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FULL LENGTH PAPER

Merit functions in vector optimization

C. G. Liu · K. F. Ng · W. H. Yang

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Abstract We study the weak domination property and weakly efficient solutions in vector optimization problems. In particular scalarization of these problems is obtained by virtue of some suitable merit functions. Some natural conditions to ensure the existence of error bounds for merit functions are also given.

Keywords Weakly efficient solution · Merit function · Weak domination property · Error bound

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1 Introduction

We consider the following vector optimization problem denoted by $\mathcal{P}(F, A, C)$ (or simply by $\mathcal{P}$):

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C. G. Liu (✉)
Department of Mathematics, Jinan University, Guangzhou 510630, People's Republic of China
e-mail: tcgliu@jnu.edu.cn

K. F. Ng
Department of Mathematics, The Chinese University of Hong Kong, Shatin, New Territory, Hong Kong
e-mail: kfng@math.cuhk.edu.hk

W. H. Yang
School of Mathematical Sciences, Fudan University, Shanghai 200433, People's Republic of China
e-mail: whyang@fudan.edu.cn

$$\begin{array}{l} \min_C\ F(x) \\ \text{s.t. } x \in A, \end{array} \tag{1.1}$$

where F is a continuous linear map from a Banach space X into a Banach space Y, A is a closed convex subset of X, and C is a closed convex cone in Y with nonempty interior, that is, $\text{int}C \neq \emptyset$. To solve a vector optimization problem such as that in (1.1) the most important methods are probably the methods of scalarization, and research in this area has been very active. Let us discuss briefly a few of them here. The first is on the linear scalarization (cf. [21, Theorem 4.2.10]) by which we know that $\bar{a} \in A$ is a weakly efficient solution (namely $F(\bar{a}) \notin F(A) + \text{int}C$) of the problem (1.1) if and only if there exists $\xi \in C^+\backslash\{0\}$ such that it is a solution of the scalarized problem

$$\begin{array}{l} \min\ (\xi \circ F)(x) \\ \text{s.t. } x \in A \end{array}, \tag{1.2}$$

where $C^+ := \{y^* \in Y^* : \langle y^*, c\rangle \geq 0, \forall c \in C\}$, Y^* denotes the dual space of Y and $\langle y^*, c\rangle := y^*(c)$. Note that, in order to find all weakly efficient solutions of $\mathcal{P}$, one has potentially to consider numerous scalarized problems of the type (1.2), and if ξ is fixed some weakly efficient solutions may very well be of large distance to the solution set of the problem (1.2). Similar remarks can be said for the scalarized methods respectively proposed by Jahn [18,19] and by Zaffaroni [29]. In the approach of Zaffaroni, he made use of the Hiriart-Urruty function Δ_C of C (see (3.1)). Then $\bar{a} \in A$ is a weakly efficient solution of (1.1) if and only if there exists $l \in Y$ such that $\bar{a}$ is a solution of the following scalarized problem $\mathcal{P}_l$:

$$\begin{array}{l} \min\ \Delta_C(l - F(a)) \\ \text{s.t. } a \in A \end{array}.$$

Though one still has to work on possibly all the problems $\mathcal{P}_l (l \in Y)$ in order to get all weakly efficient solutions of (1.1), a big advantage of Zaffaroni's method is that different types of solutions of (1.1) can be described by the same function Δ_C.

Borrowing a terminology from the Variational Inequality Problems (cf. [9]), we say that $\varphi\colon A \to [0, +\infty]$ is a merit function for problem (1.1) if $\varphi^{-1}(0) = \{a \in A : \varphi(a) = 0\}$ coincides with the set of all the weakly efficient solutions of $\mathcal{P}$. We study some merit functions (defined in terms of Hiriart-Urruty functions) and establish their dual representations. For a class of merit functions we present some necessary/sufficient conditions for them to have error bounds. In particular we show that these error bound properties are equivalent to the linear regularity of the pair $\{A, X\backslash(A+\text{int}(F^{-1}(C)))\}$, and are related to the WDP (the weak domination property) of $F(A)$.

2 Notations and preliminary results

In general for any normed vector space Z and $x \in Z$, we use $B(x, r)$ to denote the closed ball with center x and radius r. Let B_Z denote the closed unit ball in Z and let

$S_Z = \{z \in Z : \|z\| = 1\}$. Let Z^* denote the Banach dual space of Z and, for $z \in Z$ and $z^* \in Z^*$, let $\langle z^*, z\rangle := z^*(z)$. Given any subset K in a normed vector space, $S(K)$ denotes the subset of K consisting of all $k \in K$ with $\|k\| = 1$. Let int$K$, bd(K), cl(K), co(K), $\overline{\text{co}}(K)$, lin(K) and aff(K) respectively denote the interior, boundary, closure, convex hull, closed convex hull, linear hull and affine hull of K. Let rint(K) and $ri(K)$ be defined by

$$\text{rint}(K) = \{k \in K : \exists r > 0 \quad \text{s.t. aff}(K) \cap B(k, r) \subset K\}$$

and

$$ri(K) := \begin{cases} \text{rint}(K) & \text{if aff}(K) \text{ is closed} \\ \emptyset & \text{otherwise} \end{cases}$$

(cf. [31, pp. 14–15]). The negative polar of a set K in a normed vector space Z is defined by

$$K^{\ominus} := \{z^* \in Z^* : \langle z^*, z\rangle \leq 0, \quad \forall z \in K\}$$

(thus $K^{\ominus}$ is the usual polar of K if K is a cone). For convenience, we also use the notation $K^+ := -K^{\ominus}$. Let $d_K(\cdot)$ (or $d(\cdot, K)$) denote the distance function of K, i.e.,

$$d_K(z) := \inf\{\|z - y\| : y \in K\}$$

with the convention that $d_K(\cdot) = +\infty$ when K is the empty set. The indicator function of K is

$$\iota_K(z) := \begin{cases} 0 & \text{if } z \in K, \\ +\infty & \text{if } z \in Z\backslash K, \end{cases}$$

and the projection of z onto K is defined by

$$P_K(z) = \{\bar{z} \in K : \|z - \bar{z}\| = d_K(z)\}.$$

For a convex subset K, the tangent cone and the normal cone of K at $z \in K$ are respectively defined by (cf. [1, p. 166])

$$T_K(z) := \text{cl}\left(\cup_{t>0} t(K - z)\right)$$

and

$$N_K(z) := \{z^* \in Z^* : \langle z^*, h\rangle \leq 0, \quad \forall h \in T_K(z)\}.$$

For a proper function $f : Z \to \mathbb{R} \cup \{+\infty\}$, $D \subset Z$ and $\tau > 0$, we say that f has an error bound (more precisely, has an error bound τ) on D if $f \geq 0$ on D and

$$d_{D\cap f^{-1}(0)}(z) \leq \tau f(z), \quad \forall z \in D. \tag{2.1}$$

This is a condition ensuring that $z \in D$ is near to the solution set $f^{-1}(0)$ if $f(z)$ is sufficiently small.

Given a partial order $\leq_K$ defined by a closed convex cone K in Z, a function $f : D \to (-\infty, +\infty]$ is said to be monotone (K-monotone, more precisely) if $f(z_1) \leq f(z_2)$ whenever $z_1 \leq_K z_2$ and $z_1, z_2 \in D$.

