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On the structure of the solution set for the single facility location problem with average distances

Justo Puerto · Antonio M. Rodríguez-Chía

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Abstract This paper analyzes continuous single facility location problems where the demand is randomly defined by a given probability distribution. For these types of problems that deal with the minimization of average distances, we obtain geometrical characterizations of the entire set of optimal solutions. For the important case of total polyhedrality on the plane we derive efficient algorithms with polynomially bounded complexity. We also develop a discretization scheme that provides ε-approximate solutions of the original problem by solving simpler location problems with points as demand facilities.

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1 Introduction

Classical single facility location problems consider a finite set of points in a real normed space X and the goal is to minimize some function depending on the distances to those points (existing facilities or demand points). In the last years the assumption that

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J. Puerto (✉)
Facultad de Matemáticas, Universidad de Sevilla,
C/Tarfia s/n, 41012 Sevilla, Spain
e-mail: puerto@us.es

A. M. Rodríguez-Chía
Facultad de Ciencias, Universidad de Cádiz,
Polígono Río San Pedro, 11510 Puerto Real, Cádiz, Spain
e-mail: antonio.rodriguezchia@uca.es

facilities are represented by isolated points has been questioned by different authors and the natural extension of considering sets rather than points has attracted the attention of researchers, [3–11,27,34].

In the literature, we can find two different alternatives to deal with problems where the demand is modelled by sets. The first one measures the distances to the closest points in the sets, i.e., the goal is not to serve all points of the set but just to reach the set, see [3–5,27]. The second one takes the average behavior into account, so that any point in the demand facility is visited according to a given probability distribution. This approach, that induces the minimization of expected distances (see [6–11,23,34]), will be the goal of this paper. The reader may note that average-distance location problems are not simple generalizations of standard location problems with points as demand facilities. Indeed, the mathematical tools used to analyze these problems are different because the natural distance induced by the norm in X can no longer be used and measure theory plays an important role in average-distance problems.

From an application point of view, point facilities are simplifications of the more realistic dimensional-demand facilities where the demand occurs according to a given probability distribution. Models with expected distances are particularly suitable for real-world situations where a server must cover random incidents (demand) whose exact location is not known in advance. Thus, in general our model can be seen as a tool to find the best response region against any random incident since we determine the best region of estimators minimizing the expected distance to any occurrence of a random variable. This could be the case of the stationing of rescue helicopters as in the real-world situation described in Ehrgott [15], the location of planes used to extinguish fires in reserves or natural parks as well as the case of stationing helicopters used to transport organs to be transplanted. Another application is the problem of locating a read/write head of a computer hard-disk to easily access the stored data, analyzed in Vickson et al. [35] and Puerto and Rodríguez-Chía [30]. Since the position of any reading or writing operation is unknown in advance, it can be represented by a random variable with a specific probability distribution. Moreover, read/write heads only move in a fixed number of directions usually radial and angular (Distances with a finite number of moving direction on finite dimension spaces are called polyhedral). Therefore this case fits to an average location problem with a distance modeled by two moving directions on the geometrical body that represents the computer hard disk (usually a disk or a cylinder).

On the other hand, from a methodological point of view, characterizing the entire set of optimal solutions of different location problems is a subject that has attracted the attention of researchers for many years. Problems on networks have been investigated since the seminal paper by Hakimi [18] and in this framework the concept of *finite dominating set (FDS)* has proven to be essential (see [20]). In continuous location problems the tools that play the most important role are the linearity domains of the objective functions. This idea was first introduced by Durier and Michelot [13], under the name of *Elementary Convex Set*, characterizing the set of optimal solutions of the Weber problem with a finite set of demand points in $\mathbb{R}^n$ (see [12] for characterizations in a general real normed space and [29] for the convex ordered median location problem). Later, Nickel et al. [27] characterized the complete set of optimal solutions of a location problem with respect to sets using an infimal distance approach. Nevertheless, finding similar results for location problems using expected distances is still an open question.

Following this line of research, this paper provides a geometrical characterization of the entire solution set for a single facility location model with sets as demand facilities using average distances. First, we study a basic model that extends to this framework the well-known Fermat-Weber problem. In this case, we obtain an interesting geometrical characterization of its optimal solution set in terms of normal cones. Then, we concentrate on a more general model that includes as particular cases, among others, most of the standard single facility location problems, namely the Fermat-Weber, the minimax or the centdian problems. Unfortunately, for this model, it is not possible to obtain the above intuitive geometrical description. It is worth noting that previous references [6,7,9–11,23,34], dealing with expected distances, consider models that can be obtained as particular cases of the one studied in this paper. However, none of them analyzes the structure of the optimal solution set because they concentrate on the development of algorithms. In addition, we obtain a discretization result that provides ε-approximate solutions of these problems by solving location problems with points as demand facilities.

The paper is organized as follows. Section 2 contains a collection of definitions and results that will be necessary throughout the paper. Section 3 presents the basic model and some existence and uniqueness results. Section 4 obtains a geometrical characterization of the entire optimal solution set for this model. Section 5 introduces an extended model and describes geometrically its entire set of optimal solutions. In Sect. 6, we provide an efficient algorithm to characterize the entire optimal solution set for the case of total polyhedrality on the plane. In Sect. 7, we develop a discretization result that provides ε-approximate solutions for these models. The paper ends with some concluding remarks.

2 Basic tools and definitions

Throughout this paper we will consider that X is a real separable Banach space and X^* its corresponding topological dual space. The pairing between X and X^* will be indicated by $\langle \cdot, \cdot \rangle$. For the ease of understanding, the reader may replace the space X by $\mathbb{R}^n$, in this case the topological dual X^* can be identified with X and the pairing is the usual scalar product. Moreover, for the sake of completeness, we restate the definitions of some concepts which are needed in the paper, the reader is referred to [19] for further details.

1. Let $B \subset X$ be a closed, bounded, convex and symmetric set with respect to the origin which contains the origin in its interior (usually called balanced set, [21]). The norm defined by the unit ball B is

$$\gamma \ : \ X \ \to \ \mathbb{R}, \qquad \gamma(x) := \inf\{ r > 0 \ : \ x \in rB \}.$$

2. The dual norm γ^o of γ is defined as the norm of unit ball B^o where $B^o := \{ p \in X^* \ : \ \langle p, x \rangle \leq 1, \quad \forall\, x \in B\}$.

3. The normal cone to B^o at $p \in X^*$ is the set $N_{B^o}(p) := \{x \in X \ : \ \langle x, q - p \rangle \leq 0, \quad \forall\, q \in B^o\}$.
4. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. A vector $p \in X^*$ is said to be a subgradient of f at a point $x \in X$ if

$$f(y) \geq f(x) + \langle p, y - x \rangle \quad \text{for each } y \in X.$$

   The set of all subgradients of f at x is called the subdifferential of f at x, $\partial f(x)$.
5. For a closed, convex set $A \subset X$, let

$$I_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{otherwise.} \end{cases}$$

6. Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$ not identically equal to $+\infty$ and minorized by some affine function. The conjugate f^* of f is the function defined by

$$f^*(p) = \sup\{\langle p, x \rangle - f(x) : x \in \text{ dom } f\} \quad \text{for any } p \in X^*,$$

   where dom f stands for the effective domain of the function f.
7. A function $g : X \longrightarrow \mathbb{R}$ is closed if $g = cl\ g$, defined as

$$cl\ g(x) = \begin{cases} \liminf\limits_{x' \to x} g(x') & \text{if } \liminf\limits_{x' \to x} g(x') > -\infty \\ -\infty & \text{otherwise.} \end{cases}$$

After these definitions we recall the following well-known results.

Theorem 2.1

1. (*See* [31]) *Let* $\gamma(\cdot)$ *be a norm and* B^o *the unit ball of its corresponding dual norm, then*
   (*a*) $\partial I_{B^o}(x) = N_{B^o}(x) \quad \forall x \in B^o$.
   (*b*) $\partial \gamma(x) = \begin{cases} B^o & \text{if } x = 0 \\ \{p \in B^o \ : \ \langle p, x \rangle = \gamma(x)\} & \text{if } x \neq 0. \end{cases}$
   (*c*) $\gamma^*(p) = I_{B^o}(p)$ *for any* $p \in X^*$.
2. (*See [1]*) *For a closed and proper convex function:*

$$p \in \partial f(x),\ x \in X,\ p \in X^* \quad \text{if and only if} \quad x \in \partial f^*(p).$$

Finally, the last result of this section is an adaptation to our framework of a more general result that appears in [22, theorems 1,2 and 3 in Section 2].

Theorem 2.2 *Let* μ *be a positive measure* σ*-finite relative to the Borel structure on* X *and let* $T \subseteq X$ *be a Borel subset with* $\mu(T) > 0$. *If* $\varphi_t(x) := \gamma_t(x - t)$ *with* $\gamma_t(\cdot)$ *a continuous norm in* (X, γ) *for any* $t \in T$*, the following assertions hold:*

1. *The function* $\phi(x) := \int_T \varphi_t(x)\mu(\mathrm{d}t)$ *is continuous and convex on* X, *and the functions* $t \longrightarrow \varphi_t(x)$ *are summable for all* x, *hence* $\phi(x)$ *is not identically equal to* $+\infty$.
2. *Let* $F : X \longrightarrow L^1(X, \mathbb{R})$ *such that* $(F(x))(t) = \varphi_t(x)x \in X,\ t \in T$. *Then,*

$$\partial\phi(x) = \int_T \partial\varphi_t(x)\mu(\mathrm{d}t) = \int_T \partial F(x)\mu(\mathrm{d}t).$$

3. *For any* $x_o \in X$, $\partial\phi(x_o)$ *consists of all the functionals* $x' \in X^*$ *that can be represented by* $\langle x, x'\rangle = \int_T (Ax)(t)\mu(\mathrm{d}t)$ *where* $A \in \partial F(x_o)$.

3 The basic model. Existence and uniqueness results

In this section, we introduce a single facility location problem with expected distances whose structure extends in a natural way the well-known Fermat-Weber problem (see [24]). In order to define this problem, we consider a positive measure μ, σ-finite relative to the Borel structure on X, and $T \subseteq X$ a Borel subset with $\mu(T) > 0$. The location problem that we deal with is

$$\inf_{x\in X} \phi(x) := \int_T \varphi_t(x)\ \mu(\mathrm{d}t), \qquad (P_\phi(T))$$

where $\varphi_t(x) := \gamma_t(x - t)$ and $\gamma_t(\cdot)$ is a continuous norm for each $t \in T$. Its optimal solution set is denoted by $M_\phi(T)$.

We note in passing that in the case where μ is concentrated on a finite set of points, $P_\phi(T)$ reduces to the objective function of the classical Fermat-Weber problem. The interested reader can find several references in the literature dealing with particular cases of this problem, as for instance [6,7,9–11,23].

In the following, we study existence and uniqueness results for Problem $P_\phi(T)$. To do that, we recall that a function $g : X \longrightarrow \mathbb{R}$ is lower-semicontinuous (*lsc*) if the sets $\{x :\ g(x) \le \alpha\}$ are closed for all $\alpha \in \mathbb{R}$.

Theorem 3.1 (See [16, Theorem 3]) *The problem* $P_\phi(T)$ *has optimal solutions provided that any of the following conditions holds:*

1. X *is finite dimensional and* φ_t *are lsc in the* t *argument.*
2. X *is reflexive and* φ_t *are sequentially lsc in the* t *argument for the weak topology.*
3. X *is the dual space to a separable space and* φ_t *are sequentially lsc in the* t *argument for the weak topology.*
4. X *is a dual space,* φ_t *are lsc in the* t *argument for the weak* topology and* T *is* μ*-separable.*

The above result gives sufficient conditions that ensure existence of optimal solutions of $(P_\phi(T))$. In the following, we will assume without loss of generality that our problems fall into this category; although, in general, there may exist problems of this type without an optimal solution. Examples are shown in Papini [28].

