Math. Program., Ser. A (2012) 133:437–460
DOI 10.1007/s10107-010-0433-z

FULL LENGTH PAPER

A continuous-time linear complementarity system for dynamic user equilibria in single bottleneck traffic flows

Jong-Shi Pang · Lanshan Han · Gitakrishnan Ramadurai · Satish Ukkusuri

Received: 23 June 2009 / Accepted: 13 December 2010 / Published online: 6 January 2011

Abstract This paper formally introduces a linear complementarity system (LCS) formulation for a continuous-time, multi-user class, dynamic user equilibrium (DUE) model for the determination of trip timing decisions in a simplified single bottleneck model. Existence of a Lipschitz solution trajectory to the model is established by a constructive time-stepping method whose convergence is rigorously analyzed. The solvability of the time-discretized subproblems by Lemke's algorithm is also proved. Combining linear complementarity with ordinary differential equations and being a new entry to the mathematical programming field, the LCS provides a computational tractable framework for the rigorous treatment of the DUE problem in continuous time; this paper makes a positive contribution in this promising research venue pertaining to the application of differential variational theory to dynamic traffic problems.

The work of this Jong-Shi Pang is based on research supported by the National Science Foundation under grants DMS-0754374, CMMI-0969600, and EFRI-1024984.

J.-S. Pang (✉)
Department of Industrial and Enterprise Systems Engineering,
University of Illinois at Urbana-Champaign,
Urbana, IL 61801, USA
e-mail: jspang@illinois.edu

L. Han · S. Ukkusuri
School of Civil Engineering,
Purdue University, West Lafayette, IN 47906, USA
e-mail: lanshanhan@gmail.com

S. Ukkusuri
e-mail: sukkusur@purdue.edu

G. Ramadurai
Department of Civil Engineering, Indian Institute of Technology,
Madras, India
e-mail: gitakrishnan@iitm.ac.in

Mathematics Subject Classification (2000) 90C33 · 90C90

1 Introduction

Dynamic user equilibrium (DUE), an extension of the traditional Wardop's equilibrium, is the most common behavioral assumption used in determining time varying traffic flows in a transportation network. A convenient and a benchmark problem in trip timing decisions is Vickrey's bottleneck model [26]. In this model, a fixed number of homogeneous commuters travel on a single link that has a bottleneck with limited capacity. Commuters experience queue delay at the bottleneck and early or late arrival penalty, referred to as schedule delay, at the destination. The problem is to determine the departure rates such that all individuals have the least possible total cost and no individual can unilaterally shift and reduce cost. Several studies have examined the single bottleneck model [1,2,8,14,24]. The paper [14] derives expressions for equilibrium departure rates when the schedule delay cost and travel cost are linear functions of time; the paper [24] proves the existence of equilibrium solutions in a more general class of models where individuals have different arrival time preferences; the paper [8] provides the conditions under which this equilibrium is unique. All the above studies were concerned with the single-user class problem where all users have the same travel and schedule delay parameters. The single bottleneck model with heterogeneous (multi-user class) commuters has been studied by [17,25,27]. The paper [16] provides an extensive analysis of the single bottleneck model with heterogeneous commuters, proving, under general assumptions, the existence and uniqueness of a solution. However the proof is non-constructive and no solution algorithm is provided.

A key drawback of the above studies is the lack of a tractable analytical formulation. In addition to providing a rigorous mathematical framework for the unambiguous investigation of the solution properties of the problem, appropriate analytical formulations allow the development of computational algorithms for the model solution. Such formulations also allow the addition and relaxation of assumptions in a more systematic and controlled manner. For example, if an analytical model with linear cost functions is obtained, it is straightforward to extend it to capture nonlinear cost functions. Analytical formulations could also allow us to test sensitivity of assumptions. For example, [16] assumes that "a positive measure of users does not leave at one instant". In other words, the rate of departures is finite. Ideally, one would like finite departure rate to be an outcome after solving the formulation, rather than a prior assumption. Analytical formulations can be used to test the sensitivity of the model with and without such a fixed departure rate assumption.

While the single bottleneck model is a simplified representation of traffic networks, it serves as a benchmark problem for more complex network-wide problems. In particular, the single bottleneck model can provide very useful insight and necessary technique for the study of more complex DUE problems involving more realistic features. Moreover, it can serve as a test bed on which new mathematical paradigms can be demonstrated before being applied to more complex problems. As such, a clear understanding of this problem using novel, rigorous mathematical programming techniques will eventually allow for a stronger foundation of dynamic equilibrium models, which is very much needed in the area of dynamic traffic flow analysis.

Only recently, the paper [20] develops a linear complementarity formulation for a single bottleneck model with heterogeneous commuters (multi-user class) in discrete time, establishes the existence and uniqueness of a solution, and demonstrates the computability of the solution using the well established Lemke's algorithm [7]. In the current work, we develop a continuous-time based model. A continuous-time formulation is desirable for the following reasons: first, time is continuous; thus a solution to the model captures the dynamics of traffic flow throughout a given time interval of interest. Discrete systems approximate time by dividing into suitable intervals—the finer the discretization the better the approximation. However, ideally, whenever possible, time should be treated as a continuous variable. Second, discretizing time leads to approximations in travel time computations. In the formulation developed in [20], the travel time for all individuals departing in a time interval is approximated as the average travel time over all individuals departing in that interval. When several individuals depart in a short time interval, the difference between the first and the last individual could be significantly large and the average over all individuals could result in a poor approximation. In the continuous-time formulation, this problem is avoided if the departure rate function is bounded. Third, discrete-time systems could lead to inconsistent solutions when coupling the traffic flow behavior model with the route flow propagation. This could lead to violation of FIFO in particular cases (see [3]); continuous-time systems are inherently devoid of such a problem. Fourth, a discrete-time model immediately raises the question of the meaning of a time unit and the impact on the model solution if such a unit is refined; in a continuous-time model, one is naturally interested in the analysis of the limiting properties of a discrete-time model as the time step approaches zero.

In summary, from a modeling point-of-view, continuous-time models have multiple advantages over discrete-time models; from an analytical point-of-view, continuous-time models provide a rigorous framework for the understanding of many important issues of traffic flows that are otherwise neglected in a discrete-time framework. This paper is the beginning of a long-term research effort to systematically investigate continuous-time dynamic user equilibrium problems via the theory and methods of differential variational systems. Since this is a wide area of study that has so far lacks a mathematically rigorous framework, we focus on a simplified model to provide a proof of concept for this fruitful direction of research that promises a transformational view of continuous-time traffic flow analysis.

Specifically, the main goal of this paper to introduce a formal mathematical formulation of the continuous-time single bottleneck DUE model with heterogeneous commuters as a linear complementarity system (LCS) and to show how a rigorous analysis can be carried out based on the formulation. The LCS is a novel class of nonsmooth differential systems that have recently attracted a lot of research interests in mathematical programming and control theory. In general, an LCS comprises a linear ordinary differential equation (ODE) with an algebraic variable that is required to be a solution to a finite-dimensional linear complementarity problem (LCP) which in turn is parameterized by the state variable in the linear ODE. Among the growing literature on this topic, the paper [19] is of particular relevance to our study; see also [10]. For other references on the LCS and extended systems that discuss fundamental issues associated with these novel dynamical systems, such as non-Zenoness, well-posedness, and many system-theoretic properties, as well as broad applications of these systems, the reader is referred to [4–6,9,11–13,18,21–23]. The LCSs and the more general differential variational systems provide a very promising platform on which rigorous analysis of dynamic traffic flow can be performed. Specifically, the ODE component can be exploited to describe the dynamics and evolution of the traffic conditions such as the congestion level and travel time; while the complementarity conditions or the more general finite-dimensional variational inequalities have been demonstrated to be very powerful in modeling equilibrium conditions as well as certain hybrid features of the dynamics. Therefore, the combination of these two components forms an much needed modeling tool for the DUE problems. The research in this paper exhibits an example of how this tool can be applied to a simplified DUE problem. The present work is the start of a sustained effort aimed at employing the differential variational inequality formalism to study realistic dynamic traffic equilibria. In particular, an extension of the present work is already in progress.

An important lesson derived from the cited references is that as an LCS, the continuous-time single bottleneck DUE model involves many intricate details that need to be carefully investigated in order to obtain a mathematically rigorous understanding and demonstration of the model properties. Issues such as the existence and regularity of the solution trajectories, the Zeno phenomenon of these trajectories (i.e., the phenomenon of infinite number of event changes in finite time intervals), and the convergence of the discrete-time approximations to a continuous-time trajectory are all non-trivial and require a systematic treatment, which is made possible by the framework introduced in this paper. Whereas all these are important issues, due to their technical challenges and space limitation, we cannot deal with all of them in a single study like this one. Therefore, in the rest of the paper, we focus only on 2 topics: model formulation and a time-stepping method. Further study of the model and extensions to more realistic traffic problems are currently in progress. Specifically, in Sect. 2 we introduce our LCS model, formally define a solution concept, and contrast our model with past models. In Sect. 3, we prove the existence of a solution constructively by applying a numerical time-stepping scheme; this proof involves the demonstration that the time-discretized subproblems all have solutions computable by Lemke's algorithm. For a mathematical programming minded reader, our work provides a manifestation of the growing importance of the LCS, which is an outgrowth of the classical topic of the LCP to deal with systems under evolution.

2 The mathematical model

To present our formulation, we first define the parameters and the variables. In the formulation, except for the index set $\mathbb{G}$, all model parameters are positive scalars. Time is treated as continuous and vehicle flow is approximated as fluid-flow. Therefore, travelers are infinitesimal players participating in a non-atomic game. This assumption is consistent with past studies [8,16,24]. However, past studies have tended to opportunistically adopt the atomic player definition especially to illustrate concepts and draw insights. Such misleading and conflicting definitions have been consciously avoided in this discussion.

