{A}) \to \mathbb{R}$ since $\overline{B}(\mathcal{A})$ is a Banach space. But, the Choquet integral $\hat{v} : \overline{B}(\mathcal{A}) \to \mathbb{R}$ has been defined as the sup-norm continuous extension of $\hat{v} : B(\mathcal{A}) \to \mathbb{R}$. Since $f = \hat{v}$ on $B(\mathcal{A})$ by (ii), we deduce that $\bar{f} = \hat{v}$ on $\overline{B}(\mathcal{A})$, that is, $\bar{f}$ is a Choquet functional w.r.t. $v \in bv(\mathcal{A})$.

We end the proof by showing that $\bar{f}$ is subadditive. Indeed, let x, y in $\overline{B}(\mathcal{A}) = \overline{B}_0(\mathcal{A})$, then $\bar{f}(x) = \lim_n f(x^n)$, $\bar{f}(y) = \lim_n f(y^n)$ for some sequences (x^n), (y^n) in $B_0(\mathcal{A})$ converging to x and y respectively. From the subadditivity of f on $B(\mathcal{A})$ (hence also on $B_0(\mathcal{A})$) by (ii), we have $f(x^n + y^n) \leq f(x^n) + f(y^n)$. At the limit, we get

$$\bar{f}(x + y) = \lim_n f(x^n + y^n) \leq \lim_n f(x^n) + \lim_n f(y^n) = \bar{f}(x) + \bar{f}(y). \qquad \square$$

[$(ii') \implies (i)$]. The Choquet functional $\bar{f} : \overline{B}(\mathcal{A}) \to \mathbb{R}$ is subadditive if and only if it is submodular (see Marinacci and Montrucchio [9]). Consequently, from (ii'), the Choquet functional $\bar{f} : \overline{B}(\mathcal{A}) \to \mathbb{R}$ is positively homogeneous, constant additive, submodular, and sup-norm continuous on $\overline{B}(\mathcal{A})$. Thus, f, which is the restriction of $\bar{f}$ on $B(\mathcal{A})$, satisfies the properties of (i). □

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