Math. Program., Ser. B (2013) 139:71–79
DOI 10.1007/s10107-013-0660-1

FULL LENGTH PAPER

Semicontinuous limits of nets of continuous functions

Gerald Beer

Received: 25 October 2010 / Accepted: 8 September 2011 / Published online: 4 April 2013

Abstract In this paper we present a topology on the space of real-valued functions defined on a functionally Hausdorff space X that is finer than the topology of pointwise convergence and for which (1) the closure of the set of continuous functions $\mathcal{C}(X)$ is the set of upper semicontinuous functions on X, and (2) the pointwise convergence of a net in $\mathcal{C}(X)$ to an upper semicontinuous limit automatically ensures convergence in this finer topology.

Keywords Semicontinuous function · Pointwise convergence · Strong pointwise convergence · The Bartle property · Sticking topology

Mathematics Subject Classification (2000) 26A15 · 54C30 · 54C35

1 Introduction

Let $\langle X, \tau\rangle$ be a Hausdorff space. A real-valued function g on X is called *upper semicontinuous* at $x_0 \in X$ if for each $\varepsilon > 0$, $\exists$ a neighborhood V of x_0 such that $\forall v \in V,\ g(v) < g(x_0) + \varepsilon$. We denote the set of real-valued globally upper semicontinuous functions on X by $\mathcal{U}(X)$. Global upper semicontinuity of a real-valued function g can be characterized in either of these ways [5]: (1) $\forall \alpha \in \mathbb{R}$, $\{x \in X : g(x) < \alpha\}$ is open in X; (2) the *hypograph* of g defined by $\{(x, \alpha) : x \in X, \alpha \in \mathbb{R}, \text{ and } \alpha \le g(x)\}$ is a closed subset of $X \times \mathbb{R}$. The family $\mathcal{U}(X)$ forms a cone-lattice: if f, g are upper semicontinuous, so are $f \vee g$, $f \wedge g$, $f + g$, and αf (for $\alpha \ge 0$).

Dedicated to Jonathan Borwein on the occasion of his 60th birthday.

G. Beer (✉)
Department of Mathematics, California State University, Los Angeles,
Los Angeles, CA90032, USA
e-mail: gbeer@cslanet.calstatela.edu

Certain properties of the domain space can be characterized in terms of the ability to approximate elements of $\mathcal{U}(X)$ by elements of $\mathcal{C}(X)$, the continuous real-valued functions on X. For example, X is completely regular if and only if each $g \in \mathcal{U}(X)$ is the infimum of the continuous real-valued functions that majorize g [13]. This gives rise to the existence of a net of continuous functions pointwise convergent to g from above. To see this, for each finite subset $\{x_1, x_2, x_3, \ldots, x_n\}$ of X and scalars $\alpha_i > g(x_i)$ where $1 \leq i \leq n$, there exists a continuous function $f = f_{(x_1,x_2,\ldots,x_n,\alpha_1,\alpha_2,\ldots,\alpha_n)}$ majorizing g such that for each i, $f(x_i) < \alpha_i$ (f exists because $\mathcal{C}(X)$ is a lattice). Put

$$\Omega := \{(x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n) : n \in \mathbb{N} \text{ and } \forall i \leq n, \alpha_i > g(x_i)\},$$

directed by $(x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n) \preceq (w_1, w_2, \ldots, w_m, \beta_1, \beta_2, \ldots, \beta_m)$ provided each (x_i, α_i) lies on or above some (w_j, β_j). Then

$$(x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n) \mapsto f_{(x_1,x_2,\ldots,x_n,\alpha_1,\alpha_2,\ldots,\alpha_n)}$$

is the desired net.

The existence of a (decreasing) sequence in $\mathcal{C}(X)$ pointwise convergent to an arbitrary upper semicontinuous function g from above is characterized by the perfect normality of X, i.e., by the property that each closed subset of X is a G_δ subset [23]. In the special case that X is metrizable, there is an attractive geometric algorithm for constructing this sequence [5, pg. 18].

If we topologize the space $\mathbb{R}^X$ of all real-valued functions on X in different ways, we get different closures for $\mathcal{C}(X)$. Under very mild assumptions, the closure with respect to the topology of pointwise convergence is $\mathbb{R}^X$ itself, i.e., the continuous functions are dense. If we equip $\mathbb{R}^X$ with topology of uniform convergence or in the case that X is a κ-space, the topology of uniform convergence on compact subsets [15, pg. 202], then $\mathcal{C}(X)$ is in fact closed. One of course wonders if there is an intermediate topology for which the closure is exactly $\mathcal{U}(X)$. It is the purpose of this note to identify what is the weakest topology of this kind, in the sense that if $\langle f_\lambda \rangle$ is a net in $\mathcal{C}(X)$ pointwise convergent to $g \in \mathcal{U}(X)$, then it is of necessity already convergent to g in this intermediate topology.

