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Lovász–Schrijver SDP-operator, near-perfect graphs and near-bipartite graphs

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Abstract We study the Lovász–Schrijver lift-and-project operator (LS_+) based on the cone of symmetric, positive semidefinite matrices, applied to the fractional stable set polytope of graphs. The problem of obtaining a combinatorial characterization of graphs for which the LS_+-operator generates the stable set polytope in one step has been open since 1990. We call these graphs LS_+-*perfect*. In the current contribution, we pursue a full combinatorial characterization of LS_+-perfect graphs and make progress towards such a characterization by establishing a new, close relationship among LS_+-perfect graphs, near-bipartite graphs and a newly introduced concept of full-support-perfect graphs.

Keywords Stable set problem · Lift-and-project methods · Semidefinite programming · Integer programming

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1 Introduction

The notion of a perfect graph was introduced by Berge in the early 1960s [5]. A graph is called *perfect* if each of its induced subgraphs has chromatic number equal to the size of a maximum cardinality clique in the subgraph. Perfect graphs have caught the attention of many researchers in the area and inspired numerous very interesting contributions to the literature for the past fifty years. One of the main results in the seminal paper of Grötschel et al. [20] is that perfect graphs constitute a graph class where the Maximum Weight Stable Set Problem (MWSSP) can be solved in polynomial-time. Some years later, the same authors proved a very beautiful, related result which connects a purely graph theoretic notion to polyhedrality of a typically nonlinear convex relaxation and to the integrality and equality of two fundamental polytopes:

Theorem 1 (Grötschel et al. [20,21]) *For every graph G, the following are equivalent:*

(1) *G is perfect;*
(2) *$STAB(G) = CLIQUE(G)$;*
(3) *$TH(G) = STAB(G)$;*
(4) *$TH(G) = CLIQUE(G)$;*
(5) *$TH(G)$ is polyhedral.*

In the above theorem, $STAB(G)$ is the stable set polytope of G, $CLIQUE(G)$ is its clique relaxation and $TH(G)$ is the *theta body* of G defined by Lovász [27].

In the early 1990s, Lovász and Schrijver [28] introduced the semidefinite relaxation $LS_+(G)$ (LS_+ operator was originally denoted by N_+ in [28]) of $STAB(G)$ which is stronger than $TH(G)$. Following the same line of reasoning used for perfect graphs, they proved that MWSSP can be solved in polynomial-time for the class of graphs for which $LS_+(G) = STAB(G)$. We call these graphs LS_+-*perfect graphs* (originally, the authors called these graphs N_+-*perfect* [7]). The set of LS_+-perfect graphs is known to contain many rich and interesting classes of graphs (e.g., perfect graphs, t-perfect graphs, wheels, anti-holes, near-bipartite graphs) and their clique sums. However, no combinatorial characterization of LS_+-perfect graphs have been obtained so far.

There are many studies of various variants of lift-and-project operators applied to the relaxations of the stable set problem (see for instance, [9,15,16,18,19,22–24, 26,31,34]). Why study LS_+-perfect graphs? For example, if we want to characterize the largest family of graphs for which MWSSP can be solved in polynomial-time, then perhaps, we should pick a tractable relaxation of $STAB(G)$ which is as strong as possible. This reasoning would suggest that, we should focus on the strongest, tractable lift-and-project operator and reiterate it as much as possible while maintaining tractability of the underlying relaxation. Even though the (lower bound) analysis for the strongest lift-and-project operators is typically very challenging, some considerable amount of work on the behaviour of the strongest lift-and-project operators applied to the stable set problem already exists (see [1] and the references therein). In the spectrum of strong lift-and-project operators which utilize positive semidefiniteness constraints, given the above results of Grötschel et al. [20,21], it seems clear to us that we should pick an operator which is at least as strong as $TH(G)$. Among many of the convex relaxations that are closely related to $TH(G)$ but stronger, $TH(G)$ continues

to emerge as the central object with quite special mathematical properties (see [10] and the references therein). Given that the operator $LS_+(G)$ can be defined as the intersection of the matrix-space liftings of the odd-cycle polytope of G and the theta body of G, by definition, $LS_+(G)$ encodes and retains very interesting combinatorial information about the graph G. Then, the next question is why not focus on iterated (hence stronger) operator LS_+^k for $k \geq 2$ but k small enough to maintain tractability? The answer to this is related to the above; but, it is a bit more subtle: in the lifted, matrix-space representation of LS_+, if we remove the positive semidefiniteness constraint, we end up with the lifting of the operator LS (defined later). In this lifted matrix space, if we remove the restriction that the matrix be symmetric, we end up with the lifting of a weaker relaxation LS_0. LS_0^k retains many interesting combinatorial properties of G, see [24,25]. Moreover, Lovász and Schrijver proved that for every graph G, $LS_0(G) = LS(G)$. However, this property does not generalize to the iterated operators LS_0^k, LS^k, even for $k = 2$, even if we require that the underlying graph G be perfect (see [2,3]). Therefore, LS_+ has many of the desired attributes for this purpose.

One of our main goals in this line of research is to obtain a characterization of LS_+-perfect graphs similar to the one given in Theorem 1 for perfect graphs. More precisely, we would like to find an appropriate polyhedral relaxation of $STAB(G)$ playing the role of $CLIQUE(G)$ in Theorem 1, when we replace $TH(G)$ by $LS_+(G)$. In [7] we introduced the polyhedral relaxation $NB(G)$ of $STAB(G)$, which is, to the best of our knowledge, the tightest polyhedral relaxation of $LS_+(G)$. Roughly speaking, $NB(G)$ is defined by the family of facets of stable set polytopes of a family of graphs that are built from *near-bipartite graphs* by using simple operations so that the stable set polytope of the resulting graph does not have any facets outside the class of facets which define the stable set polytope of near-bipartite graphs (for a precise definition of $NB(G)$, see Sect. 2). In our quest to obtain the desired characterization of LS_+-perfect graphs, $NB(G)$ is our current best guess for replacing $CLIQUE(G)$ in Theorem 1. More specifically, we conjecture that the next four statements are equivalent.

Conjecture 2 *For every graph G, the following four statements are equivalent:*

(1) *$STAB(G) = NB(G)$;*
(2) *$LS_+(G) = STAB(G)$;*
(3) *$LS_+(G) = NB(G)$;*
(4) *$LS_+(G)$ is polyhedral.*

Verifying the validity of Conjecture 2 is equivalent to determining the validity of the following two statements:

Conjecture 3 *For every graph G, if $LS_+(G)$ is polyhedral then $STAB(G) = NB(G)$.*

Conjecture 4 *For every graph G, if $LS_+(G) = STAB(G)$ then $STAB(G) = NB(G)$.*

In [6], we made some progress towards proving Conjecture 3, by presenting an infinite family of graphs for which it holds. Recently, Conjecture 4 was verified for web graphs [17]. In this contribution, we prove that Conjecture 4 holds for a class of graphs called fs-*perfect graphs* that stand for *full-support-perfect* graphs. This graph family was originally defined in [29] and includes the set of *near-perfect* graphs previously defined by Shepherd in [32].

One of the main difficulties in obtaining a good combinatorial characterization for LS_+-perfect graphs is that the lift-and-project operator LS_+ (and many related operators) can behave sporadically under many well-studied graph-minor operations (see [16,26]). Therefore, in the study of LS_+-perfect graphs we are faced with the problem of constructing suitable graph operations and then deriving certain monotonicity or loose invariance properties under such graph operations. In this context, we present two operations which *preserve* LS_+-*imperfection* in graphs.

In the next section, we begin with notation and preliminary results that will be used throughout the paper. We also state our main characterization conjecture on LS_+-perfect graphs. In Sect. 3, we characterize fs-perfection in the family of graphs built from a minimally imperfect graph and one additional node. In Sect. 4 we prove the validity of the conjecture on fs-perfect graphs. In order to ease the reading of this contribution, the proofs of results on the LS_+ operator are presented in Sect. 5. Section 6 is devoted to the conclusions and some further results.

