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Maximal monotonicity criteria for the composition and the sum under weak interiority conditions

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Abstract The main goal of this article is to present several new results on the maximality of the composition and of the sum of maximal monotone operators in Banach spaces under weak interiority conditions involving their domains. Direct applications of our results to the structure of the range and domain of a maximal monotone operator are discussed. The last section of this note studies continuity properties of the duality product between a Banach space X and its dual X^* with respect to topologies compatible with the natural duality $(X \times X^*, X^* \times X)$.

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1 Introduction

A flourishing literature on the topic of calculus rules for maximal monotone operator has recently appeared, the trend being the use of convex analysis in the attempt of solving the celebrated Rockafellar Conjecture on the sum of maximal monotone operators under minimal constraints qualification conditions (see [1,2,16,18,21–25]).

The use of convex function representations of monotone operators, discovered by Fitzpatrick [5] and rediscovered by Martínez-Legaz–Théra [9] and Burachik–Svaiter

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[3], first showed its usefulness in the context of reflexive Banach spaces. This approach allowed that several results on maximal monotone operators be re-obtained such as those referring to the characterization of maximality (see [19]), the maximality of the sum of monotone operators or of the composition of a monotone operator with a continuous linear operator (see [11,12,20,27]), and new results to be found such as the result on the bounded Hausdorff convergence of the sum of maximal monotone operators (see [13]).

It was hoped that the use of convex representations for monotone operators would provide a solution of the Rockafellar Conjecture in the context of non-reflexive Banach spaces. This direction has already been exploited leading to additional conditions under which this conjecture holds (see [2,21–25]).

The objective of this note is to obtain general criteria for the maximality of the composition and of the sum of maximal monotone operators in general Banach spaces using a 1-dimensional subspace idea of Voisei (see [24, Theorem 5.9]) and based on a construction used by Penot and Zălinescu [12] under the reflexivity assumption and by Voisei and Zălinescu [25] under stronger representability conditions.

The plan of this paper is as follows. In Sect. 2, the main notions and results on maximal monotone operators and their convex representations in locally convex spaces are presented. The framework of this section was motivated by the facts that several results on monotone operators known in the context of Banach spaces hold in locally convex spaces and that, generally, the natural duality $(X \times X^*, X^* \times X)$ of a Banach space X comes from a non-Banach space topology, namely, when $X \times X^*$ is endowed with the non-barreled weak $\times$ weak-star topology. Section 3 contains our calculus rules for maximal monotone operators in the context of a Banach space under relative interiority conditions together with a comparison of our results with several recently published results on this topic. Section 4 deals with several equivalent conditions for the continuity of the coupling function of $X \times X^*$ with respect to topologies compatible to the natural duality $(X \times X^*, X^* \times X)$. For example the continuity of the coupling function with respect to the Mackey topology for the natural duality is equivalent to the reflexivity of X, while the continuity of the coupling function with respect to the product of the strong and bounded weak-star topologies translates into X is finite dimensional.

2 Main notions

Throughout this section, if not otherwise explicitly mentioned, (X, τ) is a separated locally convex space, X^* is its topological dual endowed with the weak-star topology w^*, and the topological dual of (X^*, w^*) is identified with X. The weak topology w on X is also considered. For $x \in X$ and $x^* \in X^*$ we set $\langle x, x^* \rangle := x^*(x)$.

For a subset A of X we denote by $\operatorname{int} A$, $\operatorname{cl} A$ (or $\operatorname{cl}_\tau A$ when we wish to emphasize on the topology τ), $\operatorname{aff} A$ and $\operatorname{conv} A$ the interior, the closure, the affine hull and the convex hull of A, respectively; moreover $\operatorname{core} A$ (or A^i) and ${}^iA$ are the algebraic interior and the relative algebraic interior (or intrinsic core) of A, while ${}^{ic}A := {}^iA$ if $\operatorname{aff} A$ is closed and ${}^{ic}A := \emptyset$ otherwise; in particular, if $\operatorname{core} A \neq \emptyset$ then ${}^{ic}A = \operatorname{core} A$. If $A, B \subset X$ we set $A + B := \{a + b \mid a \in A, \ b \in B\}$ with the convention $A + \emptyset := \emptyset + A := \emptyset$.

We consider the class $\Lambda(X)$ of proper convex functions $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and the class $\Gamma_\tau(X)$ (or simply $\Gamma(X)$) of those functions $f \in \Lambda(X)$ which are τ-lower semicontinuous (lsc for short). Recall that f is *proper* if $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ is nonempty and f does not take the value $-\infty$.

To $f : X \to \overline{\mathbb{R}}$ we associate its *convex hull* $\operatorname{conv} f : X \to \overline{\mathbb{R}}$ and its (τ-)*lsc convex hull* $\operatorname{cl}\operatorname{conv} f : X \to \overline{\mathbb{R}}$ ($\operatorname{cl}_\tau \operatorname{conv} f$ when we wish to put emphasis on the topology τ) defined by

$$(\operatorname{conv} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{conv}(\operatorname{epi} f)\},$$
$$(\operatorname{cl}\operatorname{conv} f)(x) := \inf\{t \in \mathbb{R} \mid (x, t) \in \operatorname{cl}\operatorname{conv}(\operatorname{epi} f)\},$$

where $\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}$ is the *epigraph* of f.

The *conjugate* of $f : X \to \overline{\mathbb{R}}$ with respect to the dual system (X, X^*) is given by

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(x^*) := \sup\{\langle x, x^*\rangle - f(x) \mid x \in X\}. \tag{1}$$

The conjugate f^* is a weakly-star (or w^*-) lsc convex function. For the proper function $f : X \to \overline{\mathbb{R}}$ we define the *subdifferential* of f at x by

$$\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^*\rangle \leq f(x') - f(x)\ \forall x' \in X\},$$

for $x \in \operatorname{dom} f$ and $\partial f(x) := \emptyset$ for $x \notin \operatorname{dom} f$. Recall that $N_C = \partial\iota_C$ is the *normal cone* of C, where ι_C is the *indicator function* of $C \subset X$ defined by $\iota_C(x) := 0$ for $x \in C$ and $\iota_C(x) := \infty$ for $x \in X \backslash C$.

When X^* is endowed with the topology w^* (or with any other locally convex topology σ such that $(X^*, \sigma)^* = X$), in other words, if we take conjugates for functions defined in X^* with respect to the dual system (X^*, X), then $f^{**} = (f^*)^* = \operatorname{cl}\operatorname{conv} f$ whenever $\operatorname{cl}\operatorname{conv} f$ (or equivalently f^*) is proper.

For $f, g : E \to \overline{\mathbb{R}}$ we set $[f \leq g] := \{x \in E \mid f(x) \leq g(x)\}$; the sets $[f = g]$, $[f < g]$, and $[f > g]$ are defined similarly.

Let $Z := X \times X^*$. Consider the coupling function

$$c : Z \to \mathbb{R}, \quad c(z) := \langle x, x^*\rangle \quad \text{for } z := (x, x^*) \in Z.$$

It is known that the topological dual of $(Z, \tau \times w^*)$ can be (and will be) identified with Z by the coupling

$$z \cdot z' := \langle z, z'\rangle := \langle x, x'^*\rangle + \langle x', x^*\rangle \quad \text{for } z = (x, x^*),\ \ z' = (x', x'^*) \in Z.$$

With respect to the natural dual system (Z, Z) induced by the previous coupling, the conjugate of $f : Z \to \overline{\mathbb{R}}$ is denoted by

$$f^\square : Z \to \overline{\mathbb{R}}, \quad f^\square(z) = \sup\{z \cdot z' - f(z') \mid z' \in Z\},$$

and $f^{\square\square} = \operatorname{cl}_{\tau\times w^*} \operatorname{conv} f$ whenever $f^\square$ (or $\operatorname{cl}_{\tau\times w^*} \operatorname{conv} f$) is proper.