2 Preliminaries

In the sequel all spaces are Hausdorff and assumed to consist of at least two points. A function $h \in \mathbb{R}^X$ is called *lower semicontinuous* if $-h$ is upper semicontinuous. We denote the lower semicontinuous functions on $\langle X, \tau \rangle$ by $\mathcal{L}(X)$. Evidently, $\mathcal{U}(X) \cap \mathcal{L}(X) = \mathcal{C}(X)$, and the characteristic function χ_E of a subset E of X is upper (resp. lower) semicontinuous if and only if E is closed (resp. open). A real function h is lower semicontinuous if and only if its *epigraph* $\{(x, \alpha) : x \in X, \alpha \in \mathbb{R}, \text{and } \alpha \geq h(x)\}$ is closed. The theory that we will develop of course yields a dual theory for lower semicontinuous functions.

There is a huge literature on the minimization of lower semicontinuous functions, especially lower semicontinuous convex functions, where perhaps the most celebrated result is the Ekeland Variational Principle (see, e.g., [8,19]). The most important convergence notions for lower semicontinuous functions in applications such as epi-convergence, Attouch-Wets convergence, and in the convex case, slice or Joly convergence, are derivative of set-convergence constructs, upon identifying elements of $\mathcal{L}(X)$ with their epigraphs (see, e.g., [3,5,10,19,22]). For domains that are finite dimensional Euclidean spaces where all of these notions coincide, convergence may be equivalently understood as (1) classical Kuratowski-Painlevé convergence of epigraphs, and as (2) convergence of epigraphs in the Fell topology. While epigraphical convergence from below implies pointwise convergence, this fails more generally. We invite the reader to construct a sequence of piecewise linear real functions on $\mathbb{R}$ that epigraphically converges to the zero function but that nevertheless is nowhere pointwise convergent. Thus, such notions and their duals for upper semicontinuous functions are outside the scope of the present investigation.

If g is upper semicontinuous and h is a lower semicontinuous function that majorizes g, one might seek to insert a continuous function between them. Underlying structural properties of X can also be characterized by insertion theorems (see, e.g., [15]), also called sandwich theorems [9].

If W is a set, a *quasi-uniformity* on W is a family $\mathfrak{D}$ of reflexive relations on W that form a filter and such that for each $D \in \mathfrak{D}$, $\exists D_0 \in \mathfrak{D}$ with $D_0 \circ D_0 \subseteq D$ [16,18]. A subfamily $\widehat{\mathfrak{D}}$ of $\mathfrak{D}$ is called a *base* for the quasi-uniformity if $\forall D \in \mathfrak{D}$, $\exists \widehat{D} \in \widehat{\mathfrak{D}}$ with $\widehat{D} \subseteq D$. The *conjugate quasi-uniformity* determined by $\mathfrak{D}$ is the family of relations $\{D^{-1} : D \in \mathfrak{D}\}$, and the smallest uniformity containing $\mathfrak{D}$ has as a base all sets of the form $\widehat{D} \cap \widehat{D}^{-1}$ where $\widehat{D}$ runs over a prescribed base for $\mathfrak{D}$.

Given a quasi-uniformity $\mathfrak{D}$ on W, for each $D \in \mathfrak{D}$ and $w_0 \in W$ put $D(w_0) = \{w : (w_0, w) \in D\}$. Then $\{D(w_0) : D \in \mathfrak{D}\}$ forms a neighborhood base at w_0 for a topology on W called the *topology of the quasi-uniformity*. While a topology is induced by a uniformity if and only if it is completely regular [15,24], each topology is induced by a quasi-uniformity [16], the most familiar of which is the Pervin quasi-uniformity [21].

