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FULL LENGTH PAPER

A constructive proof of Ky Fan's coincidence theorem

A. J. J. Talman · Z. Yang

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Abstract We present a constructive proof for the well-known Ky Fan's coincidence theorem through a simplicial algorithm. In a finite number of steps the algorithm generates a simplex containing an approximate coincidence point. In the limit, when the mesh size converges to zero, the sequence of approximations converges to a coincidence point.

Keywords Fan's coincidence theorem · Fixed point · Simplicial algorithm · Upper semi-continuity

1 Introduction

The fixed point theorems of Brouwer and Kakutani and their variants are powerful tools to show the existence of solutions to various problems in economics, game theory, mathematics and engineering; e.g., see [2]. Yet, for a number of more recent economic and game-theoretic problems these tools appeared to be insufficient and

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A. J. J. Talman (✉)
CentER, Department of Econometrics and Operations Research, Tilburg University,
P.O. Box 90153, 5000 LE Tilburg, The Netherlands
e-mail: talman@uvt.nl

Z. Yang
Faculty of Business Administration, Yokohama National University,
Yokohama 240-8501, Japan
e-mail: yang@ynu.ac.jp

Ky Fan's coincidence theorem, see [4], a more powerful theorem, had to be invoked; e.g., see [6,9,12,5].

It is widely known that the fixed point theorems of Brouwer and Kakutani and many of their extensions can be constructively proved through simplicial algorithms pioneered by Scarf [7]; see [8,1,11] for overviews on the subject. In this paper we show that Ky Fan's coincidence theorem can be constructively proved via a simplicial algorithm as well. Aside from the theoretical interest, the constructive proof is also of practical use in that we can also have numerical methods to compute (approximately) equilibria in the recently studied economic models as mentioned above, and thus to enable us to evaluate and analyze the policy implication of those models. Fan's theorem states: "Let X be a non-empty compact and convex set in the n-dimensional Euclidean space $\mathbb{R}^n$ and let ϕ and ψ be two upper semi-continuous mappings from X to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$ such that $\psi(x)$ is bounded for every $x \in X$. Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying $d^\top x = \max\{d^\top y \mid y \in X\}$, there exist $v \in \phi(x)$ and $w \in \psi(x)$ such that $d^\top v \geq d^\top w$. Then there exists x^* in X satisfying $\phi(x^*) \cap \psi(x^*) \neq \emptyset$." Such an intersection point x^* is called *a coincidence*. The difficulty of computing a coincidence in this theorem lies in the following aspects. First, many extensions of Brouwer's theorem such as Kakutani's theorem are essentially equivalent variants of Brouwer's theorem in the sense that they are or can be directly derived from Brouwer's theorem, whereas for Fan's coincidence theorem this is not the case. In fact, the latter theorem was proved by Fan [4] in a completely different way. Second, the boundary condition stated in Fan's theorem differs considerably from those in Brouwer's theorem and its extensions. For each element of the normal cone at a point of X, Fan's theorem requires a weak separation condition between the images of the two mappings. For different elements of the normal cone the separation condition may hold for different elements of the images. Moreover, the images of one of the two mappings are allowed to be unbounded, whereas in a fixed point theorem the images are assumed to be compact. The question is then how to deal with those complications in an algorithm. Precisely because of this, the usual simplicial approximation or other computational techniques on X may not apply here. To circumvent the computational problem, we propose to embed X into a full-dimensional compact and convex set Q containing X in its interior and having a smooth boundary in the sense that at each point on the boundary of the set Q the normal cone is just a ray. We then extend the two underlying mappings to the set Q in a proper way. This enables us to apply the simplicial algorithm of Wright [10] to a cube that contains the set Q in its interior. With this method a simplex containing an approximate coincidence is found in a finite number of iterations. The accuracy of approximation can be increased by restarting the algorithm at the previously found approximation with a simplicial subdivision having a smaller mesh size and in this way an approximate solution of any a priori given accuracy can be reached in a finite number of steps. In the limit the sequence of approximate coincidences converges to a coincidence point of the two mappings.

The rest of this paper is organized as follows. In Sect. 2 we present Fan's coincidence theorem and discuss several related results. In Sect. 3 we give a constructive proof of Fan's theorem.