2 Further definitions and preliminary results

2.1 Graphs and the stable set polytope

Throughout this work, G stands for a simple graph with node set $V(G)$ and edge set $E(G)$. The *complementary graph* of G, denoted by $\overline{G}$, is such that $V(\overline{G}) := V(G)$ and, $E(\overline{G}) := \{uv : u \neq v, u, v \in V(G), uv \notin E(G)\}$. For any positive integer n, K_n, C_n and P_n denote the graphs with n nodes corresponding to a *complete graph*, a *cycle* and a *path*, respectively. We assume that in the cycle C_n node i is adjacent to node $i + 1$ for $i \in \{1, \ldots, n - 1\}$ and n is adjacent to node 1.

Given $V' \subseteq V(G)$, we say that G' is a subgraph of G *induced by the nodes in* V' if $V(G') = V'$ and $E(G') = \{uv : uv \in E(G), \{u, v\} \subseteq V(G')\}$. When $V(G')$ is clear from the context, we say that G' is a node induced subgraph of G and write $G' \subseteq G$. Given $U \subseteq V(G)$, we denote by $G - U$ the subgraph of G induced by the nodes in $V(G) \backslash U$. For simplicity, we write $G - u$ instead of $G - \{u\}$. We say that G_E is an *edge subgraph* of G if $V(G_E) = V(G)$ and $E(G_E) \subseteq E(G)$.

Given the graph G, the set $\Gamma_G(v)$ is the *neighbourhood* of node $v \in V(G)$ and $\delta_G(v) = |\Gamma_G(v)|$. The set $\Gamma_G[v] = \Gamma_G(v) \cup \{v\}$ is the *closed neighbourhood* of node v. When the graph is clear from the context, we simply write $\Gamma(v)$ or $\Gamma[v]$. If $G' \subseteq G$ and $v \in V(G)$, $G' \ominus v$ is the subgraph of G induced by the nodes in $V(G') \backslash \Gamma[v]$. We say that $G' \ominus v$ is obtained from G' by *destruction* of $v \in V(G)$.

A *stable set* in G is a subset of mutually nonadjacent nodes in G and a *clique* is a subset of pairwise adjacent nodes in G. The cardinality of a stable set of maximum cardinality in G is denoted by $\alpha(G)$. The stable set polytope in G, STAB(G), is the convex hull of the characteristic vectors of stable sets in G. The *support* of a valid inequality of the stable set polytope of a graph G is the subgraph induced by the nodes with nonzero coefficient in the inequality. A *full-support* inequality has G as its support.

If $G' \subseteq G$, we may consider every point in STAB(G') as a point in STAB(G), although they do not belong to the same space (for the missing nodes, we take

direct sums with the interval $[0, 1]$, since originally $\mathrm{STAB}(G) \subseteq \mathrm{STAB}(G') \oplus [0, 1]^{V(G) \setminus V(G')}$). Then, given any family of graphs $\mathcal{F}$ and a graph G, we denote by $\mathcal{F}(G)$ the relaxation of $\mathrm{STAB}(G)$ defined by

$$\mathcal{F}(G) := \bigcap_{G' \subseteq G; G' \in \mathcal{F}} \mathrm{STAB}(G'). \tag{1}$$

If FRAC denotes the family of complete graphs of size two, following the definition (1), the polyhedron $\mathrm{FRAC}(G)$ is called the *edge relaxation*. It is known that G is bipartite if and only if $\mathrm{STAB}(G) = \mathrm{FRAC}(G)$. Similarly, if CLIQUE denotes the family of complete graphs, $\mathrm{CLIQUE}(G)$ is the *clique relaxation* already mentioned and a graph is perfect if and only if $\mathrm{STAB}(G) = \mathrm{CLIQUE}(G)$ [14]. Moreover, if OC denotes the family of odd cycles, as a consequence of results in [28] we have the following

Remark 5 If $G - v$ is bipartite for some $v \in V(G)$ then $\mathrm{STAB}(G) = \mathrm{FRAC}(G) \cap \mathrm{OC}(G)$.

In [33] Shepherd defined a graph G to be *near-bipartite* if $G \ominus v$ is bipartite for every $v \in V(G)$. We denote by NB the family of near-bipartite graphs. Since complete graphs and odd cycles are near-bipartite graphs, it is clear that

$$\mathrm{NB}(G) \subseteq \mathrm{CLIQUE}(G) \cap \mathrm{OC}(G).$$

2.2 Minimally imperfect, near-perfect and fs-perfect graphs

Minimally imperfect graphs are those graphs that are not perfect, but after deleting any node they become perfect. The Strong Perfect Graph Theorem [13] (also see [11]; and see [12] for the related recognition problem) states that the only minimally imperfect graphs are the odd cycles and their complements.

Given a graph G it is known that the *full-rank constraint*

$$\sum_{u \in V} x_u \leq \alpha(G)$$

is always valid for $\mathrm{STAB}(G)$. A graph is *near-perfect* if its stable set polytope is defined only by non-negativity constraints, clique constraints and the full-rank constraint [32]. Due to the results of Chvátal [14], near-perfect graphs define a superclass of perfect graphs and due to the results in [30], minimally imperfect graphs are also near-perfect. Moreover, every node induced subgraph of a near-perfect graph is near-perfect [32]. In addition, Shepherd [32] conjectured that near-perfect graphs could be characterized in terms of certain combinatorial parameters and established that the validity of his conjecture follows from the Strong Perfect Graph (then Conjecture, now) Theorem. Therefore,

Theorem 6 ([13,32]) *A graph G is near-perfect if and only if, for every $G' \subseteq G$ minimally imperfect, the following two properties hold:*

(1) $\alpha(G') = \alpha(G)$;

(2) *for all* $v \in V(G)$, $\alpha(G' \ominus v) = \alpha(G) - 1$.

As a generalization of near-perfect graphs, we consider the family of fs-*perfect* (full-support perfect) graphs. A graph is fs-perfect if its stable set polytope is defined only by non-negativity constraints, clique constraints and at most one full-support inequality. Then, every node induced subgraph of an fs-perfect graph is fs-perfect. We say that a graph is *properly* fs-perfect if it is an imperfect fs-perfect graph. Clearly, near-perfect graphs are fs-perfect, but we will see that fs-perfect graphs define a strict superclass of near-perfect graphs.

2.3 The LS$_+$ operator

In this section, we present a definition of the LS$_+$-operator [28] and some of its well-known properties when it is applied to relaxations of the stable set polytope of a graph. In order to do so, we need some more notation and definitions.

We denote by $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ the vectors of the canonical basis of $\mathbb{R}^{n+1}$ where the first coordinate is indexed zero. Given a convex set K in $[0, 1]^n$,

$$\mathrm{cone}(K) := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} : x = x_0 y; \ y \in K; \ x_0 \geq 0 \right\}.$$

Let $\mathbb{S}^n$ be the space of $n \times n$ symmetric matrices with real entries. If $Y \in \mathbb{S}^n$, $\mathrm{diag}(Y)$ denotes the vector whose ith entry is Y_{ii}, for every $i \in \{1, \ldots, n\}$. Let

$$M(K) := \Big\{ Y \in \mathbb{S}^{n+1} : \ Y\mathbf{e}_0 = \mathrm{diag}(Y),$$
$$Y\mathbf{e}_i \in \mathrm{cone}(K),$$
$$Y(\mathbf{e}_0 - \mathbf{e}_i) \in \mathrm{cone}(K), \forall i \in \{1, \ldots, n\} \Big\}.$$

Projecting this polyhedral lifting back to the space $\mathbb{R}^n$ results in

$$\mathrm{LS}(K) := \left\{ x \in [0, 1]^n : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y\mathbf{e}_0, \text{ for some } Y \in M(K) \right\}.$$

Clearly, $\mathrm{LS}(K)$ is a relaxation of the convex hull of integer solutions in K, i.e., $\mathrm{conv}(K \cap \{0, 1\}^n)$.

