

Robinson's implicit function theorem and its extensions

A. L. Dontchev · R. T. Rockafellar

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Abstract S. M. Robinson published in 1980 a powerful theorem about solutions to certain “generalized equations” corresponding to parameterized variational inequalities which could represent the first-order optimality conditions in nonlinear programming, in particular. In fact, his result covered much of the classical implicit function theorem, if not quite all, but went far beyond that in ideas and format. Here, Robinson's theorem is viewed from the perspective of more recent developments in variational analysis as well as some lesser-known results in the implicit function literature on equations, prior to the advent of generalized equations. Extensions are presented which fully cover such results, translating them at the same time to generalized equations broader than variational inequalities. Robinson's notion of first-order approximations in the absence of differentiability is utilized in part, but even looser forms of approximation are shown to furnish significant information about solutions.

Keywords Inverse and implicit function theorems · Calmness · Lipschitz modulus · First-order approximations · Semiderivatives · Variational inequalities

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Dedicated to Stephen M. Robinson with deep respect for his fundamental contributions to optimization theory and beyond.

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A. L. Dontchev (✉)
Mathematical Reviews, Ann Arbor, MI 48107-8604, USA
e-mail: ald@ams.org

R. T. Rockafellar
Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA
e-mail: rtr@math.washington.edu

1 Introduction

In the landmark paper [14], S. M. Robinson studied “generalized equations” with parameters, where an equation $f(p, x) = 0$ was replaced by a more complicated condition on $f(p, x)$ which nonetheless was to be solved for x in terms of p . Robinson focused on a particular class of conditions, but in this paper we use his “generalized equation” terminology for all conditions of the form

$$f(p, x) + F(x) \ni 0, \quad (1)$$

where p and x belong to Banach spaces P and X , f is a function from $P \times X$ to a Banach space Y , and F is a *set-valued* mapping from X to Y . In that setting, signaled by the notation $F : X \rightrightarrows Y$, the domain of F is $\text{dom } F = \{x \mid F(x) \neq \emptyset\}$ and the graph of F is $\text{gph } F = \{(x, y) \mid y \in F(x)\}$. The inverse of F is the set-valued mapping $F^{-1} : Y \rightrightarrows X$ defined by $F^{-1}(y) = \{x \mid F(x) \in y\}$. As a special case, F might just be single-valued on $\text{dom } F$, and we then could write $F : X \rightarrow Y$.

For a parameterized generalized equation (1), the central object of interest is the *solution mapping* $S : P \rightrightarrows X$ defined by

$$S(p) = \{x \mid f(p, x) + F(x) \ni 0\}, \quad (2)$$

especially questions about its possible single-valuedness and differentiability, or properties akin to differentiability. Obviously (1) reduces to the equation $f(p, x) = 0$ when F is the zero mapping from X to Y , which we indicate by writing $F \equiv 0$. The context is then the traditional one of implicit functions, handled classically through derivative assumptions on f .

The case of (1) where Robinson broke new ground in [14] is the one where Y is the dual Banach space X^* and F is the *normal cone mapping* $N_C : X \rightrightarrows X^*$ associated with a nonempty closed convex set $C \subset X$, as defined by

$$N_C(x) = \begin{cases} \{y \in X^* \mid \langle y, v - x \rangle \leq 0 \text{ for all } v \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases} \quad (3)$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between X and X^* . The generalized equation (1) comes down then to the *variational inequality*

$$\langle f(p, x), v - x \rangle \geq 0 \quad \text{for all } v \in C. \quad (4)$$

Again, a simple equation $f(p, x) = 0$ can be obtained as a specialization, namely by taking $C = X$, which makes $N_C \equiv 0$. But in this context only equations for functions $f : P \times X \rightarrow X^*$ are encompassed, not for f mapping into a space $Y \neq X^*$. That feature makes a difference already in finite dimensions, since if $X = \mathbb{R}^n$ it requires the image space for f to be \mathbb{R}^n too.

A broader formulation as in (1) is therefore needed, if Robinson’s pioneering contributions to generalized implicit function theory, in approaches as well as results, are

to be brought out fully. Putting Robinson’s ideas in this perspective and extending them further are our chief aims here.

A notion that will conveniently help us, in working with a solution mapping $S : P \rightrightarrows X$, is that of a *graphical localization of S at \bar{p} for \bar{x}* , where $(\bar{p}, \bar{x}) \in \text{gph } S$. This refers to a “submapping” s obtained by taking

$$\text{gph } s = (U \times V) \cap \text{gph } S \quad \text{for some neighborhoods } U \text{ of } \bar{p} \text{ and } V \text{ of } \bar{x},$$

so that

$$s(p) = \begin{cases} V \cap S(p) & \text{for } p \in U, \\ \emptyset & \text{for } p \notin U. \end{cases}$$

Note that $\text{dom } s$ may be different from $U \cap \text{dom } S$ and can depend on the choice of V . Of particular interest is the case where s is nonempty-valued around \bar{p} , which corresponds to \bar{p} belonging to the interior of $\text{dom } s$. Even more important is the case where, in addition, s is single-valued around \bar{p} , i.e., is a function on some neighborhood of \bar{x} . We say then that s is a *single-valued graphical localization of S at \bar{p} for \bar{x}* .