For representational convenience, there are two classes of variables: primary and derived. Primary variables include travel time, cumulative departure, and equilibrium costs for the different user groups. Derived variables are those that are derived from the primary variables.

Model parameters:

$\mathbb{G}$	user classes with elements $g \in \mathbb{G}$
T	total time duration
d_g	total demand of class g
s	bottleneck capacity (number of vehicles per unit time)
α_g	class g's value of travel time ($ per unit time)
β_g	class g's value of schedule delay when arriving early ($ per unit time)
γ_g	class g's value of schedule delay when arriving late ($ per unit time)
t_g^*	class g's preferred arrival time at destination

It is assumed that for each class g, $\beta_g < \alpha_g < \gamma_g$: commuters would rather arrive at work early than endure traffic congestion delay; arriving late to work is least preferred. Furthermore, we assume without loss of generality that $T > \max_{g \in \mathbb{G}} t_g^*$; the duration of the problem should be sufficiently long to ensure the preferred arrival time of all user classes are included.

Primary variables: all nonnegative

$TT(t)$	travel time of users departing at time instance t (note: without loss of generality, we ignore the free-flow travel time, therefore, $TT(t) + t$ is the arrival time of users departing at time t)
$N_g(t)$	class g's cumulative departures in the time interval $[0, t]$
c_g^*	equilibrium cost of class g.

Derived variables:

$u(t)$	a slack variable $\triangleq \frac{d\,TT(t)}{dt} - \frac{1}{s}\left[\sum_{g \in \mathbb{G}} r_g(t) - s\right]$
$r_g(t)$	class g's rate of departures at time t (departures per unit time) $\triangleq \frac{dN_g(t)}{dt}$
$e_g(t)$	duration between early arrival and preferred arrival time t_g^* of users in class $g \triangleq \max\left\{0, t_g^* - (TT(t) + t)\right\}$
$\ell_g(t)$	duration between late arrival and preferred arrival time t_g^* of users in class $g \triangleq \max\left\{0, (TT(t) + t) - t_g^*\right\} = -(t_g^* - t) + TT(t) + e_g(t)$
$C_g(t)$	travel cost of users in class g departing at time t $\triangleq \alpha_g\, TT(t) + \beta_g\, e_g(t) + \gamma_g\, \ell_g(t) = -\gamma_g(t_g^* - t)$ $+(\alpha_g + \gamma_g)TT(t) + (\beta_g + \gamma_g)\max\left\{0, t_g^* - (TT(t) + t)\right\}$.

Note that $C_g(t) \geq 0$ for all $g \in \mathbb{G}$ and all $t \in [0, T]$; moreover, $C_g(t) = 0$ if and only if $t = t_g^*$ and $TT(t) = 0$. Thus, $TT(t) > 0$ implies $C_g(t) > 0$ for all $g \in \mathbb{G}$.

2.1 LCS formulation

Assuming that the traffic queue starts to build up at or after the initial time $t = 0$, the multi-user class, single bottleneck DUE problem is to find nonnegative scalars $\{c_g^*\}_{g\in\mathbb{G}}$ and absolutely continuous functions $TT(t)$ and $\{N_g(t)\}_{g\in\mathbb{G}}$ such that the conditions (A)–(C) are satisfied:

(A) for almost all $t \in (0, T]$,

$$
\begin{aligned}
\frac{d\,TT(t)}{dt} &= u(t) + \frac{1}{s}\left[\sum_{g\in\mathbb{G}} r_g(t) - s\right] \\
\frac{dN_g(t)}{dt} &= r_g(t), \quad \forall\, g \in \mathbb{G} \\
0 &\leq u(t) \perp TT(t) \geq 0 \\
0 &\leq r_g(t) \perp C_g(t) - c_g^* \geq 0, \quad \forall\, g \in \mathbb{G},
\end{aligned}
$$

where for two scalars a and b, $a \perp b$ means $ab = 0$; for vectors, the $\perp$ notation denotes perpendicularity;

(B) the initial conditions:

$$
\begin{aligned}
TT(0) &= \max\left(0,\ \frac{1}{s}\sum_{g\in\mathbb{G}} N_g(0) - 1\right) \\
0 &\leq N_g(0) \perp C_g(0) - c_g^* \geq 0, \quad \forall g \in \mathbb{G},
\end{aligned}
$$

(C) the boundary conditions: $N_g(T) = d_g$ for all $g \in \mathbb{G}$.

In the above formulation, the first differential equation together with the first complementarity condition in (A) defines the dynamics of the travel time; these conditions, at their times of validity, stipulate the instantaneous rate of change of the travel time at an instance where the travel time is positive. The second complementarity condition in (A) defines the equilibrium condition; namely the total cost of a user class departing at a time instance is minimum and equal to the equilibrium cost c_g^* when there is a positive departure rate at that instance. Mathematically, two features of the above LCS are worth noting: (i) it contains the algebraic time-invariant unknowns c_g^*, and (ii) the initial conditions (B) are complementarity conditions to be satisfied by (instead of explicitly defining) the initial values $TT(0)$ and $N_g(0)$. Thus the single-bottleneck traffic model may be considered as an LCS with variational initial conditions and also with the boundary conditions expressed in (C); this extends the standard initial/boundary-value versions of the LCS studied in the literature.

2.2 Solution concept and discussion

The complementarity conditions in the LCS (A)–(C) induce event changes in the dynamical system, i.e., mode switches[1] in the terminology of switched systems, which in turn imply that discontinuities could occur in a "solution" of the system. Thus one needs to be careful in addressing the "regularity" of such a solution in order to establish its existence and to analyze its properties. As defined, for the differential variables $TT(t)$ and $N_g(t)$, we seek *absolutely continuous* functions of time; in particular, these functions must be continuous on $[0, T]$ and differentiable *almost everywhere* (but not necessarily everywhere) therein. For the algebraic variables $r_g(t)$, $u(t)$, and $C_g(t)$, we require them to be (Lebesque) integrable on the interval $[0, T]$. The latter somewhat loose requirement allows the possibility for these algebraic functions to be discontinuous; in turn, such discontinuity is the source for the lack of everywhere differentiability of the differential variables and for the almost everywhere (instead of everywhere) validity of the conditions in (A).

The LCS (A)–(C) is a switched dynamical system; this is in contrast to the model in Lindsey [16] whose definition is to a large extent based on the geometry of the scheduled delay functions (denoted $D_g(\bullet)$ in the reference, which is equal to $\beta_g e_g(t) + \gamma_g \ell_g(t)$ in our model) and does not involve differential equations. Several consequences follow from our dynamical formulation: one, the differential equations give detailed dynamics information about the travel times and the number of users in the system; such dynamics is not apparent in Lindsey's model. The LCS also reveals the dynamics about the travel costs and departure rate functions, albeit not as fully as the travel time functions, and suggests the possibility that the Zeno phenomenon could be present in the DUE problem. In particular, the departure rate or the arrival rate of a class may switch between a positive level and the zero level infinitely many times in a finite time interval. In practice, this situation represents clumping of departures at specific points in time interspersed with zero departures infinitesimal time intervals. Never been a concern in the literature of this problem, the Zeno phenomenon has theoretical, practical, and computational implications on the model; its detailed investigation, including its presence or absence, is regrettably beyond the scope of this paper. For references on the Zeno property for special classes of LCSs, see the recent articles [6,18,22,23].

From a practical standpoint, the numerical solution of the LCS is accomplished via a time-stepping scheme that involves the solution of a sequence of finite-dimensional discrete-time subproblems. The calculation of the numerical trajectories and their convergence analysis provide a constructive proof of existence of a solution to the continuous-time DUE model. This is a significant point of contrast with past work. For example, the proof of solution existence in [16] is based on the non-constructive Kakutani's fixed-point theorem. The convergence analysis of the numerical scheme presented in this paper offers insights into the limiting behavior of the discrete-time trajectories as the time step tends to zero.

[1] In general, a switched system includes several "nice" dynamics (modes), the system can change back and forth among these nice dynamics. These kind of changes are called mode switches. For an LCS, one can define modes according to the fundamental index sets associated with the complementarity condition.

In summary, the LCS (A)–(C) provides a fundamental formulation for the single-bottleneck DUE problem with heterogeneous commuters and offers a computationally tractable framework for the constructive analysis of this problem and many of its ensuing issues. Due to space limitation, this paper addresses only the issue of solution existence via a time-stepping scheme coupled with Lemke's algorithm for solving the discretized subproblems.

3 An implicit euler time-stepping method

In this section, we show the existence of nonnegative scalar equilibrium costs $\{c_g^*\}_{g\in\mathbb{G}}$ and absolutely continuous functions for travel time ($TT(t)$) and cumulative departures $\{N_g(t)\}_{g\in\mathbb{G}}$ so that the conditions (A)–(C) are satisfied. The main theorem is as follows.