2 Fan's coincidence theorem

Let Y be an arbitrary non-empty set in the n-dimensional Euclidean space $\mathbb{R}^n$. For $x \in Y$, the set

$$N(Y, x) = \left\{y \in \mathbb{R}^n \mid (x - x')^\top y \geq 0 \text{ for all } x' \in Y\right\}$$

denotes the normal cone of Y at x. Its polar cone

$$T(Y, x) = \left\{z \in \mathbb{R}^n \mid z^\top y \leq 0 \text{ for all } y \in N(Y, x)\right\}$$

denotes the tangent cone of Y at x. If Y is compact and convex, $N(Y, \cdot)$ is an upper semi-continuous, convex-valued and closed-valued mapping on Y, $T(Y, \cdot)$ is a convex-valued and closed-valued mapping on Y, and, for every $y \in Y$, both sets $N(Y, y)$ and $T(Y, y)$ are non-empty. Given a set D of $\mathbb{R}^n$, $\text{bd}(D)$ and $\text{int}(D)$ represent the sets of relative boundary and interior points of D, respectively, and $\text{co}(D)$ represents the convex hull of D. Let $B^n = \{x \in \mathbb{R}^n \mid ||x||_2 \leq 1\}$ be the unit ball. For any set $D \subset \mathbb{R}^n$ and $\epsilon > 0$, let $B(D, \epsilon) = D + \epsilon B^n$ be the extension of D with its ϵ-neighborhood.

Let X be a non-empty subset of $\mathbb{R}^n$ and let ϕ be a mapping from X to the collection of non-empty subsets of $\mathbb{R}^n$. A point $x^* \in X$ is a *zero point* of ϕ if $0^n \in \phi(x^*)$, where 0^n denotes the n-vector of zeros, a *fixed point* of ϕ if $x^* \in \phi(x^*)$, a *coincidence* of ϕ and some other mapping ψ from X to the collection of non-empty subsets of $\mathbb{R}^n$ if $\phi(x^*) \cap \psi(x^*) \neq \emptyset$. The following theorem establishes the existence of a zero point and is an equivalent form of Fan's coincidence theorem [4] in its full generality. We will give a constructive proof for this theorem and thus for Fan's theorem as well.

Theorem 2.1 *Let ϕ be an upper semi-continuous point-to-set mapping from the non-empty convex and compact set X in $\mathbb{R}^n$ to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$. Suppose that for every $x \in X$ and every $v \in N(X, x)$, there is a $y \in \phi(x)$ satisfying $v^\top y \leq 0$. Then there exists a zero point of ϕ in X.*

Note that in the theorem the vector y may depend on v, even at the same point x, and also that $\phi(x)$ may be unbounded. These are crucial points that make Fan's theorem differ from the usual fixed point theorems. From Theorem 2.1 we can immediately derive several fundamental results, including the coincidence theorem of Fan [4] next.

Theorem 2.2 *Let ϕ and ψ be two upper semi-continuous mappings from the non-empty convex and compact set X in $\mathbb{R}^n$ to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$ such that $\psi(x)$ is bounded for every $x \in X$. Suppose that for every $x \in X$ and every $d \in \mathbb{R}^n$ satisfying $d^\top x = \max\{d^\top y \mid y \in X\}$, there exist $u \in \phi(x)$ and $w \in \psi(x)$ such that $d^\top u \geq d^\top w$. Then there exists a coincidence of ϕ and ψ in X.*

Proof Define the mapping γ on X by $\gamma(x) = \psi(x) - \phi(x)$ for all $x \in X$. Clearly, being the difference of two such mappings of which one is compact-valued, γ is an upper semi-continuous mapping from X to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$. Since $d^\top x = \max\{d^\top y \mid y \in X\}$ implies $d \in N(X, x)$, γ

satisfies the conditions of Theorem 2.1. Hence, there exists a zero point of γ in X. By construction, every zero point of γ is a coincidence of the mappings ϕ and ψ. □

Reversely, Theorem 2.1 immediately follows from Theorem 2.2 if we take the mapping $\psi(x) = \{0^n\}$ for every $x \in X$. Next we derive Kakutani's fixed point theorem.