In order to review some results in this direction for generalized equations (1) in which the function f is (Fréchet) differentiable, we will use the notation that $Df(\bar{p}, \bar{x})$ is the continuous linear mapping from $P \times X$ to Y giving the derivative of f at (\bar{p}, \bar{x}) . In the same vein, $D_p f(\bar{p}, \bar{x})$ will denote the derivative of $f(\cdot, \bar{x})$ at \bar{p} , while $D_x(\bar{p}, \bar{x})$ will denote the derivative of $f(\bar{p}, \cdot)$ at \bar{x} . We say that a continuous linear mapping $A : X \rightarrow Y$ (in notation $A \in \mathcal{L}(X, Y)$) is *invertible* if its inverse A^{-1} is single-valued and continuous from Y onto X (i.e., belongs to $\mathcal{L}(Y, X)$); actually the continuity is automatic when A^{-1} is single-valued from Y onto X , according to the Banach open mapping principle. Continuous differentiability of f refers to the derivative $Df(p, x)$, as an element of $\mathcal{L}(P \times X, Y)$, depending continuously on (p, x) .

Theorem 1.1 (classical implicit function theorem). *In the case of the generalized equation (1) in which $F \equiv 0$, suppose f is continuously differentiable around (\bar{p}, \bar{x}) , and that $D_x f(\bar{p}, \bar{x})$ is invertible. Then the solution mapping S in (2) has a single-valued graphical localization s at \bar{p} for \bar{x} . Moreover, this localization is continuously differentiable around \bar{p} , and a formula for its derivative is available:*

$$Ds(p) = -D_x f(p, s(p))^{-1} D_p f(p, s(p)) \quad \text{for all } p \text{ in a neighborhood of } \bar{p}.$$

An immediate specialization of this implicit function theorem is the seemingly more basic result which is the classical inverse function theorem.

Theorem 1.2 (classical inverse function theorem). *In the case of the generalized equation (1) in which $F \equiv 0$, suppose*

$$f(p, x) = g(x) - p \quad \text{for a function } g : X \rightarrow Y = P,$$

so that $S(p) = \{x \mid g(x) = p\} = g^{-1}(p)$. Let g be continuously differentiable around \bar{x} , and suppose that the derivative $Dg(\bar{x})$ is invertible; let $\bar{p} = g(\bar{x})$. Then

the mapping $S = g^{-1}$ has a single-valued graphical localization s at \bar{p} for \bar{x} . Moreover this localization is continuously differentiable around \bar{p} , and a formula for its derivative is available:

$$Ds(p) = Dg(s(p))^{-1} \text{ for all } p \text{ in a neighborhood of } \bar{p}.$$

Historical background on these famous and universally appreciated results in analysis can be found in [12], for example. It is well known that the classical inverse function theorem is actually equivalent to the classical implicit function theorem. However, it is the implicit function theorem rather than the inverse function theorem that provides the pattern of result one might hope to get for a generalized equation (1) with $F \neq 0$. In such a context the differentiability of a graphical localization s of the solution mapping S cannot be expected, but single-valuedness can still be realistic along other properties which, to some degree, may substitute for differentiability.

Two quantitative properties related conceptually to differentiability are prominent in this respect: “calmness” and “Lipschitz continuity”. Both will be important in our developments, and even in understanding the theorem of Robinson on which we want to build. In recalling their definitions, we use $\|\cdot\|$ for the norm in any Banach space that comes up.

Calmness For $s : P \rightarrow X$, a subset D of P and a point $\bar{p} \in D \cap \text{dom } s$, the function s is said to be calm¹ at \bar{p} relative to D if there exists a constant $\kappa \geq 0$ (a calmness constant) such that

$$\|s(p) - s(\bar{p})\| \leq \kappa \|p - \bar{p}\| \quad \text{for all } p \in D \cap \text{dom } s. \quad (5)$$

When this holds for some neighborhood D of \bar{p} , s is said to be calm at \bar{p} , in which case the infimum of all the calmness constants κ for which (5) holds with respect to various choices of such a neighborhood D of \bar{p} is called the calmness modulus of s at \bar{p} and denoted by $\text{clm}(s; \bar{p})$.