For any two closed convex cones K_1, K_2 in a normed vector space Z we use $\measuredangle(K_1, K_2)$ to denote the quantity

$$\measuredangle(K_1, K_2) = \begin{cases} \inf\{d(k, K_2) : k \in S(K_1)\} & \text{if } K_1 \neq \{0\} \text{ and } K_2 \neq \{0\} \\ 1 & \text{if } K_1 = \{0\} \text{ or } K_2 = \{0\} \end{cases}. \tag{2.2}$$

So

$$\measuredangle(K_1, K_2) \in [0, 1].$$

Moreover we have the following lemma:

Lemma 1 *Let Z be a normed vector space and let K_1 and K_2 be two closed convex cones in Z. Then*

$$\frac{1}{2}\measuredangle(K_2, K_1) \leq \measuredangle(K_1, K_2) \leq 2\measuredangle(K_2, K_1) \tag{2.3}$$

Proof If K_1 or K_2 is $\{0\}$, then (2.3) holds trivially. Suppose that $K_1, K_2 \neq \{0\}$ and define

$$\gamma(K_1, K_2) := \inf\{\|k_1 - k_2\| : k_1 \in S(K_1), k_2 \in S(K_2)\}. \tag{2.4}$$

We claim that

$$\measuredangle(K_1, K_2) \leq \gamma(K_1, K_2) \leq 2\measuredangle(K_1, K_2). \tag{2.5}$$

Granting this (2.3) follows because $\gamma(K_1, K_2) = \gamma(K_2, K_1)$. The first inequality of (2.5) is obviously true by (2.2) and (2.4). Let $k_1 \in S(K_1)$. For any $\epsilon > 0$, there exists $k_\epsilon \in K_2$ such that

$$\|k_1 - k_\epsilon\| < d(k_1, K_2) + \epsilon. \tag{2.6}$$

Since $K_2 \neq \{0\}$, there exists $k_2 \in S(K_2)$ such that $k_\epsilon = \|k_\epsilon\| k_2$. Then

$$\begin{aligned} \|k_1 - k_2\| &\leq \|k_1 - k_\epsilon\| + \|k_\epsilon - k_2\| \\ &= \|k_1 - k_\epsilon\| + |\|k_\epsilon\| - 1| \\ &\leq \|k_1 - k_\epsilon\| + |\|k_\epsilon\| - \|k_1\|| \\ &\leq 2\|k_1 - k_\epsilon\|. \end{aligned}$$

By (2.6), we have that

$$\gamma(K_1, K_2) \le \|k_1 - k_2\| \le 2\|k_1 - k_\epsilon\| < 2d(k_1, K_2) + 2\epsilon.$$

Let ϵ converge to zero and take the infimum of $d(k_1, K_2)$ for $k_1 \in S(K_1)$, the second inequality of (2.5) holds and the proof is complete. □

3 Hiriart-Urruty function

We give in this section some dual representation results for Hiriart-Urruty functions Δ_D. Recall that (cf. [14,15,29]) for a set D in normed vector space Z, the function $\Delta_D : Z \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$\Delta_D(z) := d_D(z) - d_{Z\setminus D}(z) = \begin{cases} d_D(z), & z \in Z\setminus D \\ -d_{Z\setminus D}(z), & z \in D \end{cases} \tag{3.1}$$

(thus $\Delta_D \equiv +\infty$ if D is empty and $\Delta_D \equiv -\infty$ if $D = Z$). It is easy to verify that

$$\Delta_D = -\Delta_{Z\setminus D}. \tag{3.2}$$

From (3.1) we note that, if D is a non-trivial subset of Z (that is, $\emptyset \ne D \ne Z$), then, as the difference of two continuous functions, Δ_D is continuous on the whole space. The authors are indebted to the referee for pointing out the following lemma which is basically known:

Lemma 2 *Let A be a nonempty convex subset of a normed vector space Z. Then the following assertions hold for any $x \in Z$:*

- (i) $d_A(x) = d_{\mathrm{cl}(A)}(x) = \sup_{x^* \in B_{X^*}} \inf_{u \in \mathrm{cl}(A)} \langle x^*, x - u\rangle = \sup_{x^* \in B_{X^*}} \inf_{u \in A} \langle x^*, x - u\rangle$.
- (ii) $d_A(x) = \max\{0, \sup_{x^* \in S_{X^*}} \inf_{u \in A} \langle x^*, x - u\rangle\}$.
- (iii) $d_A(x) = \sup_{x^* \in S_{X^*}} \inf_{u \in A} \langle x^*, x - u\rangle \iff \sup_{x^* \in S_{X^*}} \inf_{u \in A} \langle x^*, x - u\rangle \ge 0$.
- (iv) *If* $\mathrm{int}A \ne \emptyset$ *and* $x \in Z\setminus\mathrm{int}A$ *then*

$$d_A(x) = \sup_{x^* \in S_{X^*}} \inf_{u \in A} \langle x^*, x - u\rangle.$$

- (v) *If* $A = D + K$ *with* K *a convex cone,* $\mathrm{int}A \ne \emptyset$*, and if* $x \in Z\setminus\mathrm{int}A$*, then*

$$d_A(x) = \sup_{x^* \in S(K^+)} \inf_{u \in A} \langle -x^*, x - u\rangle = \sup_{x^* \in S(K^+)} \inf_{z \in D} \langle -x^*, x - z\rangle.$$

Proof Part (i) is from [31, Theorem 3.8.2], and (ii) follows immediately because $B_{X^*} = \cup_{0 \le t \le 1} t S_{X^*}$. (iii) follows from (ii). (iv) follows from (iii) together with a classical separation theorem. Since $\inf_{k \in K} \langle x^*, k\rangle = -\infty$ if and only if $x^* \notin K^+$, (v) follows from (iv). □

Remark 1 By the usual Hahn-Banach argument, the assumption $\mathrm{int}A \neq \emptyset$ in (iv) and (v) can be replaced by the weaker assumption that $ri(A) \neq \emptyset$.

Proposition 1 *Let A be a convex subset of Z and suppose that $ri(A) \neq \emptyset$. Then*

$$\Delta_A(x) = \sup_{x^* \in S_{X^*}} \inf_{u \in A} \langle x^*, x - u\rangle, \quad \forall x \in Z. \tag{3.3}$$

Consequently, if $D + K$ is a convex subset of Z with K a convex cone such that $\mathrm{int}K \neq \emptyset$, then

$$\Delta_{D+K}(x) = \sup_{x^* \in S(K^+)} \inf_{u \in D+K} \langle -x^*, x - u\rangle = \sup_{x^* \in S(K^+)} \inf_{u \in D} \langle -x^*, x - u\rangle \tag{3.4}$$

for all $x \in Z$.

Proof As in Lemma 2(v), the second assertion follows from the first. By Lemma 2(iv) (and Remark 1), to prove the first assertion it is sufficient (since $-S_{X^*} = S_{X^*}$) to show that

$$d_{Z\setminus A}(x) = \inf_{x^* \in S_{X^*}} \sup_{u \in A} \langle x^*, x - u\rangle, \quad \forall x \in A.$$

To do this let $x \in A$ and consider any $r > d_{Z\setminus A}(x)$. Then there exists $z_1 \in Z\setminus A$ such that $r > \|z_1 - x\|$. Since $d_A(z_1) = \sup_{x^* \in S_{X^*}} \inf_{u \in A} \langle x^*, z_1 - u\rangle$ by Lemma 2(iv), for each $\epsilon > 0$ there exists $x^*_\epsilon \in S_{X^*}$ such that

$$d_A(z_1) - \epsilon < \inf_{u \in A} \langle x^*_\epsilon, z_1 - u\rangle.$$

Then, for all $u \in A$, one has

$$\begin{aligned}\langle -x^*_\epsilon, x - u\rangle = \langle -x^*_\epsilon, x - z_1\rangle + \langle -x^*_\epsilon, z_1 - u\rangle &\leq \|x - z_1\| - (d_A(z_1) - \epsilon)\\ &< r - d_A(z_1) + \epsilon.\end{aligned}$$

This implies that $\inf_{x^* \in S_{X^*}} \sup_{u \in A} \langle x^*, x - u\rangle < r - \epsilon$. Letting $\epsilon \to 0$ and $r \to d_{Z\setminus A}(x)$ it follows that $\inf_{x^* \in S_{X^*}} \sup_{u \in A} \langle x^*, x - u\rangle \leq d_{Z\setminus A}(x)$. For the proof of the converse inequality we may assume that $d_{Z\setminus A}(x) > 0$. Consider positive real numbers r', r'' such that $d_{Z\setminus A}(x) > r' > r''$. Let $x^* \in S_{X^*}$. Then there exists $w \in B(0, r')$ such that $\langle x^*, w\rangle > r''$. Moreover, since $B(x, r') \subset A$, one has $B(0, r') \subset A - x$ and so $w \in B(0, r') \subset A - x$. Consequently

$$r'' < \langle x^*, w\rangle \leq \sup_{u \in A} \langle x^*, x - u\rangle$$

and so $r'' \leq \inf_{x^* \in S_{X^*}} \sup_{u \in A} \langle x^*, x - u\rangle$. The proof is completed by letting $r'' \to d_{Z\setminus A}(x)$. □

Since $\mathrm{cl}(D + \mathrm{int}K) = \mathrm{cl}(\mathrm{int}(D + K)) = \mathrm{cl}(D + K)$ (provided that $\mathrm{int}K \neq \emptyset$), the following corollaries follow immediately from the preceding proposition.