Theorem 2.3 *Let ϕ be an upper semi-continuous point-to-set mapping from the non-empty convex and compact set X in $\mathbb{R}^n$ to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$. Suppose that for every $x \in X$ it holds that $\phi(x) \cap X \neq \emptyset$. Then there exists a fixed point of ϕ in X.*

Proof For given $x \in X$, take some $y \in \phi(x) \cap X$. Since $y \in X$ we have that $v^\top y \leq v^\top x$ for all $v \in N(X, x)$. Hence, the mappings ϕ and ψ on X, where ψ is defined by $\psi(x) = \{x\}$ for all $x \in X$, satisfies the conditions of Theorem 2.2. Therefore, there exists a coincidence of ϕ and ψ in X. Clearly, any coincidence of ϕ and ψ is a fixed point of ϕ in X. □

The next result says that if for every x in X the image $\phi(x)$ has a non-empty intersection with the tangent cone $T(X, x)$ of X at x, a zero point of ϕ must exist.

Corollary 2.4 *Let ϕ be an upper semi-continuous point-to-set mapping from the non-empty convex and compact set X in $\mathbb{R}^n$ to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$. Suppose that for every $x \in X$ it holds that $\phi(x) \cap T(X, x) \neq \emptyset$. Then there exists a zero point of ϕ in X.*

Proof For any $x \in X$ it holds that $T(X, x) \subseteq \{y \in \mathbb{R}^n \mid v^\top y \leq 0\}$ for every $v \in N(X, x)$. Hence, ϕ satisfies the conditions of Theorem 2.1. □

3 A constructive proof of Fan's theorem

This section will give a constructive proof of Theorem 2.1. To achieve this, we adapt the zero point problem to the framework in which the simplicial algorithm of Wright [10] can be applied. For $x \in \mathbb{R}^n$, let $p(x)$ be the orthogonal projection of x on X, i.e., $p(x) \in X$ is such that

$$\| x - p(x) \|_2 \quad \leq \quad \| x - y \|_2 \quad \text{for all } y \in X.$$

Since X is a closed and convex set, $p(\cdot)$ is a continuous function on $\mathbb{R}^n$. The next lemma shows that the vector $x - p(x)$ is an element of the normal cone of X at $p(x)$.

Lemma 3.1 *For every $x \in \mathbb{R}^n$ it holds that $x - p(x) \in N(X, p(x))$.*

Define the set $Q = \{q \in \mathbb{R}^n \mid \| q - p(q) \|_2 \leq 1\}$. The next result can also be easily derived.

Lemma 3.2 *The set Q is a full-dimensional, compact and convex subset of $\mathbb{R}^n$, containing X in its interior.*

For $q \in Q$, let $v(q) = q - p(q)$. By construction, $v(q)$ lies in the unit ball B^n for every $q \in Q$, $\| v(q) \|_2 = 1$ if and only if $q \in \mathrm{bd(Q)}$, and $v(q) = 0^n$ if and only if $q \in X$. For $v \in \mathbb{R}^n$, define the set $C(v)$ by $C(v) = \{y \in \mathbb{R}^n \mid y = \alpha v,\ \alpha \geq 0\}$. When $v \neq 0^n$, $C(v)$ is a half-line.

Lemma 3.3 *For every* $q \in \mathrm{bd(Q)}$*, it holds that* $N(Q, q) = C(v(q))$ *and* $N(Q, q) \subseteq N(X, p(q))$.

Proof For $y \in \mathbb{R}^n$, let $B(y) = \{x \in \mathbb{R}^n | \ \| x - y \|_2 \leq 1\}$. Take any point $q \in \mathrm{bd(Q)}$. By definition of Q, $B(p(q)) \subseteq Q$ and $q \in \mathrm{bd}(B(p(q)))$, and therefore $N(Q, q) \subseteq N(B(p(q)), q)$. However, $q \in \mathrm{bd}(B(p(q)))$ implies $N(B(p(q)), q) = C(q - p(q)) = C(v(q))$. Hence, $N(Q, q) \subseteq C(v(q))$. On the other hand, since Q is convex and $q \in \mathrm{bd(Q)}$, the normal cone $N(Q, q)$ of Q at q contains at least one half-line, which must be then $C(v(q))$. Finally, since according to Lemma 3.2 it holds that $v(q) \in N(X, p(q))$, it follows that $C(v(q)) \subseteq N(X, p(q))$ and therefore $N(Q, q) \subseteq N(X, p(q))$. □