Whenever X is a normed vector space, besides X^* we consider the bi-dual X^{**}. We identify X with the linear subspace $J(X)$ of X^{**}, where $J : X \to X^{**}$ is the canonical injection: $\langle x^*, Jx\rangle := \langle x, x^*\rangle$ for $x \in X$ and $x^* \in X^*$, and we denote Jx by $\widehat{x}$ or simply by x. In this case $Z = X \times X^*$ is seen as a normed vector space with the norm $\|z\| := \left(\|x\|^2 + \|x^*\|^2\right)^{1/2}$ for $z := (x, x^*) \in Z$. Its topological dual Z^* is identified with $X^* \times X^{**}$ by the coupling

$$\langle (x, x^*), (y^*, y^{**})\rangle := \langle x, y^*\rangle + \langle x^*, y^{**}\rangle, \quad (x, x^*) \in X \times X^*,\ (y^*, y^{**}) \in X^* \times X^{**}.$$

In this context, for $f : Z \to \overline{\mathbb{R}}$, the conjugate $f^* : Z^* \to \overline{\mathbb{R}}$ is given by (1) while for $f^{\square} : Z \to \overline{\mathbb{R}}$ one has

$$f^{\square}(x, x^*) = f^*(x^*, Jx) = f^*(x^*, x) \quad \forall (x, x^*) \in Z. \tag{2}$$

Note that $(f^{\square})^*(x^*, x) = f^{\square\square}(x, x^*) = f(x, x^*)$ when $f : X \times X^* \to \overline{\mathbb{R}}$ is a proper convex $\tau \times w^*$–lsc function.

We consider the following classes of functions:

$$\begin{aligned}
\mathcal{F} &:= \mathcal{F}(Z) := \{f \in \Lambda(Z) \mid f \geq c\},\\
\mathcal{R} &:= \mathcal{R}(Z) := \Gamma_{\tau \times w^*}(Z) \cap \mathcal{F}(Z),\\
\mathcal{D} &:= \mathcal{D}(Z) := \{f \in \mathcal{R}(Z) \mid f^{\square} \geq c\}.
\end{aligned}$$

If no confusion can occur, the multifunction (or operator) $T : X \rightrightarrows X^*$ will be identified with its graph

$$\operatorname{gph} T := \{(x, x^*) \in X \times X^* \mid x^* \in T(x)\} \subset Z;$$

as usual, the *domain* and the *image* of T are the sets

$$\operatorname{dom} T := \{x \in X \mid T(x) \neq \emptyset\} = \Pr_X(T), \quad \operatorname{Im} T := \bigcup_{x \in X} T(x) = \Pr_{X^*}(T),$$

where $\Pr_X$ and $\Pr_{X^*}$ are the projections of Z onto X and X^*, respectively. When $S : X \rightrightarrows X^*$, the multifunction $(S + T) : X \rightrightarrows X^*$ is defined by $(S + T)(x) := S(x) + T(x)$.

The multifunction $T : X \rightrightarrows X^*$ is said to be *monotone* if $c(z - z') \geq 0$ for all $z, z' \in T$ and *maximal monotone* if T is monotone and maximal in the sense of inclusion. In other terms, T is maximal monotone if T is monotone and any element $z \in Z$ which is *monotonically related to* (m.r.t. for short) T, that is, $c(z - z') \geq 0$ for every $z' \in T$, belongs to T. The classes of monotone and maximal monotone operators $T : X \rightrightarrows X^*$ are denoted by $\mathcal{M}(X)$ and $\mathfrak{M}(X)$, respectively. It is well known that

$$M_f := [f = c] = \{z \in Z \mid f(z) = c(z)\} \in \mathcal{M}(X) \quad \forall f \in \mathcal{F}$$

(see e.g. [11]). It was practically proved by Fitzpatrick [5, Theorem 2.4] that

$$f \in \mathcal{F} \Longrightarrow [f = c] \subset [f^{\square} = c], \tag{3}$$

from which, it follows that

$$[f = c] = [f^{\square} = c] \quad \forall f \in \mathcal{D}. \tag{4}$$

We say that T is *representable* if $T = M_f$ for some $f \in \mathcal{R}$; in this case f is called a *representative* of T. We denote by $\mathcal{R}_T$ the class of representatives of T. We say that T is *dual-representable* if $T = M_f$ for some $f \in \mathcal{D}$; in this case f is called a *d-representative* of T. We denote by $\mathcal{D}_T$ the class of d-representatives of T.

Whenever $T : X \rightrightarrows X^*$ is representable, $T(x)$ is w^*-closed and convex and $T^{-1}(x^*)$ is τ-closed and convex for all $x \in X$ and $x^* \in X^*$. Indeed, if $T = M_f$ for some $f \in \mathcal{R}$ then

$$T(x) = \{x^* \in X^* \mid f(x, x^*) \le c(x, x^*)\} = [f(x, \cdot) - c(x, \cdot) \le 0],$$

is w^*-closed and convex because $f(x, \cdot) - c(x, \cdot)$ is convex and w^*–lsc. The assertion about $T^{-1}(x^*)$ follows similarly.

We associate to $T : X \rightrightarrows X^*$ the functions $c_T, \psi_T, \varphi_T : Z \to \overline{\mathbb{R}}$ defined by

$$c_T := c + \iota_T, \quad \psi_T := \operatorname{cl}_{\tau \times w^*} \operatorname{conv} c_T, \quad \varphi_T := c_T^{\square} = \psi_T^{\square}.$$

The $\tau \times w^*$–lsc functions φ_T and ψ_T were first introduced in [5] and [21], respectively; φ_T is called the Fitzpatrick function of T and we call ψ_T the Penot function of T.

Notice that, as observed in [24, Remark 3.6], if $f \in \mathcal{D}_T$, that is, T is dual-representable with f a d-representative of T, then

$$\varphi_T \le f \le \psi_T, \quad \varphi_T \le f^{\square} \le \psi_T. \tag{5}$$

In fact,

$$[f \in \mathcal{R},\ A \subset [f = c]] \Longrightarrow \varphi_A \le f, f^{\square} \le \psi_A;$$

in particular,

$$f \in \mathcal{R} \Longrightarrow \varphi_{[f=c]} \le f, f^{\square} \le \psi_{[f=c]}. \tag{6}$$

Indeed, take $f \in \mathcal{R}$ and $A \subset [f = c]$. Hence $f \le c_A$, and so $f \le \psi_A$; it follows that $f^{\square} \ge \psi_A^{\square} = \varphi_A$. Taking (3) into account we have $A \subset [f^{\square} = c]$, whence $f^{\square} \le c_A$, $f^{\square} \le \psi_A$, and $f = f^{\square\square} \ge c_A^{\square} \ge \varphi_A$.

From the definition of φ_T one has (as observed in [23, Proposition 2])

$$T \subset [\operatorname{dom} T \times X^*] \cup [X \times \operatorname{Im} T] \subset [\varphi_T \ge c]. \tag{7}$$

Moreover, as observed in several places (see e.g. [11,23]), for $T \subset X \times X^*$ we have

$$T \in \mathcal{M}(X) \Longleftrightarrow \operatorname{conv} c_T \geq c \Longleftrightarrow T \subset [\varphi_T = c] \Longleftrightarrow T \subset [\varphi_T \leq c], \quad (8)$$

$$T \in \mathcal{M}(X) \Rightarrow T \subset [\psi_T = c] \subset [\varphi_T = c], \quad (9)$$

$$T \text{ representable} \Rightarrow T \text{ monotone}, \quad (10)$$

and

$$T \in \mathfrak{M}(X) \Longleftrightarrow T = [\varphi_T \leq c] \Longleftrightarrow [\varphi_T \in \mathcal{R} \text{ and } T = [\varphi_T = c]] \Longleftrightarrow \varphi_T \in \mathcal{R}_T. \quad (11)$$

Moreover,

$$T \in \mathcal{M}(X) \Longleftrightarrow \operatorname{conv} c_T \geq c \Longleftrightarrow \psi_T = \operatorname{cl}_{\tau \times w^*} \operatorname{conv} c_T \geq c \Longleftrightarrow \psi_T \geq \varphi_T. \quad (12)$$

Indeed, for $T = \emptyset$ one has $\psi_T = \operatorname{conv} c_T = \infty$, $\varphi_T = -\infty$, and so (12) holds. Let T be nonempty. Assume that T is monotone, i.e., for every $z, z' \in T$, $c(z - z') \geq 0$, or equivalently $c(z) \geq z \cdot z' - c(z')$. Taking the supremum with respect to $z' \in T$ we get $c(z) \geq \varphi_T(z)$ for every $z \in T$, that is $c_T \geq \varphi_T$ on Z. Since φ_T is $\tau \times w^*$–lsc convex this implies $\psi_T = \operatorname{cl}_{\tau \times w^*} \operatorname{conv} c_T \geq \varphi_T$ on Z. From the Fenchel inequality we have that $\psi_T(z) + \varphi_T(z) \geq z \cdot z = 2c(z)$ for $z \in Z$. Hence $\psi_T \geq c$ on Z whenever $\psi_T \geq \varphi_T$ on Z. Since $\psi_T \leq \operatorname{conv} c_T$, this yields $\operatorname{conv} c_T \geq c$ on Z if $\psi_T \geq c$. If $\operatorname{conv} c_T \geq c$, by (8), one gets $T \in \mathcal{M}(X)$.