Corollary 1 *Suppose that D is a nonempty convex set and that K is a nonempty convex cone with $\mathrm{int}K \neq \emptyset$. Then*

$$\Delta_{D+\mathrm{int}K}(z) = \sup_{z^* \in S(K^+)} \inf_{a \in D} \langle -z^*, z - a\rangle, \quad \forall z \in Z, \tag{3.5}$$

that is,

$$\Delta_{D+\mathrm{int}K}(\cdot) = \Delta_{D+K}(\cdot) \quad on\ Z. \tag{3.6}$$

Corollary 2 *For a convex cone K with $\mathrm{int}K \neq \emptyset$, we have that*

$$\Delta_K(z) = \sup_{z^* \in S(K^+)} \langle -z^*, z\rangle, \quad \forall z \in Z.$$

Remark 2 (a) This corollary is known [10] in the special case when Z ia finite dimensional.
(b) The referee kindly pointed out that formula (3.3) in Proposition 1 is known [6] for the special case when A is a proper closed convex solid subset of $\mathbb{R}^n$, and that (3.4) is known [11] for $K = \{0\}$.

For the remainder of this section we assume that K is a closed convex cone in a normed vector space Z such that $\mathrm{int}K \neq \emptyset$. Let $D \subset Z$ be such that $D + K$ is convex and let

$$\widehat{D} = Z \backslash (D + \mathrm{int}K).$$

For our convenience we list below some properties of the Hiriart-Urruty function $\Delta_{\widehat{D}}$ defined by $\widehat{D}$, that is,

$$\Delta_{\widehat{D}} = d_{\widehat{D}} - d_{D+\mathrm{int}K}. \tag{3.7}$$

Proposition 2 *The following assertions hold*:

(i) $\Delta_{\widehat{D}}(z) = d_{\widehat{D}}(z)$ *for all* $z \in D$.
(ii) $\Delta_{\widehat{D}}(z) = \inf_{z^* \in S(K^+)} \sup_{a \in D} \langle z^*, z - a\rangle = \inf_{z^* \in S(K^+)} \{\langle z^*, z\rangle - \inf_{a \in D} \langle z^*, a\rangle\}, \quad \forall z \in Z.$
(iii) $\Delta_{\widehat{D}}$ *is K-monotone.*

Proof (i) Let $z \in D$. Since $\mathrm{int}K$ is assumed nonempty, there exists $e \in \mathrm{int}K$. Since $\lim_{n\to\infty}(z + \frac{1}{n}e) = z$ it follows that $z \in \mathrm{cl}(D + \mathrm{int}K)$ and hence that $d_{D+\mathrm{int}K}(z) = 0$. Thus (i) holds by (3.7).
(ii) By (3.2), $\Delta_{\widehat{D}} = -\Delta_{D+\mathrm{int}K}$ and so the first equality in (ii) follows immediately from (3.4) and (3.6), while the second equality is obvious.
(iii) (iii) follows from (ii). □

It is well known and easy to verify that (cf. [31, Theorem 1.1.9])

$$K = \{x \in X : \langle x^*, x\rangle \geq 0, \forall x^* \in K^+\}. \tag{3.8}$$

Let $\text{rext}(K^+)$ denote the union of all extreme rays of K^+ but deleting the origin, that is, $x^* \in \text{rext}(K^+)$ if and only if $x^* \neq 0$ such that $\mathbb{R}_+ x^*$ is an extreme ray of K^+. Note that

$$\overline{\text{co}}^{w^*}[\text{rext}(K^+)] = w^*\text{-cl}\,\{\Sigma_{i=1}^n x_i^* : \text{each } x_i^* \in \text{rext}(K^+)\}.$$

Proposition 3 *Let* $e \in \text{int}K$ *with* $e + \delta B_Z \subset K$ *for some* $\delta > 0$. *Then the following assertions hold*:

(i)

$$\langle z^*, e\rangle \geq \delta\|z^*\|, \quad \forall z^* \in K^+. \tag{3.9}$$

(ii)

$$K^+ = \overline{\text{co}}^{w^*}\left[\text{rext}(K^+)\right] = \overline{\text{co}}^{w^*}\{tx^* : t > 0, x^* \in \text{rext}(K^+) \textit{ and } \|x^*\| = 1\}. \tag{3.10}$$

(iii)

$$K = \{x \in X : \langle x^*, x\rangle \geq 0, \forall x^* \in S(K^+) \cap \text{rext}(K^+)\}. \tag{3.11}$$

(iv)

$$\inf_{x^* \in S(K^+)} \langle x^*, k\rangle = \inf_{z^* \in S(K^+) \cap \text{rext}(K^+)} \langle z^*, k\rangle, \quad \forall k \in K. \tag{3.12}$$

Proof It is routine to verify (i) and the second equality in (3.10) is evident. To prove the first equality in (3.10), let $H := \{x^* \in X^* : \langle x^*, e\rangle = 1\}$. Then $H \cap K^+$ is norm-bounded by (3.9), and is clearly a w^*-closed convex set disjoint from $\{0\}$. Consequently, by an elementary result (cf. [26, Exercise 2.30]) the first equality in (3.10) holds. Combining (3.8) and (3.10), we have (3.11). To prove (3.12), we need only to show the "$\geq$" part. Let $x^* \in S(K^+)$ and $k \in K$. We have to show that

$$\langle x^*, k\rangle \geq \inf_{z^* \in S(K^+) \cap \text{rext}(K^+)} \langle z^*, k\rangle. \tag{3.13}$$

Fix x^* and k. By (3.10), there exist $\{\lambda_i\}_{i=1}^\infty \subset (0, +\infty)$ and $\{x_i^*\}_{i=1}^\infty \subset \text{co}[S(K^+) \cap \text{rext}(K^+)]$ such that $\lambda_i x_i^* \xrightarrow{w^*} x^*$. Then $\|x_i^*\| \leq 1$ for any $i \in \mathbb{N}$, and

$$\liminf_{i \to +\infty} \lambda_i \geq \liminf_{i \to +\infty} \|\lambda_i x_i^*\| \geq \|x^*\| = 1, \tag{3.14}$$

thanks to the w^*-lower semicontinuity of the norm on X^*. Note that for any $i \in \mathbb{N}$,

$$\langle \lambda_i x_i^*, k\rangle \geq \lambda_i \left[\inf_{z^* \in \mathrm{co}[S(K^+)\cap \mathrm{rext}(K^+)]} \langle z^*, k\rangle\right] = \lambda_i \left[\inf_{z^* \in S(K^+)\cap \mathrm{rext}(K^+)} \langle z^*, k\rangle\right].$$

Passing to the lower limits, it follows from (3.14) that

$$\liminf_{i\to+\infty} \langle \lambda_i x_i^*, k\rangle \geq \inf_{z^* \in S(K^+)\cap \mathrm{rext}(K^+)} \langle z^*, k\rangle,$$

which implies (3.13) because $\lambda_i x_i^* \xrightarrow{w^*} x^*$. □

Another useful function defined in terms of Hirrart-Urruty functions is ξ defined by

$$\xi_{D,K}(z) := -\inf_{a\in D} \Delta_K(z-a), \quad \forall z \in Z. \tag{3.15}$$

Proposition 4 *The following assertions hold*:

(i)

$$\xi_{D,K}(z) = \sup_{a\in D} \inf_{z^*\in S(K^+)} \langle z^*, z-a\rangle, \quad \forall z \in Z. \tag{3.16}$$

(ii) $\xi_{D,K}(z) \geq 0$ *for all* $z \in D$.
(iii) $\xi_{D,K}$ *is* K*-monotone.*
(iv) $\Delta_{\widehat{D}} \geq \xi_{D,K}$ *on* Z.
(v) $\xi_{D,K}(z) = \sup_{a\in D} \inf_{z^*\in S(K^+)\cap \mathrm{rext}(K^+)} \langle z^*, z-a\rangle, \quad \forall z \in Z.$

Proof By (3.15) and Corollary 2, (3.16) follows. It is also clear that (i) implies (ii) and (iii). Moreover, comparison of (i) with Proposition 2(ii) gives (iv). Finally (v) follows from (i) and (3.12). □