Theorem 1 *There exist nonnegative scalars* $\{c_g^*\}_{g\in\mathbb{G}}$ *and absolutely continuous functions* $TT(t)$ *and* $\{N_g(t)\}_{g\in\mathbb{G}}$, *and integrable functions* $\{r_g(t)\}_{g\in\mathbb{G}}$ *and* $u(t)$ *so that conditions (A)–(C) are all satisfied.*

The proof of the theorem is constructive. Namely, we establish the existence of a said solution by constructing approximating iterates that will be shown to converge to such a solution. In turn, the construction is by a well-known time-stepping method used in ordinary differential equations extended to the LCS. Specifically, the method is as follows. For a positive integer $\nu > 0$, we divide the time interval $[0, T]$ into ν subintervals of equal length $h_\nu \triangleq T/\nu$. We construct numerical trajectories $\widehat{TT}^{\,\nu}(t)$ and $\widehat{N}_g^\nu(t)$ for $t \in [0, T]$ and for all $g \in \mathbb{G}$ by first computing the discrete-time iterates

$$\left\{TT^{\nu,0}, TT^{\nu,1}, \ldots, TT^{\nu,\nu}\right\}, \quad \left\{N_g^{\nu,0}, N_g^{\nu,1}, \ldots, N_g^{\nu,\nu}\right\}_{g\in\mathbb{G}}, \quad \left\{r_g^{\nu,1}, \ldots, r_g^{\nu,\nu}\right\}_{g\in\mathbb{G}},$$

and the nonnegative scalars $\{c_g^{\nu,*}\}_{g\in\mathbb{G}}$. This is done via the solution of the discretization of the continuous-time conditions (A)–(C) in which we approximate each time derivative by the forward difference quotient:

$$\frac{dx(t)}{dt} \approx \frac{x(t+h_\nu) - x(t)}{h_\nu}.$$

Hence, the discrete-time approximation of conditions (A) is:

$$\frac{TT^{\nu,i} - TT^{\nu,i-1}}{h_\nu} = u^{\nu,i} + \frac{1}{s}\left[\sum_{g\in\mathbb{G}} r_g^{\nu,i} - s\right]$$
$$\frac{N_g^{\nu,i} - N_g^{\nu,i-1}}{h_\nu} = r_g^{\nu,i}$$

$$
\begin{aligned}
0 &\le u^{\nu,i} \perp TT^{\nu,i} \ge 0 \\
0 &\le r_g^{\nu,i} \perp -\gamma_g(t_g^* - i\,h_\nu) + (\alpha_g + \gamma_g)TT^{\nu,i} + (\beta_g + \gamma_g) \\
&\quad \times \max\left\{0,\ t_g^* - (TT^{\nu,i} + i\,h_\nu)\right\} - c_g^{\nu,*} \ge 0.
\end{aligned}
$$

From the first equation we get:

$$
u^{\nu,i} = \frac{TT^{\nu,i} - TT^{\nu,i-1}}{h_\nu} - \frac{1}{s}\left[\sum_{g\in\mathbb{G}} r_g^{\nu,i} - s\right]. \tag{1}
$$

Substituting it into the first complementarity condition and multiplying by h_ν, and dividing the right-hand side of the second complementarity condition by $1/(\alpha_g + \gamma_g)$, we obtain the conditions (A_ν)–(C_ν) below:

(A_ν) for all $i = 1, \ldots, \nu$,

$$
\begin{aligned}
&0 \le TT^{\nu,i} \perp TT^{\nu,i} - TT^{\nu,i-1} - \frac{h_\nu}{s}\left[\sum_{g\in\mathbb{G}} r_g^{\nu,i} - s\right] \ge 0 \\
&N_g^{\nu,i} - N_g^{\nu,i-1} = h\, r_g^{\nu,i}, \quad \forall\, g \in \mathbb{G} \\
&0 \le r_g^{\nu,i} \perp -\frac{\gamma_g(t_g^* - i\,h_\nu)}{\alpha_g + \gamma_g} + TT^{\nu,i} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g} \\
&\quad \times \max\left\{0,\ t_g^* - (TT^{\nu,i} + i\,h_\nu)\right\} - \frac{c_g^{\nu,*}}{\alpha_g + \gamma_g} \ge 0, \quad \forall\, g \in \mathbb{G},
\end{aligned}
$$

(B_ν) the initial conditions:

$$
\begin{aligned}
&0 \le TT^{\nu,0} \perp TT^{\nu,0} - \left[\frac{1}{s}\sum_{g\in\mathbb{G}} N_g^{\nu,0} - 1\right] \ge 0 \\
&0 \le N_g^{\nu,0} \perp -\frac{\gamma_g\, t_g^*}{\alpha_g + \gamma_g} + TT^{\nu,0} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}\max\left\{0,\ t_g^* - TT^{\nu,0}\right\} \\
&\quad -\frac{c_g^{\nu,*}}{\alpha_g + \gamma_g} \ge 0, \quad \forall\, g \in \mathbb{G},
\end{aligned}
$$

(C_ν) the boundary conditions: $N_g^{\nu,\nu} = d_g$ for all $g \in \mathbb{G}$.

Note that it is not necessary to impose the nonnegativity of equilibrium cost $c_g^{\nu,*}$ explicitly; this will follow from the conditions (A_ν)–(C_ν). Indeed, suppose that rate of departure for some group g, $r_g^{\nu,i} > 0$ for some $i \in \{1, \ldots, \nu\}$. Then, from the second complementarity condition above,

$$\begin{aligned} c_g^{\nu,*} &= -\gamma_g(t_g^* - i\,h_\nu) + (\alpha_g + \gamma_g) TT^{\nu,i} + (\beta_g + \gamma_g) \max\left\{0, t_g^* - (TT^{\nu,i} + i\,h_\nu)\right\} \\ &\geq -\gamma_g\, t_g^* + i\,\gamma_g\, h_\nu + \alpha_g\, TT^{\nu,i} + \gamma_g\, TT^{\nu,i} + \beta_g\ \max\left\{0, t_g^* - (TT^{\nu,i} + i\,h_\nu)\right\} \\ &\quad + \gamma_g\, t_g^* - \gamma_g\, TT^{\nu,i} - i\,\gamma_g\, h_\nu \\ &= \alpha_g\, TT^{\nu,i} + \beta_g\ \max\left\{0,\ t_g^* - (TT^{\nu,i} + i\,h_\nu)\right\}\ \geq\ 0. \end{aligned}$$

On the other hand if the departure rates $r_g^{\nu,i} = 0$ for all $i \in \{1, \ldots, \nu\}$, then all vehicles depart at $t = 0$; we have, $N_g^{\nu,0} = N_g^{\nu,\nu} = d_g > 0$, which implies

$$c_g^{\nu,*}\ \geq\ \alpha_g\, TT^{\nu,0} + \beta_g\ \max\left\{0,\ t_g^* - TT^{\nu,0}\right\}\ \geq\ 0.$$

With a solution satisfying conditions (A_ν)–(C_ν), we construct the continuous piecewise linear functions on the interval $[0, T]$: for $i = 0, 1, \ldots, \nu - 1$,

$$\left.\begin{array}{l} \widehat{TT}^{\,\nu}(t)\ \triangleq\ TT^{\nu,i} + \frac{t - i\,h_\nu}{h_\nu}(TT^{\nu,i+1} - TT^{\nu,i}) \\ \widehat{N}_g^{\nu}(t)\ \triangleq\ N_g^{\nu,i} + \frac{t - i\,h_\nu}{h_\nu}(N_g^{\nu,i+1} - N_g^{\nu,i}), \quad \forall\, g\,\in\,\mathbb{G} \end{array}\right\} \quad \text{for } t\ \in\ [i\,h_\nu, (i+1)h_\nu]$$

From the travel time function $\widehat{TT}^{\,\nu}(t)$, we can construct the schedule delay and travel cost functions, all being continuous and piecewise linear on $[0, T]$: for all $g \in \mathbb{G}$,

$$\begin{aligned} \widehat{e}_g^{\,\nu}(t)\ &\triangleq \max\left\{0, t_g^* - \left(\widehat{TT}^{\,\nu}(t) + t\right)\right\} \\ \widehat{\ell}_g^{\,\nu}(t)\ &\triangleq \max\left\{0, \left(\widehat{TT}^{\,\nu}(t) + t\right) - t_g^*\right\} \\ \widehat{C}_g^{\,\nu}(t) &\triangleq \alpha_g\, \widehat{TT}^{\,\nu}(t) + \beta_g\, \widehat{e}_g^{\,\nu}(t) + \gamma_g\, \widehat{\ell}_g^{\,\nu}(t). \end{aligned}$$

We also construct auxiliary piecewise continuous functions $\widehat{u}^{\,\nu}(t)$ and $\widehat{r}^{\,\nu}(t)$ as follows:

$$\begin{array}{ll} \widehat{u}^{\,\nu}(t)\ \triangleq\ u^{\nu,i} = \frac{TT^{\nu,i} - TT^{\nu,i-1}}{h_\nu} - \left[\frac{1}{s}\sum_{g\in\mathbb{G}} r_g^{\nu,i} - 1\right] & \text{if } t\ \in\ ((i-1)\,h_\nu, i\,h_\nu] \\ \widehat{r}_g^{\,\nu}(t)\ \triangleq\ r^{\nu,i} & \text{if } t\ \in\ ((i-1)\,h_\nu, i\,h_\nu] \end{array}$$

Our goal in the subsequent analysis is to establish the well-definedness and convergence (in a sense to be made precise below) of these functions to a solution of the single-bottleneck model defined by conditions (A)–(C). This task is carried out in several steps. We will first establish the solvability of the discretized conditions (A_ν)–(C_ν). We then derive several bounds for the discrete-time iterates. These bounds are needed for the convergence of $\widehat{TT}^{\nu}(t)$, $\widehat{N}_g^{\nu}(t)$, $\widehat{u}^{\,\nu}(t)$, and $\widehat{r}_g^{\,\nu}(t)$ as ν tends to infinity. Finally, we prove that the limiting trajectories satisfy conditions (A)–(C) and hence the main theorem holds.

It should be noted that while there are previous studies [10,19] on the convergence of numerical time-stepping methods for solving LCSs, with the exception of Theorem 7.1 in [19], which is a general result with wide applicability, and also the principal tool that we will use later, all results in these references are not applicable to the system (A)–(C). By exploiting the structure of the system, we are able to provide an independent proof of convergence that is based on the mentioned theorem. Thus, a small part of the contribution of this work to mathematical programming is to demonstrate how an extended LCP result—Lemma 2—that has so far received minimal attention in this literature plays a central role in this particular application.