From (11) and (12) we obtain that each maximal monotone operator $T : X \rightrightarrows X^*$ is dual-representable, a d-representative being φ_T (or ψ_T). Indeed, if $T \in \mathfrak{M}(X)$ we have $\psi_T = \varphi_T^\square \geq c$ (by (12)) and $\varphi_T = \psi_T^\square \geq c$ (by (11)), and so $\varphi_T, \psi_T \in \mathcal{D}_T$.

An operator $T \subset Z$ is called of *negative infimum type on* Z (NI for short) if $\varphi_T \geq c$ on Z. Let us note that this notion is different from the original definition given by S. Simons (see [16, Definiton 25.5] or [18, Definition 36.2]) in the sense that the original NI definition translates in our context as T is of NI type in Z^*.

The following result was established by Voisei [21, Theorems 2.2, 2.3] when X is a Banach space, but the same proof works for X a separated locally convex space. We give a proof for completeness.

Theorem 1 *Let $T \subset X \times X^*$. Then*

(i) *T is representable if and only if $T \in \mathcal{M}(X)$ and $T = [\psi_T = c]$, that is, $\psi_T \in \mathcal{R}_T$;*

(ii) *$T \in \mathfrak{M}(X)$ if and only if T is representable and T is of negative infimum type.*

Proof (i) The implication "⇐" is obvious from (12). Assume that $f \in \mathcal{R}$ is a representative of T. Then $T = [f = c] \in \mathcal{M}(X)$ and $c \leq f \leq \psi_T$. From (9) we get $T \subset [\psi_T = c] \subset [f = c] = T$, and so $T = [\psi_T = c]$.

(ii) The implication "⇒" follows from (11). Assume that T is representable and NI. Then $\varphi_T \geq c$ and $T = [\psi_T = c]$. By (10), (12) and (4) we have that $T = [\varphi_T = c]$ and the conclusion follows from (11). □

The simplest examples of NI operators are those with full domain or range. Indeed, if $\operatorname{dom} T = X$ or $\operatorname{Im} T = X^*$ then by (7) we get $\varphi_T \geq c$ on Z.

A commonly used method (which was first used in [21]) to prove that an operator T is of NI type is to show that whenever (x, x^*) is m.r.t. T then $x \in \operatorname{dom} T$ or $x^* \in \operatorname{Im} T$ or, in other words, $\Pr_X[\varphi_T \leq c] \subset \operatorname{dom} T$ or $\Pr_{X^*}[\varphi_T \leq c] \subset \operatorname{Im} T$ imply that T is NI. Here we use a similar argument for Theorem 4 below.

Theorem 2 *Let $T \subset X \times X^*$. Then $T \in \mathfrak{M}(X)$ if and only if every representative of T is a d-representative of T.*

Proof If T is maximal monotone and $f \in \mathcal{R}$ with $[f = c] = T$, from (6), (11) and (12) we have that $f \in \mathcal{D}_T$. Conversely, because, by Theorem 1(i), ψ_T is a representative of T, it becomes a d-representative of T which means that $\varphi_T \geq c$, that is, T is NI. The conclusion follows from Theorem 1(ii). □

It has been observed above that every maximal monotone operator is dual-representable with d-representatives φ_T or ψ_T. Also, it has been noted in several places that, in the context of reflexive spaces, the notions of maximal monotonicity and dual-representability coincide (see for instance [11], [12, Proposition 2.3]). It remains an open problem whether every dual-representable operator is maximal monotone in the non-reflexive Banach space settings.

3 Maximality criteria for the composition and the sum

First we recall the general construction used in [20,25]. For X, Y locally convex spaces and $F \subset X \times Y \times X^* \times Y^*$ we define

$$G := G(F) := \{(x, x^*) \in X \times X^* \mid \exists y^* \in Y^* : \ (x, 0, x^*, y^*) \in F\}.$$

The following lemma will be needed in the sequel; it was stated in [25, Lemma 13] and extends slightly [20, Lemma 5.3 (b)].

Lemma 1 *Let X, Y be locally convex spaces.*

(i) *If $F \in \mathcal{M}(X \times Y)$ and $Y_0 \subset Y$ is a closed linear subspace such that*

$$F(x, y) = F(x, y) + \{0\} \times Y_0^{\perp} \quad \forall (x, y) \in X \times Y, \tag{13}$$

then $\Pr_Y(\operatorname{dom} \varphi_F) \subset y + Y_0$ *for every* $y \in \Pr_Y(F)$.

(ii) *If $F \in \mathfrak{M}(X \times Y)$, then* $\Pr_Y(\operatorname{dom} \varphi_F) \subset \overline{\operatorname{aff}}(\Pr_Y(F))$.

We used the notation $A^{\perp} := \{x^* \in X^* \mid \langle x, x^* \rangle = 0 \ \forall x \in A\}$ (for $A \subset X$). Also, we use the notation $\operatorname{ri} A$ for the topological interior of A with respect to $\overline{\operatorname{aff}}\, A := \operatorname{cl}(\operatorname{aff} A)$; thus $\operatorname{ri} A$ is empty if $\operatorname{aff} A$ is not closed and one always has $\operatorname{ri} A \subset {}^{ic}A$. In the sequel, we use the facts that for C convex with ${}^{ic}C$ nonempty, we have $\operatorname{aff} C = \operatorname{aff}({}^{ic}C)$ and,

$${}^{ic}C \subset A \subset C \Longrightarrow [\operatorname{aff} C = \operatorname{aff} A \text{ and } {}^{ic}C = {}^{ic}A]. \tag{14}$$

The next result corresponds to Lemmas 3.2, 3.3 and Proposition 3.4 in [12]; its proof follows the line of the proofs of the mentioned results.

Theorem 3 *Let X, Y be Banach spaces and $F \in \mathcal{M}(X \times Y)$.*

(i) *If $f \in \mathcal{R}_F$, $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} f^{\square}))$ and $g : X \times X^* \to \overline{\mathbb{R}}$ is given by*

$$g(x, x^*) := \inf\{f^{\square}(x, 0, x^*, y^*) \mid y^* \in Y^*\}, \ (x, x^*) \in X \times X^*, \quad (15)$$

then

$$g^{\square}(u, u^*) = \min\{f(u, 0, u^*, v^*) \mid v^* \in Y^*\} \ \ \forall (u, u^*) \in X \times X^*, \quad (16)$$

and $g^{\square}$ is a representative of $G(F)$.

(ii) *If F is dual-representable and $f \in \mathcal{D}_F$ is such that $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} f))$ and $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} f^{\square}))$, then $G(F)$ is dual-representable and g defined by (15) is a d-representative of $G(F)$.*

(iii) *If $F \in \mathfrak{M}(X \times Y)$ is such that $0 \in {}^{ic}(\operatorname{conv}(\Pr_Y(F)))$ and $f \in \mathcal{R}_F$, then $G(F)$ is dual-representable and g defined by (15) is a d-representative of $G(F)$.*

(iv) *If X is reflexive, $F \in \mathfrak{M}(X \times Y)$ and $f \in \mathcal{R}_F$ then*

$$\begin{aligned} {}^{ic}(\Pr_Y(\operatorname{dom} f)) &= {}^{ic}(\Pr_Y(\operatorname{dom} \varphi_F)) = {}^{ic}(\operatorname{conv}(\Pr_Y(F))) \\ &= {}^{ic}(\Pr_Y(F)) = \operatorname{ri}(\Pr_Y(F)). \end{aligned} \quad (17)$$

Furthermore, if $0 \in {}^{ic}(\Pr_Y(F))$, or equivalently $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} f))$, then $G(F) \in \mathfrak{M}(X)$.

Proof (i) The proof is the same as that of [12, Lemma 3.2]; just observe that this time the graph of $\mathcal{C} : X \times X^* \rightrightarrows X \times Y \times X^* \times X^*$ given by

$$\mathcal{C}(x, x^*) := \{x\} \times \{0\} \times \{x^*\} \times Y^*, \quad (x, x^*) \in X \times X^*,$$

is a closed linear subspace and $\mathcal{C}^*(x^*, y^*, x^{**}, y^{**}) = \{(x^*, x^{**})\}$ if $y^{**} = 0$, $\mathcal{C}^*(x^*, y^*, x^{**}, y^{**}) = \emptyset$ otherwise.