Let $\mathbb{S}^n_+$ be the space of $n \times n$ symmetric positive semidefinite (PSD) matrices with real entries. Then

$$M_+(K) := M(K) \cap \mathbb{S}^{n+1}_+$$

yields the tighter relaxation

$$\mathrm{LS}_+(K) := \left\{ x \in [0, 1]^n : \begin{pmatrix} 1 \\ x \end{pmatrix} = Y\mathbf{e}_0, \text{ for some } Y \in M_+(K) \right\}.$$

If we let $LS^0(K) := K$, then the successive applications of the LS operator yield $LS^k(K) = LS(LS^{k-1}(K))$ for every $k \geq 1$. Similarly for the LS_+ operator. Lovász and Schrijver proved that $LS^n(K) = LS_+^n(K) = \text{conv}(K \cap \{0, 1\}^n)$.

In this paper, we focus on the behaviour of the LS_+ operator on the edge relaxation of the stable set polytope. In order to simplify the notation we write $LS_+(G)$ instead of $LS_+(\text{FRAC}(G))$ and similarly for the successive iterations of it. It is known [28] that, for every graph G,

$$\text{STAB}(G) \subseteq LS_+(G) \subseteq \text{TH}(G) \subseteq \text{CLIQUE}(G)$$

and

$$\text{STAB}(G) \subseteq LS_+(G) \subseteq \text{NB}(G).$$

2.4 LS_+-perfect graphs

Recall that a graph G is LS_+-perfect if $LS_+(G) = \text{STAB}(G)$. A graph that is not LS_+-perfect is called LS_+-*imperfect*.

Using the results in [16] and [26] we know that all imperfect graphs with at most 6 nodes are LS_+-perfect, except for the two properly near-perfect graphs depicted in Fig. 1, denoted by G_{LT} and G_{EMN}, respectively. These graphs prominently figure into our current work as the building blocks of an interesting family of graphs.

From the results in [28], it can be proved that every subgraph of an LS_+-perfect graph is also LS_+-perfect. Moreover, every graph for which $\text{STAB}(G) = \text{NB}(G)$ is LS_+-perfect. In particular, perfect and near-bipartite graphs are LS_+-perfect. Recall that in Conjecture 3 we wonder whether the only LS_+-perfect graphs are those graphs G for which $\text{STAB}(G) = \text{NB}(G)$. Obviously, G is LS_+-perfect if and only if every facet defining inequality of $\text{STAB}(G)$ is valid for $LS_+(G)$. In this context, we have Lemma 1.5 in [28] that can be rewritten in the following way:

Theorem 7 *Let $ax \leq \beta$ be a full-support valid inequality for $\text{STAB}(G)$. If, for every $v \in V(G)$, $\sum_{w \in V(G-v)} ax \leq \beta - a_v$ is valid for $\text{FRAC}(G \ominus v)$ then $ax \leq \beta$ is valid for $LS_+(G)$.*

In [6] we proved that the converse of the previous result is not true. However, it is plausible that the converse holds when the full-support valid inequality is a facet defining inequality of $\text{STAB}(G)$. Actually, the latter assertion would be a consequence

Fig. 1 The graphs G_{LT} and G_{EMN}
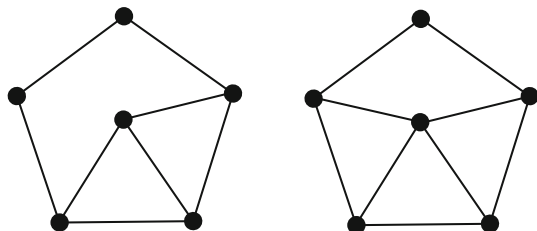
of the validity of Conjecture 3. Thus, we present an equivalent formulation of it in the following.

Conjecture 8 *If a graph is LS$_+$-perfect and its stable set polytope has a full-support facet defining inequality, then the graph is near-bipartite.*

2.5 Graph operations

In this section, we present some properties of four graph operations that will be used throughout this paper. Firstly, let us recall the *complete join* of graphs. Given two graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$, we say that a graph G is obtained after the complete join of G_1 and G_2, denoted $G = G_1 \vee G_2$, if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{vw : v \in V(G_1) \text{ and } w \in V(G_2)\}$. A simple example of a join is the *n-wheel* W_n, for $n \geq 2$ which is the complete join of the trivial graph with one node and the *n*-cycle.

It is known that every facet defining inequality of STAB$(G_1 \vee G_2)$ can be obtained by the cartesian product of facets of STAB(G_1) and STAB(G_2). Hence, odd wheels are fs-perfect but not near-perfect graphs. Moreover, it is easy to see

Remark 9 The complete join of two graphs is properly fs-perfect if and only if one of them is a complete graph and the other one is a properly fs-perfect graph.

Also, it is known that

Remark 10 The complete join of two graphs is LS$_+$-perfect if and only if both of them are LS$_+$-perfect graphs.

Now, let us recall the graph operation, *odd-subdivision of an edge* [35]. Given a graph $G = (V, E)$ and $e \in E$, we say that the graph G' is obtained from G after an *odd-subdivision of the edge e* if it is replaced in G by a path of odd length. Next, we consider the *k-stretching of a node* which is a generalization of the *type (i) stretching operation* defined in [26]. Let v be a node of G with neighborhood $\Gamma(v)$ and let A_1 and A_2 be nonempty subsets of $\Gamma(v)$ such that $A_1 \cup A_2 = \Gamma(v)$, and $A_1 \cap A_2$ is a clique of size k. A *k-stretching of a node* v is obtained as follows: remove v, introduce three nodes instead, called v_1, v_2 and u, and add an edge between v_i and every node in $\{u\} \cup A_i$ for $i \in \{1, 2\}$. Figure 2 illustrates the case $k = 1$.

The type (i) stretching operation presented in [26] corresponds to the case $k = 0$.

We also consider another graph operation defined in [4]. Given a graph G with nodes $\{1, \ldots, n\}$ and a clique $K = \{v_1, \ldots, v_s\}$ in G (not necessarily maximal), the *clique subdivision of the edge $v_1 v_2$ in K* is defined as follows: delete the edge $v_1 v_2$ from G, add the nodes v_{n+1} and v_{n+2} together with the edges $v_1 v_{n+1}$, $v_{n+1} v_{n+2}$, $v_{n+2} v_2$ and $v_{n+i} v_j$ for $i \in \{1, 2\}$ and $j \in \{3, \ldots, s\}$. Figure 3 illustrates the clique subdivision of the edge $v_1 v_2$ in the clique $K = \{v_1, v_2, v_6\}$.

Notice that if the clique is an edge this operation reduces to the odd-subdivision of it.
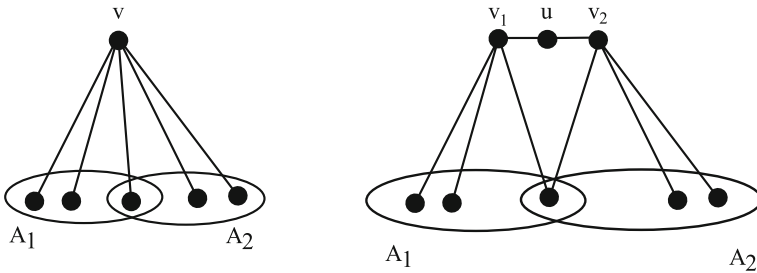
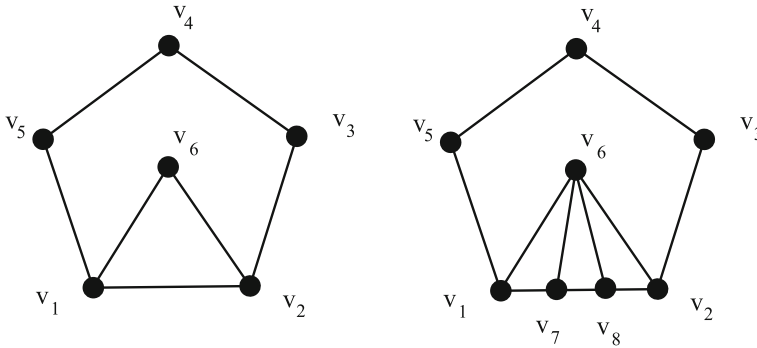**Fig. 2** A 1-stretching operation on node v

Fig. 3 The graph G' is obtained from G after the clique subdivision of edge $v_1 v_2$

3 fs-perfection on graphs in $\mathcal{F}^k$

In order to prove Conjecture 8 on fs-perfect graphs, we first consider a minimal structure that a graph must have in order to be properly fs-perfect and LS_+-imperfect. This leads us to define $\mathcal{F}^k$ for every $k \geq 2$ as the family of graphs having node set $\{0, 1, \ldots, 2k + 1\}$ and such that $G - 0$ is a minimally imperfect graph with $1 \leq \delta_G(0) \leq 2k$. Let us consider necessary conditions for a graph in $\mathcal{F}^k$ to be fs-perfect.