We now introduce three Hausdorff function space topologies on the real-valued functions defined on $\langle X, \tau\rangle$. Letting $\mathfrak{F}_0(X)$ denote the family of nonempty finite subsets of X, a base for the standard uniformity for the *topology of pointwise convergence* $\mathscr{T}_p$ on $\mathbb{R}^X$ [20,24], sometimes called the *pointwise uniformity*, consists of all entourages of the form

$$[F, \varepsilon]^p := \{(g, h) : \forall x \in F,\ |g(x) - h(x)| < \varepsilon\}\,(F \in \mathfrak{F}_0(X), \varepsilon > 0).$$

Thinking of this uniformity as a quasi-uniformity, a finer quasi-uniformity on $\mathbb{R}^X$ has as a base all sets of the form

$$\begin{aligned}[F, \varepsilon]^u := \{(g, h) : \forall x \in F,\ |g(x) - h(x)| < \varepsilon \text{ and } \exists \text{ a neighborhood } V \text{ of } F \\ \text{such that } \forall x \in V,\ g(x) < h(x) + \varepsilon\}\ (F \in \mathfrak{F}_0(X), \varepsilon > 0).\end{aligned}$$

We denote the induced topology on $\mathbb{R}^X$ by $\mathscr{T}_u$. Note that

- the neighborhoods V in the definition of $[F, \varepsilon]^u$ depend on g and h, not just on F and ε;
- $[F_1 \cup F_2, \min\{\varepsilon_1, \varepsilon_2\}]^u \subseteq [F_1, \varepsilon_1]^u \cap [F_2, \varepsilon_2]^u$;
- $[F, \frac{1}{2}\varepsilon]^u \circ [F, \frac{1}{2}\varepsilon]^u \subseteq [F, \varepsilon]^u$;
- as the induced topology $\mathscr{T}_u$ is finer than $\mathscr{T}_p$, it is Hausdorff as well;
- if g is lower semicontinuous and its graph lies above a compact subset K of the product, and h is close to g in $\mathscr{T}_u$, then the same can be said for the graph of h;
- $(f, g) \mapsto f + g$, $(f, g) \mapsto f \vee g$, and $(f, g) \mapsto f \wedge g$, are $\mathscr{T}_u$-jointly continuous on $\mathbb{R}^X \times \mathbb{R}^X$;
- $(\alpha, f) \mapsto \alpha f$ is jointly continuous on $(0, \infty) \times \mathbb{R}^X$.

In general, $(\alpha, f) \mapsto \alpha f$ is not jointly continuous on $[0, \infty) \times \mathbb{R}^X$. Points $(0, g)$ in the product where joint continuity occurs are described by the following result.

Proposition 2.1 *Let $\langle X, \tau \rangle$ be a Hausdorff space, where $\mathbb{R}^X$ is equipped with $\mathscr{T}_u$. Then $(\alpha, f) \mapsto \alpha f$ defined on $[0, \infty) \times \mathbb{R}^X$ is jointly continuous at $(0, g)$ if and only if g is locally bounded from below.*

Proof Suppose g is not bounded from below on any neighborhood V of x_0. Let h denote the zero function on X. Then the sequence $\langle (\frac{1}{n}, g) \rangle$ converges to $(0, g)$ while $\langle \frac{1}{n} g \rangle$ fails to converge to h, as $[\{x_0\}, 1]^u(h)$ contains no function of the form $\frac{1}{n} g$.

For sufficiency, assume g is locally bounded from below; it suffices to produce for each $x_0 \in X$ and $\varepsilon > 0$, a neighborhood of $(0, g)$ mapped into $[\{x_0\}, \varepsilon]^u(h)$. Choose an open neighborhood V of x_0 and $n \in \mathbb{N}$ such that $\forall v \in V,\ g(v) > -n\frac{\varepsilon}{2}$. We claim that $[0, \frac{1}{n}) \times [\{x_0\}, \frac{\varepsilon}{2}]^u(g)$ is mapped by $(\alpha, f) \mapsto \alpha f$ into $[\{x_0\}, \varepsilon]^u(h)$. To see this, fix (α, f) with $\alpha \in [0, \frac{1}{n})$ and such that for some neighborhood W of x_0 contained in V, $\forall w \in W,\ g(w) - \frac{\varepsilon}{2} < f(w)$. If $\alpha = 0$ there is nothing to prove. Otherwise, for each $w \in W$, we have

$$\alpha f(w) > \alpha g(w) - \alpha \frac{\varepsilon}{2} > \frac{1}{n}\left(-n\frac{\varepsilon}{2}\right) - \frac{1}{n}\frac{\varepsilon}{2} \geq -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon,$$

which means that $\alpha f \in [\{x_0\}, \varepsilon]^u(h)$.

We see from Proposition 2.1 and the sixth and seventh bullets above that $\mathcal{C}(X)$ or even $\mathcal{L}(X)$ equipped with $\mathscr{T}_u$ becomes a topological cone-lattice.