About the uniqueness of this problem, we must note that even if the norms γ_t are strictly convex the function $\gamma_t(\cdot - t)$ is not. Nevertheless, we can quote the following result given in Garkavi and Smatkov [16, Theorem 4].

Theorem 3.2 *Assume that $\mu(T) < +\infty$ and $\gamma_t(\cdot)$ are strictly convex norms for any $t \in T$. Problem $P_\phi(T)$ has no more than one optimal solution if and only if T does not contain two nonintersecting subsets T_1 and T_2 such that $\mu(T_1) = \mu(T_2) = \mu(T)/2$, T_1 and T_2 enclosed in nonintersecting rays r_1 and r_2, respectively, and lying in the same straight line.*

For the general case where finiteness of $\mu(T)$ is not required, we can prove the following.

Theorem 3.3 *Let $\dim(X) \geq 2$. If γ_t are strictly convex norms for any $t \in T$ and μ is absolutely continuous with respect to any measure that assigns null measure to any subspace of dimension less than or equal to 1 then Problem $P_\phi(T)$ has a unique optimal solution.*

Proof It is sufficient to prove that ϕ is a strictly convex function. Let $x \neq y$ and $Z = T \backslash Z^c$ where $Z^c \subseteq T$ is the set defined by the intersection of the line through x and y with T. Since μ assigns null measure to lines, $\mu(Z^c) = 0$. For $0 < \nu < 1$, we have

$$\begin{aligned}
\phi\left(\nu x + (1-\nu)y\right) &= \int_T \gamma_t\left(\nu x + (1-\nu)y - t\right)\ \mu(\mathrm{d}t) \\
\text{(because } \mu(Z^c) = 0\text{)} &= \int_Z \gamma_t\left(\nu(x-t) + (1-\nu)(y-t)\right)\ \mu(\mathrm{d}t) \\
\text{(since } \gamma_t \text{ are strict)} &< \int_Z \left[\nu\gamma_t(x-t) + (1-\nu)\gamma_t(y-t)\right]\ \mu(\mathrm{d}t) \\
&= \nu\phi(x) + (1-\nu)\phi(y).
\end{aligned}$$

Therefore, ϕ is strictly convex and then Problem $P_\phi(T)$ has a unique optimal solution. □

Note that in finite dimensional spaces the above result is true for any measure absolutely continuous with respect to the Lebesgue measure and in general normed spaces, for any measure absolutely continuous with respect to any Hausdorff measure with $d > 1$ (coefficient defining this measure).

Now, we prove some results concerning the structure of $M_\phi(T)$.

Proposition 3.1 *$M_\phi(T)$ is a closed, convex set. In addition, if T is a bounded set with $\mu(T) < +\infty$ then ϕ has bounded lower level sets.*

Proof $M_\phi(T)$ is closed and convex as lower-level set of a continuous convex function, see Theorem 2.2.

In order to prove the boundedness of the lower level sets, we obtain the following inequality

$$\phi(x) \leq \int\limits_T (\gamma_t(x) + \gamma_t(t))\, \mu(\mathrm{d}t) \leq \left(\sup_{u\in T} \gamma_u(x)\right) \mu(T) + \int\limits_T \gamma_t(t)\mu(\mathrm{d}t).$$

Without loss of generality, we can assume that μ is not a degenerate measure at the point 0, because in this case the lower level sets of $\phi(\cdot)$ would coincide with the ones of $\gamma_0(\cdot)$ and the result follows.

Let $R(x) := \mu(T) \sup_{u\in T} \gamma_u(x) + \int_T \gamma_t(t)\mu(\mathrm{d}t)$. Since T is bounded and $\mu(T) < +\infty$ then $R(x) < +\infty$ for any $x \in X$. Moreover,

$$R(x) \geq \phi(x) \geq \left(\inf_{t\in T} \gamma_t(x)\right) \mu(T) - \int\limits_T \gamma_t(t)\mu(\mathrm{d}t).$$

Now, since μ is not a degenerate measure at the point 0, $\int_T \gamma_t(t)\mu(\mathrm{d}t) \neq 0$, thus

$$R(x) > \left(\inf_{t\in T} \gamma_t(x)\right) \mu(T) - \int\limits_T \gamma_t(t)\mu(\mathrm{d}t).$$

Therefore, the lower level set of the function ϕ at the value $\phi(x)$, satisfies

$$L_{\leq}(\phi, \phi(x)) \subset \bigcup_{s\in T} \left\{ y : \gamma_s(y) \leq \frac{1}{\mu(T)}(R(x) + \int\limits_T \gamma_t(t)\mu(\mathrm{d}t)) \right\}$$

which is a bounded set. □

The next result is the characterization of a dominant set for Problem $P_\phi(T)$ (recall that a set is dominant for a problem if it always contains at least one optimal solution of this problem). It extends the well-known convex hull property of Wendell–Hurter [36] which states that the convex hull of the demand points always contains at least one optimal solution if $X = \mathbb{R}^2$ or if X is an inner product space when $dim(X) > 2$.

Theorem 3.4 *Let $co(T)$ be the convex hull of T and $\varphi_t(x) = \gamma(x-t)\, \forall t \in T$. The closure of $co(T)$ contains at least an optimal solution of Problem $P_\phi(T)$ if $dim(X) = 2$ or γ is a norm derived from an inner product when $dim(X) \geq 3$.*

Proof Let $cl(co(T))$ denote the closure of $co(T)$ and let $x^* \notin cl(co(T))$ be an optimal solution of $P_\phi(T)$. Then if γ is derived from an inner product space the orthogonal projection y^* of x^* onto the hyperplane that strictly separates $cl(co(T))$ from x^* satisfies $\gamma(y^* - t) < \gamma(x^* - t)$ for any $t \in T$. Then integrating over T the result follows by contradiction.

For the case of $\mathbb{R}^2$, let $R_t(x^*) = \{x \ : \ \gamma(x - t) \leq \gamma(x^* - t)\}$. If $cl(co(T)) \cap \bigcap_{t\in T} R_t(x^*) = \emptyset$, since $R_t(x^*)$ are bounded sets for any $t \in T$, these sets have no recession directions and then applying Helly's Theorem (see [31, Corollary 21.3.2]), we have that there exist $t_1, t_2, t_3 \in T$ such that $cl(co(T)) \cap \bigcap_{i=1}^{3} R_{t_i}(x^*) = \emptyset$. Hence, we obtain that $co(t_1, t_2, t_3) \cap \bigcap_{i=1}^{3} R_{t_i}(x^*) = \emptyset$. This contradicts Corollary 1 in Wendell and Hurter [36], that ensures that for a given set $\{a_1, \ldots, a_M\} \subseteq \mathbb{R}^2$, and for any $x \in \mathbb{R}^2$ there exists $\hat{x} \in co(\{a_1, \ldots, a_M\})$ such that $\gamma(\hat{x} - a_i) \leq \gamma(x - a_i)$ for all $i = 1, \ldots, M$. Therefore, we can take $\hat{x} \in cl(co(T)) \cap \bigcap_{t \in T} R_t(x^*)$ and this point satisfies $\int_T \gamma(\hat{x} - t)\mu(\mathrm{d}t) \leq \int_T \gamma(x^* - t)\mu(\mathrm{d}t)$. □

The result is not true in general as can be seen in the following example.

Example 3.1 Let T be the set containing the n elements of the natural basis in $\mathbb{R}^n$ equipped with a $\|\cdot\|_p$ norm, for $p \in (1, \infty)$. Let us consider a uniform discrete probability measure μ with support on T. Then, it is straightforward that the (unique) solution to $P_\phi(T)$ with $\gamma_t = \|\cdot\|_p$, $\forall t \in T$, is the point with all components equal to $\alpha = \frac{1}{1+(n-1)^{1/(p-1)}}$, which is not in general in the convex hull of T (for example, if $p = n = 3$, then α is around 0.414, larger than $\frac{1}{3}$ which would be needed for (α, α, α) belonging to $conv(T)$).

It is worth noting that this section extends, to the more general framework where (X, γ) is any separable Banach space, the corresponding results proved in [9] for 2-dimensional spaces and in [13] for finite sets of points in $\mathbb{R}^n$.

4 Optimality conditions

The goal of this section is to geometrically characterize $M_\phi(T)$, the entire set of optimal solutions of $P_\phi(T)$. In the following we assume that the hypotheses of Theorem 2.2 are fulfilled. In addition, we assume, without loss of generality, that $\mu(T) \neq 0$ (otherwise any $x \in X$ is an optimal solution and the objective value is 0). The main result in this section is Theorem 4.1 that gives a geometrical characterization of the set of optimal solutions of $P_\phi(T)$. To obtain that result we need the following technical lemma.

Lemma 4.1

1. *If $M_\phi(T) \neq \emptyset$, there exists at least one μ-measurable function $q \in L^1(X, X^*)$ and a Borel set $T' \subseteq T$ with $\mu(T \backslash T') = 0$ such that $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$ and $M_\phi(T) = \bigcap_{t \in T'} \partial\varphi_t^*(q(t))$.*
2. *If there exists a μ-measurable function $q \in L^1(X, X^*)$ and a Borel set $T' \subseteq T$ with $\mu(T \backslash T') = 0$ such that $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$ and $\bigcap_{t \in T'} \partial\varphi_t^*(q(t)) \neq \emptyset$ then $M_\phi(T) = \bigcap_{t \in T'} \partial\varphi_t^*(q(t))$.*

Proof

1) Let $x_o \in M_\phi(T)$, then $0 \in \partial\phi(x_o)$. By Theorem 2.2, we have that there exists $q \in L^1(X, X^*)$ such that

$$0 = \int_{T'} q(t)\mu(\mathrm{d}t) \in \int_{T'} \partial\varphi_t(x_o)\mu(\mathrm{d}t) = \int_{T} \partial\varphi_t(x_o)\mu(\mathrm{d}t) = \partial\phi(x_o),$$

and $q(t) \in \partial\varphi_t(x_o)\ \forall t \in T'$. Hence, by Theorem 2.1, $x_o \in \partial\varphi_t^*(q(t))\ \forall t \in T'$. Moreover, using the same arguments, for any $x \in \bigcap_{t\in T'} \partial\varphi_t^*(q(t))$, we have that $q(t) \in \partial\varphi_t(x)\ \forall t \in T'$ and thus also $0 \in \partial\phi(x)$. Therefore, if $M_\phi(T) \neq \emptyset$ then $M_\phi(T) = \bigcap_{t\in T'} \partial\varphi_t^*(q(t))$.

2) Let $q \in L^1(X, X^*)$ be a μ-measurable function, $T' \subseteq T$ be a Borel set with $\mu(T\backslash T') = 0$ such that $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$ and $x_o \in \bigcap_{t\in T'} \partial\varphi_t^*(q(t))$. By Theorem 2.1, we have that $q(t) \in \partial\varphi_t(x_o)\ \forall t \in T'$. In addition, by Theorem 2.2,

$$0 = \int\limits_{T'} q(t)\mu(\mathrm{d}t) \in \int\limits_{T'} \partial\varphi_t(x_o)\mu(\mathrm{d}t) = \int\limits_{T} \partial\varphi_t(x_o)\mu(\mathrm{d}t) = \partial\phi(x_o),$$

and thus $0 \in \partial\phi(x_o)$, which in turn is equivalent to $x_o \in M_\phi(T)$. Analogously, for any $x \in M_\phi(T)$, by reversing arguments, we obtain that $x \in \bigcap_{t\in T'} \partial\varphi_t^*(q(t))$, and the result follows. □

In order to give the geometrical description of the solution set of problem $P_\phi(T)$, we use a family of sets that were introduced in Location Analysis by Durier and Michelot [13] and that we adapt to our different framework.