From the lemma it follows that for every point on the boundary of Q the normal cone of Q at that point is a ray. Moreover, since Q is full-dimensional, the normal cone of Q at an interior point of Q is just the origin. In this way any point $q \in Q$ represents a unique combination of a point $x = p(q)$ in X and an element $v = q - p(q)$ in $N(X, x)$. Also, any combination of a point x in X and an element v in $N(X, x)$ with length at most equal to 1 is represented by a unique point $q = x + v$ in Q. Hence, the set Q is a full-dimensional expansion of X with smooth boundary.

Given a positive integer t, let I_t denote the set $\{1, 2, \ldots, t\}$. Let $P = \{x \in \mathbb{R}^n \mid l \leq x \leq u\}$ be a subset of $\mathbb{R}^n$ for some given vectors l and u in $\mathbb{R}^n$ with $u_i > l_i$, for all $i \in I_n$, containing Q in its interior. Let $e(i)$, $i \in I_n$, denote the ith unit vector in $\mathbb{R}^n$, and let $I = \{-n, \ldots, -1, 1, \ldots, n\}$. For $i \in I_n$, define $a^i = e(i)$ and $b_i = u_i$, and $a^{-i} = -e(i)$ and $b_{-i} = -l_i$. Then P can be reformulated as

$$P = \left\{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i \text{ for all } i \in I\right\}.$$

Clearly, P is a simple full-dimensional rectangular in $\mathbb{R}^n$, and none of the constraints is redundant. Let $\mathcal{I}$ be the collection of non-empty subsets J of I such that $|J| \leq n$ and $-j \notin J$ whenever $j \in J$. For each $J \in \mathcal{I}$, define

$$F(J) = \left\{x \in P \mid a^{i\top} x = b_i, \text{ for all } i \in J\right\}.$$

Clearly, for every $J \in \mathcal{I}$, the set $F(J)$ is an $(n - |J|)$-dimensional face of P, where $|J|$ denotes the number of elements in J.

Let q^0 be an arbitrary point in the interior of P. We will apply the simplicial variable dimension algorithm of Wright [10] on the set P with q^0 as the starting point. Notice that we allow q^0 to lie outside X or even outside Q. For any $J \in \mathcal{I}$, let $cF(J)$ be the convex hull of the point q^0 and $F(J)$. Since q^0 lies in the interior of P, the dimension of $cF(J)$, $J \in \mathcal{I}$, is equal to $n - |J| + 1$. Starting with q^0, Wright's algorithm operates in a simplicial subdivision or triangulation of the polytope P. See [8,11] on triangulations in detail. Let $\mathcal{T}$ be a triangulation of the set P with an arbitrarily given mesh size such that every subset $cF(J)$ of P is simplicially subdivided. For example, we may take the V-triangulation with arbitrary mesh size introduced by Doup and Talman [3] for triangulating a simplotope. The set P, being the Cartesian product of n intervals, is a special case of a simplotope. If τ is a $(t-1)$-dimensional simplex, called a facet, of a t-dimensional simplex σ on $vF(J)$, where $t = n - |J| + 1$, then τ is either a facet of precisely one other t-simplex on $vF(J)$ or τ lies on the boundary of $vF(J)$.