A final remark about the standard quasi-uniformity for $\mathscr{T}_u$: the conjugate quasi-uniformity has as a base all sets of the form

$$[F, \varepsilon]^l := \{(g, h) : \forall x \in F,\ |g(x) - h(x)| < \varepsilon \text{ and } \exists \text{ a neighborhood } V \text{ of } F \text{ such that } \forall x \in V,\ h(x) < g(x) + \varepsilon\}\ (F \in \mathfrak{F}_0(X), \varepsilon > 0).$$

We denote the induced topology by $\mathscr{T}_l$.

3 Results

For a (Hausdorff) space $\langle X, \tau \rangle$, since $\mathcal{C}(X)$ contains the constant functions, the following properties are obviously equivalent:

(1) whenever $x_1 \neq x_2$, $\exists f \in \mathcal{C}(X)$ such that $f(x_1) \neq f(x_2)$;
(2) whenever $x_1 \neq x_2$ and $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$, $\exists f \in \mathcal{C}(X)$ such that $f(x_1) = \alpha_1$ and $f(x_2) = \alpha_2$.

We call $\langle X, \tau\rangle$ *functionally Hausdorff* [24] if it satisfies these conditions.

Proposition 3.1 *Let $\langle X, \tau\rangle$ be a Hausdorff space. The following conditions are equivalent.*

(1) $\langle X, \tau\rangle$ is functionally Hausdorff;
(2) $\mathcal{C}(X)$ is $\mathscr{T}_p$-dense in $\mathbb{R}^X$;
(3) The $\mathscr{T}_p$-closure of $\mathcal{C}(X)$ includes $\mathcal{U}(X)$.

Proof (1) $\Rightarrow$ (2). Let $g \in \mathbb{R}^X$ be arbitrary. It suffices to show that whenever $F \in \mathfrak{F}_0(X)$, there exists $h_F \in \mathcal{C}(X)$ such that $h_F|F = g|F$, for directing $\mathfrak{F}_0(X)$ by inclusion, the net $F \mapsto h_F$ is pointwise convergent to g. We prove this by induction on the number of elements of F. Our above discussion provides adequate justification when $n = 2$. Suppose we know this to be true whenever F has $n = k$ elements and $\{x_1, x_2, \ldots, x_k, x_{k+1}\}$ are $k+1$ distinct elements of X. Put $\alpha_i = g(x_i)$ for $i = 1, 2, \ldots, k + 1$. Since $\mathcal{C}(X)$ contains the constant functions and is closed under addition, there is no loss of generality in assuming that $\alpha_{k+1} = 0$. Choose by the induction hypothesis $f \in \mathcal{C}(X)$ with $f(x_i) = \alpha_i$ for $i = 1, 2, \ldots, k$, and then choose $f_i \in \mathcal{C}(X)$ mapping x_{k+1} to zero and x_i to one. For each $i \leq k$, put $g_i = f_i \wedge 1$; then $h_{\{x_1,x_2,\ldots,x_k,x_{k+1}\}}$ defined by

$$h_{\{x_1,x_2,\ldots,x_k,x_{k+1}\}}(x) := f(x)(g_1(x) \vee g_2(x) \vee \ldots g_k(x))$$

does the job.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Suppose (1) fails; then there exists distinct x_1 and x_2 such that $\forall f \in \mathcal{C}(X)$, we have $f(x_1) = f(x_2)$. Then $[\{x_1, x_2\}, \frac{1}{3}]^p(\chi_{\{x_1\}})$ contains no continuous function, and so (3) fails.

Proposition 3.2 *Let $\langle X, \tau\rangle$ be a Hausdorff space. If $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ is a net in $\mathcal{C}(X)$ that is $\mathscr{T}_p$-convergent to $g \in \mathcal{U}(X)$, then the net is $\mathscr{T}_u$-convergent to g.*