Theorem 11 [29] *Let $G \in \mathcal{F}^k$ be an fs-perfect graph. Then, the following conditions hold:*

(1) $\alpha(G) = \alpha(G - 0)$;

(2) $1 \leq \alpha(G \ominus 0) \leq \alpha(G) - 1$;

(3) *the full-support facet defining inequality of $STAB(G)$ is the inequality*

$$(\alpha(G) - \alpha(G \ominus 0)) x_0 + \sum_{i=1}^{2k+1} x_i \leq \alpha(G).$$

Proof Let

$$\sum_{i=0}^{2k+1} a_i x_i \leq \beta \tag{2}$$

be the full-support facet defining inequality of STAB(G). We may assume that all coefficients a_i, $i \in \{0, \ldots, 2k + 1\}$ are positive integers. Clearly,

$$\text{STAB}(G - 0) = \left\{ x \in \text{CLIQUE}(G - 0) : \sum_{i=1}^{2k+1} a_i x_i \leq \beta \right\}.$$

Since $G - 0$ is a minimally imperfect graph, the inequality $\sum_{i=1}^{2k+1} a_i x_i \leq \beta$ is a positive multiple of its rank constraint, i.e., there exists a positive integer p such that $a_i = p$ for $i \in \{1, \ldots, 2k + 1\}$ and $\beta = p \, \alpha(G - 0)$. Therefore, (2) has the form

$$a_0 x_0 + p \sum_{i=1}^{2k+1} x_i \leq p \, \alpha(G - 0). \tag{3}$$

Observe that there is at least one root $\tilde{x}$ of (3) such that $\tilde{x}_0 = 1$. Clearly, $\tilde{x}$ is the incidence vector of a stable set S of G such that $S - \{0\}$ is a maximum cardinality stable set of $G \ominus 0$. Then,

$$a_0 = p \, (\alpha(G - 0) - \alpha(G \ominus 0)).$$

Since $a_0 \geq 1$, we have $\alpha(G \ominus 0) \leq \alpha(G - 0) - 1$. Moreover, since $\delta_G(0) \leq 2k$, we have $\alpha(G \ominus 0) \geq 1$. Therefore, the inequality (3) becomes

$$(\alpha(G - 0) - \alpha(G \ominus 0))x_0 + \sum_{i=1}^{2k+1} x_i \leq \alpha(G - 0). \tag{4}$$

To complete the proof, we only need to show that $\alpha(G) = \alpha(G - 0)$. Let $\bar{x}$ be the incidence vector of a maximum cardinality stable set in G, then

$$\alpha(G) = \bar{x}_0 + \sum_{i=1}^{2k+1} \bar{x}_i. \tag{5}$$

Moreover, since $\alpha(G - 0) - \alpha(G \ominus 0) \geq 1$ and $\bar{x}$ satisfies (4) we have

$$\alpha(G) = \bar{x}_0 + \sum_{i=1}^{2k+1} \bar{x}_i \leq (\alpha(G - 0) - \alpha(G \ominus 0))\bar{x}_0 + \sum_{i=1}^{2k+1} \bar{x}_i \leq \alpha(G - 0).$$

We have that $\alpha(G) \leq \alpha(G - 0)$, implying $\alpha(G) = \alpha(G - 0)$. $\square$

As a first consequence of the previous theorem we have:

Corollary 12 *Let $G \in \mathcal{F}^k$ be such that $\alpha(G) = 2$. Then, G is fs-perfect if and only if $G - 0 = \overline{C_{2k+1}}$ (the complementary graph of C_{2k+1}) and G is near-perfect.*

Proof Assume that G is fs-perfect. By the previous theorem, $\alpha(G-0) = \alpha(G) = 2$ and then $G - 0 = \overline{C_{2k+1}}$. Moreover, $1 \leq \alpha(G \ominus 0) \leq \alpha(G) - 1 = 1$ and $a_0 = 1$. Thus, G is near-perfect. The converse follows from the definition of fs-perfect graphs. □

For $k \geq 2$, let H^k denote the graph in $\mathcal{F}^k$ having $\alpha(H^k) = 2$ and $\delta_{H^k}(0) = 2k$. Using Theorem 6 it is easy to check that H^k is near-perfect. Using Corollary 12, we have the following result:

Corollary 13 *Let $G \in \mathcal{F}^k$ be such that $\alpha(G) = 2$. Then, G is* fs-*perfect if and only if G is a near-perfect edge subgraph of H^k.*

Let us now study the structure of fs-perfect graphs G in $\mathcal{F}^k$ with stability number at least 3. By Theorem 11, $\alpha(G - 0) = \alpha(G) \geq 3$ and $G - 0 = C_{2k+1}$ with $k \geq 3$. Recall that in the cycle C_{2k+1} node i is adjacent to node $i + 1$ for $i \in \{1, \ldots, 2k\}$ and $2k + 1$ is adjacent to node 1. Clearly, if $\delta(0) \leq 2$ and $0v \in E(G)$ then $G - v$ is bipartite and, by Remark 5, G is not fs-perfect.

From now on, $3 \leq s = \delta(0) \leq 2k$ and $\Gamma(0) = \{v_1, \ldots, v_s\}$ such that $1 \leq v_1 < \ldots < v_s \leq 2k + 1$. Observe that for every $i \in \{1, \ldots, s - 1\}$, the nodes in $\{w \in V(G - 0) : v_i \leq w \leq v_{i+1}\}$ together with node 0 form a chordless cycle D_i in G. Also, D_s in G is the chordless cycle induced by the nodes in $\{w \in V(G - 0) : v_s \leq w \leq 2k + 1$ or $1 \leq w \leq v_1\}$ and node 0. We refer to these cycles as *central cycles of G*. Sometimes, we refer to central cycles of length k as *k-central cycles*. It is easy to see, using parity arguments, that every $G \in \mathcal{F}^k$ has an odd number of odd central cycles. If G has only one odd central cycle, say D_1, then $G - v_1$ is bipartite and, by Remark 5, G is not fs-perfect.

We summarize the previous ideas in the following result:

Lemma 14 *Let $G \in \mathcal{F}^k$ be a* fs-*perfect graph with $\alpha(G) \geq 3$. Then, $k \geq 3$, $G - 0 = C_{2k+1}$ and G has at least three odd central cycles.*

According to the lemma above, we may focus on the structural properties of graphs in $\mathcal{F}^k$ with at least three odd central cycles. Firstly, we have:

Lemma 15 *Let $G \in \mathcal{F}^k$ with $k \geq 3$ be such that $G - 0 = C_{2k+1}$ and G has at least three odd central cycles. Then, G can be obtained after some odd subdivisions of edges in a graph $G' \in \mathcal{F}^p$ for some $2 \leq p < k$ with $\delta_{G'}(0) = \delta_G(0)$. Moreover,*

(1) *if every central cycle of G is odd, G' has one central cycle of length 5 and $2(p-1)$ central cycles of length 3;*
(2) *if G has an even central cycle, every central cycle of G' has length 3 or 4.*

Proof Let $\delta_G(0) = s$ with $s \geq 3$.