4 Weakly efficient points and weak domination property

Given a subset A of a normed vector space Z with a closed convex cone K such that $\mathrm{int}K \neq \emptyset$, we use $\mathrm{WMin}(A, K)$ to denote the set of all weakly efficient points of A, that is

$$\mathrm{WMin}(A, K) := \{z \in A : z \notin A + \mathrm{int}K\}. \tag{4.1}$$

It is well known (and easy to verify) that

$$\mathrm{WMin}(A, K) \subset \mathrm{bd}(A). \tag{4.2}$$

For $x \in Z$, A_x denotes the section of A at x (cf. [21, p. 43]), that is, $A_x := A \cap (x - K) = \{a \in A : a \leq_K x\}$, where $\leq_K$ denotes the partial ordering induced by K while $\ll_K$ will serve to denote the stronger relation defined by

$$a_1 \ll_K a_2 \Longleftrightarrow a_2 - a_1 \in \text{int}K.$$

For any $a \in A_x$, we note that $z \in A_x$ whenever $z \leq_K a$ and $z \in A$. We recall a well known concept that A is said to have the weak domination property (WDP) if for any $z \in A \backslash \text{WMin}(A, K)$ there exists $z' \in \text{WMin}(A, K)$ such that $z' \leq_K z$. This implies a (formally) stronger property that for any $z \in A \backslash WMIN(A, K)$ there exists

$$z'' \in \text{WMin}(A, K) \quad \text{s.t.} \quad z'' \ll_K z.$$

Indeed if $z \in A \backslash \text{WMin}(A, K)$ then by (4.1) there exists $y \in A$ such that $z \in y + \text{int}K$ (that is $y \ll_K z$). If this y is not already in $\text{WMin}(A, K)$ then $y \in A \backslash \text{WMin}(A, K)$ and it follows from the assumption (applied to y in place of x) that there exists $y' \in \text{WMin}(A, K)$ such that $y' \leq_K y$. Since $y \ll_K z$ this implies that $y' \ll_K z$ and so y' has the desired property stated for z''.

It is known [21, Proposition 2.4.10] that if A is compact then it has the WDP. Another sufficient condition is provided by Atlouch-Riahi Theorem (cf. [12, Theorem 3.2.36]). If Z is a Banach space and there exists $x^* \in Z^*$ such that $K \subset \{x \in Z : \langle x^*, z\rangle \geq \delta\|x\|\}$ ($\delta > 0$, e.g., if $x^* \in \text{int}C^+$) and x^* is bounded below on A then A has the WDP. Theorem 2 will present further sufficient conditions.

For the remainder of this section, let X, Y, C, A, F be as explained in Sect. 1, and we consider $\mathcal{P} := \mathcal{P}(F, A, C)$ as in (1.1). Let $E_w(\mathcal{P}(F, A, C))$ (or $E_w(\mathcal{P})$ for short if no confusion can arise) denote the set of all weakly efficient solutions of the problem (1.1), that is

$$E_w(\mathcal{P}(F, A, C)) = \{a \in A : F(a) \in \text{WMin}(F(A), C)\}.$$

Thus, for any $a \in A$, a is a weakly efficient solution if and only if there does not exist $a' \in A$ such that $F(a') \ll_C F(a)$. To avoid the triviality we always assume that

$$F(X) \cap \text{int}C \neq \emptyset \tag{4.3}$$

(otherwise $E_w(\mathcal{P}) = A$). Let

$$C_X := F^{-1}(C).$$

It is known and easy to verify that

$$\text{int}C_X = F^{-1}(\text{int}C) \neq \emptyset, \tag{4.4}$$

thanks to the assumption (4.3). Let $F^* : Y^* \to X^*$ denote the adjoint of F, that is,

$$\langle F^*(y^*), x\rangle = \langle y^*, F(x)\rangle, \quad \forall y^* \in Y^*, x \in X.$$

It is also known [30] that

$$F^*(C^+) = C_X{}^+. \tag{4.5}$$

Theorem 1 *It holds that*

$$A\backslash(A + \text{int}C_X) = \text{WMin}(A, C_X) = E_w(\mathcal{P}) \subset \text{bd}(A), \tag{4.6}$$

and for any $a \in A$, *the following assertions are equivalent*:

(i) $a \in E_w(\mathcal{P})$, (ii) $N_A(a) \cap [-F^*(C^+)] \neq \{0\}$,
(iii) $N_A(a) \cap [-C_X{}^+] \neq \{0\}$.

Proof Note that for any $a \in A$,

$$\begin{aligned} a \notin \text{WMin}(A, C_X) &\Leftrightarrow \exists a' \in A \text{ s.t. } a - a' \in \text{int}C_X \\ &\Leftrightarrow \exists a' \in A \text{ s.t. } F(a - a') \in \text{int}C \\ &\Leftrightarrow a \notin E_w(\mathcal{P}). \end{aligned}$$

This together with (4.2) makes (4.6) clear.
Moreover by (1.2), we have the following equivalences:

$$\begin{aligned} a \in \text{WMin}(A, C_X) &\Leftrightarrow \exists \xi \in C_X{}^+\backslash\{0\} \text{ s.t. } \xi(a) \leq \xi(x), \quad \forall x \in A \\ &\Leftrightarrow \exists \xi \in C_X{}^+\backslash\{0\} \text{ s.t. } -\xi \in N_A(a). \end{aligned}$$

Thus (i)⇔(iii) by (4.6). The equivalence (ii)⇔(iii) follows from (4.5). □

From the proof for (4.6), the first assertion of the following proposition is clear. The second assertion then follows easily.

Corollary 3 *Let* $A \subset X$ *be a nonempty convex set,* $F \in L(X, Y)$ *and* $C \subset Y$ *a closed cone with* $ImF \cap \text{int}C \neq \emptyset$. *Then*

$$\text{WMin}(A, F^{-1}(C)) = A \cap F^{-1}(\text{WMin}(F(A), C)),$$

and A *has the WDP with respect to* $F^{-1}(C)$ *if and only if* $F(A)$ *has the WDP with respect to* C.

Theorem 2 $F(A)$ *has the WDP if (at least) one of the following conditions is satisfied*:

(i) *A is a polyhedron, and there exists* $x^* \in C_X{}^+\backslash\{0\}$ *such that* x^* *is bounded below on* A.
(ii) *A is a polyhedron and* $E_w(\mathcal{P}) \neq \emptyset$.

Proof (i) Let $x_0^* \in C_X{}^+\backslash\{0\}$ be bounded below on A, and suppose that

$$A = \left[\cap_{i\in I}\{\langle x_i^*, \cdot\rangle \geq b_i\}\right] \cap \left[\cap_{j\in J}\{\langle x_j^*, \cdot\rangle = b_j\}\right], \tag{4.7}$$

where I and J are disjoint finite sets. Denote by $|I|$ the number of elements in I. To prove (i) we apply the mathematical induction on $|I|$. First, for the case when $I = \emptyset$, we have $A = \cap_{j\in J}\{\langle x_j^*, \cdot\rangle = b_j\}$. For any $a \in A$, we have $N_A(a) = \mathrm{lin}\{x_j^* : j \in J\}$ and $A = a + H$, where $H := \cap_{j\in J}\{\langle x_j^*, \cdot\rangle = 0\}$. Since $\inf_A\langle x_0^*, \cdot\rangle > -\infty$, x_0^* must vanish at the vector subspace H. Thus $x_0^* \in \mathrm{lin}\{x_j^* : j \in J\}$, that is, $-x_0^* \in N_A(a)$. Since $x_0^* \in {C_X}^+\backslash\{0\}$ we have $N_A(a) \cap [-{C_X}^+] \neq \{0\}$ and it follows from Theorem 1 that $a \in E_w(\mathcal{P})$ for all $a \in A$. Therefore we have $E_w(\mathcal{P}) = A$ and so $\mathrm{WMin}(F(A), C) = F(A)$ by (4.1). In particular $F(A)$ has the weak domination property provided that $|I| = 0$ in (4.7).