3.1 Solvability of the discrete-time system

We show that for each $\nu = 1, 2, \ldots$, there exist $\left(TT^{\nu,i}\right)_{i=0}^{\nu}$, $\left\{\left(N_g^{\nu,i}\right)_{i=0}^{\nu}\right\}_{g\in\mathbb{G}}$, $\left\{\left(r_g^{\nu,i}\right)_{i=0}^{\nu}\right\}_{g\in\mathbb{G}}$, and $\left(c_g^{\nu,*}\right)_{g\in\mathbb{G}}$ satisfying the discrete-time conditions (A_ν)–(C_ν); moreover these iterates can be computed by the well-known Lemke's algorithm [7]. For this purpose, we derive an equivalent linear complementarity problem (LCP) formulation for these conditions. Note that

$$N_g^{\nu,\nu} - N_g^{\nu,0} = h_\nu \sum_{i=1}^{\nu} r_g^{\nu,i}, \quad \text{or equivalently } N_g^{\nu,\nu} = N_g^{\nu,0} + h_\nu \sum_{i=1}^{\nu} r_g^{\nu,i};$$

thus,

$$N_g^{\nu,0} = d_g - h_\nu \sum_{i=1}^{\nu} r_g^{\nu,i}, \quad \forall\, g \in \mathbb{G}. \tag{2}$$

Moreover, to convert the discrete system into an LCP in the standard form, we need to introduce a new variable for each group g. Letting

$$f_g^{\nu} \triangleq -\gamma_g\, t_g^* + (\alpha_g + \gamma_g) TT^{\nu,0} + (\beta_g + \gamma_g) \max\left\{0, t_g^* - TT^{\nu,0}\right\} - c_g^{\nu,*},$$

we deduce

$$c_g^{\nu,*} = -\gamma_g\, t_g^* + (\alpha_g + \gamma_g) TT^{\nu,0} + (\beta_g + \gamma_g) \max\left\{0, t_g^* - TT^{\nu,0}\right\} - f_g^{\nu}, \tag{3}$$

which we can substitute into condition (A_ν). We also let $e_g^{\nu,i} \triangleq \max\left\{0, t_g^* - (TT^{\nu,i} + i\, h_\nu)\right\}$. Substituting the latter definition, (2), and (3) into the discrete-time system (A_ν)–(C_ν), we obtain the following (LCP_ν) in which the variables are $\left(TT^{\nu,i}\right)_{i=0}^{\nu}$ and $\left\{\left(r_g^{\nu,i}\right)_{i=1}^{\nu}, f_g^{\nu}, \left(e_g^{\nu,i}\right)_{i=0}^{\nu}\right\}_{g\in\mathbb{G}}$:

$$
\begin{array}{lll}
0 \leq TT^{\nu,0} & \perp TT^{\nu,0} + \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} \sum_{i=1}^{\nu} r_g^{\nu,i} - \frac{1}{s} \sum_{g \in \mathbb{G}} d_g + 1 \geq 0 & \\
0 \leq TT^{\nu,i} & \perp TT^{\nu,i} - TT^{\nu,i-1} - \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} r_g^{\nu,i} + h_\nu \geq 0, \quad i = 1, \ldots, \nu & \\
0 \leq r_g^{\nu,i} & \perp \frac{i\, h_\nu\, \gamma_g}{\alpha_g + \gamma_g} + TT^{\nu,i} - TT^{\nu,0} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}(e_g^{\nu,i} - e_g^{\nu,0}) + \frac{f_g^\nu}{\alpha_g + \gamma_g} \geq 0, & \\
 & \quad i = 1, \ldots, \nu;\; g \in \mathbb{G} & \\
0 \leq f_g^\nu & \perp d_g - h_\nu \sum_{i=1}^{\nu} r_g^{\nu,i} \geq 0, \quad g \in \mathbb{G} & \\
0 \leq e_g^{\nu,i} & \perp -(t_g^* - i\, h_\nu) + TT^{\nu,i} + e_g^{\nu,i} \geq 0, \quad i = 0, 1, \ldots, \nu;\; g \in \mathbb{G}. &
\end{array}
$$

The following lemma formally states the equivalence of LCP_ν to the original discrete-time system (A_ν)–(C_ν). The proof is obvious and hence omitted.

Lemma 1 *Suppose* $\left(TT^{\nu,i}\right)_{i=0}^{\nu}$ *and* $\left\{ \left(r_g^{\nu,i}\right)_{i=1}^{\nu}, f_g^\nu, \left(e_g^{\nu,i}\right)_{i=0}^{\nu} \right\}_{g \in \mathbb{G}}$ *solve* LCP_ν. *Let* $\left\{ \left(N_g^{\nu,i}\right)_{i=1}^{\nu} \right\}_{g \in \mathbb{G}}$ *and* $\left(c_g^{\nu,*}\right)_{g \in \mathbb{G}}$ *be given by*

$$
\begin{aligned}
N_g^{\nu,0} &= d_g - h_\nu \sum_{i=1}^{\nu} r_g^{\nu,i}, \quad \forall\, g \in \mathbb{G} \\
N_g^{\nu,i} &= h_\nu r_g^{\nu,i} + N_g^{\nu,i-1}, \quad \forall\, i = 1, \ldots, \nu, \ \forall\, g \in \mathbb{G} \\
c_g^{\nu,*} &= -\gamma_g\, t_g^* + (\alpha_g + \gamma_g) TT^{\nu,0} + (\beta_g + \gamma_g) \max\left\{0, t_g^* - TT^{\nu,0}\right\} \\
&\quad - f_g^\nu, \quad \forall\, g \in \mathbb{G}
\end{aligned}
$$

Then $\left(TT^{\nu,i}\right)_{i=0}^{\nu}$, $\left\{ \left(N_g^{\nu,i}\right)_{i=0}^{\nu} \right\}_{g \in \mathbb{G}}$, $\left\{ \left(r_g^{\nu,i}\right)_{i=0}^{\nu} \right\}_{g \in \mathbb{G}}$ *and* $\left(c_g^{\nu,*}\right)_{g \in \mathbb{G}}$ *satisfy conditions* (A_ν)–(C_ν). □

Thus to show that the discrete-time system (A_ν)–(C_ν) has a solution, it suffices to show the same for the LCP_ν. This LCP can be written in the standard form of an LCP [7] of finding a vector x satisfying $0 \leq x \perp q^\nu + M^\nu x \geq 0$, where

$$
x \triangleq \begin{pmatrix} TT^{\nu,0} \\ \left(TT^{\nu,i}\right)_{i=1}^{\nu} \\ \left\{ \left(r_g^{\nu,i}\right)_{i=1}^{\nu} \right\}_{g \in \mathbb{G}} \\ \left(f_g^\nu\right)_{g \in \mathbb{G}} \\ \left\{ \left(e_g^{\nu,i}\right)_{i=0}^{\nu} \right\}_{g \in \mathbb{G}} \end{pmatrix}, \quad q^\nu \triangleq \begin{pmatrix} -\frac{1}{s} \sum_{g \in \mathbb{G}} d_g + 1 \\ h_\nu\, \mathbf{1}_\nu \\ \left\{ \left(\frac{i\, h_\nu\, \gamma_g}{\alpha_g + \gamma_g}\right)_{i=1}^{\nu} \right\}_{g \in \mathbb{G}} \\ \left(d_g\right)_{g \in \mathbb{G}} \\ -\left\{ \left(t_g^* - i\, h_\nu\right)_{i=0}^{\nu} \right\}_{g \in \mathbb{G}} \end{pmatrix},
$$

with $\mathbf{1}_\nu$ being the ν-vector of all ones, and

$$M^\nu x = \begin{pmatrix} TT^{\nu,0} + \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} \sum_{i=1}^{\nu} r_g^{\nu,i} \\ \left(TT^{\nu,i} - TT^{\nu,i-1} - \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} r_g^{\nu,i} \right)_{i=1}^{\nu} \\ \left\{ \left(TT^{\nu,i} - TT^{\nu,0} + \frac{\beta_g+\gamma_g}{\alpha_g+\gamma_g} (e_g^{\nu,i} - e_g^{\nu,0}) + \frac{f_g^\nu}{\alpha_g+\gamma_g} \right)_{i=1}^{\nu} \right\}_{g \in \mathbb{G}} \\ -h_\nu \left(\sum_{i=1}^{\nu} r_g^{\nu,i} \right)_{g \in \mathbb{G}} \\ \left\{ \left(TT^{\nu,i} + e_g^{\nu,i} \right)_{i=0}^{\nu} \right\}_{g \in \mathbb{G}} \end{pmatrix}.$$

In turn, to show the solvability of LCP$_\nu$, we employ an existence result from LCP theory. Let the solution set of the LCP: $0 \leq z \perp q + Mz \geq 0$ be denoted by SOL(q, M). We recall that a matrix M is a *semimonotone* if for every vector $x \gneqq 0$, there exists a component $x_i > 0$ such that $(Mx)_i \geq 0$ [7]. To present our proof, we also briefly review the well-known Lemke's method for solving LCPs. For an LCP $0 \leq z \perp q + Mz \geq 0$, Lemke's method works with the following augmented LCP:

$$0 \leq z \perp w = q + d\, z_0 + Mz \geq 0$$

where $d > 0$ is a so-called *covering vector*. The algorithm is initialized with $z = 0$ and $z_0 = \max_i\{-q_i/d_i\}$, and goes through, via linear programming pivoting, basic feasible solutions of the system of equations: $w - Mz - dz_0 = q$ until it reaches a solution to the original LCP ($z_0 = 0$) or terminates on a so-called *secondary ray*. Therefore this method will successfully compute a solution to the original LCP if it can be shown that a secondary ray cannot exist. The following lemma provides sufficient conditions for this to occur; its proof is based on combining the proofs of Theorems 4.4.9 and 4.4.11 in [7].