If \bar{p} is an isolated point of $\text{dom } s$, trivially $\text{clm}(s; \bar{p}) = 0$, but otherwise the calmness modulus is given by the formula

$$\text{clm}(s; \bar{p}) = \limsup_{\substack{p \in \text{dom } s \\ p \rightarrow \bar{p}, p \neq \bar{p}}} \frac{\|s(p) - s(\bar{p})\|}{\|p - \bar{p}\|}.$$

Calmness at \bar{x} corresponds to this limit being finite, but we denote the limit by $\text{clm}(s; \bar{p})$ even when it is not finite, so that

$$s \text{ is calm at } \bar{p} \iff \bar{p} \in \text{dom } s \text{ and } \text{clm}(s; \bar{p}) < \infty.$$

¹ The name “calmness” was coined by Francis Clarke in [3], see also [16].

Lipschitz continuity For $s : P \rightarrow X$ and a subset $D \subset P$, the function s is said to be Lipschitz continuous² relative to D if $D \subset \text{dom } s$ and there exists $\kappa \geq 0$ (a Lipschitz constant) such that

$$\|s(p') - s(p)\| \leq \kappa \|p' - p\| \quad \text{for all } p, p' \in D. \tag{6}$$

When this holds for some neighborhood D of \bar{p} , s is said to be Lipschitz continuous around \bar{p} , in which case the infimum of the set of all values $\kappa \geq 0$ for which (6) holds relative to various choices of such a neighborhood D of \bar{p} is called the Lipschitz modulus of s at \bar{p} and denoted by $\text{lip}(s; \bar{p})$.

Note that, in contrast to the calmness modulus, the Lipschitz modulus is only defined when $\bar{p} \in \text{int dom } s$. For such points \bar{p} it is given equivalently by the formula

$$\text{lip}(s; \bar{p}) = \limsup_{\substack{p, p' \rightarrow \bar{p}, \\ p \neq p'}} \frac{\|s(p') - s(p)\|}{\|p' - p\|}.$$

Lipschitz continuity around \bar{p} corresponds to this limit being finite, but again we adopt the tactic of using it to define $\text{lip}(s; \bar{p})$ even when the limit is not finite, so that

$$s \text{ is Lipschitz continuous around } \bar{p} \iff \bar{p} \in \text{int dom } s \text{ and } \text{lip}(s; \bar{p}) < \infty.$$

What did we mean in saying that calmness and Lipschitz continuity are related to differentiability? In the first place, both properties provide estimates of the ‘‘rate of change’’ of $s(p)$ with respect to changes in p . In calmness at \bar{p} , comparisons are made only between p and \bar{p} , whereas in Lipschitz continuity around \bar{p} comparisons are made between arbitrary pairs of points p and p' in some neighborhood of \bar{p} . For Lipschitz continuity there is also the famous theorem of Rademacher, according to which this property of s on an open set D in P when P is finite-dimensional entails s being differentiable at all but a negligible set of points in D .

But calmness and Lipschitz continuity are also connected with differentiability through notions of approximation. Its instructive to observe that the very definition of the (Fréchet) differentiability can be captured in the following manner:

$$\begin{cases} s \text{ is differentiable at } \bar{p} \text{ if and only if } \bar{p} \in \text{int dom } s \\ \text{and there exists } A \in \mathcal{L}(P, X) \text{ such that } \text{clm}(s - A; \bar{p}) = 0. \end{cases} \tag{7}$$

Then $A = Ds(\bar{p})$, of course. Lipschitz continuity serves in a parallel way in the definition of *strict* (Fréchet) differentiability:

$$\begin{cases} s \text{ is strictly differentiable at } \bar{p} \text{ if and only if} \\ \text{there exists } A \in \mathcal{L}(P, X) \text{ such that } \text{lip}(s - A; \bar{p}) = 0. \end{cases} \tag{8}$$

² This is named after Rudolf Otto Sigismund Lipschitz (1832–1903).

Just as differentiability at \bar{p} entails continuity at \bar{p} , strict differentiability at \bar{p} entails Lipschitz continuity around \bar{p} . Strict differentiability also has an important tie to continuous differentiability. If s is continuously differentiable around \bar{p} , then s is strictly differentiable at \bar{p} . On the other hand, if s is strictly differentiable on some neighborhood of \bar{p} , then s is continuously differentiable around \bar{p} . Thus, strict differentiability is the pointwise property that captures continuous differentiability.

The conditions $\text{clm}(s - A; \bar{p}) = 0$ and $\text{lip}(s - A; \bar{p}) = 0$ in (7) and (8) express the quality of the “linearization” of s at \bar{p} afforded by A , or more exactly by the affine function $h(p) = s(\bar{p}) + A(p - \bar{p})$. (These conditions concern differences, so the constant terms in the formula for $h(p)$ drop out of them.) Later, we will work with *nonaffine* approximations h based on such conditions. For now, before getting on with the formulation of Robinson’s main theorem from [14], we mention two results which indicate how the classical inverse function theorem can be complemented, or sharpened, in terms of the concepts just reviewed.