Definition 4.1 Let $q \in L^1(X, X^*)$ be a μ-measurable function such that $q(t) \in B_t^o$ for each $t \in T'$ and let $T' \subset T$ be a Borel set such that $\mu(T\backslash T') = 0$. Let $\mathcal{C}_q(T') := \bigcap\limits_{t\in T'} (t + N_t(q(t)))$, where $N_t(q(t))$ stands for the normal cone to B_t^o at the point $q(t)$. The sets $\mathcal{C}_q(T')$, when nonempty, are called elementary convex sets.

Theorem 4.1

(1) *If $M_\phi(T) \neq \emptyset$ then it coincides with an elementary convex set $\mathcal{C}_q(T')$, associated with a measurable function $q \in L^1(X, X^*)$ and a Borel set $T' \subseteq T$ with $\mu(T\backslash T') = 0$, such that $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$.*

(2) *Conversely, let $\mathcal{C}_q(T')$ be an elementary convex set associated with a Borel set $T' \subseteq T$ with $\mu(T\backslash T') = 0$ and a function $q \in L^1(X, X^*)$. If $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$, then $\mathcal{C}_q(T') = M_\phi(T)$.*

Proof

(1) Let $\varphi_t(x) = \gamma_t(x - t)$. From point (1) of Lemma 4.1 there exists a function $q \in L^1(X, X^*)$ and a Borel set T' such that $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$, $\mu(T\backslash T') = 0$ and $M_\phi(T) = \bigcap_{t\in T'} \partial\varphi_t^*(q(t))$.
In addition, $\varphi_t^*(q(t))$, the conjugate of φ_t at $q(t)$ for $t \in T'$, is given by

$$\varphi_t^*(q(t)) = \sup_{x\in X}\{\langle q(t), x\rangle - \gamma_t(x - t)\}.$$

By the change of variable $y = x - t$, we obtain $\varphi_t^*(q(t)) = \langle q(t), t\rangle + \gamma_t^*(q(t))$ and by Theorem 2.1, $\gamma_t^*(q(t)) = I_{B_t^o}(q(t))$. Thus, since $\partial\varphi_t^*(q(t)) = t + \partial\gamma_t^*(q(t))$ we also obtain that $\partial\varphi_t^*(q(t)) = t + N_t(q(t))$. Hence, we conclude that

$$\bigcap_{t \in T'} \partial \varphi_t^*(q(t)) = \bigcap_{t \in T'} (t + N_t(q(t))) = \mathcal{C}_q(T'),$$

and the result follows.

(2) The function q and the set T' satisfy the assumptions of point (2) of Lemma 4.1, i.e, $\int_{T'} q(t)\mu(\mathrm{d}t) = 0$ and $\bigcap_{t \in T'} \partial \varphi_t^*(q(t)) \neq \emptyset$. Then $M_\phi(T) = \mathcal{C}_q(T')$. □

To illustrate the result we present the following examples.

Example 4.1 (i) Let X be the real line and μ be given by the density $f(t) = e^{-t}$ for $t \geq 0$. The location problem is:

$$\min_{x \in \mathbb{R}} \int_{\mathbb{R}_+} |x - t| e^{-t} \, \mathrm{d}t.$$

Consider $q(t) = \begin{cases} 1 & \text{if } \ t \leq \log 2 \\ -1 & \text{if } \ t > \log 2 \end{cases}$. It is straightforward that $\int_{\mathbb{R}_+} q(t) e^{-t}$ $\mathrm{d}t = 0$. Moreover, $t + N_{B^o}(q(t)) = \begin{cases} t + \mathbb{R}_+ & \text{if } \ t \leq \log 2 \\ t - \mathbb{R}_+ & \text{if } \ t > \log 2 \end{cases}$; and thus $\bigcap_{t \in \mathbb{R}_+} (t + N_{B^o}(q(t)) = \{\log 2\}$. Applying Theorem 4.1, we get that the unique optimal solution is $x^* = \log 2$.

(ii) Let $X = \mathbb{R}^2$ with the rectilinear $\|\cdot\|_1$-norm. Consider a measure μ given by the density $f(t) = 1/2\mathcal{I}_{C_1}(t) + 1/2\mathcal{I}_{C_2}(t)$, where $t = (t_1, t_2) \in \mathbb{R}^2$, $C_1 = co\{(1, 1), (2, 1), (1, 2), (2, 2)\}$ and $C_2 = co\{(-1, -1), (-2, -1), (-1, -2), (-2, -2)\}$ and for a set A, $\mathcal{I}_A(t) = 1$ if $t \in A$ and 0 otherwise. The location problem is:

$$\min_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \|x - t\|_1 f(t) \, \mathrm{d}t.$$

Consider $q(t) = \begin{cases} (-1, -1) & \text{if } \ t \in C_1 \\ (1, 1) & \text{if } \ t \in C_2 \\ (0, 0) & \text{if } \ t \notin C_1 \cup C_2 \end{cases}$. It is straightforward that $\int_{\mathbb{R}^2} q(t) f(t) \, \mathrm{d}t = (0, 0)$. Moreover, $t + N_{B^o}(q(t)) = \begin{cases} t - \mathbb{R}^2_+ & \text{if } \ t \in C_1 \\ t + \mathbb{R}^2_+ & \text{if } \ t \in C_2 \end{cases}$; and thus $\bigcap_{t \in C_1 \cup C_2} (t + N_{B^o}(q(t)) = co\{(-1, -1), (-1, 1), (1, -1), (1, 1)\} := C^*$. Applying Theorem 4.1, we get that the complete set of optimal solutions is C^*.

(iii) Let $X = \ell_1$ be the space that consists of all sequences of scalars $(\xi_i)_{i=0}^{\infty} := \{\xi_0, \xi_1, \xi_2, \ldots\}$ for which

$$\|\xi\|_1 := \sum_{i=0}^{\infty} |\xi_i| < +\infty.$$

Note that in this case $X^* = \ell_0$, see [25], the space of bounded sequences and $\|\xi^*\|_\infty = \max_{i=0,1,2,\ldots} |\xi_i^*|$ for any $\xi^* \in X^*$. Let μ be a probability measure given by the truncated exponential distribution in $(1, +\infty)$ of average $\frac{1}{a}$ $(a > 0)$, i.e. its corresponding density function is

$$f(t) = \begin{cases} \frac{ae^{-at}}{e^{-a}}, & \text{if } t > 1 \\ 0, & \text{otherwise.} \end{cases}$$

For $T = \left\{ \left(\frac{1}{t^k}\right)_{k=0}^{\infty} : t \in (1, +\infty) \right\}$, the location problem is

$$\min_{x \in \ell_1} \int_1^{+\infty} \left\| x - \left(\frac{1}{t^k}\right)_{k=0}^{\infty} \right\|_1 f(t)\,\mathrm{d}t. \tag{1}$$

Consider $q(t) = \begin{cases} (1)_{k=0}^{\infty}, & \text{if } \ 1 \le t \le 1 + \frac{1}{a}\log 2 \\ (-1)_{k=0}^{\infty}, & \text{if } \ t > 1 + \frac{1}{a}\log 2. \end{cases}$
It is straightforward that $\int_1^{+\infty} q(t) f(t)\,\mathrm{d}t = (0)_{k=0}^{+\infty}$. Moreover,

$$N_{B^o}(q(t)) = \begin{cases} ([0, +\infty))_{k=0}^{\infty}, & \text{if } 1 < t \le 1 + \frac{1}{a}\log 2 \\ ((-\infty, 0])_{k=0}^{\infty}, & \text{if } t \ge 1 + \frac{1}{a}\log 2 \end{cases}$$

and thus $\bigcap_{t\in(1,+\infty)} \left(\left(\frac{1}{t^k}\right)_{k=0}^{\infty} + N_{B^o}(q(t)) \right) = \left\{ \left(\frac{1}{\left(1+\frac{1}{a}\log 2\right)^k} \right)_{k=0}^{+\infty} \right\} := C^*$.
Applying Theorem 4.1, we get that the complete set of optimal solutions of Problem (1) is C^*.

We can derive simpler optimality conditions in the particular case that $X = \mathbb{R}^n$ and total polyhedrality, i.e. the norm $\gamma_t = \gamma$ for any $t \in T$ and γ is a polyhedral norm. Assume that the extreme points of the dual norm γ^o are $v^i, i = 1, \ldots, g$. Then we can obtain the gradient $\nabla\phi(x)$ at any point.

Proposition 4.1 *If μ is absolutely continuous with respect to the Lebesgue measure, then $\nabla\phi(x) = \sum_{i=1}^{g} \mu((x - N_{B^0}(v^i)) \cap T)v^i$.*

Proof We know that convex functions in $\mathbb{R}^n$ only have a countable set of non-differentiability points (see [31, Corollary 25.5.1]). Therefore, $\phi(x)$ is differentiable and its derivative is given by: $\nabla\phi(x) = \int_T \nabla\gamma(x-t)\mu(\mathrm{d}t)$. By James' Theorem, in any reflexive space $X = \bigcup_{u\in B^o} N_{B^o}(u)$ (see [33]). Then, we have $T = T \cap \bigcup_{i=1}^{g} N_{B^0}(v^i)$, and $\mu(N_{B^0}(v^i)) = \mu(int(N_{B^0}(v^i))),\ i = 1, \ldots, g$, because μ is absolutely continuous with respect to the Lebesgue measure. Hence

$$\nabla\phi(x) = \int_{T\cap\bigcup_{i=1}^{g} N_{B^0}(v^i)} \nabla\gamma(x-t)\,\mu(\mathrm{d}t) = \sum_{i=1}^{g} \int_{(x-N_{B^0}(v^i))\cap T} \nabla\gamma(y)\mu(\mathrm{d}y)$$
$$= \sum_{i=1}^{g} v^i \mu((x - N_{B^0}(v^i)) \cap T).$$

□

The application of the above proposition allows us to rewrite the optimality condition:

$$x^* \text{ is an optimal solution iff } \sum_{i=1}^{g} v^i \mu((x^* - N_{B^0}(v^i)) \cap T) = 0.$$

Remark 4.1 The extension of this condition to more general spaces seems to be a hard task. The main problem is that in general Banach spaces continuous convex functions may exist having nowhere a Gâteaux differential. A possible extension may be obtained considering Asplund spaces (See [17]).

Remark 4.2 One can develop estimates on the size of the set $M_\phi(T)$. In order to do that we need a measure of the size, since many different norms $\{\gamma_t\}_{t\in T}$ are defined on X. Assuming that $\int_T \gamma_t(x)\mu(\mathrm{d}t) < \infty,\ \forall x \in X$, we define the following seminorm $|||x||| = \int_T \gamma_t(x)\mu(\mathrm{d}t)$. Then, if we denote by ϕ^* the optimal value of problem $P_\phi(T)$ and x^* and y^* belong to $M_\phi(T)$ then $\phi^* = \int_T \gamma_t(x^* - t)\mu(\mathrm{d}t) \geq |||x^* - y^*||| - \int_T \gamma_t(y^* - t)\mu(\mathrm{d}t)$. Hence, $|||x^* - y^*||| \leq 2\phi^*$.

The following example proves that this bound is sharp.

Example 4.2 Consider $X = \mathbb{R}$, $T = \{0, 1\}$ and the measure

$$\mu(-\infty, t] = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } 0 \leq t < 1 \\ 1 & \text{otherwise.} \end{cases}$$

The optimal solution set is $[0, 1]$ with value $\phi^* = 1/2$.

5 The extended model

The model analyzed so far (in previous sections) is the natural reformulation of the single facility Fermat-Weber problem with respect to expected distances. In recent years, characterizing optimal solution sets of wider families of location problems has been an issue attracting the interest of location theory researchers. Attempts to deal with more general models than the Weber problem, have been considered in standard location analysis with points as demand facilities (see e.g. [12]) as well as with sets as demand facilities (see [27]). In this section, we extend the model in Sect. 4 to deal with more general problems involving different objective functions over average distances and simultaneously, several probability distributions. In addition, this approach also extends the results in [12] and [27] to location problems with expected distances. The structure of these problems, being more complex than the model considered in previous sections, does not allow us to obtain the simple and elegant geometrical description presented there for the basic model. In spite of that, we provide a different geometrical characterization of the solution set for the extended model using alternative tools.