Lemma 2 *Let M be a semimonotone matrix. If $d > 0$ is such that for every $\tau > 0$, SOL$(q + \tau d, M)$ is bounded, then the LCP (q, M) has a solution that can be computed by Lemke's algorithm with d as the covering vector.*

Proof We use the same notation as in the proof of [7, Theorem 4.4.9]. As shown therein, if the algorithm terminates at an secondary ray, then there exists a tuple $(w^*, \tilde{w}, z_0^*, \tilde{z}_0, z^*, \tilde{z})$ such that for all $\lambda \geq 0$

$$w^* + \lambda \tilde{w} = q + d(z_0^* + \lambda \tilde{z}_0) + M(z^* + \lambda \tilde{z}) \tag{4}$$

and

$$(w_i^* + \lambda \tilde{w}_i)(z_i^* + \lambda \tilde{z}_i) = 0, \quad \forall i = 1, \ldots, n, \tag{5}$$

where $\tilde{z} \neq 0$. It is clear that if $z_0^* = 0$ then z^* is a solution of the original LCP. Therefore, without loss of generality we assume $z_0^* > 0$. As pointed out in the proof of Theorem 4.4.11, the semimonotonicity of M implies $\tilde{z}_0 = 0$. Thus by (4) and (5) we deduce that $z^* + \lambda \tilde{z}$ is a solution to the LCP$(q + z_0^* d, M)$ for every $\lambda \geq 0$. This contradicts the boundedness assumption of the Lemma. Hence we have the desired result. □

Specializing the above lemma to the LCP$_\nu$, we have the following result.

Lemma 3 *For each integer $\nu = 1, 2, \ldots,$ the LCP$_\nu$ has a solution that can be computed by Lemke's algorithm with any positive vector as the covering vector.*

Proof Fix an integer $\nu > 0$. To show the semimonotonicity of M^ν, let $x \gneqq 0$ be given. There are five components to the vector x. We examine each in turn here. If $e_g^{\nu,i} > 0$ for some $g \in \mathbb{G}$ and some $i \in \{0, 1, \ldots, \nu\}$, then $TT^{\nu,i} + e_g^{\nu,i} > 0$. Hence we may assume that $e_g^{\nu,i} = 0$ for all $g \in \mathbb{G}$ and all $i \in \{0, 1, \ldots, \nu\}$. If $TT^{\nu,0} > 0$, then the corresponding component in $M^\nu x$ is

$$TT^{\nu,0} + \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} \sum_{i=1}^{\nu} r_g^{\nu,i} > 0.$$

Hence we may further assume $TT^{\nu,0} = 0$. If $r_g^{\nu,i} > 0$ for some $g \in \mathbb{G}$ and some $i \in \{0, 1, \ldots, \nu\}$, then the component of $M^\nu x$ corresponding to $r_g^{\nu,i}$ is

$$TT^{\nu,i} + \frac{f_g^\nu}{\alpha_g + \gamma_g} \geq 0.$$

Thus the semimonotonicity of M^ν follows in this case. Therefore, we may assume that $r_g^{\nu,i} = 0$ for all $g \in \mathbb{G}$ and all $i \in \{0, 1, \ldots, \nu\}$. If $TT^{\nu,i} > 0$ for some $i \in \{0, 1, \ldots, \nu\}$, we take i_* to be the first index i such that $TT^{\nu,i} > 0$. In this case, the element of $M^\nu x$ corresponding to TT^{ν,i_*} is simply TT^{ν,i_*}; thus the semimonotonicity of M^ν also holds. Therefore, we may assume that $TT^{\nu,i} = 0$ for all i. Since x is nonzero, we must have $f_g^\nu > 0$ for some g, and the semimonotonicity of M^ν follows readily too.

To complete the proof of the theorem, we consider only the case where we use the vector $\mathbf{1}$ of all ones as the covering vector; a similar proof applies to any positive vector. Suppose that for some scalar $\tau > 0$, the LCP $(q^\nu + \tau\,\mathbf{1}, M^\nu)$ has a sequence of solutions:

$$\left(TT^{\nu,0,k}, \left(TT^{\nu,i,k} \right)_{i=1}^{\nu}, \quad \left\{ \left(r_g^{\nu,i,k} \right)_{i=1}^{\nu} \right\}_{g \in \mathbb{G}}, \quad \left(f_g^{\nu,k} \right)_{g \in \mathbb{G}}, \quad \left\{ \left(e_g^{\nu,i,k} \right)_{i=1}^{\nu} \right\}_{g \in \mathbb{G}} \right),$$

which satisfy the following complementarity conditions:

$$0 \leq TT^{\nu,0,k} \perp TT^{\nu,0,k} + \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} \sum_{i=1}^{\nu} r_g^{\nu,i,k} - \frac{1}{s} \sum_{g \in \mathbb{G}} d_g + 1 + \tau \; \geq 0 \tag{6}$$

$$0 \leq TT^{\nu,i,k} \perp TT^{\nu,i,k} - TT^{\nu,i-1,k} - \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} r_g^{\nu,i,k} + h_\nu + \tau \geq 0, \quad i = 1, \dots, \nu \tag{7}$$

$$0 \leq r_g^{\nu,i,k} \perp \frac{i\, h_\nu\, \gamma_g}{\alpha_g + \gamma_g} + TT^{\nu,i,k} - TT^{\nu,0,k} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}(e_g^{\nu,i,k} - e_g^{\nu,0,k}) + \frac{f_g^{\nu,k}}{\alpha_g + \gamma_g} + \tau \geq 0, \quad i = 1, \dots, \nu; \; g \in \mathbb{G} \tag{8}$$

$$0 \leq f_g^{\nu,k} \perp d_g - h_\nu \sum_{i=1}^{\nu} r_g^{\nu,i,k} + \tau \geq 0, \quad g \in \mathbb{G} \tag{9}$$

$$0 \leq e_g^{\nu,i,k} \perp -(t_g^* - i\, h_\nu) + TT^{\nu,i,k} + e_g^{\nu,i,k} + \tau \geq 0, \quad i = 0, 1, \dots, \nu; \; g \in \mathbb{G}. \tag{10}$$

We will show the desired boundedness of these solutions in several steps: first $TT^{\nu,0,k}$, next $e_g^{\nu,i,k}$, then $f_g^{\nu,k}$, followed by $r_g^{\nu,i,k}$, and last $TT^{\nu,i,k}$.

Boundedness of $TT^{\nu,0,k}$. Suppose that this is not true. We may assume without loss of generality that

$$\lim_{k \to \infty} TT^{\nu,0,k} = \infty.$$

Therefore, we may assume further that $TT^{\nu,0,k} > 0$ for all $k = 1, \dots, \infty$. If follows from (6) that

$$TT^{\nu,0,k} + \frac{h_\nu}{s} \sum_{g \in \mathbb{G}} \sum_{i=1}^{\nu} r_g^{\nu,i,k} - \frac{1}{s} \sum_{g \in \mathbb{G}} d_g + 1 + \tau = 0,$$

for all $k = 1, \dots, \infty$. By the nonnegativity of $r_g^{\nu,i,k}$, we deduce

$$TT^{\nu,0,k} \leq \frac{1}{s} \sum_{g \in \mathbb{G}} d_g.$$

This is a contradiction. So the boundedness of $TT^{\nu,0,k}$ follows.

Boundedness of $e_g^{\nu,i,k}$. Suppose that this is not true for some g_* and i_*. We may assume without loss of generality that

$$\lim_{k \to \infty} e_{g_*}^{\nu,i_*,k} = \infty.$$

Similar to above reasoning, we deduce from (10) that

$$-(t_{g_*}^* - i_*\, h_\nu) + TT^{\nu,i_*,k} + e_{g_*}^{\nu,i_*,k} + \tau = 0,$$

for all $k = 1, 2, \ldots$. By the nonnegativity of $TT^{\nu,i_*,k}$, it follows that

$$e_{g_*}^{\nu,i_*,k} \;\leq\; t_{g_*}^* - i_*\, h_\nu \;\leq\; t_{g_*}^*.$$

This is again a contradiction. So the boundedness of $e_g^{\nu,i,k}$ follows for all $i = 1, \ldots, \nu$ and all $g \in \mathbb{G}$.

Boundedness of $f_g^{\nu,k}$. Suppose that this is not true for some g_*. We may assume that

$$\lim_{k\to\infty} f_{g_*}^{\nu,k} = \infty.$$

In turn, we may assume that $f_{g_*}^{\nu,k} > 0$ for all $k = 1, 2, \ldots$. Thus, from (9), we have

$$d_{g_*} - h_\nu \sum_{i=1}^{\nu} r_{g_*}^{\nu,i,k} + \tau_k = 0, \tag{11}$$

for all $k = 1, 2, \ldots$. On the other hand, due to the boundedness of $TT^{\nu,0,k}$ and $e_{g^*}^{\nu,0,k}$, we have

$$\frac{i\, h_\nu\, \gamma_{g_*}}{\alpha_{g_*} + \gamma_{g_*}} + TT^{\nu,i,k} - TT^{\nu,0,k} + \frac{\beta_{g_*} + \gamma_{g_*}}{\alpha_{g_*} + \gamma_{g_*}}(e_{g_*}^{\nu,i,k} - e_{g_*}^{\nu,0,k}) + \frac{f_{g_*}^{\nu,k}}{\alpha_{g_*} + \gamma_{g_*}} + \tau \;>\; 0$$

for all $i = 1, \ldots, \nu$ and all k sufficiently large. From (8) we have $r_{g_*}^{\nu,i,k} = 0$ for all $i = 1, \ldots, \nu$ and all k sufficiently large. But this contradicts (11). Therefore, the boundedness of $f_g^{\nu,k}$ holds readily.