Notice that $g(x, x^*) = \inf\{f^{\square}(u, v, u^*, v^*) \mid (u, v, u^*, v^*) \in \mathcal{C}(x, x^*)\}$ for $(x, x^*) \in X \times X^*$ and

$$\operatorname{dom} f^{\square} - \operatorname{Im} \mathcal{C} = X \times \Pr_Y(\operatorname{dom} f^{\square}) \times X^* \times Y^*,$$

from which $0 \in {}^{ic}(\operatorname{dom} f^{\square} - \operatorname{Im} \mathcal{C})$; by the fundamental duality formula (see e.g. [26, Theorem 2.8.6(v)]) we get

$$g^*(u^*, u^{**}) = \min\{(f^{\square})^*(u^*, v^*, u^{**}, 0) \mid v^* \in Y^*)\} \ \ \forall (u^*, u^{**}) \in X^* \times X^{**}.$$

For $u^{**} = u \in X$ and $u^* \in X^*$ we get (see (2))

$$\begin{aligned} g^{\square}(u, u^*) &= g^*(u^*, u) = \min\{(f^{\square})^*(u^*, v^*, u, 0) \mid v^* \in Y^*\} \\ &= \min\{f^{\square\square}(u, 0, u^*, v^*) \mid v^* \in Y^*\} = \min\{f(u, 0, u^*, v^*) \mid v^* \in Y^*\}, \end{aligned}$$

which makes $g^{\square}$ a representative of G since f is a representative of F.

(ii) The conclusion follows from (i) applied to $f^{\square}$.
(iii) Because $f \in \mathcal{R}_F$, by Theorem 2 we obtain that f is a d-representative of F,

and so, by (5), we have that $\varphi_F \le f \le \psi_F$. It follows that

$$F \subset \operatorname{conv} F \subset \operatorname{dom} \psi_F \subset \operatorname{dom} f \subset \operatorname{dom} \varphi_F,$$

whence

$$\begin{aligned} \Pr_Y(F) &\subset \Pr_Y(\operatorname{conv} F) = \operatorname{conv}(\Pr_Y(F)) \\ &\subset \Pr_Y(\operatorname{dom} \psi_F) \subset \Pr_Y(\operatorname{dom} f) \subset \Pr_Y(\operatorname{dom} \varphi_F). \end{aligned} \tag{18}$$

This yields

$$\begin{aligned} \operatorname{aff}(\Pr_Y(F)) &= \operatorname{aff}(\Pr_Y(\operatorname{conv} F)) \subset \operatorname{aff}(\Pr_Y(\operatorname{dom} \psi_F)) \\ &\subset \operatorname{aff}(\Pr_Y(\operatorname{dom} f)) \subset \operatorname{aff}(\Pr_Y(\operatorname{dom} \varphi_F)) \subset \overline{\operatorname{aff}}(\Pr_Y(F)), \end{aligned} \tag{19}$$

the last inclusion being obtained using Lemma 1(ii).

Because $0 \in {}^{ic}(\Pr_Y(\operatorname{conv} F))$, $\operatorname{aff}(\Pr_Y(\operatorname{conv} F)) = \operatorname{aff}(\Pr_Y(F))$ is closed, and so all inclusions in (19) become equalities. Hence

$$\begin{aligned} {}^{ic}(\Pr_Y(F)) &\subset {}^{ic}(\Pr_Y(\operatorname{conv} F)) \subset {}^{ic}(\Pr_Y(\operatorname{dom} \psi_F)) \\ &\subset {}^{ic}(\Pr_Y(\operatorname{dom} f)) \subset {}^{ic}(\Pr_Y(\operatorname{dom} \varphi_F)). \end{aligned} \tag{20}$$

Because $f^{\square}$ is also a representative of F, (20) also holds for f replaced by $f^{\square}$. Hence $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} f))$ and $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} f^{\square}))$. The conclusion follows using (ii).

(iv) As in (iii), (18) and (19) hold. Let us prove that

$$ {}^{ic}(\Pr_Y(\operatorname{dom} \varphi_F)) \subset \Pr_Y(F). \tag{21}$$

Observe first that if $0 \in {}^{ic}(\Pr_Y(\operatorname{dom} \varphi_F))$, then from (i) applied for $f = \psi_F$ we obtain that $G(F)$ is dual-representable in the reflexive space X, with a d-representative given by

$$g^{\square}(u, u^*) = \min\{\psi_F(u, 0, u^*, v^*) \mid v^* \in Y^*\}, \quad (u, u^*) \in X \times X^*,$$

since $g \geq c$ and $g^{\square\square} = \overline{g} \geq c$. Therefore, by [12, Proposition 2.3], $G(F)$ is maximal monotone. Hence $G(F)$ is non-empty, and so $0 \in \Pr_Y(F)$. Similarly, if $y \in {}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F))$ then $0 \in {}^{ic}(\Pr_Y(\operatorname{dom}\varphi_{F'}))$, where $F' := F - (0, y, 0, 0)$ is maximal monotone. By the previous case it follows that $0 \in \Pr_Y(F')$, that is, $y \in \Pr_Y(F)$. Hence (21) holds.

Assume now that ${}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F)) \neq \emptyset$. Then, by (21),

$$\operatorname{aff}(\Pr_Y(\operatorname{dom}\varphi_F)) = \operatorname{aff}({}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F))) \subset \operatorname{aff}(\Pr_Y(F)).$$

Since $\operatorname{aff}(\Pr_Y(\operatorname{dom}\varphi_F))$ is closed, we obtain again that all inclusions in (19) are in fact equalities. Taking into account (18) and (21) we obtain that the first three equalities in (17) hold; the last one follows from the fact that $\operatorname{ri}(\Pr_Y(\operatorname{dom}\varphi_F)) = {}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F))$ (see [26, Proposition 2.7.2]).

To complete the proof it is sufficient to observe that when ${}^{ic}(\Pr_Y(F))$, or ${}^{ic}(\Pr_Y(\operatorname{conv} F))$, or ${}^{ic}(\Pr_Y(\operatorname{dom} f))$ is nonempty then ${}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F))$ is nonempty.

In the first two cases we obtain that $\operatorname{aff}(\Pr_Y(F))$ is closed, and so, as seen above, (20) holds; hence ${}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F))$ is nonempty. Assume now that the set ${}^{ic}(\Pr_Y(\operatorname{dom} f))$ is nonempty. Then $\operatorname{aff}(\Pr_Y(\operatorname{dom} f))$ is closed. From (19) we obtain that $\operatorname{aff}(\Pr_Y(\operatorname{dom} f)) = \operatorname{aff}(\Pr_Y(\operatorname{dom}\varphi_F))$ is closed, and so, from (18), we get ${}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F)) \neq \emptyset$. As seen above ${}^{ic}(\Pr_Y(\operatorname{dom}\varphi_F)) \neq \emptyset$ implies that $G(F)$ is maximal monotone. □

For $F : X \times Y \rightrightarrows X^* \times Y^*$ and $A : X \to Y$ a continuous linear operator, we consider $F_A : X \times Y \rightrightarrows X^* \times Y^*$ defined by

$$\operatorname{gph} F_A := \{(x, y, x^*, y^*) \in X \times Y \times X^* \times Y^* \mid (x^* - A^\top y^*, y^*) \in F(x, Ax + y)\},$$

where $A^\top : Y^* \to X^*$ is the adjoint of A, or $F_A(x, y) = B^\top F B(x, y)$ with $B(x, y) := (x, y + Ax)$ for $(x, y) \in X \times Y$.

Since $B : X \times Y \to X \times Y$ is an isomorphism of normed vector spaces (with $B^\top(x^*, y^*) = (x^* + A^\top y^*, y^*)$), if F is dual-representable, (maximal) monotone then F_A is dual-representable, (maximal) monotone. Moreover, if f is a (d-) representative of F then $f_A := f \circ L$ is a (d-) representative of F_A, where $L := B \times (B^{-1})^\top$. Using the previous result for F_A we get the next two consequences; note that we need the reflexivity of X in order to apply Theorem 3(iv).