(1) If every central cycle of G is odd then s is odd and G has a central cycle with length at least 5. Let $p = \frac{s+1}{2}$ and $G' \in \mathcal{F}^p$ with $\delta_{G'}(0) = s$ such that 0 is adjacent to all nodes in C_{2p+1} except to two nodes, e.g., nodes s and $s + 1$. Clearly, G' has one central cycle of length five, and $s - 1 = 2(p - 1)$ central cycles of length three. Hence G is obtained after some odd subdivisions of edges of G'.

(2) If G has $r \geq 1$ even central cycles then $s + r$ is odd. Let D_i with $i \in \{1, \ldots, s\}$ be the central cycles of G. Let $p = \frac{s+r-1}{2}$ and $G' \in \mathcal{F}^p$ such that $\delta_{G'}(0) = s$ and for $i \in \{1, \ldots, s\}$ the central cycle D_i' in G' is:
   (a) a 3-cycle if D_i is odd,
   (b) a 4-cycle If D_i is even.
   It is straightforward to check that G is obtained from G' after some odd subdivisions of edges.   □

In addition, we have

Lemma 16 *Let $G \in \mathcal{F}^k$ with $k \geq 3$ be such that $G - 0 = C_{2k+1}$ and $\delta_G(0) \geq 3$. Let $t(G)$ be the number of 3-central cycles and $r(G)$ the number of 4-central cycles in G.*

(1) *if G has three consecutive 3-central cycles then G can be obtained after the clique subdivision of an edge in a graph $G' \in \mathcal{F}^{k-1}$ with $t(G') = t(G) - 2$ and $\delta_{G'}(0) = \delta_G(0) - 2$;*
(2) *if $r(G) \geq 2$ then G can be obtained after the 1-stretching operation on a node in a graph $G' \in \mathcal{F}^{k-1}$ with $r(G') = r(G) - 1$ and $\delta_{G'}(0) = \delta_G(0) - 1$.*

Proof (1) By the hypothesis, we may consider that $\{2k - 1, 2k, 2k + 1, 1\} \subseteq \Gamma(0)$. Let $G' \in \mathcal{F}^{k-1}$ be a graph having $\Gamma_{G'}(0) = \Gamma(0) \setminus \{2k, 2k + 1\}$. Clearly, G' has $t(G) - 2$ 3-central cycles and $\delta_{G'}(0) = \delta_G(0) - 2$. Moreover, it is easy to see that G is the clique subdivision of the edge in G' having endpoints $\{1, 2k - 1\}$.

(2) Since there is a 4-central cycle, without loss of generality, we may assume that the nodes in $\{0, 1, 2k, 2k + 1\}$ induce a 4-central cycle in G. Consider $G' \in F^{k-1}$ such that $\Gamma_{G'}(0) = \Gamma_G(0) \setminus \{2k\}$, then G is a 1-stretching of G' performed at node 1 and where the new nodes are $2k$ and $2k + 1$. Clearly, $r(G') = r(G) - 1$ and $\delta_{G'}(0) = \delta_G(0) - 1$.   □

Utilizing the previous lemmas we obtain the following result.

Theorem 17 *Let $G \in \mathcal{F}^k$ with $k \geq 3$ and suppose G has at least three odd central cycles. Then, G can be obtained from G_{LT} or G_{EMN} after successively applying odd-subdivision of an edge, 1-stretching of a node and clique-subdivision of an edge operations.*

Proof Using Lemma 15, by successively performing the odd subdivision of an edge operation, we may restrict ourselves to consider the following cases:

(a) G has one 5-central cycle and $2(k - 1)$ 3-central cycles.
(b) G has at least one even central cycle and every central cycle has length 3 or 4.

Consider $r(G)$, the number of 4-cycles in G. If $r(G) = 0$ then G is a graph described in case (a). Since $k \geq 3$, we have $2(k - 1) \geq 4$, and by Lemma 16 (1), we can conclude that G is obtained from G_{LT} after successive clique-subdivisions of edges. For graphs in case (b), we have that $r(G) \geq 1$ and then by Lemma 15 (2) we have that $2k = r(G) + \delta(G) - 1$. If $r(G) = 1$ and $\delta(G)$ is even, since $k \geq 3$ it follows that G has $t(G) = \delta(G) - 1 \geq 5$ number of 3-cycles. Using Lemma 16 (1) it is not hard to see that G can be obtained from G_{EMN} by successive clique subdivisions of edges. If $r(G) \geq 2$, Lemma 16 (2) implies that G can be obtained by successive 1-stretching of nodes from a graph $G' \in \mathcal{F}^{k'}$ with $r(G') = 1$ and $2k' = \delta(G')$. If $2k' = 4$, then $G' = G_{EMN}$; otherwise, we can refer to the previous case and the proof is complete.   □

4 The conjecture on fs-perfect graphs

In this section, we prove the validity of Conjecture 8 on the family of fs-perfect graphs. We start by proving it on graphs in the family $\mathcal{F}^k$ with $k \geq 2$. Recall that when $G \in \mathcal{F}^k$ is a fs-perfect graph with $\alpha(G) = 2$, Corollary 13 states that G is a near-perfect edge subgraph of H^k. Observe that $H^2 = G_{EMN}$ and then, due to the results in [16], H^k is LS_+-imperfect when $k = 2$.

Next, we prove the imperfection property for the whole family of graphs H^k.

Theorem 18 *For $k \geq 2$, the graph H^k is LS_+-imperfect.*

In order to ease the reading of this paper we postpone the proof of Theorem 18 to Sect. 5. Regarding the behaviour of the LS_+ operator on edge subgraphs, we have the following result:

Lemma 19 *Let G_E be an edge subgraph of G and $ax \leq \beta$ be a valid inequality for $STAB(G_E)$. Then, if $ax \leq \beta$ is not valid for $LS_+^r(G)$, then $STAB(G_E) \neq LS_+^r(G_E)$.*

Proof Clearly, by definition $LS_+^r(G) \subseteq LS_+^r(G_E)$. Thus, if there exists $\hat{x} \in LS_+^r(G)$ such that $a\hat{x} > \beta$ then $ax \leq \beta$ is not valid for $LS_+^r(G_E)$. Moreover, by hypothesis, $ax \leq \beta$ is valid for $STAB(G_E)$ and the result follows. □

As a consequence of the above, we have:

Theorem 20 *Let $G \in \mathcal{F}^k$ be an fs-perfect graph with $\alpha(G) = 2$. Then, G is LS_+-imperfect.*

Proof By Corollary 13 we know that G is a near-perfect edge subgraph of H^k. Theorem 18 states that H^k is LS_+-imperfect; thus, the full rank constraint is not valid for $LS_+(H^k)$. Since $\alpha(H^k) = \alpha(G) = 2$, the full rank constraint is not valid for $LS_+(G)$ and G is LS_+-imperfect. □

Let us consider the fs-perfect graphs G in $\mathcal{F}^k$ with $\alpha(G) \geq 3$. Due to the structural characterization in Theorem 17, we are interested in the behavior of the LS_+-operator under the odd subdivision of an edge, k-stretching of a node and clique subdivision of an edge operations. In this context, a related earlier result is:

Theorem 21 ([26]) *Let G be a graph and $r \geq 1$ such that $LS_+^r(G) \neq STAB(G)$. Further assume that $\tilde{G}$ is obtained from G by using the odd subdivision operation on one of its edges. Then, $LS_+^r(\tilde{G}) \neq STAB(\tilde{G})$.*

Concerning the remaining operations, we present the following results whose proofs are included in Sect. 5 for the sake of clarity.