In order to give the formulation of this model, we consider a globalizing function $\Phi(\cdot)$ which is a monotone norm on $\mathbb{R}^M$ (Recall that a norm Φ is said to be monotone on $\mathbb{R}^M$ if $\Phi(u) \leq \Phi(v)$ for every u, v verifying $|u_i| \leq |v_i|$ for each $i = 1, \dots, M$, see [2]). In addition, let μ_i be a positive measure σ-finite relative to the Borel structure on X, and $T \subseteq X$ a Borel subset with $\mu_i(T) > 0$ for $i = 1, \dots, M$; $\bar{d}_i(x)$ represents the average-distance from x to T weighted with μ_i, i.e. $\bar{d}_i(x) := \int_T \varphi_t(x)\mu_i(\mathrm{d}t)$, where $\varphi_t(x) = \gamma_t(x - t)$ and $\gamma_t(\cdot)$ is a continuous norm for each $t \in T$. Hence, the formulation of the extended model is:

$$\inf_{x \in X} F(x) := \Phi(\overline{D}(x)), \qquad (P_\Phi(\Upsilon))$$

where $\overline{D}(x) = (\bar{d}_1(x), \dots, \bar{d}_M(x))$ and $\Upsilon = \{\mu_1, \dots, \mu_M\}$. Its optimal solution set is denoted by $M_\Phi(\Upsilon)$.

It is worth noting that for particular choices of the monotone norm Φ, we get well-known problems in Location Analysis such as the center, cent-dian, k-centrum, etc. (See [26] for a description of these functions). In addition, the reader can see that the function $F = \Phi \circ \overline{D}$ is convex on $\mathbb{R}^M$ provided that Φ is monotone (see [19, Proposition IV.2.1.8]).

The existence and uniqueness results obtained for the basic model are still valid for the extended model. Thus, in this section we concentrate on obtaining a geometrical characterization of the set of optimal solutions $M_\Phi(\Upsilon)$. In order to do that we first characterize the subdifferential of $F(x)$.

Lemma 5.1 *Let $x \in X$ be such that $\overline{D}(x) \neq 0 \in \mathbb{R}^M$. Then, $\hat{\delta} \in \partial F(x)$ iff there exist $p_i \in \partial \bar{d}_i(x)$ such that $p_i = \int_T q_i(t)\mu(\mathrm{d}t)$ with $q_i(t) \in \partial\varphi_t(x)$, for $i = 1, \dots, M$ and $\delta \in \partial\Phi(y)$ for $y = \overline{D}(x)$ with $\delta \in \mathbb{R}^M_+$, such that $\hat{\delta} = \sum_{i=1}^M \delta_i p_i$.*

Proof By Theorem 2.2, we have that $p_i \in \partial \bar{d}_i(x)$ if and only if there exist $q_i(t) \in \partial\varphi_t(x)$ almost everywhere in T, such that $p_i = \int_T q_i(t)\mu_i(\mathrm{d}t)$ and $\bar{d}_i(x) = \int_T \langle q_i(t), x - t\rangle \mu_i(\mathrm{d}t)$, for $i = 1, \dots, M$. Moreover, since Φ is a norm, and $y > 0$, we have

$$\partial\Phi(y) = \left\{ \delta \in \mathbb{R}^M_+ \ : \ \Phi^o(\delta) = 1; \ \Phi(y) = \sum_{i=1}^M \delta_i y_i \right\},$$

where Φ^o denotes the dual norm of Φ. In addition, by [22, Theorem 2, Sect. 8], the subdifferential of the composition of a nondecreasing convex function with several convex functions is given by

$$\begin{aligned}\partial F(x) &= \partial\Phi(\overline{D}(x))\\ &= \left\{\sum_{i=1}^{M}\delta_i p_i \;:\; (\delta_1,\ldots,\delta_M)\in\partial\Phi(\overline{D}(x)),\ (p_1,\ldots,p_M)\in\partial\overline{D}(x)\right\},\end{aligned}$$

and the result follows. □

In the following we introduce the families of sets that will be used to obtain the geometrical characterization of the optimal solution set $M_\Phi(\Upsilon)$.

Definition 5.1 Given $p=(p_1,\ldots,p_M)\in(X^*)^M$ and $I\subseteq\{1,\ldots,M\}$. Let

$$\overline{C}_I(p) := \bigcap_{i\in I}\partial\bar{d}_i^*(p_i),$$

where $\bar{d}_i^*$ is the conjugate function of $\bar{d}_i(x)$, and for any $\delta=(\delta_1,\ldots,\delta_M)\geq 0$ let

$$\overline{D}_I(\delta) := \left\{x \;:\; \sum_{i\in I}\delta_i\bar{d}_i(x)=F(x)\right\}.$$

It is useful to observe that $\overline{C}_I(p)$ is nonvoid only for some choices of I and p. The sets $\overline{C}_I(p)$ are called generalized elementary convex sets (g.e.c.s.). The reader may note the similarity with the concept of elementary convex set introduced in Definition 4.1 which justifies its name. It is straightforward to see that the g.e.c.s. are convex because they are defined by a finite intersection of convex sets (recall that subdifferential sets are convex).

Definition 5.2 We call (I,δ,p) an optimizing triplet if

1. $I\neq\emptyset$, $I\subseteq\{1,\ldots,M\}$,
2. $\delta=(\delta_1,\ldots,\delta_M)$ with $\delta_i>0$ $(i\in I)$, and $\delta_i=0$ $(i\notin I)$ satisfying $\Phi^o(\delta)=1$ and
3. $p=(p_1,\ldots,p_M)$ such that $p_i\in\partial\bar{d}_i(x)$ for $i=1,\ldots,M$ and for some $x\in X$, with $\sum_{i=1}^M\delta_i p_i=0$.

Note that there may be triplets (I,δ,p) not being optimizing. The rationale behind the above definition is that a triplet is optimizing if it can be used to construct a zero element in $\partial F(x)$ (following the description given in Lemma 5.1) to show optimality.

A different definition of optimizing triplet, based on infimal distances to sets, was used by Nickel et al. [27] (there it was called "suitable") for characterizing optimal solution sets of location problems in a different framework, namely using sets and inf-distances.

In order to give a complete characterization of $M_\Phi(\Upsilon)$, we prove the following theorem.

Theorem 5.1

1. *If* $M_\Phi(\Upsilon) \neq \emptyset$, *then there exists an optimizing triplet* (I, δ, p) *such that* $M_\Phi(\Upsilon) = \overline{C}_I(p) \cap \overline{D}_I(\delta)$.
2. $M_\Phi(\Upsilon) = \overline{C}_I(p) \cap \overline{D}_I(\delta)$, *for any optimizing triplet* (I, δ, p) *such that* $\overline{C}_I(p) \cap \overline{D}_I(\delta) \neq \emptyset$.

Proof Assume $x_0 \in M_\Phi(\Upsilon)$ which in turn is equivalent to $0 \in \partial F(x_0)$. Then, by Lemma 5.1 applied to $\hat{\delta} = 0 \in X$, we have that there exists an optimizing triplet $(I_{x_0}, \delta_{x_0}, p_{x_0})$ which may depend on x_0, satisfying $0 = \sum_{i=1}^{M} \delta_i p_i \in \partial F(x_0)$. Using Theorems 2.1 and 2.2, and since Φ is a norm we have that $x_0 \in \overline{C}_{I_{x_0}}(p_{x_0}) \cap \overline{D}_{I_{x_0}}(\delta_{x_0})$. Moreover, any $x \in \overline{C}_{I_{x_0}}(p_{x_0}) \cap \overline{D}_{I_{x_0}}(\delta_{x_0})$ also satisfies that $0 \in \partial F(x)$. Therefore, we have just proved that $\overline{C}_{I_{x_0}}(p_{x_0}) \cap \overline{D}_{I_{x_0}}(\delta_{x_0}) \subseteq M_\Phi(\Upsilon)$.

Hence, in order to complete the proof we have to prove that any $\bar{x} \in M_\Phi(\Upsilon)$ also satisfies that $\bar{x} \in \overline{C}_{I_{x_0}}(p_{x_0}) \cap \overline{D}_{I_{x_0}}(\delta_{x_0})$.

Observe that, since $x_0 \in \overline{D}_{I_{x_0}}(\delta_{x_0})$, then

$$F(x_0) = \sum_{i=1}^{M} \delta_{x_0,i} \int_T \varphi_t(x_0)\mu_i(\mathrm{d}t). \tag{2}$$

Moreover, since $x_0 \in \overline{C}_{I_{x_0}}(p_{x_0})$, we have that $x_0 \in \partial \bar{d}_i^*(p_{x_0,i})$ for all $i \in I_{x_0}$, which by Theorem 2.1 implies that $p_{x_0,i} \in \partial \bar{d}_i(x_0)$ for all $i \in I_{x_0}$. Next, we use Theorem 2.2 to ensure that there exist $q_i(t) \in \partial\varphi_t(x_0)$ such that $p_{x_0,i} = \int_T q_i(t)\mu_i(\mathrm{d}t)$ $\forall i \in I$. Thus, $\varphi_t(x_0) = \langle q_i(t), x_0 - t\rangle$ and we can rewrite (2) in the following way

$$F(x_0) = \sum_{i=1}^{M} \delta_{x_0,i} \int_T \langle q_i(t), x_0 - t\rangle \mu_i(\mathrm{d}t) = -\sum_{i=1}^{M} \delta_{x_0,i} \int_T \langle q_i(t), t\rangle \mu_i(\mathrm{d}t). \tag{3}$$

Observe that the last equality follows from the third property of the optimizing triplet, i.e. $\sum_{i=1}^{M} \delta_{x_0,i} p_{x_0,i} = 0$, and then $\sum_{i=1}^{M} \delta_{x_0,i} \int_T q_i(t)\mu_i(\mathrm{d}t) = 0$. Next, using again this representation of 0, we get from (3) for any $x \in X$

$$F(x_0) = -\sum_{i=1}^{M} \delta_{x_0,i} \int_T \langle q_i(t), t\rangle \mu_i(\mathrm{d}t) = \sum_{i=1}^{M} \delta_{x_0,i} \int_T \langle q_i(t), x - t\rangle \mu_i(\mathrm{d}t).$$

Now, since γ and Φ are norms, we have that

$$\gamma_t(y) = \sup_{x'\,;\,\gamma^o(x')=1} \langle x', y\rangle \quad \text{and} \quad F(x) = \Phi(\overline{D}(x)) = \sup_{\delta\,:\,\Phi^o(\delta)=1} \langle \delta, \overline{D}(x)\rangle.$$

Hence, we obtain the following inequalities

$$F(x_0) = \sum_{i=1}^{M} \delta_{x_0,i} \int_T \langle q_i(t), x - t\rangle \mu_i(\mathrm{d}t) \leq \sum_{i=1}^{M} \delta_{x_0,i} \int_T \gamma_t(x - t)\mu_i(\mathrm{d}t) \leq F(x).$$

The last inequalities are valid for any $x \in X$. Thus, taking $x = \bar{x}$ and observing that $\bar{x} \in M_\Phi(\Upsilon)$, we have that $F(x_0) = F(\bar{x})$. This implies that $\gamma_t(\bar{x} - t) = \langle q_i(t), \bar{x} - t\rangle$ almost everywhere in T. Hence, $q_i(t) \in \partial\varphi_t(\bar{x})$ almost everywhere in T and $\bar{x} \in \overline{C}_{I_{x_0}}(p_{x_0}) \cap \overline{D}_{I_{x_0}}(\delta_{x_0})$. Thus, the choice of the triplet $(I_{x_0}, \delta_{x_0}, p_{x_0})$ does not depend on x_0. Therefore, we have proved that there exists an optimizing triplet (I, δ, p) such that $M_\Phi(\Upsilon) = \overline{C}_I(p) \cap \overline{D}_I(\delta)$.