We suppose that $F(A)$ has the weak domination property whenever A is a subset of X expressible in the form (4.7) with $|I| \leq m$ (and with any finite set J). Now consider the case that A is defined by (4.7) but with $|I| = m + 1$. For contradiction we suppose that $F(A)$ does not have the WDP; that is, there exists $\hat{x} \in A\backslash E_w(\mathcal{P})$ such that for any $x' \in A$,

$$F(x') \ll_C F(\hat{x}) \Rightarrow x' \notin E_w(\mathcal{P}). \tag{4.8}$$

For any $k \in I$, define A_k by

$$A_k := \left[\cap_{i\in I\backslash\{k\}}\{\langle x_i^*, \cdot\rangle \geq b_i\}\right] \cap \left[\cap_{j\in J\cup\{k\}}\{\langle x_j^*, \cdot\rangle = b_j\}\right]. \tag{4.9}$$

Thus $A_k \subset A$. We claim that for any $x \in A\backslash E_w(\mathcal{P})$, there exist $k \in I$ and $a \in A_k$ such that $F(a) \in \mathrm{WMin}(F(A_k), C)$ and $F(a) \ll_C F(x)$. Indeed, fix $x \in A\backslash E_w(\mathcal{P})$; then $x \notin \mathrm{WMin}(A, C_X)$, and there exists $c \in \mathrm{int}C_X$ such that $x - c \in A$. Let $\mathcal{T} := \{t > 0 : x - tc \in A\}$ and $\bar{t} := \sup\mathcal{T}$. Then $\bar{t} \geq 1$. Moreover,

$$\langle x_0^*, x - tc\rangle \geq \inf_A\langle x_0^*, \cdot\rangle > -\infty, \quad \forall t \in \mathcal{T}.$$

Noting $\langle x_0^*, c\rangle > 0$ by (3.9), it follows that

$$t \leq \frac{\langle x_0^*, x\rangle - \inf_A\langle x_0^*, \cdot\rangle}{\langle x_0^*, c\rangle} < +\infty, \quad \forall t \in \mathcal{T},$$

and so $\bar{t} < +\infty$. Let $\bar{x} = x - \bar{t}c$. Then $\bar{x} \in A$ and $\bar{x} \ll_{C_X} x$. We show next that there exists $k \in I$ such that

$$\langle x_k^*, \bar{x}\rangle = b_k. \tag{4.10}$$

In fact if there does not exist such k then for each $i \in I$, $\langle x_i^*, \cdot\rangle > b_i$ at $\bar{x}$ and so there exists $\epsilon > 0$ such that

$$\langle x_i^*, \bar{x} - \epsilon c\rangle > b_i, \quad \forall i \in I. \tag{4.11}$$

On the other hand, since $\bar{x}, x, x - c \in A \subset \cap_{j\in J}\{\langle x_j^*, \cdot\rangle = b_j\}$, it is easy to verify that $\langle x_j^*, \bar{x} - \epsilon c\rangle = b_j$ for each $j \in J$. Together with (4.11) and (4.7), this implies that $x - (\bar{t} + \epsilon)c = \bar{x} - \epsilon c \in A$, which contradicts the definition of $\bar{t}$. Therefore there must exist some $k \in I$ such that (4.10) holds and so $\bar{x} \in A_k$ by (4.9). By the mathematical induction assumption, $F(A_k)$ has the weak domination property. Thus there exists $a \in A_k$ such that $F(a) \in \mathrm{WMin}(F(A_k), C)$ and $F(a) \ll_C F(\bar{x})$. Since $\bar{x} \ll_{C_X} x$ it follows that $F(a) \ll_C F(x)$. Therefore our claim is true.

In particular applying this established claim to $\hat{x}$ in place of x, there exist $k_1 \in I$ and $a_1 \in A_{k_1}$ with $F(a_1) \in \mathrm{WMin}(F(A_{k_1}), C)$ such that $F(a_1) \ll_C F(\hat{x})$. Since $A_{k_1} \subset A$ it follows from (4.8) that $a_1 \in A \backslash E_w(\mathcal{P})$. Applying the established claim (to a_1 in place of x) there exist $k_2 \in I$ and $a_2 \in A_{k_2}$ with $F(a_2) \in \mathrm{WMin}(F(A_{k_2}), C)$ such that $F(a_2) \ll_C F(a_1)$ ($\ll_C F(\hat{x})$). Repeating this process, we obtain sequences $\{k_n\}_{n=1}^{+\infty} \subset I$ and $\{a_n\}_{n=1}^{+\infty} \subset A \backslash E_w(\mathcal{P})$ with each $a_n \in A_{k_n}$ such that

$$F(a_n) \in [\mathrm{WMin}(F(A_{k_n}), C)], \quad \forall n \in \mathbb{N},$$

and

$$F(a_{n+1}) \ll_C F(a_n), \quad \forall n \in \mathbb{N}. \tag{4.12}$$

Since $|I| < +\infty$, there must exist distinct natural numbers n_1, n_2 (say $n_1 < n_2$) such that $k_{n_1} = k_{n_2}$. So

$$F(a_{n_1}) \in \mathrm{WMin}(F(A_{k_{n_1}}), C) = \mathrm{WMin}(F(A_{k_{n_2}}), C).$$

This contradicts the facts that $a_{n_2} \in A_{k_{n_2}}$ and $F(a_{n_2}) \ll_C F(a_{n_1})$ (see (4.12)). Thus $F(A)$ has the weak domination property and the proof is completed by the mathematical induction.

(ii) Let $a \in E_w(\mathcal{P})$. By Theorem 1, there exists $x^* \in [(-N_A(a)) \cap {C_X}^+] \backslash \{0\}$. Then $\min_A \langle x^*, \cdot \rangle = x^*(a)$ is finite and so (ii) follows immediately form (i). □

5 Merit functions

We continue our discussion on problem (1.1), with notation being the same as in the preceding section. For the remainder of this paper, let

$$\widehat{A} := X \backslash (A + \mathrm{int} C_X) \tag{5.1}$$

and suppose that the set $E_w(\mathcal{P}) = E_w(\mathcal{P}(F, A, C))$ of weakly efficient solutions of (1.1) is nonempty. Recalling (4.6), this means that

$$E_w(\mathcal{P}) = A \cap \widehat{A} \neq \emptyset. \tag{5.2}$$

By virtue of the function ξ studied in Sect. 3, we define $\theta : A \to [0, +\infty]$ by

$$\begin{aligned}\theta(x) &:= \xi_{F(A),C}(F(x)) = -\inf_{a\in A} \triangle_C(F(x) - F(a))\\ &= \sup_{a\in A} \inf_{y^*\in S(C^+)} \langle y^*, F(x) - F(a)\rangle\\ &= \sup_{a\in A} \inf_{y^*\in S(C^+)\cap \mathrm{rext}(C^+)} \langle y^*, F(x) - F(a)\rangle, \quad \forall x \in X. \end{aligned} \tag{5.3}$$

(see Proposition 4).

Example 1 Let $(Y, C) = (\mathbb{R}^m, \mathbb{R}^m_+)$; thus (Y^*, C^+) is identified with (Y, C) and $S(C^+) \cap \mathrm{rext}(C^+)$ is simply $\{e_1, \ldots, e_m\}$, where e_i is the i^{th} unit vector in $\mathbb{R}^m$, that is, its i^{th} coordinate is 1 and the other coordinates are zero. Represent $F : X \to \mathbb{R}^m$ in the form $F = (f_1, \ldots, f_m)$ with each f_i continuous linear functional on X:

$$F(x) = (f_1(x), \ldots, f_m(x)), \quad x \in X.$$

Thus, $\langle e_i, F(x) - F(a)\rangle = f_i(x) - f_i(a)$ and

$$\theta(x) = \sup_{a\in A} \min_{1\le i\le m} \{f_i(x) - f_i(a)\}, \quad \forall x \in A.$$

The importance of the function θ define in (5.3) lies in the following result.