Theorem 22 *Let G be a graph and $r \geq 1$ such that $LS_+^r(G) \neq STAB(G)$. Further assume that $\tilde{G}$ is obtained from G by using the k-stretching operation on one of its nodes. Then, $LS_+^r(\tilde{G}) \neq STAB(\tilde{G})$.*

Theorem 23 *Let G be a graph and $r \geq 1$ such that $LS^r_+(G) \neq STAB(G)$. Further assume that $\tilde{G}$ is obtained from G by using the clique subdivision operation on one of its edges. Then, $LS^r_+(\tilde{G}) \neq STAB(\tilde{G})$.*

In summary, we can conclude that the odd-subdivision of an edge, the 1-stretching of a node and the clique-subdivision of an edge are operations that *preserve* LS_+-imperfection. Then, the behavior of these operations under the LS_+ operator together with the fact that graphs G_{LT} and G_{EMN} are LS_+-imperfect, Lemma 14 and Theorem 17 allow us to deduce:

Theorem 24 *Let $G \in \mathcal{F}^k$ be an fs-perfect graph with $\alpha(G) \geq 3$. Then, G is LS_+-imperfect.*

Finally, we are able to present the main result of this contribution.

Theorem 25 *Let G be a properly fs-perfect graph which is also LS_+-perfect. Then, G is the complete join of a complete graph (possibly empty) and a minimally imperfect graph.*

Proof Since G is a properly fs-perfect graph, G has a $(2k + 1)$-minimally imperfect node induced subgraph G'. If $G' = G$ the theorem follows. Otherwise, let $v \in V(G) \setminus V(G')$ and let G_v be the subgraph of G induced by $\{v\} \cup V(G')$. Clearly, G_v is properly fs-perfect as well as LS_+-perfect. Then, by Theorem 20 and Theorem 24, $G_v \notin \mathcal{F}^k$. So, $\delta_{G_v}(v) = 2k + 1$ and $G_v = \{v\} \vee G'$. Therefore, if G'' is the subgraph of G induced by $V(G) - V(G')$, $G = G' \vee G''$. By Remark 9, G'' is a complete graph and by Remark 10 the result follows.   □

Since complete joins of complete graphs and minimally imperfect graphs are near-bipartite, they satisfy $NB(G) = STAB(G)$. Therefore, based on the results obtained so far, we can conclude that Conjecture 8 holds for fs-perfect graphs.

5 Results concerning the LS_+-operator

In this section we include the proofs of some results on the LS_+-operator that were stated without proof in the previous sections.

5.1 The LS_+-imperfection of the graph H^k

Recall that $V(H^k) = \{0, 1, \ldots, 2k+1\}$, $H^k - 0 = \overline{C_{2k+1}}$ and $\delta(0) = 2k$. Without loss of generality, we may assume that the node $2k + 1$ in H^k is the only one not connected with node 0. Let us introduce the point $x(k, \gamma) := \frac{1}{2k+2+\gamma}(2, 2, \ldots, 2, 4)^\top \in \mathbb{R}^{2k+2}$ where for $i \in \{1, \ldots, 2k + 2\}$, the ith component of $x(k, \gamma)$ corresponds to the node $i - 1$ in H^k. In what follows, we show that $x(k, \gamma) \in LS_+(H^k) \setminus STAB(H^k)$ for some $\gamma \in (0, 1)$ thus proving Theorem 18. We first consider $\beta_k = \frac{1}{2k+2}$, $\gamma \in (0, 1)$ and the

$(2k + 3) \times (2k + 3)$ matrix given by

$$
Y(k, \gamma) := \begin{bmatrix}
(2k+2+\gamma) & 2 & 2 & 2 & 2 & 2 & \cdots & 2 & 4 \\
2 & 2 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 \\
2 & 0 & 2 & 1-\beta_k & 0 & 0 & \cdots & 0 & 1+\beta_k \\
2 & 0 & 1-\beta_k & 2 & 1+\beta_k & 0 & \cdots & 0 & 0 \\
2 & 0 & 0 & 1+\beta_k & 2 & 1-\beta_k & \cdots & 0 & 0 \\
2 & 0 & 0 & 0 & 1-\beta_k & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2 & 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 1+\beta_k \\
4 & 2 & 1+\beta_k & 0 & 0 & 0 & \cdots & 1+\beta_k & 4
\end{bmatrix}.
$$

Lemma 26 *For $k \geq 3$, there exists $\gamma \in (0, 1)$ such that $Y(k, \gamma)$ is PSD.*

Proof Let us denote by $\tilde{Y}(k)$ the $(2k + 2) \times (2k + 2)$ submatrix of $Y(k, \gamma)$ obtained after deleting the first row and column. Also consider $\hat{Y}(k)$, the Schur Complement of the $(1, 1)$ entry of $\tilde{Y}(k)$, then

$$
\hat{Y}(k) = \begin{bmatrix}
2 & 1-\beta_k & 0 & 0 & \cdots & 0 & 1+\beta_k \\
1-\beta_k & 2 & 1+\beta_k & 0 & \cdots & 0 & 0 \\
0 & 1+\beta_k & 2 & 1-\beta_k & \cdots & 0 & 0 \\
0 & 0 & 1-\beta_k & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & 1+\beta_k \\
1+\beta_k & 0 & 0 & 0 & \cdots & 1+\beta_k & 2
\end{bmatrix}.
$$

Claim 27 *For every $k \geq 2$, $\tilde{Y}(k)$ is positive definite.*

Proof Let us first show that $\hat{Y}(k)$ is positive definite. For this purpose, we only need to verify that every leading principal minor of $\hat{Y}(k)$ is positive.

Let us define $A_0(k) := 1$, $B_0(k) := 2$ and for $\ell \geq 1$,

$$
A_\ell(k) := \det \begin{bmatrix}
2 & 1-\beta_k & 0 & 0 & \cdots & 0 & 0 \\
1-\beta_k & 2 & 1+\beta_k & 0 & \cdots & 0 & 0 \\
0 & 1+\beta_k & 2 & 1-\beta_k & \cdots & 0 & 0 \\
0 & 0 & 1-\beta_k & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & 1-\beta_k \\
0 & 0 & 0 & 0 & \cdots & 1-\beta_k & 2
\end{bmatrix},
$$

where the matrix in the definition is $2\ell \times 2\ell$ and

$$B_\ell(k) := \det \begin{bmatrix} 2 & 1-\beta_k & 0 & 0 & \cdots & 0 & 0 \\ 1-\beta_k & 2 & 1+\beta_k & 0 & \cdots & 0 & 0 \\ 0 & 1+\beta_k & 2 & 1-\beta_k & \cdots & 0 & 0 \\ 0 & 0 & 1-\beta_k & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1+\beta_k \\ 0 & 0 & 0 & 0 & \cdots & 1+\beta_k & 2 \end{bmatrix},$$

where the matrix in the definition is $(2\ell + 1) \times (2\ell + 1)$. Using the determinant expansion on $A_\ell(k)$ and $B_\ell(k)$ we have that for every $\ell \geq 1$,

$$A_\ell(k) = 2B_{\ell-1}(k) - (1 - \beta_k)^2 A_{\ell-1}(k),$$
$$B_\ell(k) = 2A_\ell(k) - (1 + \beta_k)^2 B_{\ell-1}(k),$$

and

$$\det\left(\hat{Y}(k)\right) = 2\left[A_k(k) - (1 + \beta_k)^2 B_{k-1}(k) + (1 - \beta_k)^k(1 + \beta_k)^{k+1}\right]$$
$$= 2\left[B_k(k) - A_k(k) + (1 - \beta_k)^k(1 + \beta_k)^{k+1}\right]. \tag{6}$$

Using these recursive formulas, we find that for every $\ell \geq 1$,

$$A_\ell(k) = (2\ell + 1 - \beta_k)(1 - \beta_k)^{\ell-1}(1 + \beta_k)^\ell, \tag{7}$$
$$B_\ell(k) = (2\ell + 2)(1 - \beta_k)^\ell(1 + \beta_k)^\ell. \tag{8}$$

Note that $A_\ell(k)$ and $B_\ell(k)$ are positive for every ℓ and every k. Moreover, every leading principal minor of $\hat{Y}(k)$, except itself, is either $A_\ell(k)$ or $B_\ell(k)$ for some ℓ. Using (6), (7) and (8) we compute

$$\det\left(\hat{Y}(k)\right) = 2\left[2 - (2k + 1)\beta_k - \beta_k^2\right](1 - \beta_k)^{k-1}(1 + \beta_k)^k \tag{9}$$

which is indeed positive. Since we verified that every principle minor of $\hat{Y}(k)$ is positive, we deduce that $\hat{Y}(k)$ is positive definite. Finally, after applying the Schur Complement Lemma we have that $\tilde{Y}(k)$ is positive definite. $\qquad\square$

Using this claim we have:

Claim 28 *Let u be the (unique) vector such that*

$$\tilde{Y}(k)u = 2(\mathbf{1} + \mathbf{e}_{2k+2}). \tag{10}$$

Then $Y(k, \gamma)$ is PSD if and only if $\gamma \geq 1 - \beta_k u_{2k+2}$.