Corollary 1 *Assume that X, Y are Banach spaces with X reflexive, $F \in \mathfrak{M}(X \times Y)$, $A \in L(X, Y)$ and $f \in \mathcal{R}_F$. Then*

$$\begin{aligned} {}^{ic}\{y - Ax \mid (x, y) \in \operatorname{dom} F\} &= {}^{ic}\{y - Ax \mid (x, y) \in \operatorname{conv}(\operatorname{dom} F)\} \\ &= {}^{ic}\{y - Ax \mid (x, y) \in \Pr_{X\times Y}(\operatorname{dom} f)\} \\ &= \operatorname{ri}(\{y - Ax \mid (x, y) \in \operatorname{dom} F\}). \end{aligned}$$

Assume that $0 \in {}^{ic}\{y - Ax \mid (x, y) \in \Pr_{X\times Y}(\operatorname{dom} f)\}$ *(or equivalently* $0 \in {}^{ic}\{y - Ax \mid (x, y) \in \operatorname{dom} F\}$*). Then the multifunction $G(F_A)$ whose graph is* $\{(x, x^*) \in$

$X \times X^* \mid \exists y^* \in Y^* : (x^* - A^\top y^*, y^*) \in F(x, Ax)\}$ is maximal monotone with a representative $g : X \times X^* \to \overline{\mathbb{R}}$ given by

$$g(x, x^*) = \min\{f(x, Ax, x^* - A^\top y^*, y^*) \mid y^* \in Y^*\}, (x, x^*) \in X \times X^*.$$

Corollary 2 *Assume that X, Y are Banach spaces with X reflexive, $M \in \mathfrak{M}(X)$ with representative f, $N \in \mathfrak{M}(Y)$ with representative g, and $A \in L(X, Y)$. Then*

$$\begin{aligned}{}^{ic}(\operatorname{dom} N - A(\operatorname{dom} M)) &= {}^{ic}(\operatorname{conv}(\operatorname{dom} N - A(\operatorname{dom} M))) \\ &= {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} g) - A(\operatorname{Pr}_X(\operatorname{dom} f))) \\ &= \operatorname{ri}(\operatorname{dom} N - A(\operatorname{dom} M)).\end{aligned}$$

If, in addition, $0 \in {}^{ic}(\operatorname{dom} N - A(\operatorname{dom} M))$ then $M + A^\top N A$ is maximal monotone with a representative given by

$$h : X \times X^* \to \overline{\mathbb{R}}, \ h(x, x^*) := \inf\{f(x, x^* - A^\top y^*) + g(Ax, y^*) \mid y^* \in Y^*\}, \quad (22)$$

and the infimum in the expression of h is attained.

The maximal monotonicity of $A^\top N A$ when X, Y are Banach spaces with X reflexive and $0 \in \operatorname{core}(\operatorname{Im} A + \operatorname{conv}(\operatorname{dom} N))$ is obtained in [1, Theorem 5.5] (see also [11, Theorem 14] and [27, Theorem 7]).

Taking X an arbitrary reflexive space, gph $M = \{0\} \times X^*$ and $A = 0$ in Corollary 2 we get the next result that covers [17, Theorem 2.2] which states that $\operatorname{int}(\operatorname{dom} N) = \operatorname{int}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_N))$ and [14, Theorem 1] which states that $\operatorname{cl}(\operatorname{dom} N)$ is convex when the interior of $\operatorname{conv}(\operatorname{dom} N)$ is nonempty. It is known that for an operator which is maximal monotone locally the closure of its domain is convex (see [16, Theorem 26.3] or [18, Theorem 44.2]); in fact we shall prove in Corollary 5 below that $N \in \mathfrak{M}(Y)$ is maximal monotone locally whenever ${}^{ic}(\operatorname{dom} N) \neq \emptyset$.

Corollary 3 *Let Y be a Banach space and $N \in \mathfrak{M}(Y)$. If g is a representative of N then*

$$\operatorname{ri}(\operatorname{dom} N) = {}^{ic}(\operatorname{dom} N) = {}^{ic}(\operatorname{conv}(\operatorname{dom} N)) = {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} g)). \quad (23)$$

In particular ${}^{ic}(\operatorname{dom} N)$ is convex; moreover, if ${}^{ic}(\operatorname{dom} N)$ is nonempty then $\operatorname{cl}(\operatorname{dom} N)$ is convex, too.

Proof As mentioned above, (23) follows from Corollary 2 taking gph $M = \{0\} \times X^*$ and $A = 0$. From (23) it is clear that ${}^{ic}(\operatorname{dom} N)$ is convex. Assume that ${}^{ic}(\operatorname{dom} N) \neq \emptyset$. Using again (23) we get

$$\emptyset \neq {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} g)) \subset \operatorname{dom} N \subset \operatorname{Pr}_Y(\operatorname{dom} g).$$

Hence $\operatorname{cl}(\operatorname{dom} N) = \operatorname{cl}(\operatorname{Pr}_Y(\operatorname{dom} g))$, and so $\operatorname{cl}(\operatorname{dom} N)$ is convex. □

Remark 1 When X, Y are Banach spaces, $A : X \to Y$ is a continuous linear operator, and f, g are representatives for the operators $M \in \mathfrak{M}(X)$ and $N \in \mathfrak{M}(Y)$ with $0 \in {}^{ic}(\operatorname{conv}(\operatorname{dom} N - A(\operatorname{dom} M)))$, applying Theorem 3(iii) for F_A we obtain that h defined by (22) has

$$h^{\square}(x,x^*) = \min\left\{f^{\square}(x, x^* - A^{\top}y^*) + g^{\square}(Ax, y^*) \mid y^* \in Y^*\right\}, \quad \forall (x,x^*) \in X \times X^*,$$

and $M + A^{\top}NA$ admits $h^{\square}$ as a d-representative, in particular, $M + A^{\top}NA$ is dual-representable and has w^*-closed values.

When applied for the sum, that is, $X = Y$ and $A = \mathrm{Id}_X$, Corollary 2 reduces to a known criterion for the maximality of the sum in reflexive spaces. However, using an idea of Voisei [24], we can obtain criteria for the maximality of the composition and sum in general Banach spaces.

Theorem 4 *Let X, Y be Banach spaces, let $A : X \to Y$ be a continuous linear operator and $N \in \mathfrak{M}(Y)$. Assume that $0 \in {}^{ic}(\operatorname{Im} A - \operatorname{conv}(\operatorname{dom} N))$ and ${}^{ic}(\operatorname{conv}(\operatorname{dom} N)) \neq \emptyset$ (or, equivalently, $0 \in {}^{ic}(\operatorname{Im} A - \operatorname{Pr}_Y(\operatorname{dom} \varphi_N))$ and ${}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_N)) \neq \emptyset$). Then $A^{\top}NA \in \mathfrak{M}(X)$. In particular the conclusion holds if $\operatorname{Im} A \cap \operatorname{core}(\operatorname{conv}(\operatorname{dom} N)) \neq \emptyset$.*

Proof First let us recall that for convex subsets C, D of a real linear space we have that

$${}^{i}C \neq \emptyset,\ {}^{i}D \neq \emptyset \Longrightarrow {}^{i}(C - D) = {}^{i}C - {}^{i}D$$

(for a complete proof see [28, Lemma 2(iii)]).

Without loss of generality we assume that $0 \in {}^{ic}(\operatorname{dom} N) = {}^{ic}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_N))$ (see Corollary 3 for the last equality). Otherwise, since ${}^{ic}(\operatorname{conv}(\operatorname{dom} N)) = {}^{ic}(\operatorname{dom} N) = {}^{i}(\operatorname{dom} N) \neq \emptyset$, ${}^{i}(\operatorname{Im} A) = \operatorname{Im} A$ and

$$\begin{aligned} 0 \in {}^{ic}(\operatorname{Im} A - \operatorname{conv}(\operatorname{dom} N)) &= {}^{i}(\operatorname{Im} A - \operatorname{conv}(\operatorname{dom} N)) \\ &= \operatorname{Im} A - {}^{i}(\operatorname{dom} N) = \operatorname{Im} A - {}^{ic}(\operatorname{dom} N), \end{aligned}$$

we can take $\overline{x} \in X$ such that $A\overline{x} \in {}^{ic}(\operatorname{dom} N) = {}^{ic}(\operatorname{conv}(\operatorname{dom} N))$; then, for $\operatorname{gph} N' := \operatorname{gph} N - (A\overline{x}, 0)$, we have $0 \in {}^{ic}(\operatorname{dom} N')$ and $\operatorname{gph}(A^{\top}N'A) = \operatorname{gph}(A^{\top}NA) - (\overline{x}, 0)$.