Theorem 3 *Consider problem* (1.1) *and let* θ *be defined as in* (5.3). *Then* θ *is a* C_X *monotone merit function for* (1.1) *such that*

$$\theta(x) \le \|F\| \cdot d_{\widehat{A}}(x), \quad \forall x \in A, \tag{5.4}$$

and, for some constant $r > 0$,

$$r \cdot d_{\widehat{A}}(x) \le \theta(x), \quad \forall x \in A. \tag{5.5}$$

Proof Let $x_1 \le_{C_X} x_2$. Since $C_X := F^{-1}(C)$, it follows that $F(x_1) \le_C F(x_2)$. By the definition of θ given in (5.3), we then have $\theta(x_1) \le \theta(x_2)$. Moreover, for any $x \in A$ we have from (5.3) that $\theta(x) > 0$ if and only if there exists $a_x \in A$ such that $\Delta_C(F(x) - F(a_x)) < 0$, that is, by (3.1), there exists $a_x \in A$ such that

$$F(x) - F(a_x) \in C \quad \text{and} \quad d_{Y\setminus C}(F(x) - F(a_x)) > 0,$$

equivalently $F(x) - F(a_x) \in \mathrm{int}C$, that is $x \notin E_w(\mathcal{P})$. Since, as already noted, $\theta \ge 0$ on A. This implies that $\theta^{-1}(0) = E_w(\mathcal{P})$; thus θ is a merit function for problem (1.1). To prove (5.4), let $x \in A$ and we may suppose further that $\theta(x) > 0$. By (5.3) and (3.5) we note that

$$-\theta(x) = \inf_{a\in A} \sup_{y^*\in S(C^+)} \langle -y^*, F(x) - F(a)\rangle$$

and

$$\Delta_{F(A)+\mathrm{int}C}(F(x)) = \sup_{y^*\in S(C^+)} \inf_{a\in A} \langle -y^*, F(x) - F(a)\rangle.$$

Comparing the right-hand members of the above two equalities we have

$$\Delta_{F(A)+\mathrm{int}C}(F(x)) \le -\theta(x).$$

Since $\theta(x) > 0$ and by (3.1), this means that

$$-d_{Y\backslash(F(A)+\mathrm{int}C)}(F(x)) \le -\theta(x). \tag{5.6}$$

On the other hand, since $\widehat{A} := X\backslash(A + \mathrm{int}C_X)$ and thanks to (4.4), one can easily check that

$$F(\widehat{A}) \subseteq Y\backslash(F(A) + \mathrm{int}C)$$

and so (5.6) entails that

$$\theta(x) \le d_{F(\widehat{A})}(F(x)) \le \|F\| \cdot d_{\widehat{A}}(x),$$

which proves (5.4).

By (4.4), there exists $h \in X$ such that

$$0 \ne h \in \mathrm{int}C_X = F^{-1}(\mathrm{int}C). \tag{5.7}$$

Let $\alpha := \|h\|$ and let $c := F(h)$. Then there exist $\delta > 0$ such that $c + \delta B_Y \subset C$ and it follows from Proposition 3 (i) that

$$\delta \le \inf_{y^*\in S(C^+)} \langle y^*, c\rangle. \tag{5.8}$$

Let $r := \frac{\delta}{\alpha}$. Since $\theta \ge 0$ on A, to prove (5.5) we may assume that $d_{\widehat{A}}(x) > 0$. Take $r' > 0$ such that $r' < \alpha^{-1} d_{\widehat{A}}(x)$. Then

$$d_{\widehat{A}}(x - r'h) \ge d_{\widehat{A}}(x) - \|r'h\| = d_{\widehat{A}}(x) - r'\alpha > 0$$

and hence we have that $x - r'h \in A + \mathrm{int}C_X$. Thus there exists $a' \in A$ such that $x - r'h \in a' + \mathrm{int}C_X$, and it follows from (5.7) that $F(x) - F(a') \in r'c + \mathrm{int}C$. Consequently, (5.8) entails that

$$r'\delta \le \inf_{y^*\in S(C^+)} \langle y^*, r'c\rangle \le \inf_{y^*\in S(C^+)} \langle y^*, F(x) - F(a')\rangle$$

and it follows from (5.3) that $r'\delta \le \theta(x)$. Letting $r' \to \alpha^{-1} d_{\widehat{A}}(x)$, (5.5) is seen to hold with $r = \delta\alpha^{-1}$. □

Motivated by (5.4) and (5.5), we say that a merit function g for problem (1.1) is said to be regular if there exist some constants $\tau_1, \tau_2 > 0$ such that

$$\tau_1 d_{\widehat{A}}(x) \le g(x) \le \tau_2 d_{\widehat{A}}(x) \quad \forall x \in A \tag{5.9}$$

(thus, e.g., θ is regular). By (5.9) and (2.1), clearly a regular merit function g for problem (1.1) has an error bound on A if and only if $d_{\widehat{A}}$ does.

Example 2 By (3.1), (3.2) and (3.5), we note that, for all $x \in X$,

$$d_{\widehat{A}}(x) - d_{A+\mathrm{int}C_X}(x) = \Delta_{\widehat{A}}(x) = -\Delta_{A+\mathrm{int}C_X}(x) = \inf_{x^* \in S(C_X^+)} \sup_{a \in A} \langle x^*, x - a\rangle. \tag{5.10}$$

We define $g(x) = d_{\widehat{A}}(x)$ if $x \in A$ and $g(x) = +\infty$ otherwise. Since $A \subseteq \mathrm{cl}(A + \mathrm{int}C_X)$, it follows from (5.10) that

$$g(x) = d_{\widehat{A}}(x) = \inf_{x^* \in S(C_X^+)} \sup_{a \in A} \langle x^*, x - a\rangle, \quad \forall x \in A. \tag{5.11}$$

In particular g is C_X-monotone, $g^{-1}(0) = A \cap \widehat{A} = E_w(\mathcal{P})$. It is now easily seen that g is a regular C_X-monotone merit function for problem (1.1).

Recall that two sets A_1, A_2 in X are said to be linearly regular if there exists a constant $\tau > 0$ such that

$$d_{A_1 \cap A_2}(x) \le \tau \max\left\{d_{A_1}(x), d_{A_2}(x)\right\} \quad \forall x \in X.$$

Theorem 4 *Let $\widehat{A}$ be defined as in (5.1). Then the following statements are equivalent*:

(i) *The pair $\{A, \widehat{A}\}$ is linearly regular, that is, there exists $\tau > 0$ such that*

$$d_{A \cap \widehat{A}}(x) \le \tau \max\{d_A(x), d_{\widehat{A}}(x)\} \quad \forall x \in X.$$

(ii) *$d_{\widehat{A}}$ has an error bound on A.*

Proof (ii) $\Rightarrow$ (i). By (5.2), $A \cap \widehat{A} = A \cap d_{\widehat{A}}^{-1}(0)$ and so (ii) implies that there exists $\tau > 0$ such that

$$d_{A \cap \widehat{A}}(x) \le \tau d_{\widehat{A}}(x) \quad \forall x \in A. \tag{5.12}$$

By [23, Theorem 3.1(b)], this implies that (i) holds. Similarly (i) $\Rightarrow$ (ii) can be proved with ease. □

6 Error bounds

To prepare for our main results of this section, we first establish a preparatory proposition. Regarding its assumptions, note that, by Theorem 2(ii), if A is a polyhedron then $F(A)$ has the weak domination property.

Proposition 5 *Suppose that $F(A)$ has the WDP. Then $d_{\widehat{A}}$ has an error bound if and only if there exists $\tau > 0$ such that for any $x^* \in S(C_X{}^+) \cap \mathrm{bd}(C_X{}^+)$,*

$$d(x, E_w(\mathcal{P})) \leq \tau \sup_{a\in A}\langle x^*, x-a\rangle = \tau\left[\langle x^*, x\rangle - \inf_A \langle x^*, \cdot\rangle\right], \quad \forall x \in A. \tag{6.1}$$

((6.1) *holds trivially for those x^* with* $\inf_A\langle x^*, \cdot\rangle = -\infty$.)

Proof The second equality in (6.1) is obvious. Suppose that $d(\cdot, \widehat{A})$ has an error bound on A, that is, there exists $\tau > 0$ such that (5.12) holds. Then (6.1) holds for the same τ and for any $x^* \in S(C_X{}^+)$, thanks to (5.2) and (5.11).