Proof Fix $F \in \mathfrak{F}_0(X)$ and $\varepsilon > 0$. We will show that $f_\lambda \in [F, \varepsilon]^u(g)$ eventually. By $\mathscr{T}_p$-convergence, choose $\lambda_0 \in \Lambda$ such that $\lambda \succeq \lambda_0 \Rightarrow f_\lambda \in [F, \frac{\varepsilon}{3}]^p(g)$. Fix $\lambda_1 \succeq \lambda_0$. Clearly if $x \in F$, then $|g(x) - f_{\lambda_1}(x)| < \frac{\varepsilon}{3}$. By continuity, for each $x \in F$, $\exists U_x \in \tau$ such that $x \in U_x$ and whenever $w \in U_x$, we have $|f_{\lambda_1}(x) - f_{\lambda_1}(w)| < \frac{\varepsilon}{3}$. By upper semicontinuity, there exists for each $x \in F$ a second open neighborhood W_x such that $w \in W_x \Rightarrow g(w) < g(x) + \frac{\varepsilon}{3}$. Put $V := \cup_{x\in F}(U_x \cap W_x)$, a neighborhood of F. Let $v \in V$ be arbitrary; choosing x with $v \in U_x \cap W_x$, we compute

$$g(v) < g(x) + \frac{\varepsilon}{3} < f_{\lambda_1}(x) + \frac{2}{3}\varepsilon < f_{\lambda_1}(v) + \varepsilon,$$

and so $f_{\lambda_1} \in [F, \varepsilon]^u(g)$ as required.

The following example shows that $\mathscr{T}_p$-convergence in $\mathcal{U}(X)$ does not ensure $\mathscr{T}_u$-convergence, even if the limit is continuous.

Example 3.3 Let $g : \mathbb{R} \to \mathbb{R}$ be the constant function $g(x) \equiv 1$, and $\forall n \in \mathbb{N}$, let $f_n = \chi_{E_n}$ where $E_n := \{0\} \cup \{x \in \mathbb{R} : |x| \geq \frac{1}{n}\}$. Since each E_n is closed, each f_n is upper semicontinuous. While the sequence $\langle f_n \rangle$ is pointwise convergent to g, for each n, $f_n \notin [\{0\}, \frac{1}{2}]^u(g)$.

A point is in the closure of a set if and only if there exists a net in the set convergent to the point. Putting together Propositions 3.1 and 3.2, we obtain the following proposition which says that in a functionally Hausdorff space, the $\mathscr{T}_u$-closure of $\mathcal{C}(X)$ includes $\mathcal{U}(X)$.

Proposition 3.4 *Let $\langle X, \tau \rangle$ be a functionally Hausdorff space. Then for each $g \in \mathcal{U}(X)$, there exists a net $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ in $\mathcal{C}(X)$ that is $\mathscr{T}_u$-convergent to g.*

We next show that $\mathcal{U}(X)$ is $\mathscr{T}_u$-closed.

Proposition 3.5 *Let $\langle X, \tau \rangle$ is Hausdorff space. Suppose $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is a net in $\mathcal{U}(X)$ $\mathscr{T}_u$-convergent to $g \in \mathbb{R}^X$. Then $g \in \mathcal{U}(X)$.*

Proof To prove upper semicontinuity, fix $x_0 \in X$ and $\varepsilon > 0$. There exists $\lambda_0 \in \Lambda$ such that $\lambda \succeq \lambda_0 \Rightarrow f_\lambda \in [\{x_0\}, \frac{\varepsilon}{3}]^u(g)$. By definition, there exists V such that $x_0 \in V \in \tau$ and $\forall v \in V$, $g(v) < f_{\lambda_0}(v) + \frac{\varepsilon}{3}$ and $|g(x_0) - f_{\lambda_0}(x_0)| < \frac{\varepsilon}{3}$. By upper semicontinuity, we may also assume that for each $v \in V$, $f_{\lambda_0}(v) < f_{\lambda_0}(x_0) + \frac{1}{3}\varepsilon$. Fixing $v \in V$, we compute

$$g(v) < f_{\lambda_0}(v) + \frac{1}{3}\varepsilon < f_{\lambda_0}(x_0) + \frac{2}{3}\varepsilon < g(x_0) + \varepsilon,$$

establishing upper semicontinuity of g at x_0.

We can associate with each closed subset C of X its upper semicontinuous characteristic function χ_C. If a net of such characteristic functions is $\mathscr{T}_u$-convergent to some function g, then by pointwise convergence, the range of g is contained in $\{0, 1\}$, i.e., g is a characteristic function of a (possibly empty) subset of X, which, by the last result, must be a closed subset. Denoting the family of closed subsets by $\mathscr{C}$, we see that $\{\chi_C : C \in \mathscr{C}\}$ is $\mathscr{T}_u$-closed. Identifying each closed subset with its characteristic function, we get a so-called *hyperspace topology* [5,19] on $\mathscr{C}$. It is left to the reader to show that a net of closed subsets $\langle C_\lambda \rangle_{\lambda \in \Lambda}$ is convergent to a closed set C in the hyperspace if and only if both of the following conditions are satisfied:

- $\{x \in X : x \in C_\lambda \text{ residually}\} = C = \{x \in X : x \in C_\lambda \text{ cofinally}\}$;
- $\forall c \in C$, $\exists \lambda_0 \in \Lambda$ such that $\forall \lambda \succeq \Lambda_0 \,\exists\, V(c, \lambda) \in \tau$ with $c \in V(c, \lambda) \cap C \subseteq C_\lambda$.