Finally, by reversing the arguments one also proves the second assertion. □

The characterization obtained in the above theorem is rather important from several points of view. On the one hand, from a theoretical point of view in order to obtain a complete description of the optimal solution set of Problem $P_\Phi(\Upsilon)$, we only need to find an optimizing triplet (I, δ, p) such that $\overline{C}_I(p) \cap \overline{D}_I(\delta) \neq \emptyset$. On the other hand, from an application point of view in the case of total polyhedrality, i.e., if the demand sets and the unit balls of the norms are polygons, Theorem 5.1 is particularly useful (see Sect. 6 for a complete description). These cases are important because, in real-world applications, demand regions and unit balls of the norms are sometimes approximated by polygons. This is for instance the way that current GPS units display areas or regions. Finally, Theorem 5.1 also advances significantly the knowledge in the field of location analysis. Up to date, there were known geometrical characterizations of location problems with points and sets (using infimal distances) as demand facilities. However, the characterization of location problems with sets and expected distances was an open problem and the above theorem closes this gap.

We illustrate the elements used in Theorem 5.1 with the following examples where the elements of different optimizing triplets are shown in full detail.

Example 5.1 (i) Consider a location problem in $\mathbb{R}^2$ with the squared Euclidean norm $\|\cdot\|_2^2$. Let $T = \mathbb{R}^2$ and let Φ be the $\|\cdot\|_1$-norm in $\mathbb{R}^2$ and the measures $\mu_i, i = 1, 2$ are given, respectively, by the densities

$$f_1(t_1, t_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(t_1^2 + t_2^2)\right\} \quad \forall (t_1, t_2) \in \mathbb{R}^2$$

$$f_2(t_1, t_2) = \begin{cases} \frac{1}{4} \exp\left\{\frac{-1}{4} t_1 t_2\right\}, & \text{if } t_1, t_2 \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The extended problem to be solved is:

$$\min_{x \in \mathbb{R}^2} \bar{d}_1(x) + \bar{d}_2(x).$$

Set $q_1(t) = q_1(t_1, t_2) = (1 - t_1, 1 - t_2)$ and $q_2(t) = q_2(t_1, t_2) = (1 - t_1, 1 - t_2)$ for all $t \in T$. Consider the triplet (I, δ, p) given by $I = \{1, 2\}$, $\delta = (1, 1)$ and $p = (p_1, p_2)$ with $p_1 = (1, 1)$ and $p_2 = (-1, -1)$. For $x^* = (1, 1)$ the above triplet is an optimizing triplet since

1. $\Phi^o(\delta) = \|\delta\|_\infty = 1$.

2. $q_i(t) \in \partial\|x^*-t\|_2^2$, a.e. for $i=1,2$. Since, $\int_{\mathbb{R}^2} q_1(t) f_1(t)\,\mathrm{d}t=(1,1)=p_1$ and $\int_{\mathbb{R}^2} q_2(t) f_2(t)\,\mathrm{d}t=(-1,-1)=p_2$, by Theorem 2.2, $p_i \in \partial\bar{d}_i(x^*)$, for $i=1,2$.
3. $p_1+p_2=0$.

Applying Theorem 2.1 $x^* \in \partial\bar{d}_i^*(p_i)$ for $i=1,2$ and $F(x^*)=\sum_{i=1}^{2}\bar{d}_i(x^*)$, that is, $x^* \in \overline{C}_I(p) \cap \overline{D}_I(\delta)$. Moreover, since $\{q_i(t)\}=\partial\varphi_t(x^*)$, we have that $\overline{C}_I(p)=\{x^*\}$. Hence, x^* is the unique optimal solution of this location problem.

(ii) Let $X=\ell_1$ (see Example 4.1 iii) and let $f(\cdot)$ be the density function of a truncated exponential probability distribution in $(1,+\infty)$ of average $\frac{1}{a}$ $(a>0)$, i.e.

$$f(x)=\begin{cases} \dfrac{ae^{-ay}}{e^{-a}}, & \text{if } y>1 \\ 0, & \text{otherwise.} \end{cases}$$

Consider $T_1, T_2 \subset X$ defined as $T_1=\left\{\left(\frac{1}{y^k}\right)_{k=0}^{\infty} : y \in (1,+\infty)\right\}$, $T_2=\left\{\left(\frac{1}{k^y}\right)_{k=0}^{\infty} : y \in (1,+\infty)\right\}$ and $T=\{T_1,T_2\}$.

Let μ_1 and μ_2 be two positive mesures defined over the Borel sets of X as

$$\mu_1(B):=\int \mathcal{I}_{B\cap T_1}(t)\mu_1(\mathrm{d}t):=\int\limits_{\left\{y:\left(\frac{1}{y^k}\right)_{k=0}^{\infty}\in B\right\}} f(y)\,\mathrm{d}y,$$

$$\mu_2(B):=\int \mathcal{I}_{B\cap T_2}(t)\mu_2(\mathrm{d}t):=\int\limits_{\left\{y:\left(\frac{1}{k^y}\right)_{k=0}^{\infty}\in B\right\}} f(y)\,\mathrm{d}y.$$

(Recall that for a given set A, $\mathcal{I}_A(t)=1$ if $t\in A$ and 0 otherwise).

Taking $\Phi(y)=\|y\|_1$ for any $y\in\mathbb{R}^2$ the extended location problem to be solved is

$$\min_{x\in\ell_1} F(x)=\Phi(\overline{D}(x)):=\bar{d}_1(x)+\bar{d}_2(x),$$

where

$$\bar{d}_1(x)=\int\limits_T \|x-t\|_1\mu_1(\mathrm{d}t)=\int\limits_1^{+\infty}\left\|x-\left(\frac{1}{y^k}\right)_{k=0}^{\infty}\right\|_1 f(y)\,\mathrm{d}y,$$

$$\bar{d}_2(x)=\int\limits_T \|x-t\|_1\mu_2(\mathrm{d}t)=\int\limits_1^{+\infty}\left\|x-\left(\frac{1}{k^y}\right)_{k=0}^{\infty}\right\|_1 f(y)\,\mathrm{d}t.$$

For

$$q_1(t) = \begin{cases} (\overbrace{-1,\dots,-1}^{\lfloor y \rfloor}, 1, 1, \dots), & \text{if } t = (\frac{1}{y^k})_{k=0}^{\infty} \in T_1 \\ 0 & \text{otherwise} \end{cases}$$

$$q_2(t) = \begin{cases} (\overbrace{1,\dots,1}^{\lfloor y \rfloor}, -1, -1, \dots), & \text{if } t = (\frac{1}{k^y})_{k=0}^{\infty} \in T_2 \\ 0, & \text{otherwise,} \end{cases}$$

and $x^* = \left(\frac{1}{k^k}\right)_{k=0}^{\infty}$, we have that:

1. $$p_1 = \int_T q_1(t)\mu_1(\mathrm{d}t) = \int_1^{\infty} (\overbrace{-1,\dots,-1}^{\lfloor y \rfloor}, 1, 1, \dots) f(y)\,\mathrm{d}y$$
$$= \sum_{k=1}^{\infty} \int_k^{k+1} (\overbrace{-1,\dots,-1}^{k}, 1, 1, \dots) f(y)\,\mathrm{d}y; \text{ and}$$
$$p_2 = \int_T q_2(t)\mu_2(\mathrm{d}t) = \int_1^{\infty} (\overbrace{1,\dots,1}^{\lfloor y \rfloor}, -1, -1, \dots) f(y)\,\mathrm{d}y$$
$$= \sum_{k=1}^{\infty} \int_k^{k+1} (\overbrace{1,\dots,1}^{k}, -1, -1, \dots) f(y)\,\mathrm{d}y.$$

2. $p_1 + p_2 = 0.$

Taking $I = \{1, 2\}, \delta = (1, 1)$ and $p = (p_1, p_2), (I, \delta, p)$ is an optimizing triplet. Indeed, we can see that $q_1(t) \in \partial \left\| x^* - \left(\frac{1}{t^k}\right)_{k=0}^{\infty} \right\|_1$ and $q_2(t) \in \partial \left\| x^* - \left(\frac{1}{k^t}\right)_{k=0}^{\infty} \right\|_1$, or equivalently, $p_i \in \partial \bar{d}_i(x^*)$ for $i = 1, 2$. Therefore, $x^* \in \partial \bar{d}_i^*(p_i)$ for $i = 1, 2$ and $F(x^*) = \sum_{i=1}^{2} \bar{d}_i(x^*)$, that is, $x^* \in \overline{C}_I(p) \cap \overline{D}_I(\delta)$. Moreover, since $\{q_1(t)\} = \partial \left\| x^* - \left(\frac{1}{t^k}\right)_{k=0}^{\infty} \right\|_1$ and $\{q_2(t)\} = \partial \left\| x^* - \left(\frac{1}{k^t}\right)_{k=0}^{\infty} \right\|_1$, we have that $\overline{C}_I(p) = \{x^*\}$. Hence, x^* is the unique optimal solution of this location problem.

We conclude this section giving easier optimality conditions for the extended model under additional hypotheses.

Proposition 5.1 *Let $X = \mathbb{R}^n$ and $\varphi_t(x) = \gamma(x - t)$, where γ is a polyhedral norm. If Φ is differentiable everywhere except at the origin, then F is differentiable and its* *derivative equals:*

$$\nabla F(x) = \sum_{i=1}^{M} \sum_{u \in Ext(B^o)} u \frac{d\Phi(\overline{D}(x))}{dx_i} \mu_i(T \cap (x - N_{B^o}(u))),$$

where $Ext(B^o)$ stands for the set of extreme points of B^o.

6 The polyhedral planar case

In this section we restrict ourselves to $\mathbb{R}^2$ and total polyhedrality, that is, the norms defined are polyhedral and the sets where the demand occurs are convex polygons. This reduction allows us to describe the geometrical characterization given in previous sections in an easier way. For a better understanding of this section we consider that $\gamma_t(\cdot - t) = \gamma(\cdot - t)$ for all $t \in T$ where $\gamma(\cdot)$ is a polyhedral norm with unit ball B having g extreme points. Recall that the fundamental directions of γ are those defined by the vectors connecting the origin with the extreme points of B, [13].

By Theorem 5.1, in order to describe the entire set of solutions of these problems, we only need to identify an optimizing triplet, (I, δ, p), such that, $\overline{C}_I(p) \cap \overline{D}_I(\delta) \neq \emptyset$. Therefore, we first characterize the two families of sets $\overline{C}_I(p)$ and $\overline{D}_I(\delta)$.

Lemma 6.1 *Let γ be a polyhedral norm and let μ be an absolutely continuous measure with respect to the Lebesgue measure restricted to a planar convex polygon T. There exists a finite subdivision of $\mathbb{R}^2$ such that $\bar{d}(x, T)$ has a common closed form expression on each element of the subdivision. Moreover, this expression is linear or quadratic in x.*

Proof Since γ is polyhedral with unit ball B and $X = \mathbb{R}^2$, the dual unit ball B^o is also a polygon with g extreme points. Then, using the evaluation of γ through the dual ball we have:

$$\gamma(x - t) = \sup_{u \in B^o} \langle u, x - t\rangle = \max_{u \in Ext(B^o)} \langle u, x - t\rangle = \langle u_{k_0}, x - t\rangle \quad \forall t \in x - N_{B^o}(u_{k_0})$$

and some $k_0 \in \{1, \ldots, g\}$. Now, the evaluation of the expected distance is:

$$\bar{d}(x, T) = \int_T \gamma(x - t)\mu(\mathrm{d}t) = \sum_{j=1}^{g} \int_{T \cap (x - N_{B^o}(u_j))} \langle u_j, x - t\rangle \mu(\mathrm{d}t).$$

Next, we observe that in the above integral x is fixed and thus the resulting expression is either linear, whenever the integration domain does not depend on x, or quadratic whenever the domain is a polyhedron that depends on x. □

Clearly, the complexity for obtaining $\bar{d}(x, T)$ for a given x and μ being the Legesgue measure is $O(kg)$, where k is the number of facets of T.