Let ϕ be a mapping on X satisfying the conditions of Theorem 2.1, and let Q and P be the sets as constructed above. For $v \in B^n$, let $\pi(v)$ be defined by $\pi(v) = \mathbb{R}^n$ if $v = 0^n$ and $\pi(v) = \{y \in \mathbb{R}^n \mid y^\top v \leq 0\}$ otherwise. Notice that π is an upper semi-continuous mapping from B^n to the collection of non-empty closed and convex subsets of $\mathbb{R}^n$. Now we consider the point-to-set mapping $\bar{\phi}$ from P to $\mathbb{R}^n$ defined by

$$\bar{\phi}(x) = \begin{cases} \{p(x) - x\}, & \text{if } x \in P\backslash Q, \\ \mathrm{co}(\{p(x) - x\} \cup [\phi(p(x)) \cap \pi(x - p(x))]), & \text{if } x \in \mathrm{bd(Q)}, \\ \phi(p(x)) \cap \pi(x - p(x)), & \text{if } x \in \mathrm{int}(Q). \end{cases}$$

One can easily verify that since X and ϕ satisfy the conditions of Theorem 2.1, the mapping $\bar{\phi}$ is an upper semi-continuous mapping on P with non-empty convex and closed images in $\mathbb{R}^n$. To any vertex x of a simplex σ of $\mathcal{T}$ we assign the vector label $f(x)$, where $f(x)$ is an arbitrarily chosen element in $\bar{\phi}(x)$. For instance, we can choose $f(x)$ to be any intersection point of $\phi(p(x))$ and $\pi(x - p(x))$ for $x \in \mathrm{int}(Q)$ and choose $f(x)$ to be $p(x) - x$ for $x \notin \mathrm{int}(Q)$. Now we extend f piecewise linearly on each simplex of $\mathcal{T}$, i.e., if $x = \sum_{i=1}^{n+1} \lambda_i x^i$ in some simplex $\sigma(x^1, \ldots, x^{n+1})$ of $\mathcal{T}$ for some $\lambda_i \geq 0$, $i = 1, \ldots, n+1$, with $\sum_{i=1}^{n+1} \lambda_i = 1$, then $f(x) = \sum_{i=1}^{n+1} \lambda_i f(x^i)$. Clearly, $f(\cdot)$ is affine on each simplex of $\mathcal{T}$ and is a continuous function from P to $\mathbb{R}^n$. We call f the *piecewise linear approximation* of $\bar{\phi}$ with respect to $\mathcal{T}$.

A row vector is *lexicopositive* if it is a non-zero vector and its first non-zero entry is positive. A matrix is *lexicopositive* if all its rows are lexicopositive.

Let τ with vertices $x^1, \ldots, x^t$ be a facet of a t-dimensional simplex on $cF(J)$, where $J \in \mathcal{I}$ with $J = \{j_{t+1}, \ldots, j_{n+1}\}$, $t = n - |J| + 1$. With respect to τ, the $(n+1) \times (n+1)$ matrix

$$A_{\tau,J} = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ -f(x^1) & \cdots & -f(x^t) & a^{j_{t+1}} & \cdots & a^{j_{n+1}} \end{bmatrix}$$

is called the *label matrix* of τ. The simplex τ is said to be *J-complete* if $A_{\tau,J}^{-1}$ exists and is lexicopositive. Note that the label matrix is the same as defined by Wright [10], except that we put a zero above each vector a^{j_k}, $k = t+1, \ldots, n+1$ instead of a one. This modification simplifies the convergence proof of Wright [10].

Let σ be an n-simplex of $\mathcal{T}$ with vertices $x^1, \ldots, x^{n+1}$. With respect to σ, the $(n+1) \times (n+1)$ matrix

$$A_\sigma = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -f(x^1) & -f(x^2) & \cdots & -f(x^{n+1}) \end{bmatrix}$$

is called the *label matrix* of σ. The simplex σ is said to be *complete* if A_σ^{-1} exists and is lexicopositive.

Clearly, if, for some $J \in \mathcal{I}$, a simplex τ is a J-complete facet of a t-simplex σ with vertices $x^1, \ldots, x^{t+1}$ on $cF(J)$, where $t = n - |J| + 1$, then the system of $n+1$ linear equations with $n+2$ variables