Returning to the main line of discussion, our last three results yield the result we are after.

Theorem 3.6 *Let $\langle X, \tau \rangle$ be a functionally Hausdorff space. Then $\mathcal{U}(X)$ is the $\mathscr{T}_u$-closure of $\mathcal{C}(X)$ in $\mathbb{R}^X$. Further, any net in $\mathcal{C}(X)$ that is pointwise convergent to an upper semicontinuous limit is already $\mathscr{T}_u$-convergent.*

We now introduce a property $\diamond$ of a net $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ of real-valued functions with respect to a prospective limit function g that plays a significant role in our investigation.

$\diamond$ Whenever C is a nonempty compact subset of X, $\lambda_0 \in \Lambda$ and $\varepsilon > 0$, there exist a neighborhood V of C and indices $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that for each $i \leq n,\ \lambda_i \succeq \lambda_0$ and $\forall x \in V,\ \exists i \leq n$ with $g(x) < f_{\lambda_i}(x) + \varepsilon$.

Our next result constitutes a separate argument for the fact that $\mathcal{U}(X)$ is $\mathscr{T}_u$-closed in $\mathbb{R}^X$.

Proposition 3.7 *Let $\langle X, \tau \rangle$ be a Hausdorff space, and suppose $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is a net in $\mathbb{R}^X$ pointwise convergent to $g \in \mathbb{R}^X$.*

(1) If the net is actually $\mathscr{T}_u$-convergent, then property $\diamond$ holds;
(2) If the net lies in $\mathcal{U}(X)$ and property $\diamond$ holds, then $g \in \mathcal{U}(X)$ also holds.

Proof For (1), by $\mathscr{T}_u$-convergence to g, for each $c \in C$ we can choose $\lambda_c \succeq \lambda_0$ such that $(g, f_{\lambda_c}) \in [\{c\}, \varepsilon]^u$. Choose for each $c \in C$ an open neighborhood V_c of c such that $x \in V_c \Rightarrow g(x) < f_{\lambda_c}(x) + \varepsilon$. Extracting a finite subcover $\{V_{c_1}, V_{c_2}, \ldots, V_{c_n}\}$ of C, the prescriptions

$$V := \cup_{i=1}^n V_{c_i} \text{ and } \lambda_i := \lambda_{c_i} \text{ for } i \leq n$$

satisfy condition $\diamond$.

For (2), to prove g is upper semicontinuous at $x_0 \in X$, we apply condition $\diamond$ where $C = \{x_0\}$. Let $\varepsilon > 0$ be arbitrary and choose $\lambda_0 \in \Lambda$ such that $\lambda \succeq \lambda_0 \Rightarrow |f_\lambda(x_0) - g(x_0)| < \frac{\varepsilon}{3}$. Next choose by $\diamond$ an open neighorhood V of $\{x_0\}$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that for each $i \leq n,\ \lambda_i \succeq \lambda_0$ and $\forall x \in V,\ \exists i \leq n$ with $g(x) < f_{\lambda_i}(x) + \frac{\varepsilon}{3}$. Choose by upper semicontinuity an open set V_1 such that $x_0 \in V_1 \subseteq V$ and $\forall x \in V_1,\ \forall i \leq n,\ f_{\lambda_i}(x) < f_{\lambda_i}(x_0) + \frac{\varepsilon}{3}$. Fixing $x \in V_1$ and choosing $i \leq n$ with $g(x) < f_{\lambda_i}(x) + \frac{\varepsilon}{3}$, we compute

$$g(x) < f_{\lambda_i}(x) + \frac{\varepsilon}{3} < f_{\lambda_i}(x_0) + \frac{2}{3}\varepsilon < g(x_0) + \varepsilon.$$

This establishes upper semicontinuity at x_0.

The converse of (1) fails, that is, pointwise convergence plus $\diamond$ do not ensure $\mathscr{T}_u$-convergence, even if the net is in $\mathcal{U}(X)$ and the limit is continuous.