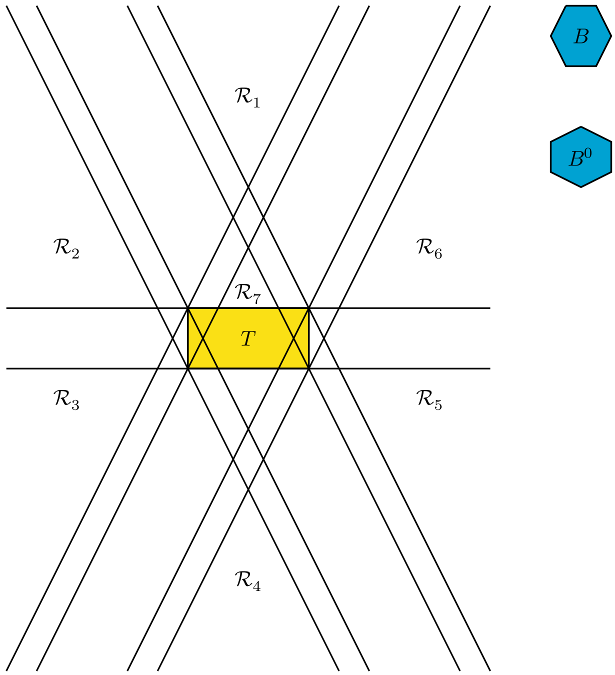

Fig. 1 Subdivision of $\mathbb{R}^2$ generated by the norm γ

Remark 6.1 From the construction above, the subdivision mentioned in Lemma 6.1 is generated by all the lines that are parallel to the fundamental directions of $\gamma(\cdot)$ through each vertex of the set T. Let $\mathcal{R} := \{\mathcal{R}_j\}_{j\in J}$ be such a subdivision (See Fig. 1).

The reader may note that if we are given a problem with M polyhedral demand sets, $\{T_i\}_{i=1}^M$, T_i having k_i facets, the number of lines defining the subdivision mentioned in Lemma 6.1 is $O(Mgk_{\max})$, where $k_{\max} = \max_{i=1,\dots,M} k_i$. Therefore, the cardinality of J is $O((Mgk_{\max})^2)$.

Example 6.1 Let $T = co\{(-2,-1), (2,-1), (2,1), (-2,1)\}$ be a set and let γ be a hexagonal norm with unit ball B defined by $B = co\{(1,0), (0.5,1), (-0.5,1), (-1,0), (-0.5,-1), (0.5,-1)\}$. Let us assume that μ is a uniform probability density on T.

First, we obtain that $B^o = co\{(1,0.5), (0,1), (-1,0.5), (-1,-0.5), (0,-1), (1,-0.5)\}$. The subdivision, $\{\mathcal{R}_j\}_{j\in J}$, generated by the fundamental directions can be seen in Fig. 1. By Lemma 6.1, the expression of $\bar{d}(x,T)$ has a common closed form expression (either linear or quadratic) for all x in the same element of that subdivision. In general, for each $x \in \mathbb{R}^2$, the expression of the expected distance to T is given by

$$\bar{d}(x,T) = \int_T \gamma(x-t)\mu(\mathrm{d}t) = \frac{1}{\mu(T)} \int_T \gamma(x-t)\,\mathrm{d}t.$$

For the particular case of $x \in \mathcal{R}_j$ with $j \in J$ we can obtain a simpler expression. In the following, we give some examples of the common analytical expression of $\bar{d}(x, T)$ for $x \in \mathcal{R}_j$ with $j = 1, 2, 7$ (See Fig. 1).

Case $x \in \mathcal{R}_1$: In this case, $\gamma(x - t) = \langle (0, -1), (t_1 - x_1, t_2 - x_2)\rangle$. Thus,

$$\begin{aligned}\bar{d}(x, T) &= \frac{1}{8}\int\limits_T \gamma(x - t)\,\mathrm{d}t\\ &= \frac{1}{8}\int\limits_{-2}^{2}\int\limits_{-1}^{1}\langle (0, 1), (x_1 - t_1, x_2 - t_2)\rangle\,\mathrm{d}t_1\,\mathrm{d}t_2 = x_2.\end{aligned}$$

Case $x \in \mathcal{R}_2$: In this case, $\gamma(x - t) = \langle (1, -0.5), (t_1 - x_1, t_2 - x_2)\rangle$.

$$\begin{aligned}\bar{d}(x, T) &= \frac{1}{8}\int\limits_T \gamma(x - t)\,\mathrm{d}t\\ &= \frac{1}{8}\int\limits_{-2}^{2}\int\limits_{-1}^{1}\langle (-1, 0.5), (x_1 - t_1, x_2 - t_2)\rangle\,\mathrm{d}t_1\,\mathrm{d}t_2 = -x_1 + \frac{x_2}{2}.\end{aligned}$$

Case $x \in \mathcal{R}_7$: In this case,

$$\gamma(x - t) = \begin{cases}\langle (1, 0.5), (x_1 - t_1, x_2 - t_2)\rangle & \text{if } t \in T \text{ and } t_1 \le \dfrac{t_2 + 2x_1 - x_2}{2}\\ \langle (0, 1), (x_1 - t_1, x_2 - t_2)\rangle & \text{if } t \in T \text{ and } \dfrac{t_2 + 2x_1 - x_2}{2}\\ & \quad \le t_1 \le \dfrac{-t_2 + 2x_1 + x_2}{2}\\ \langle (-1, 0.5), (x_1 - t_1, x_2 - t_2)\rangle & \text{if } t \in T \text{ and } t_1 \ge \dfrac{-t_2 + 2x_1 + x_2}{2}.\end{cases}$$

Thus,

$$\bar{d}(x, T) = \frac{1}{8}\int\limits_T \gamma(x - t)\,\mathrm{d}t = \frac{1}{8}\left(2x_1^2 + \frac{1}{2}x_2^2 + 4x_2 + \frac{49}{6}\right).$$

Note that for some elements of the subdivision $\mathcal{R}$, namely $\mathcal{R}_j$, $j = 1, \ldots, 6$, the function $\bar{d}(x, T)$ is linear in x.

Once we have characterized the subdivision $\mathcal{R}$ induced by the family of sets $\overline{C}_I(p)$, we study the one generated by the family $\overline{D}_I(\delta)$, that is, the subdivision generated by the function $F = \Phi \circ \overline{D}$ on $\mathbb{R}^2$. We assume that Φ is a polyhedral norm (in $\mathbb{R}^n$) with unit ball having r extreme points and the extreme points of the unit ball of its dual norm are $\{\delta^1, \ldots, \delta^{r^0}\}$ (Note that r^0 the number of extreme points of the dual unit ball is equal to the number of facets of the primal unit ball). Therefore,

$$\Phi(\overline{D}(x)) = \max_{i=1,\ldots,r^0} \langle \delta^i, \overline{D}(x)\rangle.$$

Clearly, the linearity domains of this maximum define the sets $\overline{D}_I(\delta)$. Let $\mathcal{R}' := \{\mathcal{R}'_j\}_{j\in J'}$ be the subdivision generated by $\overline{D}_I(\delta)$.

Therefore, in order to define a finer subdivision where the function F has a common linear or quadratic expression of x, we should overlap the subdivision $\mathcal{R}$ and $\mathcal{R}'$, namely $\mathcal{P} := \{\mathcal{R}_j \cap \mathcal{R}'_{j'}\}_{(j,j')\in J\times J'}$.

The subdivision $\mathcal{R}$ can be generated by using the algorithm to find planar arrangements induced by a finite set of lines (See, e.g. [14, Chapter 7]). Since in our case $\mathcal{R}$ is generated by $O(Mgk_{\max})$ lines, the overall complexity of the procedure to obtain $\mathcal{R}$ is $O((Mgk_{\max})^2)$.

Remark 6.2 Recall that $\lambda_s(n)$ is the maximum length of a Davenport-Schinzel sequence of order s on n symbols. The reader is referred to Chapter 3 in Sharir and Agarwal [32] for the exact definitions and properties of the functions $\lambda_s(n)$. We note that $\lambda_4(n) = \theta(n2^{\alpha(n)})$, where $\alpha(n)$ is the inverse of the Ackermann function.

Within $\mathcal{R}_{j_0}$ with $j_0 \in J$, the upper envelope defining Φ has a complexity of at most $O(\lambda_4(r^0))$ (see [32, Theorem 6.1]), that is, the subdivision $\{\mathcal{R}'_j\}_{j\in J'} \cap \mathcal{R}_{j_0}$ has $O(\lambda_4(r^0))$ elements. Hence, the number of elements in the subdivision $\mathcal{P}$ is $O((Mgk_{\max})^2\lambda_4(r^0))$. Moreover, it can be computed in $O((Mgk_{\max})^2\lambda_4(r^0)\log(r^0))$, (see [32, Theorem 6.1]).

ALGORITHM 6.1

1. *INPUT: Globalizing function Φ, polyhedral norm γ and polyhedral sets $\{T_i\}_{i=1}^M$.*
2. *Generate the subdivision $\mathcal{P} := \{\mathcal{P}_j\}_{j\in\overline{J}}$ (including two dimensional cells, one dimensional boundaries and zero dimensional elements -extreme points-).*
3. *While $\overline{J} \neq \emptyset$,*
   Select $j \in \overline{J}$.
   (a) *Identify $p_{ij}(x) \in \partial\bar{d}_i(x)$ and $\delta \in \partial\Phi(\overline{D}(x))$ for each $x \in \mathcal{P}_j$.*
   (b) *Solve the system of linear equations $\sum_{i=1}^M \delta_i p_{ij}(x) = 0$, let $S(x)$ be the set of solutions of this system.*
      i. *If $S(x) \cap \mathcal{P}_j \neq \emptyset$ for some $x \in \mathcal{P}_j$ then $M_\Phi(\Upsilon) = S(x) \cap \mathcal{P}_j$. STOP.*
      ii. *Otherwise, remove j from $\overline{J}$.*
   End While.
4. *OUTPUT: $M_\Phi(\Upsilon)$, the optimal solution set of $P_\Phi(\Upsilon)$.*

The above algorithm consists of two main steps. The first one is a preprocessing step necessary to compute the subdivision $\mathcal{P}$. According to the discussion above this is doable in $O((Mgk_{max})^2\lambda_4(r^0)\log r^0)$. This arrangement provides information on which facets of the units balls of γ and Φ are active for each T_i, $i = 1, \ldots, M$ (Recall that a facet of a unit ball is active for a point if this point is included in the cone generated by that facet). This is sufficient to compute the elements p_{ij} and δ_i. We remark that with the above information computing the element p_{ij} for a given $i = 1, \ldots, M$ and $j \in \overline{J}$ may require the evaluation of at most g terms. Therefore, once the preprocessing step is done, computing each p_{ij} is doable in $O(g)$ time. Hence, the overall complexity of the preprocessing step is $O((Mgk_{max})^2\lambda_4(r^0)\log r^0 + (Mgk_{max})^2\lambda_4(r^0)g)$.