Conversely suppose that there exists $\tau > 0$ such that (6.1) holds for any $x^* \in S(C_X{}^+) \cap \mathrm{bd}(C_X{}^+)$. Let $\epsilon > 0$. It suffices to show that

$$d_{E_w(\mathcal{P})}(x) \leq \max\{1, \tau\} d_{\widehat{A}}(x) + \epsilon, \quad \forall x \in A. \tag{6.2}$$

To show this, let $x \in A \backslash E_w(\mathcal{P})$. Then $x \notin \widehat{A}$ and hence there exists $x_1 \in \mathrm{bd}(\widehat{A})$ such that

$$\|x - x_1\| < d_{\widehat{A}}(x) + \epsilon. \tag{6.3}$$

If x_1 happens to belong to $E_w(\mathcal{P})$ then (6.2) holds. We may therefore assume that $x_1 \notin E_w(\mathcal{P})$. Let $r := d(x_1, E_w(\mathcal{P}))$. Then $r > 0$. Moreover by (5.1) we have $\mathrm{bd}(\widehat{A}) = \mathrm{bd}(A + \mathrm{int} C_X)$, and so x_1 is a boundary point of $A + \mathrm{int}C_X$. Thus by the Separation Theorem (cf. [16, p. 63]), there exists $x_0^* \in S_{X^*}$ such that

$$\langle x_0^*, x_1\rangle = \inf_{A+\mathrm{int}C_X} \langle x_0^*, \cdot\rangle. \tag{6.4}$$

This implies that $x_0^* \in C_X{}^+$, $\inf_{c\in \mathrm{int}C_X}\langle x_0^*, c\rangle = 0$ and

$$\langle x_0^*, x_1\rangle = \inf_A \langle x_0^*, \cdot\rangle. \tag{6.5}$$

We claim that $x_0^* \in \mathrm{bd}(C_X{}^+)$ (and so the proof will be complete because (6.2) follows from (6.1), (6.3), (6.4) and (6.5)). Let us assume on the contrary that $x_0^* \in \mathrm{int}(C_X{}^+)$, that is, $x_0^* + \delta B_{X^*} \subset C_X{}^+$ for some $\delta > 0$. This implies that

$$\langle x_0^*, c\rangle - \delta\|c\| = \inf_{x_0^*+\delta B_{X^*}} \langle \cdot, c\rangle \geq \inf_{C_X{}^+}\langle \cdot, c\rangle = 0, \quad \forall c \in C_X. \tag{6.6}$$

On the other hand, since $x_1 \in \mathrm{bd}(A + \mathrm{int}C_X)$, there exists $x_2 \in A + \mathrm{int}C_X$ such that

$$\|x_2 - x_1\| < \frac{r\delta}{1+\delta}. \tag{6.7}$$

Let $a \in A$ be such that $x_2 \in a + \mathrm{int}C_X$, that is, $F(x_2) - F(a) \in \mathrm{int}C$ (see (4.4)). Since $F(A)$ has the WDP, we may suppose that $a \in \mathrm{WMin}(F(A), C)$, that is, $a \in E_w(\mathcal{P})$. Applying (6.6) to $x_2 - a$ in place of c, we have

$$\langle x_0^*, x_2 - a\rangle \geq \delta\|x_2 - a\| \geq \delta\|x_1 - a\| - \delta\|x_1 - x_2\|,$$

and hence that

$$\langle x_0^*, x_1 - a\rangle = \langle x_0^*, x_1 - x_2\rangle + \langle x_0^*, x_2 - a\rangle \geq \delta\|x_1 - a\| - (1+\delta)\|x_1 - x_2\|.$$

Since $\|x_1 - a\| \geq d(x_1, E_w(\mathcal{P})) = r$, it follows from (6.7) that $\langle x_0^*, x_1 - a\rangle > 0$. Therefore

$$\langle x_0^*, x_1\rangle > \langle x_0^*, a\rangle \geq \inf_A \langle x_0^*, \cdot\rangle.$$

This contradicts (6.5). Thus our claim $x_0^* \in \mathrm{bd}(C_X{}^+)$ stands and the proof is complete. □

The following result is known (see, for example, [17, Prop 2.1], with $S = \mathrm{cl}(D)$).

Lemma 3 *Let $f : X \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function which is bounded from below, $D \subset \mathrm{dom} f$ and $r > 0$. Assume that for any $u \in \mathrm{dom} f \setminus \mathrm{cl}(D)$, there exists $v \in X\setminus\{u\}$ such that $\gamma\|u - v\| \leq f(u) - f(v)$. Then D is nonempty and*

$$\gamma d_D(x) \leq f(x) - \inf f \quad \forall x \in X.$$

Recall that the notation $\measuredangle$ in the following theorem is defined in (2.2).

Theorem 5 *Suppose $F(A)$ has the WDP and that there exists γ with $0 < \gamma \leq 1$ such that for any $x \in \mathrm{bd}(A)\setminus E_w(\mathcal{P})$ one has that $\measuredangle(-C_X{}^+, N_A(x)) \geq \gamma$, that is*

$$\inf_{x^* \in S(C_X{}^+)} d(-x^*, N_A(x)) \geq \gamma. \tag{6.8}$$

Then $d_{\widehat{A}}$ has an error bound on A.

Proof We suppose that $\mathrm{bd}(A) \neq E_w(\mathcal{P})$ (otherwise the result holds trivially). By (6.8), the condition $\gamma \leq 1$ is automatically satisfied. Let $\tau = \gamma^{-1}$, $x^* \in S(C_X{}^+) \cap \mathrm{bd}(C_X{}^+)$ and $\alpha := \inf_A \langle x^*, \cdot\rangle$. By Proposition 5, it suffices to show that for all $x \in A$

$$d(x, E_w(\mathcal{P})) \leq \tau[\langle x^*, x\rangle - \alpha]. \tag{6.9}$$

We may assume that $\alpha > -\infty$. Let $\epsilon_1 \in (0, \frac{\gamma}{2})$. We claim that

$$d(x, E_w(\mathcal{P})) \leq \frac{1+\epsilon_1}{\gamma - 2\epsilon_1}[\langle x^*, x\rangle - \alpha] \quad \forall x \in \mathrm{bd}(A). \tag{6.10}$$

Granting this and letting $\epsilon_1 \to 0$, (6.9) is seen to hold provided that x is a boundary point of A.

Recall from (4.6) that $E_w(\mathcal{P})$ is contained in $\mathrm{bd}(A)$. In view of Lemma 3 (applied to $x^* + \iota_{\mathrm{bd}(A)}$, $\frac{\gamma-2\epsilon_1}{1+\epsilon_1}$, $E_w(\mathcal{P})$ in place of f, γ, and D), (6.10) will follow if one can show that for any $u \in \mathrm{bd}(A)\backslash E_w(\mathcal{P})$ there exists $v \in \mathrm{bd}(A)\backslash\{u\}$ such that

$$\frac{\gamma - 2\epsilon_1}{1+\epsilon_1}\|u - v\| \leq \langle x^*, u - v\rangle. \tag{6.11}$$

To do this let $u \in \mathrm{bd}(A)\backslash E_w(\mathcal{P})$. Then by (6.8), $-x^* + (\gamma - \epsilon_1)B_{X^*}$ is disjoint from $N_A(u)$. Hence, by the Alaoglu Theorem and the Separation Theorem (cf. [16, p. 70 and p. 63]), there exists an element $h \in X$ of norm 1 such that

$$\sup_{x^*+N_A(u)} \langle \cdot, h\rangle \leq \inf_{(\gamma-\epsilon_1)B_{X^*}} \langle \cdot, h\rangle = -(\gamma - \epsilon_1). \tag{6.12}$$

Since $N_A(u)$ is a cone it follows that $\sup_{N_A(u)}\langle \cdot, h\rangle \leq 0$ and so $h \in T_A(u)$ because $T_A(u)$ and $N_A(u)$ are polars to each other (see [31, p. 87]). Hence there exists $h_1 \in \cup_{t>0}\frac{1}{t}[A - u]$ such that $\|h - h_1\| < \epsilon_1$. Then $\|h_1\| \leq 1 + \epsilon_1$ and $\mathcal{T}_1 := \{t > 0 : u + th_1 \in A\}$ is nonempty. Moreover (6.12) implies that

$$\langle x^*, h_1\rangle = \langle x^*, h\rangle + \langle x^*, h_1 - h\rangle < -(\gamma - \epsilon_1) + \epsilon_1 = -\gamma + 2\epsilon_1 < 0$$

and hence that $\lim_{t\to+\infty}\langle x^*, th_1\rangle = -\infty$. Since $\inf_A\langle x^*, \cdot\rangle$ is finite it follows that $\mathcal{T}_1$ must be bounded and hence has a maximum element which will be denoted by t_1. Then $0 < t_1 < +\infty$ and $u + t_1h_1 \in \mathrm{bd}(A)$. Consequently $v := u + t_1h_1$ has the desired property stated in (6.11) because

$$\frac{\gamma - 2\epsilon_1}{1+\epsilon_1}\|u - v\| \leq \frac{\|u - v\|}{\|h_1\|}(\gamma - 2\epsilon_1) = t_1(\gamma - 2\epsilon_1) < t_1\langle x^*, -h_1\rangle = \langle x^*, u - v\rangle.$$

Therefore (6.10) is established and so is (6.9) for $x \in \mathrm{bd}(A)$.