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} + \sum_{j \in J} \mu_j \begin{pmatrix} 0 \\ a^j \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \tag{$*$}$$

has a solution $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_{t+1}, (\mu_j)_{j \in J})$ satisfying $\lambda_i \geq 0$ for all $i = 1, \ldots, t+1$, $\sum_{i=1}^{t+1} \lambda_i = 1$, and $\mu_j \geq 0$ for all $j \in J$. Let x be defined by $x = \sum_{i=1}^{t+1} \lambda_i x^i$ at a nonnegative solution (λ, μ) of $(*)$. Then x lies in σ and $f(x) = \sum_{j \in J} \mu_j a^j$. Similarly, if a simplex $\sigma(x^1, \ldots, x^{n+1})$ is a complete n-simplex on P, then the system of $n+1$ linear equations with $n+1$ variables

$$\sum_{i=1}^{n+1} \lambda_i \begin{pmatrix} 1 \\ -f(x^i) \end{pmatrix} = \begin{pmatrix} 1 \\ 0^n \end{pmatrix} \tag{$**$}$$

has a unique solution $\lambda^* = (\lambda_1^*, \ldots, \lambda_{n+1}^*)$ satisfying $\lambda_i^* \geq 0$ for all $i = 1, \ldots, n+1$ and $\sum_{i=1}^{n+1} \lambda_i^* = 1$. Let x^* be defined by $x^* = \sum_{i=1}^{n+1} \lambda_i^* x^i$. Then x^* lies in σ and $f(x^*) = 0^n$, i.e., x^* is a zero point of f in P. One can easily show that $\{q^0\}$ is a J-complete 0-dimensional simplex for a unique index set $J \in \mathcal{I}$ containing precisely n indices.

Theorem 3.4 *There exists at least one complete n-dimensional simplex in $\mathcal{T}$.*

Proof Construct a graph $G = (N, E)$ where N denotes a set of nodes and E denotes a set of edges. A simplex σ is a node in N if in $\mathcal{T}$ it is either a J-complete $(n - |J|)$-dimensional facet of an $(n - |J| + 1)$-dimensional simplex on $cF(J)$ for some $J \in \mathcal{I}$ or a complete n-dimensional simplex. Two nodes are adjacent if both are J-complete facets of a same $(n - |J| + 1)$-dimensional simplex on $cF(J)$ for some $J \in \mathcal{I}$, or if one is a J-complete facet of the other and the latter one is a J'-complete $(n - |J| + 1)$-dimensional simplex on $cF(J)$ for some $J,\ J' \in \mathcal{I}$, or if one is a $\{j\}$-complete facet of the other and the latter one is a complete n-dimensional simplex on $cF(\{j\})$ for some $j \in I$.

Following Wright [10], it can be shown that for $G = (N, E)$ every node is adjacent to at most two other nodes and that both $\{q^0\}$ and each complete n-dimensional simplex on the set P is adjacent to precisely one other node. The other nodes in N having only one adjacent node are J-complete $(n - |J|)$-dimensional facets on $cF(J)$ lying on the boundary of P for some $J \in \mathcal{I}$. Since the total number of nodes is finite, there exists a unique finite sequence of adjacent nodes from $\{q^0\}$ to some other node having also one adjacent node. This node is either a complete n-dimensional simplex or a J-complete $(n - |J|)$-dimensional facet on $cF(J)$ lying on the boundary of P for some $J \in \mathcal{I}$. Suppose the latter is the case and let τ be this node with vertices $x^1, \ldots, x^t$, where

$t = n - |J| + 1$. Since τ is on the boundary of P, there exists $1 \le k \le n$ satisfying either $x_k^j = u_k$ for all $j = 1, \ldots, t$ or $x_k^j = l_k$ for all $j = 1, \ldots, t$. Without loss of generality we assume that the former holds. Clearly, this implies that k is an element of J. Hence, at the corresponding solution of $(*)$ it holds that $f_k(x) = \mu_k$ with $\mu_k \ge 0$, where $x = \sum_{j=1}^{t} \lambda_j x^j$, and therefore $f_k(x) \ge 0$. On the other hand, since τ is on the boundary of P we have that all vertices of τ are outside the set Q and so $f(x) = \sum_{j=1}^{t} \lambda_j (p(x^j) - x^j)$. Since $\sum_{j=1}^{t} \lambda_j = 1$ and $\lambda_k \ge 0$ and $x_k^j = u_k > p_k(x^j)$ for all $j = 1, \ldots, t$, this implies $f_k(x) = \sum_{j=1}^{t} \lambda_j (p_k(x^j) - u_k) < 0$, which contradicts that $f_k(x) \ge 0$. Hence, there exists a unique finite sequence of adjacent simplices from $\{q^0\}$ to a complete n-dimensional simplex in $\mathcal{T}$. □