The main step of the algorithm is a while loop for all the elements in $\overline{J}$. Thus, there are $O((Mgk_{max})^2\lambda_4(r^0))$ iterations. In each iteration we solve a system of

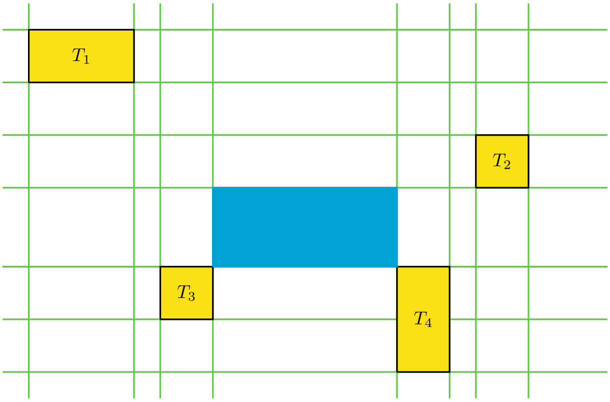

Fig. 2 Minisum problem

two linear equations in two variables to obtain the set $S(x)$. This is done in constant time. Then, we intersect $S(x)$ with the active element, $\mathcal{P}_j$, of the subdivision $\mathcal{P}$. Since $S(x)$ is either a point or a line or the whole plane, checking the intersection with $\mathcal{P}_j$ is doable in $O(\log(Mgk_{max} + r^0))$ time. Hence, the overall complexity for solving the expected distance location problem is $O((Mgk_{max})^2\lambda_4(n)\log r^0 + (Mgk_{max})^2\lambda_4(n)g + (Mgk_{\max})^2\lambda_4(r^0)\log(Mgk_{max} + r_0))$.

Example 6.2 Let T_1, T_2, T_3, T_4 and T_5 be the sets defined as follows: $T_1 = co\{(0,11),(0,13),(4,13),(4,11)\}$, $T_2 = co\{(17,7),(17,9),(19,9),(19,7)\}$, $T_3 = co\{(5,2),(5,4),(7,4),(7,2)\}$, $T_4 = co\{(14,0),(14,4),(16,4),(16,0)\}$, and $T_5 = co\{(10,11),(10,13),(12,13),(12,11)\}$.

We consider $\gamma(\cdot) = \|\cdot\|_1$-norm and the following objective functions:

$$\Phi_1(\overline{D}(x)) = \sum_{i=1}^{4} \bar{d}_i(x), \quad \Phi_2(\overline{D}(x)) = \sum_{i=1}^{5} \bar{d}_i(x)$$

$$\Phi_3(\overline{D}(x)) = \max_{i=1,\ldots,4} \bar{d}_i(x), \quad \Phi_4(\overline{D}(x)) = \sum_{i=3}^{4} \bar{d}_{(i)}(x)$$

where $d_{(i)}(x) = d_{\sigma_i}(x)$ with σ a permutation of $\{1,\ldots,4\}$, such that, $d_{\sigma_1}(x) \le \cdots \le d_{\sigma_4}(x)$.

Case $\Phi_1(\cdot)$: Taking $I = \{1,2,3,4\}$, $\delta = (1,1,1,1)$, $p_1 = (1,-1)$, $p_2 = (-1,-1)$, $p_3 = (1,1)$, and $p_4 = (-1,1)$, we can prove that (I,δ,p) is an optimizing triplet. Moreover, we have that $\overline{D}_I(\delta) = \mathbb{R}^2$ and $\overline{C}_I(p)$ is the shaded region in Fig. 2. Thus, the solution is the rectangle of vertices $(7,4)$, $(14,4)$, $(14,7)$, and $(7,7)$.

Case $\Phi_2(\cdot)$: Taking $I = \{1,2,3,4,5\}$, $\delta = (1,1,1,1,1)$, $p_1 = (1,-1)$, $p_2 = (-1,0)$, $p_3 = (1,1)$, $p_4 = (-1,1)$ and $p_5 = (0,-1)$, we can prove that (I,δ,p) is an optimizing triplet. Moreover, we have that $\overline{D}_I(\delta) = \mathbb{R}^2$ and $\overline{C}_I(p)$ is the point $x^* = (11,8)$ indicated as a dot in Fig. 3.

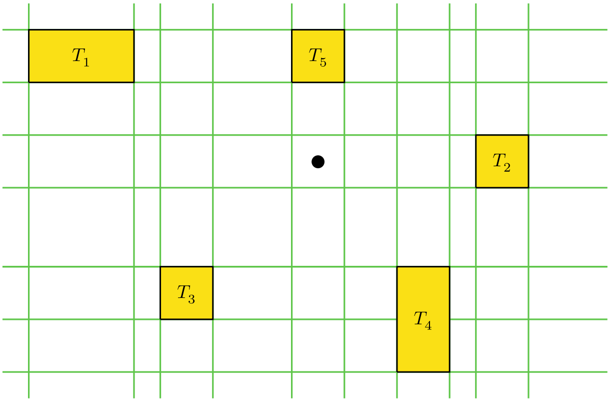

Fig. 3 Minisum problem

Case $\Phi_3(\cdot)$: The regions $\mathcal{R}_{(i)}$ for i=1,2,3,4 are the sets of points, such that, the maximum average distance is attained with respect to set T_i, that is,

$$\max_{i=1,2,3,4} \bar{d}_i(x) = \bar{d}_{i_0}(x), \quad \forall x \in \mathcal{R}_{(i_0)}.$$

Taking $I = \{1, 4\}$, $\delta = (\frac{1}{2}, 0, 0, \frac{1}{2})$, $p_1 = (1, -1)$, and $p_4 = (-1, 1)$, we can prove that (I, δ, p) is an optimizing triplet. Moreover, we have that $\overline{D}_I(\delta)$ is the segment with endpoints (8, 6.5) and (11, 9.5) (the thick line in Fig. 4) and $\overline{C}_I(p)$ the rectangle defined by the two closest vertices of T_1 and T_4.

Case $\Phi_4(\cdot)$: The regions $\mathcal{R}_{(i,j)}$ for $i, (\neq) j \in \{1, 2, 3, 4\}$, are defined such that,

$$\sum_{i=3}^{4} \bar{d}_{(i)}(x) = \bar{d}_{i_0}(x) + \bar{d}_{j_0}(x), \quad \forall x \in \mathcal{R}_{(i_0, j_0)}.$$

Taking $I = \{1, 4\}$, $\delta = (1, 0, 0, 1)$, $p_1 = (1, -1)$, and $p_4 = (-1, 1)$, we have that $\overline{D}_I(\delta)$ is the shaded region in Fig. 5 and $\overline{C}_I(p)$ is the rectangle defined by the two closest vertices of T_1 and T_4.

This example shows different shapes of optimal solution sets for average distance location problems on the $\|\cdot\|_1$-plane with globalizing functions Φ_i, $i = 1, \ldots, 4$. In the first two cases, the same globalizing function Φ_1 gives a full dimensional solution in Case 1 (the rectangle in Fig. 2) and a point in Case 2 (see Fig. 3). Case 3 shows a line segment as the optimal solution for the problem with Φ_3. Finally, Case 4 shows that one can also obtain solutions sets that are not polytopes (see Fig. 5).

7 Discretization

In this section we present some discretization results for the problems studied in previous sections, that allow us to obtain ε-approximate solution sets of these models by

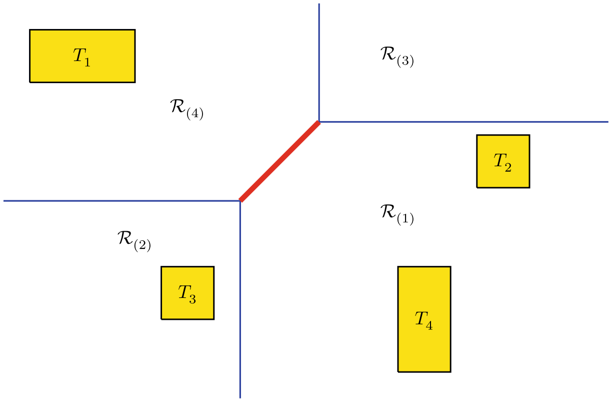

Fig. 4 Center problem

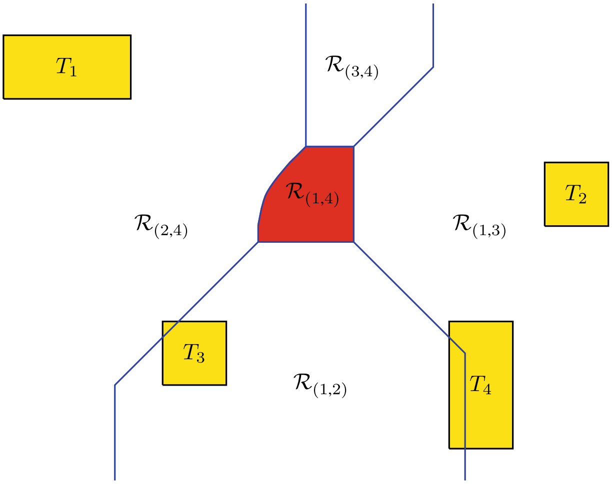

Fig. 5 Two-centrum problem

solving location problems with points, rather than sets, as demand facilities. These discretization results reduce the problem to a more friendly framework that permits us to use the large battery of tools available in the literature of facility location with respect to point facilities: exact, approximate and heuristic algorithms. Needless to say, one can not avoid the intrinsic difficulty of the original problem since the better the accuracy the finer the discretization required. In any case, under total polyhedrality, these auxiliary problems are rather simple since computation of integrals is not necessary. Thus, we can avoid the computation of the integration domain of the expected distances, i.e, $T \cap (x - N_{B^o}(u))$. Moreover, the subdivision generated by the family of sets $\overline{C}_I(p)$ induces actual linearity domains of the distances, e.g. distances cannot be quadratic in any of the elements of the subdivision.

Our discretization results are obtained by solving location problems with a countable number of points as demand facilities defined by a set A (at times A is also used itself for indexing summations extended over its own elements). For the ease of presentation, in this section we assume that $\gamma_t = \gamma$ for all $t \in T$. We consider the following auxiliary problems:

1. For $P_\phi(T)$, we solve the Weber problem,

$$\min_{x\in X} F_{W,A}(x) := \sum_{a\in A} w_a \gamma_a(x-a) \qquad (P_W(A))$$

   where $W = \{w_a\}_{a\in A}$ is a set of weights.
2. For $P_\Phi(\Upsilon)$, we solve

$$\min_{x\in X} \Phi(F_{W_1,A}(x), \dots, F_{W_M,A}(x)), \qquad (P_\Phi(A))$$

   where $F_{W_i,A}(x)$ is the objective function defined above, for the set of weights $W_i = \{w_{i,a}\}_{a\in A}$ with $i = 1, \dots, M$.

Notice that $P_W(A)$ and $P_\Phi(A)$ can be obtained as particular cases of problems $P_\phi(T)$ and $P_\Phi(\Upsilon)$, respectively, by taking μ and μ_i, for $i = 1, \dots, M$, as discrete measures, such that, $\mu(a) = w_a$ and $\mu_i(a) = w_{i,a}$ for $i = 1, \dots, M$. For these discrete versions, we have a characterization of their corresponding solution sets by Theorems 4.1 and 5.1. In the following result, we use these characterizations to obtain ε-approximate solutions for problems $P_\phi(T)$ and $P_\Phi(\Upsilon)$.