Proof Using the Schur Complement Lemma for $\tilde{Y}(k)$ we have that $Y(k, \gamma)$ is PSD if and only if

$$\tilde{Y}(k) - \frac{4}{2k+2+\gamma}(\mathbf{1} + \mathbf{e}_{2k+2})(\mathbf{1} + \mathbf{e}_{2k+2})^\top \text{ is PSD.}$$

Using the automorphism $\left[\tilde{Y}(k)\right]^{-1/2} \cdot \left[\tilde{Y}(k)\right]^{-1/2}$ of the PSD cone, the latter is true if and only if the following matrix

$$I - \frac{4}{2k+2+\gamma}\left[\tilde{Y}(k)\right]^{-1/2}(\mathbf{1} + \mathbf{e}_{2k+2})(\mathbf{1} + \mathbf{e}_{2k+2})^\top\left[\tilde{Y}(k)\right]^{-1/2} \qquad (11)$$

is PSD. Since

$$\frac{4}{2k+2+\gamma}\left[\tilde{Y}(k)\right]^{-1/2}(\mathbf{1} + \mathbf{e}_{2k+2})(\mathbf{1} + \mathbf{e}_{2k+2})^\top\left[\tilde{Y}(k)\right]^{-1/2}$$

is a rank one matrix, using (10) we have that the matrix in (11) is PSD if and only if

$$1 \geq \frac{4}{2k+2+\gamma}(\mathbf{1} + \mathbf{e}_{2k+2})^\top[\tilde{Y}(k)]^{-1}(\mathbf{1} + \mathbf{e}_{2k+2}) = \frac{2(\mathbf{1} + \mathbf{e}_{2k+2})^\top u}{2k+2+\gamma}. \qquad (12)$$

To conclude the above, we used the observation that for every $h \in \mathbb{R}^n \setminus \{0\}$,

$$I - hh^\top \text{ is PSD iff } \frac{h^\top}{\|h\|_2}\left(I - hh^\top\right)\frac{h}{\|h\|_2} \geq 0.$$

To see the latter, one can apply a spectral decomposition to the matrix $\left(I - hh^\top\right)$, where one of the eigenvectors is $\frac{h}{\|h\|_2}$.

Now, using the definition of $\tilde{Y}(k)$ we have that

$$\tilde{Y}(k)\mathbf{1} = 4(\mathbf{1}) + (4 + 2\beta_k)\mathbf{e}_{2k+2} = 2\tilde{Y}(k)u + 2\beta_k\mathbf{e}_{2k+2},$$

and then

$$u = \frac{1}{2}\mathbf{1} - \beta_k[\tilde{Y}(k)]^{-1}\mathbf{e}_{2k+2}. \qquad (13)$$

Therefore,

$$2(\mathbf{1} + \mathbf{e}_{2k+2})^\top u = 2(\mathbf{1} + \mathbf{e}_{2k+2})^\top\left(\frac{1}{2}\mathbf{1} - \beta_k[\tilde{Y}(k)]^{-1}\mathbf{e}_{2k+2}\right)$$

$$= (2k+3) - 2\beta_k(\mathbf{1} + \mathbf{e}_{2k+2})^\top[\tilde{Y}(k)]^{-1}\mathbf{e}_{2k+2}$$

and again using (10), we obtain

$$2(\mathbf{1} + \mathbf{e}_{2k+2})^\top u = (2k+3) - \beta_k u_{2k+2}. \qquad (14)$$

Hence, using (12) and (14) we can conclude that the matrix in (11) is PSD if and only if

$$1 \geq \frac{(2k+3) - \beta_k u_{2k+2}}{2k+2+\gamma}$$

or equivalently, if and only if

$$\gamma \geq 1 - \beta_k u_{2k+2}. \qquad \qquad \square$$

By the previous claims, to prove that $Y(k, \gamma)$ is PSD for some $\gamma \in (0, 1)$, it suffices to prove that there exists $\gamma \in (0, 1)$ such that

$$\gamma \geq 1 - \beta_k u_{2k+2},$$

where u is the unique solution of (10).

Thus, as long as $u_{2k+2} > 0$, we may have $\gamma < 1$ as desired. Using (13), we have that

$$u_{2k+2} = \mathbf{e}_{2k+2}^\top u = \frac{1}{2} - \beta_k \mathbf{e}_{2k+2}^\top \tilde{Y}(k)^{-1} \mathbf{e}_{2k+2}.$$

Based on the above expression, we may interpret $\mathbf{e}_{2k+2}^\top \tilde{Y}(k)^{-1} \mathbf{e}_{2k+2}$ as the $(2k+2)$nd entry of the unique solution h of the linear system $\tilde{Y}(k)h = \mathbf{e}_{2k+2}$. Now using Cramer's rule to express this entry as a ratio of two determinants and the definitions of $\tilde{Y}$, $\hat{Y}$, and A_k, we conclude

$$u_{2k+2} = \frac{1}{2} - \beta_k \frac{A_k(k)}{\det\left(\hat{Y}(k)\right)}.$$

Then, using the identities (7) and (9), we obtain

$$u_{2k+2} = \frac{1}{2}\left[1 - \beta_k \frac{2k+1-\beta_k}{2 - (2k+1)\beta_k - \beta_k^2}\right].$$

The numerator of the last ratio above is strictly less than $(2k+1)$ and the denominator is strictly larger than one. Whence, $u_{2k+2} > \frac{1}{4(k+1)} > 0$. This completes the proof. $\square$

Utilizing the previous lemma we are able to prove Theorem 18.

Proof of Theorem 18 Recall, we may assume that in H^k the node $2k+1$ is not connected to node 0. Let $\gamma \in (0, 1)$ and $x(k, \gamma) = \frac{1}{2k+2+\gamma}(2, 2, \ldots, 2, 4)^\top \in \mathbb{R}^{2k+2}$ where the ith component of $x(k, \gamma)$ corresponds to node $i-1$ in H^k for $i \in \{1, \ldots, 2k+2\}$. Let $Y^*(k, \gamma) := \frac{1}{2k+2+\gamma}Y(k, \gamma)$. Then, $Y^*(k, \gamma)$ is a symmetric

matrix that clearly satisfies that $Y^*(k, \gamma)\mathbf{e}_0 = \mathrm{diag}(Y^*(k, \gamma)) \in \mathrm{FRAC}(H^k)$. Moreover, it is not hard to check that, for $i \in \{1, \ldots, 2k + 2\}$,

$$Y^*(k, \gamma)\mathbf{e}_i \in \mathrm{cone}(\mathrm{FRAC}(H^k)) \quad \text{and} \quad Y^*(k, \gamma)(\mathbf{e}_0 - \mathbf{e}_i) \in \mathrm{cone}(\mathrm{FRAC}(H^k)).$$

This proves that $Y^*(k, \gamma) \in M(H^k)$. By the previous lemma, there exists $\bar{\gamma} \in (0, 1)$ for which $Y^*(k, \bar{\gamma}) \in M_+(H^k)$. Hence, $x(k, \bar{\gamma}) \in \mathrm{LS}_+(H^k)$. It only remains to observe that $x(k, \bar{\gamma})$ violates the rank inequality of H^k. Thus, $x(k, \bar{\gamma}) \notin \mathrm{STAB}(H^k)$. $\qquad\square$

5.2 Operations that preserve LS$_+$-imperfection

Firstly, we prove Theorem 22 on the k-stretching operation for $k \geq 1$, already stated in Sect. 4. Actually, we will see that the same proof given in [26] for the case $k = 0$ can be used for the case $k \geq 1$. Assume that $\tilde{G}$ is obtained from G after the k-stretching operation on node v and let u, v_1 and v_2 be as in the definition of the operation in Sect. 2.5. For any $x \in \mathbb{R}^{V(G)}$, we write $x = \begin{pmatrix} \bar{x} \\ x_v \end{pmatrix}$ where $\bar{x} \in \mathbb{R}^{V(G-v)}$.