Example 3.8 Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, equipped with topology it inherits from the usual topology of $\mathbb{R}$. Write $\mathbb{N}$ as a countable disjoint union of infinite subsets:

$$\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3 \cup \ldots.$$

For each $k \in \mathbb{N}$, put $E_k := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N} \backslash \mathbb{N}_k\}$, a closed subset of X, so that $f_k := \chi_{E_k}$ is upper semicontinuous. The sequence of characteristic functions $\langle f_k \rangle$ is pointwise convergent to the constant function $g(x) \equiv 1$, as for each $x \in X,\ f_k(x) = 1$ eventually. The sequence satisfies $\diamond$ with respect to g, for if C is any compact subset of X and $k_0 \in \mathbb{N}$, then taking $V = X$, we see that for each $x \in V$, either $f_{k_0}(x) = 1$ or $f_{2k_0}(x) = 1$ or both. But $\forall k \in \mathbb{N}$, each neighborhood of the origin contains points not in E_k, which means that $[\{0\}, \frac{1}{2}]^u(g)$ contains no f_k.

Not only is $\mathscr{T}_u$-convergence usually properly stronger on $\mathcal{U}(X)$ than pointwise convergence plus $\diamondsuit$, but further, the conjunction is not a convergence on $\mathcal{U}(X)$ as the term is normally understood [17, pg. 73], that is, a set-valued mapping Γ assigning to each net ϕ in $\mathcal{U}(X)$ a (possibly empty) subset $\Gamma(\phi)$ of $\mathcal{U}(X)$ corresponding to the set of limits of ϕ that satisfies some simple properties. One of these properties is invariably the following: whenever ψ is a subnet of ϕ, then $\Gamma(\phi) \subseteq \Gamma(\psi)$. This fails for pointwise convergence plus $\diamondsuit$ (consider the sequence $f_1, g, f_2, g, f_3, g, \ldots$ for the functions of Example 3.3). However, the conjunction is a convergence—in fact, a topological convergence—restricted to $\mathcal{C}(X)$, as we shall now see.

Theorem 3.9 *Let $\langle X, \tau \rangle$ be a Hausdorff space, let $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ be a net in $\mathcal{C}(X)$ and let $g \in \mathbb{R}^X$. The following are equivalent:*

(1) $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is $\mathscr{T}_u$-convergent to g;
(2) $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is pointwise convergent to g and $\diamondsuit$ holds;
(3) $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is pointwise convergent to g and g is upper semicontinuous.

Proof (1) $\Rightarrow$ (2) follows from statement (1) of Proposition 3.7, (2) $\Rightarrow$ (3) follows from statement (2) of Proposition 3.7, and (3) $\Rightarrow$ (1) is established by Proposition 3.2.

Each of the above results has its counterpart for lower semicontinuous functions which the reader can easily formulate. If we take the uniformity generated by our standard uniformity for $\mathscr{T}_u$ and its conjugate, we get a uniformity having as a base all entourages of the form

$$[F, \varepsilon]^{\square} := \{(g, h) : \exists \text{ a neighborhood } V \text{ of } F \text{ such that } \forall x \in V, \\ |g(x) - h(x)| < \varepsilon\} \ (F \in \mathfrak{F}_0(X), \varepsilon > 0).$$

The topology $\mathscr{T}_{\square}$ induced by this uniformity, called the *sticking topology* by Bouleau [11,12] and the *topology of strong pointwise convergence* by Beer and Levi [7] in the context of metric spaces, is intrinsic to the preservation of continuity: $\mathcal{C}(X)$ is $\mathscr{T}_{\square}$-closed in $\mathbb{R}^X$ and $\mathscr{T}_{\square}$-convergence reduces to pointwise convergence on $C(X)$ itself. While the following summary result appropriately modified is valid when $\mathbb{R}$ is replaced by a Hausdorff uniform space [6, Theorem 4.11], we content ourselves with a more restrictive statement. The details can be easily worked out by the reader from the prior results of this paper.

Theorem 3.10 *Let $\langle X, \tau \rangle$ be a Hausdorff space, let $g \in \mathbb{R}^X$, and let $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ be a net in $\mathcal{C}(X)$ pointwise convergent to g. The following conditions are equivalent:*

(1) $g \in \mathcal{C}(X)$;
(2) $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is $\mathscr{T}_{\square}$-convergent to g;
(3) Whenever C is a nonempty compact subset of X, $\lambda_0 \in \Lambda$ and $\varepsilon > 0$, there exists a neighborhood V of C and indices $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that $\forall i \leq n$, $\lambda_i \succeq \lambda_0$, and $\forall x \in V$, $\exists i \leq n$ with $|g(x) - f_{\lambda_i}(x)| < \varepsilon$.