It remains to show (6.9) for $x \in \mathrm{int}A$. Let $x \in \mathrm{int}A$ and let $\epsilon \in (0, \frac{1}{2})$. Then there exists $h_\epsilon \in X$ of norm 1 such that

$$\langle x^*, h_\epsilon\rangle > (1 - \epsilon) > \frac{1}{2}. \tag{6.13}$$

Let $\mathcal{T}_\epsilon := \{t \geq 0 : x - th_\epsilon \in A\}$. Recalling $\alpha = \inf_A\langle x^*, \cdot\rangle$ it follows that

$$t \leq \frac{\langle x^*, x\rangle - \alpha}{\langle x^*, h_\epsilon\rangle} \leq 2\left[\langle x^*, x\rangle - \alpha\right] \quad \forall t \in \mathcal{T}_\epsilon.$$

Thus $\mathcal{T}_\epsilon$ is a nonempty bounded closed set contained in the interval $[0, 2\langle x^*, x\rangle - 2\alpha]$. Let t_ϵ denote the maximum element of $\mathcal{T}_\epsilon$, and let $x_\epsilon := x - t_\epsilon h_\epsilon$. Then $x_\epsilon \in \mathrm{bd}(A)$ and by what has already been proved,

$$d(x_\epsilon, E_w(\mathcal{P})) \le \tau[\langle x^*, x_\epsilon\rangle - \alpha]. \tag{6.14}$$

Moreover,

$$\langle x^*, x - x_\epsilon\rangle = \langle x^*, t_\epsilon h_\epsilon\rangle \le t_\epsilon \le 2[\langle x^*, x\rangle - \alpha]. \tag{6.15}$$

On the other hand, since $\tau = \gamma^{-1} \ge 1$ and by (6.13), we have

$$\frac{1}{1-\epsilon} = \frac{\epsilon}{1-\epsilon} + 1 \le \frac{\epsilon}{1-\epsilon} + \tau,$$

and $(1-\epsilon) < \langle x^*, h_\epsilon\rangle = \left\langle x^*, \frac{x-x_\epsilon}{\|x-x_\epsilon\|}\right\rangle$. Thus $\|x - x_\epsilon\| \le (\frac{\epsilon}{1-\epsilon} + \tau)\langle x^*, x - x_\epsilon\rangle$ and it follows from (6.14) and (6.15) that

$$\begin{aligned}
d(x, E_w(\mathcal{P})) &\le \|x - x_\epsilon\| + d(x_\epsilon, E_w(\mathcal{P}))\\
&\le \left(\frac{\epsilon}{1-\epsilon} + \tau\right)\langle x^*, x - x_\epsilon\rangle + \tau[\langle x^*, x_\epsilon\rangle - \alpha]\\
&= \frac{\epsilon}{1-\epsilon}\langle x^*, x - x_\epsilon\rangle + \tau[\langle x^*, x\rangle - \alpha]\\
&\le \frac{2\epsilon}{1-\epsilon}[\langle x^*, x\rangle - \alpha] + \tau[\langle x^*, x\rangle - \alpha].
\end{aligned}$$

Letting $\epsilon \to 0$, we see that (6.9) is satisfied for any interior point x in A. The proof is complete. □

Remark 3 Recall (cf. [31, p. 5]) that $x \in A$ is called a support point of A if $N_A(x) \ne \{0\}$, that is, there exists $x^* \in X^*\backslash\{0\}$ such that

$$\langle x^*, x\rangle = \sup_A \langle x^*, \cdot\rangle.$$

Denote by $\mathrm{supp}(A)$ the set of all support points of A. Note that $\mathrm{supp}(A) \subset \mathrm{bd}(A)$.

Theorem 5 remains true if one assumes (6.8) to hold for all $x \in \mathrm{supp}(A)\backslash E_w(\mathcal{P})$ rather than for all $x \in \mathrm{bd}(A)\backslash E_w(\mathcal{P})$. In fact if $x \in \mathrm{bd}(A)\backslash \mathrm{supp}(A)$, then $N_A(x) = \{0\}$ and so the infimum in (6.8) is simply 1 and hence (6.8) is automatically satisfied for this x.

Corollary 4 *Suppose $F(A)$ has the WDP and that there exists γ with $0 < \gamma \le 1$ such that for any $x \in \mathrm{bd}(A)\backslash E_w(\mathcal{P})$ one has that $\measuredangle(N_A(x), -C_X{}^+) \ge \gamma$, that is, for any $x \in \mathrm{supp}(A)\backslash E_w(\mathcal{P})$,*

$$\inf_{x^* \in S(N_A(x))} d(-x^*, C_X{}^+) \ge \gamma. \tag{6.16}$$

Then $d_{\widehat{A}}$ has an error bound on A.

Proof By Lemma 1 and (6.16), (6.8) is satisfied with $\frac{\gamma}{2}$ in place of γ, and one can apply Theorem 5 to conclude the proof. □

In view of Example 1 and Theorems 3 and 5, the following corollary (when restricted to the finite dimensional case) immediately implies a main result in [7] (see Theorem 2.5 therein).

Corollary 5 *Suppose $E_w(\mathcal{P}) \neq \emptyset$ and that A is a polyhedron (thus $F(A)$ has the WDP; see Theorem 2). Then $d_{\widehat{A}}$ has an error bound on A.*

Proof In view of Corollary 4, it suffices to prove that (6.16) holds for some $\gamma > 0$. Let $A = \cap_{i=1}^{m}\{\langle x_i^*, \cdot\rangle \geq b_i\}$, where $b_i \in \mathbb{R}$ and $x_i^* \in X^*\backslash\{0\}$ for all $i \in \overline{1,m} := \{1, 2, \ldots, m\}$. For each nonempty subset I of $\overline{1,m}$, define $N(I)$ to be the convex hull of $\{-x_i^* : i \in I\}$. Let $\mathcal{N}_A := \{N(I) : \emptyset \neq I \subset \overline{1,m}\}$. Clearly $\mathcal{N}_A$ is a finite family.

For any $x \in \mathrm{bd}(A)$, let I_x denote the set of "active indices" at x, that is, $I_x := \{i \in \overline{1,m} : \langle x_i^*, x\rangle = b_i\}$. Clearly I_x is nonempty and

$$N_A(x) = \mathrm{co}\{-x_i^* : i \in I_x\}.$$

Thus

$$N_A(x) \in \mathcal{N}_A \quad \forall x \in \mathrm{bd}(A).$$

Moreover, by Theorem 1 we have

$$N_A(x) \cap (-C_X{}^+) = \{0\} \quad \forall x \in \mathrm{bd}(A)\backslash E_w(\mathcal{P}). \tag{6.17}$$

Let $\mathcal{F}$ denote the family $\{N_A(x) : x \in \mathrm{supp}(A)\backslash E_w(\mathcal{P})\}$. Then $\mathcal{F}$ is a subfamily of $\mathcal{N}_A$. We claim that for any $N \in \mathcal{F}$ there exists $\gamma_N > 0$ such that

$$\inf_{x^* \in S(N)} d(-x^*, C_X{}^+) \geq \gamma_N. \tag{6.18}$$

Granting this and letting $\gamma := \min\{\gamma_N : N \in \mathcal{F}\}$, one sees that $\gamma > 0$ satisfies (6.16) for each $x \in \mathrm{supp}(A)\backslash E_w(\mathcal{P})$. Thus, in view of Corollary 4, it remains to prove (6.18) for some $\gamma_N > 0$. To do this let us suppose on the contrary that there exists a sequence $\{y_j^*\}_{j=1}^{\infty} \subset S(N)$ such that $d(-y_j^*, C_X{}^+)$ converges to 0. Note that N is a closed cone in a finite dimensional subspace of X^* and hence that $S(N)$ is compact. Hence we may suppose without generality that y_j^* converges to some $y_0^* \in S(N)$. Then $d(-y_0^*, C_X{}^+) = 0$ and hence $y_0^* \in -C_X{}^+$; that is, y_0^* is a nonzero element of $N \cap (-C_X{}^+)$. This contradicts (6.17), and completes the proof. □

7 Conclusion

For vector optimization problem (1.1) defined by the data F, A and C, we considered a class of what we called regular merit functions. The importance of such functions lies in the fact that the set of minimizers of each such a function coincides with the set of all weakly efficient solutions of the problem (1.1), and their error bound properties are shown to be equivalent to the linear regularity of the pair $\{A, X\backslash(A+\text{int}(F^{-1}(C)))\}$. Two functions defined in terms of Hiriart-Urruty functions are examples regular merit functions and are provided with dual representations.

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