By Remark 1 we have that $S := A^{\top}NA$ is representable. By Theorem 1(ii) (or [21, Theorem 2.3]), it is sufficient to show that S is NI, that is, $\varphi_S \geq c$. Let $z_0 := (x_0, x_0^*) \in X \times X^*$ be such that $\varphi_S(z_0) \leq c(z_0)$. If $y_0 := Ax_0 = 0$, then $x_0 \in \operatorname{dom} S$, and so $\varphi_S(z_0) \geq c(z_0)$ by (7) (or [23, Proposition 2]). Let $y_0 \neq 0$ and take $T : \mathbb{R} \to Y$, $T(t) := ty_0$ for $t \in \mathbb{R}$. Then $T^{\top}(y^*) = \langle y_0, y^* \rangle$, for every $y^* \in Y^*$. As noticed above, ${}^{i}(\operatorname{Im} T - \operatorname{conv}(\operatorname{dom} N)) = \operatorname{Im} T - {}^{i}(\operatorname{conv}(\operatorname{dom} N))$. Since $\operatorname{Im} T$ is finite-dimensional and $\operatorname{aff}(\operatorname{conv}(\operatorname{dom} N))$ $(= \operatorname{aff}(\operatorname{dom} N))$ is a closed linear subspace of Y, it follows that $\operatorname{aff}(\operatorname{Im} T - \operatorname{conv}(\operatorname{dom} N))$ $(= \operatorname{Im} T - \operatorname{aff}(\operatorname{dom} N))$ is closed, and so $0 \in {}^{ic}(\operatorname{conv}(\operatorname{dom} N) - \operatorname{Im} T)$. By Corollary 2 with $M = 0$ we obtain that $T^{\top}NT$ is maximal monotone.

Consider $y_0^* \in Y^*$ such that $t_0^* := \langle x_0, x_0^* \rangle = \langle y_0, y_0^* \rangle$. Let $(t, t^*) \in T^\top NT$, that is $t^* = \langle y_0, y^* \rangle$ for some $y^* \in N(ty_0) = N(A(tx_0))$. Because z_0 is monotonically related to S, $A^\top y^* \in S(tx_0)$, and $\langle x_0, A^\top y_0^* - x_0^* \rangle = 0$, we have

$$\begin{aligned}(1-t)(t_0^* - t^*) &= \langle y_0 - ty_0, y_0^* - y^* \rangle = \langle x_0 - tx_0, A^\top y_0^* - A^\top y^* \rangle \\ &= \langle x_0 - tx_0, x_0^* - A^\top y^* \rangle \geq 0.\end{aligned}$$

Hence $(1, t_0^*)$ is monotonically related to $T^\top NT$, and so $(1, t_0^*) \in T^\top NT$. Therefore, $1 \in \operatorname{dom}(T^\top NT)$, that is, $y_0 \in \operatorname{dom} N$, or, equivalently, $x_0 \in \operatorname{dom} S$. Again, by (7) (or [23, Proposition 2]), $\varphi_S(z_0) \geq c(z_0)$. Hence $\varphi_S \geq c$, and so S is maximal monotone. □

Corollary 4 *Let X be a Banach space and $M, N \in \mathfrak{M}(X)$. If* ${}^{ic}(\operatorname{dom} M)$, ${}^{ic}(\operatorname{dom} N)$ *are nonempty and*

$$0 \in {}^{ic}(\operatorname{dom} M - \operatorname{dom} N), \tag{24}$$

then $M + N$ is maximal monotone. In particular, if $\operatorname{core}(\operatorname{dom} M) \cap \operatorname{core}(\operatorname{dom} N) \neq \emptyset$ *then $M + N$ is maximal monotone.*

Proof Apply the preceding theorem for A replaced by the linear operator $X \ni x \to (x, x) \in X \times X$ and for N replaced by $X \times X \ni (x, x') \rightrightarrows M(x) \times N(x') \subset X^* \times X^*$. □

By Corollary 3 condition $\operatorname{core}(\operatorname{dom} M) \cap \operatorname{core}(\operatorname{dom} N) \neq \emptyset$ is equivalent to each one of the following: $\operatorname{core}(\operatorname{conv}(\operatorname{dom} M)) \cap \operatorname{core}(\operatorname{conv}(\operatorname{dom} N)) \neq \emptyset$, $\operatorname{int}(\operatorname{dom} M) \cap \operatorname{int}(\operatorname{dom} N) \neq \emptyset$, $\operatorname{core}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_M)) \cap \operatorname{core}(\operatorname{Pr}_Y(\operatorname{dom} \varphi_N)) \neq \emptyset$.

The most general results for the maximality of $M + N$ when $M, N \in \mathfrak{M}(X)$ with X a reflexive Banach space states that the interiority condition (24) implies the maximality of $M + N$. Apparently, this condition is not sufficient when X is non-reflexive, leading to additional conditions to be considered.

Assume that $M, N \in \mathfrak{M}(X)$ and X is a Banach space. We list the main additional conditions in the literature under which the Rockafellar Conjecture holds in connection and in comparison to our interiority conditions.

In [23, Theorem 2] one has: if (24) is verified and $\operatorname{dom} M$, $\operatorname{dom} N$ are closed and convex then $M + N \in \mathfrak{M}(X)$. This results extends [21, Theorem 1.1] in which condition (24) is replaced by $0 \in \operatorname{core}(\operatorname{dom} M - \operatorname{dom} N)$. In [24, Theorem 5.13($\beta$)] one has: if (24) is verified and $\operatorname{gph} M$, $\operatorname{gph} N$ are convex then $M + N \in \mathfrak{M}(X)$. This result extends [22, Theorem 1] in which the condition $\operatorname{gph} M$, $\operatorname{gph} N$ are convex is replaced by $\operatorname{gph} M$, $\operatorname{gph} N$ are linear spaces; of course, in this case (24) is equivalent to $\operatorname{dom} M - \operatorname{dom} N$ is closed.

In [2, Theorem 10] one has $\operatorname{int}(\operatorname{dom} M) \cap \operatorname{int}(\operatorname{dom} N) \neq \emptyset$ while in [24, Theorem 5.13(γ)] one has $\operatorname{core}(\operatorname{dom} M) \cap \operatorname{core}(\operatorname{dom} N) \neq \emptyset$. These are clearly particular cases of Corollary 4.

Therefore our additional interiority conditions improve upon all known interiority conditions needed for the chain and the sum rule for maximal monotone operators to hold.

Recall that an operator $M : X \rightrightarrows X^*$ is called *maximal monotone locally (or of type (FPV))* if for every open convex set $U \subset X$ with $U \cap \operatorname{dom} M \neq \emptyset$ and every $(x, x^*) \in (U \times X^*) \setminus \operatorname{gph} M$ there exists $(u, u^*) \in \operatorname{gph} M \cap (U \times X^*) =: \operatorname{gph} M_U$ such that $\langle x - u, x^* - u^* \rangle < 0$, or equivalently

$$[\varphi_{M_U} \leq c] \cap (U \times X^*) \subset M. \tag{25}$$

The convexity of $\operatorname{cl}_s(\operatorname{Im} M)$ for $M \in \mathfrak{M}(X)$ is not necessarily fulfilled even under strong additional conditions (such as coercivity, full-space domain, etc.; see e.g. [7]). However, the additional interiority condition ${}^{ic}(\operatorname{conv}(\operatorname{dom} M)) \neq \emptyset$, guarantees the convexity of $\operatorname{cl}_{w^*}(\operatorname{Im} M)$ as we shall see in our next result. The following corollary improves upon the result [6, Proposition 1.4] which was stated for operators $M \in \mathfrak{M}(X)$ with $\operatorname{dom} M = X$.

Corollary 5 *Let X be a Banach space and $M \in \mathfrak{M}(X)$ with ${}^{ic}(\operatorname{conv}(\operatorname{dom} M)) \neq \emptyset$. Then M is maximal monotone locally and $\operatorname{cl}_{w^*}(\operatorname{Im} M)$ is convex.*

Proof The condition ${}^{ic}(\operatorname{conv}(\operatorname{dom} M)) \neq \emptyset$ allows the use of Corollary 4 for M and N_C with $C \subset X$ a closed convex set such that $M \cap \operatorname{int} C \neq \emptyset$ to get that $M + N_C$ is maximal monotone; then from [16, Theorem 26.1] or [18, Theorem 44.1] we find that M is maximal monotone locally.