Theorem 7.1

1. *If $M_\phi(T) \neq \emptyset$ and $\mu(T) < +\infty$ then for any $\varepsilon > 0$ there exist countable sets $A \subseteq T$, $\{w_a \geq 0\}_{a\in A}$, and $\pi = \{p_a\}_{a\in A} \subseteq B^o$, such that, $\sum_{a\in A} w_a p_a = 0$ and $C_\pi(A) := \bigcap_{a\in A}(a + N_{B^o}(p_a))$ is an ε-solution set of Problem $P_\phi(T)$.*
2. *If $M_\Phi(\Upsilon) \neq \emptyset$ and $\mu_i(T) < +\infty$ for all $i = 1, \dots, M$ then for any $\varepsilon > 0$ there exist a countable set $A \subseteq T$, $W_i = \{w_{i,a} \geq 0\}_{a\in A}$ for any $i = 1, \dots, M$, and an optimizing triplet (I, δ, p) for Problem $P_\Phi(A)$, such that, $\overline{C}_I(p) \cap \overline{D}_I(\delta)$ is an ε-solution set of Problem $P_\Phi(\Upsilon)$.*

Proof

(1) Since (X, γ) is a separable Banach space, it contains a de Possel net (See [22, Lemma 1, Section 3]). This means that for any $\varepsilon > 0$ there exists a countable set $A \subset X$ and a partition $\mathcal{E} = \{E_a\}_{a\in A}$ of Borel subsets, verifying: (1) $E_a \cap E_{a'} = \emptyset$, $a \neq a'$, and $\bigcup_{a\in A} E_a = X$, (2) $int(E_a) \neq \emptyset$, $E_a \subset cl(int(E_a))$, $a \in A$, (3) $\sup_{a\in A} diam\,(E_a) < \varepsilon/(2\mu(T))$ (recall that $diam(E_a) = \sup_{y,z\in E_a} \gamma(y-z)$), and (4) $a \in E_a$ such that $E_a \subset B(a, \varepsilon/(2\mu T))$ ($\forall a \in A$). It is clear that for each $E_a \in \mathcal{E}$ there exists $b \in E_a$ such that $E_a \subset B(b, \varepsilon/(2\mu(T))$. Let $w_a := \mu(T \cap E_a)$, we bound $\phi(x)$ from below, for any $x \in X$, in the following

way:

$$\begin{aligned}\phi(x) &= \int\limits_{T} \gamma(x-t)\mu(\mathrm{d}t) = \sum_{a\in A} \int\limits_{T\cap E_a} \gamma(x-t)\mu(\mathrm{d}t)\\ &\geq \sum_{a\in A}\left(\int\limits_{T\cap E_a} \gamma(x-a)\mu(\mathrm{d}t) - \frac{\varepsilon\mu(T\cap E_a)}{2\mu(T)}\right)\\ &= \sum_{a\in A}\gamma(x-a)\mu(T\cap E_a) - \varepsilon/2.\end{aligned}$$

Thus, $\phi(x) \geq F_{W,A}(x) - \varepsilon/2$, for any $x \in X$. Analogously, $\phi(x) \leq F_{W,A}(x) + \varepsilon/2$. Hence, $|\phi(x) - F_{W,A}(x)| \leq \varepsilon/2, \forall x \in X$. In addition, this implies that $|\inf_{x\in X}\phi(x) - \inf_{x\in X} F_{W,A}(x)| \leq \varepsilon/2$. If we apply Theorem 4.1 to Problem $\mathrm{P}_W(A)$ then there exists $\pi = \{p_i\}_{i\geq 0}$ with $p_a \in B^o, a \in A$, satisfying $\sum_{a\in A} w_a p_a = 0$, such that the set of optimal solutions of $\mathrm{P}_W(A)$ is given by$C_\pi(A) := \bigcap_{a\in A}(a + N_{B^o}(p_a))$. Therefore, we get for any $y^* \in C_\pi(A)$.

$$\left|\inf_{x\in X}\phi(x) - \phi(y^*)\right| \leq \left|\inf_{x\in X}\phi(x) - F_{W,A}(y^*)\right| + \left|F_{W,A}(y^*) - \phi(y^*)\right| \leq \varepsilon$$

(2) By the continuity of Φ we have that for all $\varepsilon > 0$ there exist a $\delta > 0$, such that, $|\Phi(\bar{d}_1(x),\ldots,\bar{d}_M(x)) - \Phi(F_{W_1,A}(x),\ldots,F_{W_M,A}(x))| < \frac{\varepsilon}{2}\ \ \forall x \in X$, when $|\bar{d}_i(x) - F_{W_i,A}(x)| \leq \delta, \forall i = 1,\ldots,M$. Therefore, following a similar argument as for statement (1), there exist a countable set A and a partition $\mathcal{E} = \{E_a\}_{a\in A}$ of Borel subsets, verifying: (1) $E_a \cap E_{a'} = \emptyset, a \neq a'$, and $\bigcup_{a\in A} E_a = X$; (2) $int(E_a) \neq \emptyset, E_a \subset cl(int(E_a)), a \in A$; and, (3) $\sup_{a\in A} diam\,(E_a) < \delta/(\max_{i=1,\ldots,M}\mu_i(T))$; such that, for $w_{i,a} := \mu_i(T \cap E_a), a \in A, i = 1,\ldots,M$; $|\bar{d}_i(x) - F_{W_i,A}(x)| \leq \delta, \forall i = 1,\ldots,M$ and $x \in X$. In addition, by continuity of Φ, this implies that

$$\left|\inf_{x\in X}\Phi(\bar{d}_1(x),\ldots,\bar{d}_M(x)) - \inf_{x\in X}\Phi(F_{W_1,A}(x),\ldots,F_{W_M,A}(x))\right| \leq \varepsilon/2.$$

Applying Theorem 5.1 to Problem $P_\Phi(A)$, we have that there exists an optimizing triplet (I,δ,p) such that $M_\Phi(A) = \overline{D}_I(\delta) \cap \overline{C}_I(p)$. Therefore, for any $y^* \in \overline{D}_I(\delta) \cap \overline{C}_I(p)$, we get $|\inf_{x\in X}\Phi(\bar{d}_1(x),\ldots,\bar{d}_M(x)) - \Phi(\bar{d}_1(y^*),\ldots,\bar{d}_M(y^*))| \leq |\inf_{x\in X}\Phi(\bar{d}_1(x),\ldots,\bar{d}_M(x)) - \Phi(F_{W_1,A}(y^*),\ldots,F_{W_M,A}(y^*))| + |\Phi(F_{W_1,A}(y^*),\ldots,F_{W_M,A}(y^*)) - \Phi(\bar{d}_1(y^*),\ldots,\bar{d}_M(y^*))| \leq \varepsilon$, and the result follows. □

It is worth noting that if T were a compact set, the cardinality of A would be finite. Therefore, the number of demand points defining the Weber problem $P_W(A)$ would be finite as well.

8 Concluding remarks

Most of the results in this paper extend to more general situations with weaker hypotheses as for instances removing separability or considering σ-fields different from the Borel one. However, to improve readability we have restricted ourselves to the common framework of separable Banach spaces. It is also possible to extend the results in this paper to location problems where the solution x is restricted to belong to a convex set. In this case all the characterizations also depend on the normal cone to the constraint set. Finally, the approach in this paper also applies to the location of a measurable set, say S (whose shape is fixed), rather than to a single point. Indeed, the problem:

$$\inf_{x\in X} F(x) := \int_T \int_S \gamma(x+s-t)\mu(\mathrm{d}t)\nu(\mathrm{d}s),$$

can be interpreted as the minimization of the average distance of a shape S, with the associated measure ν, with respect to T. By considering the change of variable $b = s - t$, and denoting by η the measure defined on the set $B = S - T$ (see [9]), we get $\inf_{x\in X} F(x) = \inf_{x\in X} \int_B \gamma(x-b)\eta(\mathrm{d}b)$, which falls into the class of problems considered in the paper.

References

1. Barbu, V., Precupanu, T.: Convexity and Optimization in Banach Spaces. Sijthoff & Noordhoff Publishers, Leiden (1975)
2. Bauer, F.L., Stoer, J., Witzgall, C.: Absolute and monotonic norms. Numer. Math. **3**, 257–264 (1961)
3. Brimberg, J., Wesolowsky, G.O.: Note: facility location with closest rectangular distances. Nav. Res. Logist. **47**, 77–84 (2000)
4. Brimberg, J., Wesolowsky, G.O.: Locating facilities by minimax relative to closest points of demand areas. Comp. Oper. Res. **29**, 625–636 (2002)
5. Brimberg, J., Wesolowsky, G.O.: Minisum location with closest Euclidean distances. Ann. Oper. Res. **111**, 151–165 (2002)
6. Carrizosa, E., Muñoz, M., Puerto, J.: The Weber problem with regional demand. Eur. J. Oper. Res. **104**, 358–365 (1998)
7. Carrizosa, E., Muñoz, M., Puerto, J.: Location and shape of a rectangular facility in $\mathbb{R}^n$. Convexity properties. Math. Program. **83**, 277–290 (1998)
8. Carrizosa, E., Muñoz, M., Puerto, J.: A note on the optimal positioning of service units. Oper. Res. **46**(1), 155–156 (1998)
9. Carrizosa, E., Conde, E., Muñoz, M., Puerto, J.: The generalized Weber problem with expected distances. RAIRO Rech. Oper. **29**, 35–57 (1995)
10. Drezner, Z.: Location of regional facilities. Nav. Res. Logist. Q. **33**, 523–529 (1986)
11. Drezner, Z., Wesolowsky, G.: Optimal location of a demand facility relative to area demand. Nav. Res. Logist. Q. **27**, 199–206 (1980)
12. Durier, R.: The general center location problem. Math. Oper. Res. **20**, 400–414 (1995)
13. Durier, R., Michelot, C.: Geometrical properties of the Fermat-Weber problem. Eur. J. Oper. Res. **20**, 332–343 (1985)
14. Edelsbrunner, H.: Algorithms in Combinatorial Geometry. Springer, Berlin (1987)
15. Ehrgott, M.: Location of rescue helicopters in South Tyrol. Int. J. Ind. Eng. Theory Appl. Pract. **9**, 16–22 (2002)
16. Garkavi, A.L., Smatkov, V.A.: On the Lamé point and its generalizations in a normed space. Math. USSR Sb. **24**(2), 267–286 (1974)

17. Giles, J.R.: Convex analysis with applications in differentiation of convex functions. Research Notes in Math. n. 58, Pitman Advanced Publishing Program (1982)
18. Hakimi, S.L.: Optimum location of switching centers and the absolute centers and medians of a graph. Oper. Res. **12**, 450–459 (1964)
19. Hiriart-Urruty, J.B., Lemarechal, C.: Convex Analysis and Minimization Algorithms I. Springer, Berlin (1993)
20. Hooker, J.N., Garfinkel, R.S., Chen, C.K.: Finite dominating sets for network location problems. Oper. Res. **39**(1), 100–118 (1991)
21. Horváth, J.: Topological Vector Spaces and Distributions, vol. I. Addison-Wesley, Reading (1966)
22. Ioffe, A.D., Levin, V.L.: Subdifferentials of convex functions. Trans. Mosc. Math. Soc. **26**, 1–72 (1972)
23. Koshizuka, T., Kurita, O.: Approximate formulas of average distances associated with regions and applications to location problems. Math. Program. **52**, 99–123 (1991)
24. Love, R.F., Morris, J.G., Wesolowsky, G.O.: Facilities Location: Models and Methods. North Holland, New York (1988)
25. Luenberger, D.G.: Optimization by Vector Space Methods. Wiley, New York (1969)
26. Nickel, S., Puerto, J.: Location Theory: A Unified Approach. Springer, Heidelberg (2005)
27. Nickel, S., Puerto, J., Rodríguez-Chía, A.M.: An approach to location models involving sets as existing facilities. Math. Oper. Res. **28**, 693–715 (2003)
28. Papini, P.L.: Two new examples of sets without Centers. Top **13**(2), 315–320 (2005)
29. Puerto, J., Fernández, F.R.: Geometrical properties of the symmetrical single facility location problem. J. Nonlinear Convex Anal. **1**(3), 321–342 (2000)
30. Puerto, J., Rodríguez-Chía, A.M.: Robust positioning of service units. Stoch. Models **19**(1), 125–147 (2003)
31. Rockafellar, R.T.: Convex Analysis. Princeton Press, Princeton (1970)
32. Sharir, M., Agarwal, P.: Davenport-Schinzel Sequences and Their Geometric Applications. Cambridge University Press, Cambridge (1995)
33. Tapia, R.A.: A characterization of inner product spaces. Proc. AMS **41**, 569–574 (1973)
34. Valero-Franco, C., Rodríguez-Chía, A.M., Espejo-Miranda, I.: The single facility location problem with average-distances. Top **16**, 164–194 (2008)
35. Vickson, R.G., Gerchak, Y., Rotem, D.: Optimal positioning of read/write head in mirrored disks. Locat. Sci. **3**(2), 125–132 (1995)
36. Wendell, R., Hurter, A.: Location theory, dominance and convexity. Oper. Res. **21**, 314–320 (1973)