Boundedness of $r_g^{\nu,i,k}$. This is obvious because by (9), we have $\sum_{i=1}^{\nu} r_g^{\nu,i,k} \leq h_\nu^{-1}(d_g + \tau)$.

Boundedness of $\boldsymbol{TT}^{\nu,i,k}$. Let $i_* \in \{1, 2, \ldots, \nu\}$ be the smallest i such that $TT^{\nu,i,k}$ is unbounded. We may assume

$$\lim_{k\to\infty} TT^{\nu,i_*,k} = \infty,$$

and $TT^{\nu,i_*,k} > 0$ for all $k = 1, 2, \ldots$. Thus,

$$TT^{\nu,i_*,k} - TT^{\nu,i_*-1,k} - \frac{h_\nu}{s} \sum_{g\in\mathbb{G}} r_g^{\nu,i_*,k} + h_\nu + \tau = 0$$

for all $k = 1, 2, \ldots$. This implies that

$$TT^{\nu,i_*,k} - TT^{\nu,i_*-1,k} \leq \frac{h_\nu}{s} \sum_{g\in\mathbb{G}} r_g^{\nu,i_*,k}$$

for all $k = 1, 2, \ldots$. Due to the boundedness of $r_g^{\nu,i_*,k}$, we can conclude that $TT^{\nu,i_*-1,k}$ is also unbounded. This contradicts the definition of i_*. Now, applying Lemma 2, we get the desired result. □

By combining Lemmas 1 and 2, the solvability of the time-discretized system (A_ν)–(C_ν) follows.

3.2 Boundedness of the discrete-time iterates

We next establish that the solutions to conditions (A_ν)–(C_ν) are uniformly bounded in norm by a constant that is independent of ν. First, for all $g \in \mathbb{G}$ and all ν, since $N_g^{\nu,i} - N_g^{\nu,i-1} = hr_g^{\nu,i} \geq 0$ for all $i = 1, \ldots, \nu$, we have

$$N_g^{\nu,0} \leq N_g^{\nu,1} \leq \cdots N_g^{\nu,\nu} = d_g.$$

Thus

$$TT^{\nu,0} = \max\left(0, \frac{1}{s}\sum_{g\in\mathbb{G}} N_g^{\nu,0} - 1\right) \leq \max\left(0, \frac{1}{s}\sum_{g\in\mathbb{G}} d_g - 1\right), \quad \forall\, \nu.$$

We next bound $|TT^{\nu,i} - TT^{\nu,i-1}|$ and $r_g^{\nu,i}$. Consider an index $i \geq 1$, we bound $TT^{\nu,i} - TT^{\nu,i-1}$ from both above and below. To bound it from below, from the first complementary condition in (A_ν), we clearly have

$$TT^{\nu,i} - TT^{\nu,i-1} \geq \frac{h_\nu}{s}\left[\sum_{g\in\mathbb{G}} r_g^{\nu,i} - s\right] \geq -h_\nu.$$

To bound $TT^{\nu,i} - TT^{\nu,i-1}$ from above we consider two cases:

Case 1 $TT^{\nu,i} = 0$. In this case, we simply have $TT^{\nu,i} - TT^{\nu,i-1} \leq 0$ and

$$-TT^{\nu,i-1} - \frac{h_\nu}{s}\left[\sum_{g\in\mathbb{G}} r_g^{\nu,i} - s\right] \geq 0,$$

which implies $\sum_{g\in\mathbb{G}} r_g^{\nu,i} \leq s$.

Case 2 $TT^{\nu,i} > 0$. In this case, we have

$$TT^{\nu,i} - TT^{\nu,i-1} = \frac{h_\nu}{s}\left[\sum_{g\in\mathbb{G}} r_g^{\nu,i} - s\right].$$

Clearly, if $r_g^{\nu,i} = 0$ for all $g \in \mathbb{G}$ then $TT^{\nu,i} - TT^{\nu,i-1} = -h_\nu$. Thus, we may assume there exists a class $g \in \mathbb{G}$ for which $r_g^{\nu,i} > 0$. We then have, by the second complementarity condition in (A_ν) and the fact that $e_g^{\nu,i} = \max\{0, t_g^* - (TT^{\nu,i} + i h_\nu)\}$,

$$\begin{aligned}
&-\frac{\gamma_g(t_g^* - i\,h_\nu)}{\alpha_g + \gamma_g} + TT^{\nu,i} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}\, e_g^{\nu,i} - \frac{c_g^{\nu,*}}{\alpha_g + \gamma_g} \\
&= 0 \;\le\; -\frac{\gamma_g(t_g^* - i\,h_\nu + h_\nu)}{\alpha_g + \gamma_g} + TT^{\nu,i-1} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}\, e_g^{\nu,i-1} - \frac{c_g^{\nu,*}}{\alpha_g + \gamma_g},
\end{aligned}$$

which yields

$$TT^{\nu,i} - TT^{\nu,i-1} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}(e_g^{\nu,i} - e_g^{\nu,i-1}) \le -\frac{\gamma_g}{\alpha_g + \gamma_g}\, h_\nu. \tag{12}$$

Hence, if $e_g^{\nu,i-1} = 0$, then

$$TT^{\nu,i} - TT^{\nu,i-1} = \frac{h_\nu}{s}\left[\sum_{g' \in \mathbb{G}} r_{g'}^{\nu,i} - s\right] \le -h_\nu\, \frac{\gamma_g}{\alpha_g + \gamma_g}.$$

On the other hand, if $e_g^{\nu,i-1} > 0$, then

$$-(t_g^* - (i-1)\,h_\nu) + TT^{\nu,i-1} + e_g^{\nu,i-1} = 0 \;\le\; -(t_g^* - i\,h_\nu) + TT^{\nu,i} + e_g^{\nu,i},$$

which yields

$$TT^{\nu,i} - TT^{\nu,i-1} + e_g^{\nu,i} - e_g^{\nu,i-1} \;\ge\; -h_\nu,$$

or equivalently,

$$e_g^{\nu,i} - e_g^{\nu,i-1} \;\ge\; -h_\nu - (TT^{\nu,i} - TT^{\nu,i-1}).$$

This, together with (12), gives

$$\begin{aligned}
&(TT^{\nu,i} - TT^{\nu,i-1}) - \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g} h_\nu - \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}(TT^{\nu,i} - TT^{\nu,i-1}) \\
&\le\; TT^{\nu,i} - TT^{\nu,i-1} + \frac{\beta_g + \gamma_g}{\alpha_g + \gamma_g}(e_g^{\nu,i} - e_g^{\nu,i-1}) \le -\frac{\gamma_g}{\alpha_g + \gamma_g}\, h_\nu.
\end{aligned}$$

Recalling that $\beta_g < \alpha_g < \gamma_g$, we therefore have

$$\frac{\alpha_g - \beta_g}{\alpha_g + \gamma_g}\,(TT^{\nu,i} - TT^{\nu,i-1}) \;\le\; \frac{\beta_g}{\alpha_g + \gamma_g}\, h_\nu.$$

Summarizing, we deduce that, for all $i = 1, \ldots, \nu$,

$$-\, h_\nu \;\leq\; TT^{\nu,i} - TT^{\nu,i-1} \;\leq\; h_\nu \max_{g \in \mathbb{G}} \frac{\beta_g}{\alpha_g - \beta_g}. \tag{13}$$

and

$$\sum_{g \in \mathbb{G}} r_g^{\nu,i} \;\leq\; s \left[1 + \max_{g \in \mathbb{G}} \frac{\beta_g}{\alpha_g - \beta_g} \right], \tag{14}$$

which in turn implies that

$$\frac{N_g^{\nu,i} - N_g^{\nu,i-1}}{h_\nu} \;\leq\; s \left[1 + \max_{g \in \mathbb{G}} \frac{\beta_g}{\alpha_g - \beta_g} \right]. \tag{15}$$

Thus, for all $i = 1, \ldots, \nu$,

$$\begin{aligned} TT^{\nu,i} \;&=\; TT^{\nu,0} + \sum_{j=1}^{i} (TT^{\nu,j} - TT^{\nu,j-1}) \;\leq\; \max\left(0,\, \frac{1}{s} \sum_{g \in \mathbb{G}} d_g - 1 \right) \\ &\quad + T \max_{g \in \mathbb{G}} \frac{\beta_g}{\alpha_g - \beta_g}. \end{aligned} \tag{16}$$

By (1), we have for all $i = 1, \ldots, \nu$,

$$u^{\nu,i} = \frac{TT^{\nu,i} - TT^{\nu,i-1}}{h_\nu} - \frac{1}{s} \left[\sum_{g \in \mathbb{G}} r_g^{\nu,i} - s \right] \;\leq\; \max_{g \in \mathbb{G}} \frac{\beta_g}{\alpha_g - \beta_g} + 1. \tag{17}$$

Moreover, similar to the proof of the nonnegativity of $c_g^{\nu,*}$, we can deduce that, for all $g \in \mathbb{G}$,

$$\begin{aligned} c_g^{\nu,*} &\leq \max_{0 \leq i \leq \nu} \Big[-\gamma_g (t_g^* - i\, h_\nu) + (\alpha_g + \gamma_g) TT^{\nu,i} + (\beta_g + \gamma_g) \\ &\qquad \times \max \left\{ 0,\, t_g^* - (TT^{\nu,i} + i\, h_\nu) \right\} \Big] \\ &\leq T\, \gamma_g + \max_{0 \leq i \leq \nu} \left[(\alpha_g + \gamma_g) TT^{\nu,i} + (\beta_g + \gamma_g)\, \max \left\{ 0,\, t_g^* - TT^{\nu,i} \right\} \right] \\ &\leq \text{a constant independent of } \nu. \end{aligned}$$

With the above established bounds, we can complete the convergence analysis of the numerical time-stepping scheme, by following a similar analysis as done in [19, Section 7], which can be somewhat simplified due to the special structure of the conditions (A)–(C).