Wright's algorithm [10] generates precisely the finite sequence of adjacent simplices as described in the proof of the above theorem. In summary, in a given triangulation the algorithm finds in a finite number of steps a complete simplex with vertices $x^1, \ldots, x^{n+1}$ such that $\sum_{j=1}^{n+1} \lambda_j f(x^j) = 0^n$ and $\sum_{j=1}^{n+1} \lambda_j = 1$ for some $\lambda_j \ge 0$ and $f(x^j) \in \bar{\phi}(x^j)$, $j = 1, \ldots, n+1$. Let $x = \sum_{j=1}^{n+1} \lambda_j x^j$. The point x is an approximate zero point of $\bar{\phi}$ in the sense that $0^n \in B(\bar{\phi}(x), \epsilon)$ for some $\epsilon > 0$. The number ϵ can be made arbitrarily small by taking a small enough mesh size of the triangulation. Let $\mathbb{N}$ denote the set of natural numbers. Now take a sequence of triangulations $\mathcal{T}^k$, $k \in \mathbb{N}$, with mesh size δ^k, $k \in \mathbb{N}$, strictly decreasing to zero. For each $k \in \mathbb{N}$, the algorithm finds in a finite number of iterations a simplex σ^k with vertices $x^{1,k}, x^{2,k}, \ldots, x^{n+1,k}$, such that $0^n \in \mathrm{co}(\{f(x^{1,k}), f(x^{2,k}), \ldots, f(x^{n+1,k})\})$. Hence, the algorithm finds $n+1$ points lying at most δ^k from each other and satisfying that the origin lies in the convex hull of elements of their images, i.e. $0^n = \sum_{j=1}^{n+1} \lambda_j^k f(x^{j,k})$ for some $\lambda_j^k \ge 0$ with $\sum_{j=1}^{n+1} \lambda_j^k = 1$ and $f(x^{j,k}) \in \bar{\phi}(x^{j,k})$, $j = 1, \ldots, n+1$. Let $x^k = \sum_{j=1}^{n+1} \lambda_j^k x^{j,k}$. If the approximation at x^k as a zero point of $\bar{\phi}$ is not satisfactory, i.e., the distance from the origin to $\bar{\phi}(x^k)$ is not small enough, the algorithm can be restarted right from x^k (seen as q^0) in the triangulation $\mathcal{T}^{k+1}$ with a smaller mesh size δ^{k+1}. By restarting at x^k a better approximation may be found in a few number of iterations. Due to the upper semi-continuity of $\bar{\phi}$, the finer the triangulation becomes, the better the approximation will be, in the sense that $0^n \in B(\bar{\phi}(x^k), \epsilon^k)$ with $\epsilon^k > 0$ converging to zero as k goes to infinity. In this way an approximate zero point of $\bar{\phi}$ for any a priori given level of accuracy is reached in a finite number of iterations.

We still have to show that in the limit a zero point in X of the mapping ϕ is obtained. Take any sequence of triangulations $\mathcal{T}^k$, $k \in \mathbb{N}$, of the set P, with mesh size strictly decreasing and converging to zero. For each $k \in \mathbb{N}$, let $\sigma^k = \langle x^{1,k}, x^{2,k}, \ldots, x^{n+1,k} \rangle$ be the n-dimensional complete simplex found by the algorithm in triangulation $\mathcal{T}^k$ and let x^k be as defined above. Then we have $0^n \in \mathrm{co}(\{f(x^{1,k}), f(x^{2,k}), \ldots, f(x^{n+1,k})\})$ and $x^k \in \sigma^k$. Since P is compact, the sequence x^k, $k \in \mathbb{N}$, has a subsequence that converges to some point $q^* \in P$. Without loss of generality, we assume that the sequence itself converges to q^*. Since the mesh size of the triangulation converges to zero, also the sequence $x^{j,k}$, $k \in \mathbb{N}$, converges to q^* for all j. By the upper semi-continuity of $\bar{\phi}$, for any $\epsilon > 0$, there exists an integer $K_\epsilon > 0$ such that for all $k \ge K_\epsilon$, we have $f(x^{j,k}) \in \bar{\phi}(x^{j,k}) \subset B(\bar{\phi}(q^*), \epsilon)$. Since $0^n \in \mathrm{co}(\{f(x^{1,k}), f(x^{2,k}), \ldots, f(x^{n+1,k})\})$ and $B(\bar{\phi}(q^*), \epsilon)$ is convex, $0^n \in B(\bar{\phi}(q^*), \epsilon)$.