For the case $k = 0$, the authors in [26] prove that if a point $x = \begin{pmatrix} \bar{x} \\ x_v \end{pmatrix} \in \mathrm{LS}_+^r(G)$ then the point $\tilde{x}$ given by

$$x_w = \begin{cases} \tilde{x}_w & \text{if } w \in \{u, v_1, v_2\}, \\ \bar{x}_w & \text{otherwise,} \end{cases}$$

satisfies $\tilde{x} \in \mathrm{LS}_+^r(\tilde{G})$. In order to do so they prove that if

$$Y = \left[\begin{array}{c|c|c} 1 & \bar{x}^\top & x_v \\ \hline \bar{x} & \bar{X} & \bar{y} \\ \hline x_v & \bar{y}^\top & x_v \end{array} \right] \in M_+(\mathrm{LS}_+^{r-1}(G))$$

then

$$\tilde{Y} = \left[\begin{array}{c|c|c|c|c} 1 & \bar{x}^\top & x_v & x_v & (1 - x_v) \\ \hline \bar{x} & \bar{X} & \bar{y} & \bar{y} & \bar{x} - \bar{y} \\ \hline x_v & \bar{y}^\top & x_v & x_v & 0 \\ \hline x_v & \bar{y}^\top & x_v & x_v & 0 \\ \hline (1 - x_v) & (\bar{x} - \bar{y})^\top & 0 & 0 & (1 - x_v) \end{array} \right] \in M_+(\mathrm{LS}_+^{r-1}(\tilde{G})).$$

On the other hand, they show that if $\sum_{j \in V(G)} a_j x_j \leq \beta$ is a valid inequality for STAB(G), defining $\tilde{\beta} = \beta + a_v$ and

$$
\tilde{a}_j = \begin{cases} a_v & \text{if } j \in \{v_1, v_2, u\}, \\[2mm] a_j & \text{otherwise,} \end{cases}
$$

the inequality $\sum_{j \in V(\tilde{G})} \tilde{a}_j x_j \leq \tilde{\beta}$ is valid for STAB($\tilde{G}$). Moreover, if x^* violates $\sum_{j \in V(G)} a_j x_j \leq \beta$ then $\tilde{x}^*$ violates $\sum_{j \in V(\tilde{G})} \tilde{a}_j x_j \leq \tilde{\beta}$.

Proof of Theorem 22 It is enough to observe that $\tilde{Y} \in M_+\left(\mathrm{LS}^{r-1}(\tilde{G})\right)$ and the inequality $\sum_{j \in V(\tilde{G})} \tilde{a}_j x_j \leq \tilde{\beta}$ is valid for STAB($\tilde{G}$) even for the case that $\tilde{G}$ is obtained after the k-stretching on node v in G, for $k \geq 1$. $\qquad\square$

Let us now consider the clique-subdivision operation defined in [4]. For $x \in \mathbb{R}^n$, let $\bar{x} \in \mathbb{R}^{n+2}$ such that $\bar{x}_i = x_i$ for every $i \in \{1, \dots, n\}$, $\bar{x}_{n+1} = x_2$ and $\bar{x}_{n+2} = x_1$, and write $\bar{x} = \begin{pmatrix} x \\ x_2 \\ x_1 \end{pmatrix}$. In [4] the authors prove that if $\tilde{G}$ is obtained from G by the clique subdivision of the edge $v_1 v_2$ in the clique K and $x \in \mathrm{LS}_+^k(G)$ then $\bar{x} \in \mathrm{LS}^k(\tilde{G})$. In order to do so, they show that if $Y e_0 = \begin{pmatrix} 1 \\ x \end{pmatrix}$ for
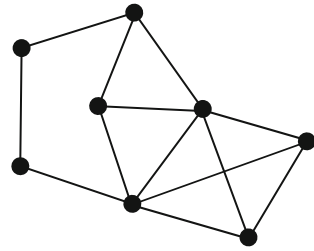
$$
Y = \left[\begin{array}{c|c|c|c}
1 & x_1 & x_2 & \bar{x}^\top \\ \hline
x_1 & x_1 & 0 & y_1^\top \\ \hline
x_2 & 0 & x_2 & y_2^\top \\ \hline
\bar{x} & y_1 & y_2 & \bar{X}
\end{array} \right] \in M(\mathrm{LS}^{r-1}(G)) \tag{15}
$$

then

$$
\tilde{Y} = \left[\begin{array}{c|c|c|c|c|c}
1 & x_1 & x_2 & \bar{x}^\top & x_2 & x_1 \\ \hline
x_1 & x_1 & 0 & y_1^\top & 0 & x_1 \\ \hline
x_2 & 0 & x_2 & y_2^\top & x_2 & 0 \\ \hline
\bar{x} & y_1 & y_2 & \bar{X} & y_2 & y_1 \\ \hline
x_2 & 0 & x_2 & y_2^\top & x_2 & 0 \\ \hline
x_1 & x_1 & 0 & y_1^\top & 0 & x_1
\end{array} \right] \in M(\mathrm{LS}^{r-1}(\tilde{G})). \tag{16}
$$

Proof of Theorem 23 It is enough to observe that if the matrix Y is PSD then so is the matrix in (16). $\qquad\square$

6 Conclusions and further results

In this work, we face the problem of characterizing the stable set polytope of LS_+-perfect graphs, a graph class where the Maximum Weight Stable Set Problem is polynomial-time solvable. This class strictly includes many well-known graph classes such as perfect graphs, t-perfect graphs, wheels, anti-holes, near-bipartite graphs and the graphs obtained from various suitable compositions of these. The stable set polytope of either a perfect or a near-bipartite graph only needs the inequalities associated with the stable set polytopes of its near-bipartite subgraphs. We have conjectured that the same holds for all LS_+-perfect graphs. In this paper, we prove the validity of this conjecture for fs-perfect graphs, a superclass of near-perfect graphs. Moreover, if $\mathcal{FS}$ denotes the class of fs-perfect graphs, using the definition in (1), we actually prove that the conjecture holds for a superclass of fs-perfect graph defined as those graphs for which $\mathcal{FS}(G) = \text{STAB}(G)$. Observe that the graph in Fig. 4 satisfies $\mathcal{FS}(G) = \text{STAB}(G)$ and it is not fs-perfect.

Also, the results used in the proof of the Theorem 25 allow us to conclude the following:

Corollary 29 *Let G be a graph such that $V(G) = \{0, 1, \ldots, 2k + 1\}$ with $k \geq 2$ and $G - 0$ is minimally imperfect. Then:*

- *If $G - 0 = C_{2k+1}$ then G is LS_+-perfect if and only if either $\delta_G(0) \geq 2$ and G has only one odd central cycle or $\delta_G(0) \in \{0, 1, 2k + 1\}$.*
- *If $G - 0 = \overline{C_{2k+1}}$ with $\alpha(G) = 2$ then G is LS_+-perfect if and only $\delta_G(0) = 2k+1$.*

From the above characterization, we identify some of the forbidden structures in the family of LS_+-perfect graphs:

Corollary 30 *Let G be an LS_+-perfect graph. Then, there is no subgraph G' of G such that*

- *$G' - v_0 = C_{2k+1}$, $2 \leq \delta_{G'}(v_0) \leq 2k$ and G' has at least two odd central cycles, or*
- *$G' - v_0 = \overline{C_{2k+1}}$ and $k + 1 \leq \delta'_G(v_0) \leq 2k$ and $\alpha(G') = 2$,*

for some $v_0 \in V(G')$ and $k \geq 2$.

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