3.3 Convergence of the numerical trajectories

We show that the numerical trajectories converge subsequentially in a certain sense to limiting trajectories that satisfy the conditions (A)–(C), which include differential equations, complementarity conditions, and boundary conditions. We shall verify them one by one.

To establish the convergence result, we need to apply well-known convergence theorems in functional spaces. These theorems can be found in any standard functional analysis text. In fact, the bounds (13) and (15) show that the functions $\widehat{TT}^{\nu}(t)$ and $\widehat{N}_g^{\nu}(t)$ are in fact *equi-Lipschitz continuous* on the interval $[0, T]$; i.e., they are Lipschitz continuous with a common Lipschitz constant that is independent of ν. Thus, by the Arzelá-Ascoli theorem (see, e.g., [15, pp. 57–59]), there is an infinite subset κ of positive integers ν such that the subsequences of functions $\left\{\widehat{TT}^{\nu}\right\}_{\nu\in\kappa}$ and $\left\{\widehat{N}_g^{\nu}\right\}_{\nu\in\kappa}$ converge, respectively, in the supremum (i.e., L_∞)-norm to Lipschitz functions $\widehat{TT}^{\infty}$ and $\widehat{N}_g^{\infty}$ on $[0, T]$; hence so does the sequence of cost functions $\left\{\widehat{C}_g^{\nu}\right\}_{\nu\in\kappa}$ to a Lipschitz function $\widehat{C}_g^{\infty}$. Clearly, $\widehat{TT}^{\infty}(t) \geq 0$ for all $t \in [0, T]$. Due to the boundedness of $c_g^{\nu,*}$, without loss of generality, we may assume that the subsequence of scalars $\{c_g^{\nu,*}\}_{\nu\in\kappa}$ converges to a scalar $c_g^{\infty,*}$ for each $g \in \mathbb{G}$. By (13) and (14), we know that $\widehat{u}^{\nu}(t)$ and $\widehat{r}_g^{\nu}(t)$ are uniformly bounded in the L_∞-norm on $[0, T]$. Therefore, by the argument used in the proof of Theorem 7.1 in [19], it follows that, by working with an appropriate infinite subset of κ if necessary, the sequences of functions $\{\widehat{u}^{\nu}\}_{\nu\in\kappa}$ and $\{\widehat{r}_g^{\nu}\}_{\nu\in\kappa}$ are weakly convergent with limits $\widehat{u}^{\infty}$ and $\widehat{r}_g^{\infty}$, respectively, that are square integrable functions on $[0, T]$; moreover, the latter limit functions are nonnegative almost everywhere on $[0, T]$. Furthermore, as proved in the reference, the differential equations:

$$\frac{d\,\widehat{TT}^{\infty}(t)}{dt} = \widehat{u}^{\infty}(t) + \frac{1}{s}\left[\sum_{g\in\mathbb{G}} \widehat{r}_g^{\infty}(t) - s\right] \quad \text{and} \quad \frac{d\,\widehat{N}_g^{\infty}(t)}{dt} = \widehat{r}_g^{\infty}(t)$$

hold for almost all $t \in [0, T]$.

It is not difficult to see, by passing the limit $\nu(\in \kappa) \to \infty$, that the initial and boundary conditions (B) and (C) must be satisfied by the limits $\widehat{TT}^{\infty}(0)$, $\widehat{N}_g^{\infty}(0)$, $\widehat{N}_g^{\infty}(T)$, $\widehat{C}_g^{\infty}(0)$, and $c_g^{\infty,*}$. We then show that the complementarity conditions in (A) are satisfied. The nonnegativity of $\widehat{TT}^{\infty}(t)$, $\widehat{u}^{\infty}(t)$ and $\widehat{r}_g^{\infty}$ is clear. To show that $\widehat{C}_g^{\infty}(t) - c_g^{\infty,*}$ is also nonnegative, we first prove the following lemma.

Lemma 4 Let $\widehat{C}_g^{\nu}(t)$ and $c_g^{\nu,*}$ be as defined above. There exists a constant $\sigma > 0$ such that

$$\widehat{C}_g^{\nu}(t) \geq c_g^{\nu,*} - \sigma\, h_\nu$$

for all ν, $g \in \mathbb{G}$ and all $t \in [0, T]$.

Proof Let $t \in (ih_\nu, (i+1)h_\nu)$ for some $i \in \{0, 1, \ldots, \nu - 1\}$. We can write $t = (i+\tau)h_\nu = (1-\tau)\,ih_\nu + \tau(i+1)h_\nu$ for some $\tau \in (0, 1)$. Thus

$$\widehat{TT}^{\nu}(t) = TT^{\nu,i} + \frac{(i+\tau)h_\nu - ih_\nu}{h_\nu}(TT^{\nu,i+1} - TT^{\nu,i}) = (1-\tau)TT^{\nu,i} + \tau TT^{\nu,i+1}.$$

For an arbitrary $g \in \mathbb{G}$, first consider the case when $t_g^* - (TT^{\nu,i} + ih_\nu) \geq 0 > t_g^* - (TT^{\nu,i} + (i+1)h_\nu)$. We have

$$\begin{aligned} t_g^* - \left(\widehat{TT}^{\,\nu}(t) + t\right) &= t_g^* - \left[(1-\tau)TT^{\nu,i} + \tau TT^{\nu,i+1}\right] - (i+\tau)h_\nu \\ &= (1-\tau)[t_g^* - (TT^{\nu,i} + i\,h_\nu)] + \tau\,[t_g^* - (TT^{\nu,i+1} + i\,h_\nu + h_\nu)] \\ &= [t_g^* - (TT^{\nu,i} + i\,h_\nu)] - \tau(TT^{\nu,i+1} - TT^{\nu,i} + h_\nu) \\ &\geq (1-\tau)[\,t_g^* - (TT^{\nu,i} + i\,h_\nu)] - h_\nu \left[\max_{g' \in \mathbb{G}} \frac{\beta_{g'}}{\alpha_{g'} - \beta_{g'}} + 1\right], \end{aligned}$$

where the last inequality is due to (13). Notice that in this case we have

$$\begin{aligned} \widehat{C}_g^\nu(ih_\nu) &= -\gamma_g(t_g^* - ih_\nu) + (\alpha_g + \gamma_g)TT^{\nu,i} + (\beta_g + \gamma_g)(t_g^* - TT^{\nu,i} - ih_\nu), \\ \widehat{C}_g^\nu((i+1)h_\nu) &= -\gamma_g(t_g^* - ih_\nu - h_\nu) + (\alpha_g + \gamma_g)TT^{\nu,i+1}. \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{C}_g^{\,\nu}(t) &= -\gamma_g(t_g^* - t) + (\alpha_g + \gamma_g)\widehat{TT}^{\,\nu}(t) + (\beta_g + \gamma_g)\max\left\{0, t_g^* - \left(\widehat{TT}^{\,\nu}(t) + t\right)\right\} \\ &\geq (1-\tau)C_g^\nu(ih_\nu) + \tau\,C_g^\nu(ih_\nu + h_\nu) - h_\nu(\beta_g + \gamma_g)\left[\max_{g' \in \mathbb{G}} \frac{\beta_{g'}}{\alpha_{g'} - \beta_{g'}} + 1\right] \\ &\geq c_g^{\nu,*} - \sigma\,h_\nu, \end{aligned}$$

with

$$\sigma \triangleq \left[\max_{g \in \mathbb{G}}(\beta_g + \gamma_g)\right]\left[\max_{g \in \mathbb{G}} \frac{\beta_g}{\alpha_g - \beta_g} + 1\right].$$

Next we consider the case when $t_g^* - (TT^{\nu,i+1} + (i+1)h_\nu) \geq 0 > t_g^* - (TT^{\nu,i} + ih_\nu)$. Then, applying (13), we have

$$\begin{aligned} t_g^* - \left(\widehat{TT}^{\,\nu}(t) + t\right) &= t_g^* - TT^{\nu,i+1} + TT^{\nu,i+1} - (1-\tau)TT^{\nu,i} - \tau TT^{\nu,i+1} \\ &\quad - (i+1)h_\nu + (1-\tau)h_\nu \\ &= [t_g^* - (TT^{\nu,i+1} + i\,h_\nu + h_\nu)] \\ &\quad + (1-\tau)(TT^{\nu,i+1} - TT^{\nu,i} + h_\nu) \\ &\geq \tau\,[t_g^* - (TT^{\nu,i+1} + i\,h_\nu + h_\nu)]. \end{aligned}$$

Hence

$$\begin{aligned}\widehat{C}_g^{\,\nu}(t) &= -\gamma_g(t_g^* - ih_\nu - \tau h_\nu) + (\alpha_g + \gamma_g)[\tau TT^{\nu,i+1} + (1-\tau)TT^{\nu,i}] \\ &\quad + \tau(\beta_g + \gamma_g)\max\left\{0, t_g^* - \left(\widehat{TT}^{\,\nu}(t) + t\right)\right\} \\ &\geq -\tau\gamma_g(t_* - ih_\nu - h_\nu) - (1-\tau)\gamma_g \\ &\quad \times(t_g^* - ih_\nu) + (\alpha_g + \gamma_g)[\tau TT^{\nu,i+1} + (1-\tau)TT^{\nu,i}] \\ &\quad + [t_g^* - (TT^{\nu,i+1} + ih_\nu + h_\nu)] \\ &= (1-\tau)\widehat{C}_g(ih_\nu) + \tau\widehat{C}_g(ih_\nu + h_\nu) \geq c_g^{\nu,*} \geq c_g^{\nu,*} - \sigma h_\nu.\end{aligned}$$