Thus, $0^n \in B(\bar{\phi}(q^*), \epsilon)$ for every $\epsilon > 0$. Because $\{0^n\}$ is convex and compact and $\bar{\phi}(q^*)$ is convex and closed, it follows from the separation theorem that $0^n \in \bar{\phi}(q^*)$, i.e., q^* is a zero point of $\bar{\phi}$. Let $v^* = q^* - p(q^*)$. From Lemma 3.3 it follows that $v^* \in N(X, p(q^*))$. We will show that $p(q^*)$ is a zero point of the original mapping ϕ.

1. In case $q^* \in P \backslash Q$, $f^* = 0^n$ implies $p(q^*) = q^*$. Since $p(q^*)$ is in X and q^* is not in X, we obtain a contradiction.
2. In case $q^* \in \mathrm{bd}(Q)$, we have $f^* = \mu^*(p(q^*) - q^*) + (1 - \mu^*)f = 0^n$ for some μ^*, $0 \leq \mu^* \leq 1$, and some $f \in \phi(p(q^*)) \cap \pi(v^*)$. For $\mu^* = 1$ this case reduces to case 1). For $0 \leq \mu^* < 1$, we obtain $f = \lambda^* v^*$ with $\lambda^* = \mu^*/(1 - \mu^*) \geq 0$. Hence, $f \in \pi(v^*) \cap C(v^*)$. The latter intersection is equal to $\{0^n\}$ and therefore $f = 0^n$. Consequently, $x^* = p(q^*)$ is a zero point of ϕ.
3. In case $q^* \in \mathrm{int(Q)\backslash X}$, we have $0^n = f^* \in \phi(p(q^*))$. Hence, $x^* = p(q^*)$ is a zero point of ϕ.
4. In case $q^* \in X$, we have $q^* = p(q^*)$. This implies that $0^n = f^* \in \phi(q^*)$. Hence, $x^* = q^*$ is a zero point of ϕ.

This completes a constructive proof for Theorem 2.1.

References

1. Allgower, E.L., Georg, K.: Numerical Continuation Methods: An Introduction. Springer, Berlin (1990)
2. Arrow, K.J., Hahn, F.H.: General Competitive Analysis. Holden-Day, San Francisco (1971)
3. Doup, T., Talman, A.J.J.: A new variable dimension simplicial algorithm to find equilibria on the product space of unit simplices. Math. Program. **37**, 319–355 (1987)
4. Fan, K.: A minimax inequality and applications. In: Shisha, O. (ed.) Inequalities III, pp. 103–113. Academic, New York (1972)
5. Florenzano, M.: General Equilibrium Analysis: Existence and Optimality Properties of Equilibra. Kluwer, Boston (2003)
6. Ichiishi, T.: Game Theory for Economic Analysis. Academic, New York (1983)
7. Scarf, H.: The approximation of fixed points of a continuous mapping. SIAM J. Appl. Math. **15**, 1328–1343 (1967)
8. Todd, M.J.: The Computation of Fixed Points and Applications. Springer, Berlin (1976)
9. Vohra, R.: An existence theorem for a bargaining set. J. Math. Econ. **20**, 19–34 (1991)
10. Wright, A.H.: The octahedral algorithm, a new simplicial fixed point algorithm. Math. Program. **21**, 47–69 (1981)
11. Yang, Z.: Computing Equilibria and Fixed Points. Kluwer, Boston (1999)
12. Yang, Z.: An intersection theorem on an unbounded set and its application to the fair allocation problem. J. Optim. Theory Appl. **110**, 429–443 (2001)