Following Arzelà [2] who worked with functions of a real variable, Bartle [4] showed that if X is a Hausdorff space then a necessary and sufficient condition for the continuity of a pointwise limit g of a net of continuous real functions is the following: $\forall \lambda_0 \in \Lambda$ and $\varepsilon > 0$, there exist indices $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that for each $i \leq n$, $\lambda_i \succeq \lambda_0$ and $\forall x \in X$, $\exists i \leq n$ with $|g(x) - f_{\lambda_i}(x)| < \varepsilon$. He called this condition *quasi-uniform convergence*. Evidently, condition (3) above is a strengthening of *quasi-uniform convergence on compacta*, which may be used in lieu of (3) provided X is a compactly generated space (see [24, p. 285] and a little more precisely [20, p. 74]). But without further assumptions, this replacement cannot be made, in the same way that one cannot assert in complete generality that the limit of a net of continuous functions that is uniformly convergent on compacta is continuous. For an alternative to condition (3) also valid in arbitrary Hausdorff spaces, the reader may consult [1, pg. 266]. These ideas and more are considered in the recent article by Caserta et al. [14] on the preservation of continuity of functions between metric spaces.

References

1. Alexandroff, P.: Einführing in die Mengenlehre und die theorie der rellen Funktionen. Deutscher Verlag der Wissenschaften, Berlin (1964)
2. Arzelà, C. : Intorno alla continuá della somma di infinitá di funzioni continue. Rend. R. Accad. Sci. Istit. Bologna, 79–84 (1883/1884)
3. Attouch, H.: Variational Convergence for Functions and Operators. Pitman, Boston (1984)
4. Bartle, R.: On compactness in functional analysis. Trans. Am. Math. Soc. **79**, 35–57 (1955)
5. Beer, G.: Topologies on Closed and Closed Convex Sets. Kluwer, Dordrecht (1993)
6. Beer, G.: The Alexandroff property and the preservation of strong uniform continuity. Appl. Gen. Top. **11**, 117–133 (2010)
7. Beer, G., Levi, S.: Strong uniform continuity. J. Math. Anal. Appl. **350**, 568–589 (2009)
8. Borwein, J., Lewis, A.: Convex Analysis and Nonlinear Optimization, CMS Books in Mathematics. Springer, New York (2006)
9. Borwein, J., Thera, M.: Sandwich theorems for semicontinuous operators. Can. Math. Bull. **35**, 463–474 (1992)
10. Borwein, J., Vanderwerff, J.: Epigraphical and uniform convergence of convex functions. Trans. Am. Math. Soc. **348**, 1617–1631 (1996)
11. Bouleau, N.: Une structure uniforme sur un espace $F(E, F)$. Cahiers Topologie Géom. Diff. **11**, 207–214 (1969)
12. Bouleau, N., On the coarsest topology preserving continuity (preprint) (2006)
13. Bourbaki, N.: Topologie générale, Chapter IX. Hermann, Paris (1948)
14. Caserta, A., Di Maio, G., Holá, L.: Arzelà's theorem and strong uniform convergence on bornologies. J. Math. Anal. Appl. **371**, 384–392 (2010)
15. Engelking, R.: General Topology. Polish Scientific Publishers, Warsaw (1977)
16. Fletcher, P., Lindgren, W.: Quasi-uniform spaces, Lecture Notes Pure Appl. Math. No. 77, Marcel Dekker, New York (1982)
17. Kelley, J.: General Topology. Van-Nostrand, Princeton (1955)
18. Künzi, H.: Quasi-uniform spaces—eleven years later. Topol. Proc. **18**, 143–171 (1993)
19. Lucchetti, R.: Convexity and Well-Posed Problems CMS Books in Mathematics. Springer, New York (2006)
20. McCoy, R., Ntantu, I.: Topological Properties of Spaces of Continuous Functions. Springer, Berlin (1988)
21. Pervin, W.: Quasi-uniformization of topological spaces. Math. Ann. **147**, 316–317 (1962)
22. Rockafellar, R., Wets, R.: Variational analysis. Springer, Berlin (1998)
23. Tong, H.: Some characterizations of normal and perfectly normal space. Duke Math. J. **19**, 289–292 (1952)
24. Willard, S.: General Topology. Addison-Wesley, Reading (1970)