We give an alternative proof for getting M maximal monotone locally. Let $U \subset X$ be an open convex set such that $U \cap \operatorname{dom} M \neq \emptyset$. Then for $K \subset U$ a nonempty closed convex set we have

$$\varphi_{M+N_K}(x, x^*) \leq \varphi_{M_U}(x, x^*) \quad \forall (x, x^*) \in K \times X^*. \tag{26}$$

Indeed, since for $x, u \in K$ and $v^* \in N_K(u)$ we have $\langle x - u, v^* \rangle \leq 0$, we get

$$\begin{aligned}
\varphi_{M+N_K}(x, x^*) &= \sup\{\langle x, u^* + v^* \rangle + \langle u, x^* \rangle - \langle u, u^* + v^* \rangle \mid u \in K \cap \operatorname{dom} M, \\
&\qquad u^* \in M(u),\ v^* \in N_K(u)\} \\
&= \sup\{\langle x, u^* \rangle + \langle u, x^* \rangle - \langle u, u^* \rangle + \langle x - u, v^* \rangle \mid u \in K \cap \operatorname{dom} M, \\
&\qquad u^* \in M(u),\ v^* \in N_K(u)\} \\
&\leq \sup\{\langle x, u^* \rangle + \langle u, x^* \rangle - \langle u, u^* \rangle \mid u \in K \cap \operatorname{dom} M,\ u^* \in M(u)\} \\
&\leq \sup\{\langle x, u^* \rangle + \langle u, x^* \rangle - \langle u, u^* \rangle \mid u \in U \cap \operatorname{dom} M,\ u^* \in M(u)\} \\
&= \varphi_{M_U}(x, x^*).
\end{aligned}$$

Let $(\overline{x}, \overline{x}^*) \in [\varphi_{M_U} \leq c] \cap (U \times X^*)$. Take a closed convex $K \subset U$ whose interior contains $\overline{x}$ and intersects $\operatorname{dom} M$. We know from Corollary 4 that $M + N_K$ is maximal monotone and from (26) we find

$$(\overline{x}, \overline{x}^*) \in [\varphi_{M_U} \leq c] \cap (K \times X^*) \subset [\varphi_{M+N_K} \leq c] = M + N_K.$$

Since $N_K(\overline{x}) = \{0\}$ this yields $(\overline{x}, \overline{x}^*) \in M$. Therefore (25) holds and M is maximal monotone locally.

To prove that $\mathrm{cl}_{w^*}(\mathrm{Im}\, M)$ is convex it suffices to show that $\mathrm{conv}(\mathrm{Im}\, M) \subset \mathrm{cl}_{w^*}(\mathrm{Im}\, M)$. To this end we adapt the proof in [6, Proposition 1.4]. Assume, by contradiction, that $\mathrm{conv}(\mathrm{Im}\, M)\setminus \mathrm{cl}_{w^*}(\mathrm{Im}\, M) \neq \emptyset$. Without loss of generality (doing a translation if necessary), we may assume that $0 \in \mathrm{conv}(\mathrm{Im}\, M)\setminus \mathrm{cl}_{w^*}(\mathrm{Im}\, M)$, that is,

$$0 = \sum_{i=1}^{n} t_i u_i^*, \quad u_i^* \in M(u_i), \ t_i \geq 0 \text{ for } 1 \leq i \leq n, \ \sum_{i=1}^{n} t_i = 1, \tag{27}$$

and there is $\varepsilon > 0$ and $x_1, x_2, \ldots, x_m \in X$ such that the weak * neighborhood of zero

$$V := V_{x_1,x_2,\ldots,x_m;\varepsilon} := \{x^* \in X^* \mid |\langle x_j, x^*\rangle| < \varepsilon \ \forall j, \ 1 \leq j \leq m\} \subset X^*$$

does not intersect $\mathrm{Im}\, M$. Without any loss of generality we assume that $x_1 \in {}^{ic}(\mathrm{dom}\, M)$.

Let $F = \mathrm{span}\{u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_m\}$ and $A : F \to X$ defined by $A(x) := x$ for $x \in F$.

Because $F \cap {}^{ic}(\mathrm{dom}\, M) \neq \emptyset$ and $F = \mathrm{Im}\, A$ is finite dimensional, as in the proof of Theorem 4, we obtain that $0 \in {}^{ic}(\mathrm{Im}\, A - \mathrm{dom}\, M)$. Therefore, by the same Theorem 4, $M_F = A^\top M A : F \rightrightarrows F^*$ is maximal monotone; recall that $F^* = \{x^*|_F \mid x^* \in X^*\}$. Since $(M_F)^{-1}$ is maximal monotone and F^* is finite dimensional (hence reflexive), by the first part we have that $\mathrm{cl}_{F^*}(\mathrm{Im}\, M_F) = \mathrm{cl}_{F^*}(\mathrm{dom}(M_F)^{-1})$ is convex. (Here cl_{F^*} stands for the closure of subsets in F^*.)

Using (27) we get $0 \in \mathrm{conv}(\mathrm{Im}\, M_F) \subset \mathrm{cl}_{F^*}(\mathrm{Im}\, M_F)$. Setting $V_F := \{x^*|_F \mid x^* \in V\}$, we have that V_F is a neighborhood of 0 in F^*, and so $V_F \cap \mathrm{Im}\, M_F \neq \emptyset$, or equivalently $V \cap M(F) \neq \emptyset$. This yields the contradiction $V \cap \mathrm{Im}\, M \neq \emptyset$. Hence $\mathrm{conv}(\mathrm{Im}\, M) \subset \mathrm{cl}_{w^*}(\mathrm{Im}\, M)$ and consequently $\mathrm{cl}_{w^*}(\mathrm{Im}\, M)$ is convex. □

As seen in the proof of Theorem 4 the condition ${}^{ic}(\mathrm{dom}\, N) \neq \emptyset$, or more precisely $0 \in {}^{ic}(\mathrm{dom}\, N)$ was essentially used to show that $S := A^\top N A$ is maximal monotone in the very particular case $X = \mathbb{R}$. One can ask if only the condition $0 \in \mathrm{dom}\, N$ or $0 \in {}^{i}(\mathrm{dom}\, N)$ is sufficient for the same conclusion under the assumption that $X = \mathbb{R}$.

The condition $0 \in \mathrm{dom}\, N$ is sufficient for $\dim Y = 1$ but not sufficient for $\dim Y \geq 2$. Indeed, if $\dim Y = 1$ then A is either 0 or an isomorphism. In the second case it is clear that $S \in \mathfrak{M}(X)$, while in the first case $\mathrm{gph}\, S = \mathbb{R} \times \{0\}$.

Take $Y = \mathbb{R}^2$ and $N = \partial f$, where $f(y_1, y_2) := \max\{1 - \sqrt{y_1}, |y_2 - 1|\}$ for $(y_1, y_2) \in [0, \infty[\times \mathbb{R}$ and $f(y_1, y_2) := \infty$ otherwise (f coincides, up to a translation, with the function given in [15, p. 218]). We have that $\mathrm{dom}\, N = ([0, \infty[\times \mathbb{R})\setminus(\{0\} \times]0, 2[)$. Hence $(0, 0) \in \mathrm{dom}\, N$. For $A : \mathbb{R} \to \mathbb{R}^2$ defined by $At := (0, 2t)$ for each $t \in \mathbb{R}$, we have that $\mathrm{dom}\, S = \mathbb{R}\setminus]0, 1[$. Hence S is not maximal monotone because the interior of its domain is not convex.

Of course the conditions $0 \in {}^{i}(\mathrm{dom}\, N)$ and $0 \in {}^{ic}(\mathrm{dom}\, N)$ are equivalent if $\dim Y < \infty$. If $\dim Y = \infty$ the condition $0 \in {}^{i}(\mathrm{dom}\, N)$ is not sufficient to get $S \in \mathfrak{M}(\mathbb{R})$. For this take $T : \ell_2 \to \ell_2$ defined by $T\left((x_n)_{n\geq 1}\right) := \left((n^{-1}x_n)_{n\geq 1}\right)$. It is clear that T is a positive self-adjoint and one-to-one continuous linear operator; hence T is maximal monotone. Take $N = T^{-1}$, that is, $\mathrm{gph}\, N = \{(x, Tx) \mid x \in \ell_2\}$; hence $N \in \mathfrak{M}(\ell_2)$. Of course, $\mathrm{dom}\, N = \mathrm{Im}\, T$ is a dense linear subspace of ℓ_2, and so $0 \in {}^{i}(\mathrm{dom}\, N)$.