In the remaining two cases when both $t_g^* - (TT^{\nu,i+1} + ih_\nu + h_\nu)$ and $t_g^* - (TT^{\nu,i} + ih_\nu)$ are nonnegative or both are nonpositive, it is easy to show that $\widehat{C}_g^{\,\nu}(t) = c_g^{\nu,*} \geq c_g^{\nu,*} - \sigma h_\nu$. This establishes our claim that $\widehat{C}_g^{\,\nu}(t) \geq c_g^{\nu,*} - \sigma\, h_\nu$ for all ν, $g \in \mathbb{G}$ and all $t \in [0, T]$. □

Notice that σ in the above lemma is independent of ν and t. We may pass to limit $\nu(\in \kappa) \to \infty$, yielding $\widehat{C}_g^{\,\infty}(t) \geq c_g^{\infty,*}$ for all $t \in [0, T]$. To complete the convergence analysis, and thus the proof of Theorem 1, it remains to verify the almost everywhere complementarity conditions. Clearly, it suffices to show the following two integral conditions, by the nonnegativity of the integrands:

$$\int_0^T \widehat{u}^{\,\infty}(t)\, \widehat{TT}^{\,\infty}(t)\, dt = 0,$$

$$\int_0^T \widehat{r}_g^{\,\infty}(t)(\widehat{C}_g^{\,\infty}(t) - c_g^{\infty,*})dt = 0, \quad \forall\, g \in \mathbb{G}.$$

Since $\left\{\widehat{TT}^{\,\nu}\right\}_{\nu\in\kappa}$ converges uniformly to $\widehat{TT}^{\,\infty}$ and $\widehat{u}^{\,\infty}$ is a weak limit of the sequence $\{\widehat{u}^{\,\nu}\}_{\nu\in\kappa}$, it follows that

$$\int_0^T \widehat{u}^{\,\infty}(t)\, \widehat{TT}^{\,\infty}(t)\, dt = \lim_{\nu(\in\kappa)\to\infty} \int_0^T \widehat{u}^{\,\nu}(t)\, \widehat{TT}^{\,\nu}(t)\, dt.$$

On the other hand, we have

$$\begin{aligned}&\int_0^T \widehat{u}^{\,\nu}(t)\, \widehat{TT}^{\,\nu}(t)\, dt \\ &\quad = \sum_{i=0}^{\nu-1} \int_{ih_\nu}^{(i+1)h_\nu} u^{\nu,i+1}\left[TT^{\nu,i} + \frac{t - i\,h_\nu}{h_\nu}(TT^{\nu,i+1} - TT^{\nu,i})\right] dt\end{aligned}$$

$$= \sum_{i=0}^{\nu-1} \int_{ih_\nu}^{(i+1)h_\nu} u^{\nu,i+1} \left[\left(\frac{t - i\,h_\nu}{h_\nu} - 1 \right) (TT^{\nu,i+1} - TT^{\nu,i}) + TT^{\nu,i+1} \right] dt$$

$$= \sum_{i=0}^{\nu-1} \int_{ih_\nu}^{(i+1)h_\nu} \frac{t - i\,h_\nu - h_\nu}{h_\nu}\, u^{\nu,i+1} (TT^{\nu,i+1} - TT^{\nu,i}) dt$$

$$= \frac{1}{2} \sum_{i=0}^{\nu-1} h_\nu\, u^{\nu,i+1} (TT^{\nu,i+1} - TT^{\nu,i}) = O(h_\nu),$$

where the last equality holds because $TT^{\nu,i+1} - TT^{\nu,i}$ is of order $O(h_\nu)$ and $u^{\nu,i+1}$ is uniformly bounded. Consequently,

$$\int_0^T \widehat{u}^\infty(t)\, \widehat{TT}^\infty(t)\, dt = 0.$$

In a similar fashion, we can establish that

$$\int_0^T \widehat{r}_g^\infty(t)(\widehat{C}_g^\infty(t) - c_g^{\infty,*}) dt = 0,$$

for all $g \in \mathbb{G}$. Therefore, the complementarity conditions are satisfied. This completes the proof of Theorem 1.

4 Concluding remarks

In this paper, we use a novel model paradigm, the Linear Complementarity System (LCS), to formulate the single bottleneck DUE problem and provide a rigorous analysis of the solution existence. Our constructive proof also provides a numerical scheme to compute a solution to the problem. Most importantly, the LCS formulation provides a rigorous mathematical foundation for the single bottleneck problem which is the simplest model to examine departure time decisions in dynamic traffic equilibrium. While this model is simple, it is a benchmark problem for transportation decision making and as such a rigorous understanding of this problem in addition to the foundational contributions to the dynamic equilibrium literature, will further the development of network-wide dynamic equilibrium problems. Given the nature of the DTA problems, the LCS is a promising methodology to formulate more complex network level models and obtain solution properties. Extensions of the analysis herein to more general DUE problems are under investigation.

References

1. Arnott, R., De Palma, A., Lindsey, R.: Schedule delay and departure time decisions with heterogeneous commuters. Trans. Res. Rec. **1197**, 56–67 (1988)
2. Arnott, R., De Palma, A., Lindsey, R.: A structural model of peak-period congestion: a traffic bottleneck with elastic demand. Am. Econ. Rev. **83**, 161–179 (1993)
3. Blumberg, M., Bar-Gera, H.: Consistent node arrival order in dynamic network loading models. Trans. Res. B Methodol. (in press)
4. Camlibel, M.K.: Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems. Ph.D. thesis, Center for Economic Research, Tilburg University, The Netherlands (2001)
5. Camlibel, M.K., Iannelli, L., Vasca, F.: Passivity and complementarity. Math. Program. Ser. A (in press)
6. Camlibel, M.K., Pang, J.S., Shen, J.: Lyapunov stability of complementarity and extended systems. SIAM J. Optim. **17**, 1056–1101 (2006)
7. Cottle, R.W., Pang, J.S., Stone, R.E.: The Linear Complementarity Problem. Academic Press, Cambridge (1992)
8. Daganzo, C.: The uniqueness of a time-dependent equilibrium distribution of arrivals at a single bottleneck. Trans. Sci. **19**, 29–37 (1985)
9. Han, L., Pang, J.S.: Non-zenoness of a class of differential quasi-variational inequalities. Math. Program. Ser. A **121**, 171–199 (2010)
10. Han, L., Tiwari, A., Camlibel, K., Pang, J.S.: Convergence of time-stepping schemes for passive and extended linear complementarity systems. SIAM J. Numer. Anal. **47**, 1974–1985 (2009)
11. Heemels, W.P.M.H.: Linear Complementarity Systems: A Study in Hybrid Dynamics. Ph.D. thesis, Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands (1999)
12. Heemels, W.P.M.H., Schumacher, J.M., Weiland, S.: Well-posedness of linear complementarity systems. In: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, Arizona, pp. 3037–3042 (1999)
13. Heemels, W.P.M.H., Schumacher, J.M., Weiland, S.: Linear complementarity systems. SIAM J. Appl. Math. **60**, 1234–1269 (2000)
14. Hendrickson, C., Kocur, G.: Schedule delay and departure time decisions in a deterministic model. Trans. Sci. **15**, 62–77 (1981)
15. Lang, S.: Real and Functional Analysis, 3rd edn. Springer, Berlin (1993)
16. Lindsey, R.: Existence, uniqueness, and trip cost function properties of user equilibrium in the bottleneck model with multiple user class. Trans. Sci. **38**, 293–314 (2004)
17. Newell, G.: The morning commute for nonidentical travelers. Trans. Sci. **21**, 74–88 (1987)
18. Pang, J.S., Shen, J.: Strongly regular differential variational systems. IEEE Trans. Autom. Control **52**, 242–255 (2007)
19. Pang, J.S., Stewart, D.E.: Differential variational inequalities. Math. Program. Ser. A **113**, 345–424 (2008)
20. Ramadurai, G., Ukkusuri, S., Zhao, J., Pang, J.S.: A linear complementarity formulation for a single bottleneck model with heterogeneous commuters. Trans. Rese. B Methodol. **44**(2), 193–214 (2010)
21. Schumacher, J.M.: Complementarity systems in optimization. Math. Program. Ser. B **101**, 263–296 (2004)
22. Shen, J., Pang, J.S.: Linear complementarity systems: Zeno states. SIAM J. Control Optim. **44**, 1040–1066 (2005)
23. Shen, J., Pang, J.S.: Semicopositive linear complementarity systems. Int. J. Robust Nonlinear Control **17**, 1367–1386 (2007)
24. Smith, M.J.: The existence of a time-dependent equilibrium distribution of arrivals at single bottleneck. Trans. Sci. **18**, 385–394 (1984)
25. Van Der Zijpp, N., Koolstra, K.: Multiclass continuous-time equilibrium model for departure time choice on single-bottleneck network. Trans. Res. Rec. **1783**, 134–141 (2002)
26. Vickrey, W.S.: Congestion theory and transport investment. Am. Econ. Rev. **59**, 251–261 (1969)
27. Vickrey, W.S.: Pricing, metering, and efficiently using urban transportation facilities. Highw. Res. Rec. **476**, 36–48 (1973)