Take $\overline{x} \in \ell_2 \setminus \operatorname{dom} N$ and $A : \mathbb{R} \to \ell_2$ with $At := t\overline{x}$ for $t \in \mathbb{R}$. Since $t\overline{x} \in \operatorname{dom} N$ iff $t = 0$, we find $\operatorname{gph} S = \{(0, 0)\}$, thus S is not maximal monotone.

4 Appendix

Several results concerning monotone operators and their representative functions would have simpler proofs if the coupling function c were continuous with respect to $s \times \tau$, where τ is a locally convex topology on X^* such that $(X^*, \tau)^* = X$ and s denotes the strong topology on X. Notice that Penot [10, Proposition 6] proved that X is finite dimensional if c is $s \times w^*$ continuous.

For a Banach space X, a natural question is whether there is a topology τ on X^* with $(X^*, \tau)^* = X$, such that c is continuous on $X \times X^*$ endowed with the $s \times \tau$ topology.

If such a topology exists then c is also continuous with respect to $s \times \tau_M^*$, where τ_M^* is the Mackey topology on X^* with respect to the duality (X^*, X). Recall that the Mackey topology on X^* has for a base of neighborhoods of 0 the family

$$\{C^0 \mid C \in \mathcal{K}_w\} \quad \text{with} \quad \mathcal{K}_w := \{C \subset X \mid C \ w\text{-compact, convex, and balanced}\},$$

where $C^0 := \{x^* \in X^* \mid \langle x, x^* \rangle \leq 1 \ \forall x \in C\}$. Another attractive topology on X^*, compatible with the duality (X^*, X), is the bounded weak-star topology bw^* which has for a base of neighborhoods of 0 the family

$$\{C^0 \mid C \in \mathcal{K}_s\} \quad \text{with} \quad \mathcal{K}_s := \{C \subset X \mid C \ s\text{-compact, convex, and balanced}\}$$

(see [4,8]). Our goal in the following results is to characterize the Banach spaces for which c is $s \times \tau_M^*$ or $s \times bw^*$ continuous.

Proposition 1 *Let X be a Banach space.*

(i) *There exists a norm-bounded τ_M^*-neighborhood of* 0 *in X^* iff X is reflexive.*
(ii) *There exists a norm-bounded bw^*-neighborhood of* 0 *in X^* iff X is finite-dimensional.*

Proof When X is reflexive the Mackey topology τ_M^* coincides with the strong topology, therefore the converse implication in (i) is clear. Similarly if X is finite-dimensional the converse in (ii) is straightforward.

For the direct implication in (i), assume that there exists a norm-bounded τ_M^*-neighborhood of 0 in X^*. Then $U_{X^*} := \{x^* \in X^* \mid \|x^*\| \leq 1\}$ is a τ_M^*-neighborhood of 0, and so there exists $C \in \mathcal{K}_w$ such that $C^0 \subset U_{X^*}$. Hence $U_X \subset C^{00} = C$, where $U_X := \{x \in X \mid \|x\| \leq 1\}$. Because U_X is weakly closed, it follows that U_X is weakly compact, and so X is reflexive.

A similar argument works for the direct implication in (ii) with C replaced by $K \in \mathcal{K}_s$; then $U_X \subset K$, and so U_X is strongly compact, making X finite-dimensional. □

Theorem 5 *Let X be a Banach space.*

(i) *The following statements are equivalent: (a) X is reflexive, (b) c is $s \times \tau_M^*$ continuous, (c) c is bounded above on a nonempty $s \times \tau_M^*$-open set, (d) c is bounded below on a nonempty $s \times \tau_M^*$-open set.*

(ii) *The following statements are equivalent: (a) X is finite dimensional, (b) c is $s \times bw^*$ continuous, (c) c is bounded above on a nonempty $s \times bw^*$-open set, (d) c is bounded below on a nonempty $s \times bw^*$-open set.*

Proof In both assertions (i), (ii) the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) are obvious.

(i) (c) $\Rightarrow$ (a) Assume that (c) is verified. Then there exist $\overline{x} \in X, \overline{x}^* \in X^*$, $\rho, m > 0$, and $C \in \mathcal{K}_w$ such that

$$c(\overline{x} + x, \overline{x}^* + x^*) \leq m,$$

for all $x \in \rho U_X$ and $x^* \in C^0$. It follows that

$$\langle x, \overline{x}^* + x^* \rangle + \langle \overline{x}, \overline{x}^* + x^* \rangle \leq m, \quad \forall x \in \rho U_X, \ \forall x^* \in C^0,$$

so after passing to supremum for $x \in \rho U_X$ we get

$$\rho \left\| x^* \right\| - \rho \left\| \overline{x}^* \right\| + \langle \overline{x}, x^* \rangle - \left| \langle \overline{x}, \overline{x}^* \rangle \right| \leq \rho \left\| \overline{x}^* + x^* \right\| + \langle \overline{x}, x^* \rangle + \langle \overline{x}, \overline{x}^* \rangle \leq m,$$
$$\rho \left\| x^* \right\| \leq m + \rho \left\| \overline{x}^* \right\| + \left| \langle \overline{x}, \overline{x}^* \rangle \right| =: m'$$

for all $x^* \in C^0$. Therefore, C^0 is a norm-bounded τ_M^*-neighborhood of 0 in X^*. According to Proposition 1 X is reflexive.

The implication (c) $\Rightarrow$ (a) in (ii) follows a similar argument. □

Corollary 6 *Let X be a Banach space. Then X is reflexive if one of the following conditions holds*:

(i) *there exists $h : X \times X^* \to \overline{\mathbb{R}}$ such that $h \geq c$ and h is $s \times \tau_M^*$-continuous at some $(\overline{x}, \overline{x}^*) \in \operatorname{dom} h$,*

(ii) *there exists a convex function $h : X \times X^* \to \overline{\mathbb{R}}$ with $h \geq c$ and $(\overline{x}, \overline{x}^*) \in \operatorname{dom} h$ with $h(\overline{x}, \cdot)$ τ_M^*-continuous at $\overline{x}^*$ and $h(\cdot, \overline{x}^*)$ s-continuous at $\overline{x}$,*

(iii) *there exists a locally convex barreled topology τ_B on $X \times X^*$ such that $(X \times X^*, \tau_B)^* = X^* \times X$.*

Proof If (i) holds then from $h \geq c$ and h is $s \times \tau_M^*$-continuous at $(\overline{x}, \overline{x}^*)$ we obtain that c is bounded above on an $s \times \tau_M^*$-open neighborhood of $(\overline{x}, \overline{x}^*)$. The conclusion follows immediately from Theorem 5(i).

Assume that (ii) holds. If we replace h by $g(x, x^*) = h(x + \overline{x}, x^* + \overline{x}^*) - \langle \overline{x}, x^* \rangle - \langle x, \overline{x}^* \rangle - \langle \overline{x}, \overline{x}^* \rangle$ for $(x, x^*) \in X \times X^*$, we may assume without loss of generality that $(\overline{x}, \overline{x}^*) = (0, 0)$. Therefore, there exist $M, r > 0$ and V a τ_M^*-neighborhood of 0 in X^* such that

$$h(x, 0) \leq M, \ h(0, x^*) \leq M \quad \forall x \in r U_X, \ \forall x^* \in V.$$

We have

$$\langle x, x^* \rangle \le h(x, x^*) \le \tfrac{1}{2}h(2x, 0) + \tfrac{1}{2}h(0, 2x) \le M, \quad \forall x \in \tfrac{r}{2}U_X, \ \forall x^* \in \tfrac{1}{2}V,$$

that is, c is bounded above in an $s \times \tau_M^*$-neighborhood of 0 in $X \times X^*$; hence X is reflexive by Theorem 5(i).

Suppose (iii) holds. Then

$$\tau_B \prec \tau_{\mathcal{M}} \prec s_{X \times X^*},$$

where $\tau_{\mathcal{M}} = s \times \tau_M^*$ denotes the Mackey topology on $X \times X^*$ relative to the natural duality and $s_{X \times X^*}$ stands for the strong topology on $X \times X^*$. The identity mapping $I : (X \times X^*, s_{X \times X^*}) \to (X \times X^*, \tau_B)$ is continuous and surjective. Using [26, Theorem 1.3.7] we obtain that I is an open mapping and so $\tau_B = \tau_{\mathcal{M}} = s_{X \times X^*}$, whence $\tau_M^* = s$. □

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