Theorem 1.3 (derivative invertibility from inverse calmness). *Let $g : X \rightarrow P$ be differentiable at $\bar{x} \in \text{int dom } g$, and let $\bar{p} := g(\bar{x})$. If the (generally set-valued) inverse mapping g^{-1} has a single-valued graphical localization s at \bar{p} for \bar{x} that is calm at \bar{p} , then the derivative $Dg(\bar{x}) \in \mathcal{L}(X, P)$ must be invertible.*

Theorem 1.4 (inverse functions under strict differentiability). *Let $g : X \rightarrow P$ be strictly differentiable at $\bar{x} \in \text{int dom } g$, and let $\bar{p} = g(\bar{x})$. Then the invertibility of $Df(\bar{x})$ is both necessary and sufficient for g^{-1} to have a single-valued graphical localization s at \bar{p} for \bar{x} which is strictly differentiable at \bar{p} . In that case, moreover,*

$$Ds(\bar{p}) = Dg(\bar{x})^{-1}.$$

As far as we know, Theorem 1.3 has not previously been noted, but it can readily be deduced from the circumstances in the classical proofs for Theorem 1.2. Theorem 1.4 is essentially due to Leach [11], although he did not emphasize the symmetry in his formulation; that was brought out later in [4, Corollary 3.3]. Leach did not offer a corresponding version of the implicit function theorem, but one can be obtained from his inverse function theorem by the same argument that relates the two classical results, as noted by Nijenhuis [13].

Theorem 1.5 (implicit functions under strict differentiability). *In classical form of (1) with $F \equiv 0$, let \bar{p} and \bar{x} be such that $\bar{x} \in S(\bar{p})$ for the solution mapping S in (2). Let f be strictly differentiable at (\bar{p}, \bar{x}) and suppose $D_x f(\bar{p}, \bar{x})$ is invertible. Then S has a single-valued graphical localization s at \bar{p} for \bar{x} which is strictly differentiable at \bar{p} (in particular Lipschitz continuous around \bar{p}), with*

$$Ds(\bar{p}) = -D_x f(\bar{p}, \bar{x})^{-1} D_p f(\bar{p}, \bar{x}).$$

The relationship between strict differentiability and continuous differentiability spelled out above reveals that Theorem 1.5 is a more powerful result from which Theorem 1.1 can be deduced. The latter corresponds in effect to assuming strict differentiability of f on a neighborhood of (\bar{p}, \bar{x}) and thereby concluding the strict differentiability of s on a neighborhood of \bar{p} .

We are ready now to state the important result of Robinson from [14]. Up to some rewording, it goes as follows.

Theorem 1.6 (Robinson’s implicit function theorem). *In the variational inequality case of the generalized equation (1), with $Y = X^*$ and F the normal cone mapping N_C associated with a nonempty, closed, convex set $C \subset X$ as in (3), consider a pair (\bar{p}, \bar{x}) such that $\bar{x} \in S(\bar{p})$ for the solution mapping S in (2). Assume that*

- (a) *f is (Fréchet) differentiable with respect to x at all points (p, x) in a neighborhood of (\bar{p}, \bar{x}) , and that both $f(p, x)$ and $D_x f(p, x)$ depend continuously on (p, x) in this neighborhood;*
- (b) *the inverse G^{-1} of the set-valued mapping $G : X \rightrightarrows Y$ defined by*

$$G(x) = f(\bar{p}, \bar{x}) + D_x f(\bar{p}, \bar{x})(x - \bar{x}) + N_C(x) \tag{9}$$

has a single-valued graphical localization σ at 0 for \bar{x} which is Lipschitz continuous around 0 with associated Lipschitz constant κ .

Then S has a single-valued graphical localization s at \bar{p} for \bar{x} which is continuous at \bar{p} and such that for every $\varepsilon > 0$ the estimate holds that for all p', p in a neighborhood of \bar{p}

$$\|s(p') - s(p)\| \leq (\kappa + \varepsilon) \|f(p', s(p)) - f(p, s(p))\|. \tag{10}$$

Corollary 1.7 (calmness of solutions). *In the framework of Theorem 1.6, if $f(\cdot, \bar{x})$ is calm at \bar{p} with $\text{clm}(f(\cdot, \bar{x}); \bar{p}) \leq \lambda$, then s is calm at \bar{p} with $\text{clm}(s; \bar{p}) \leq \kappa\lambda$.*

Corollary 1.8 (Lipschitz continuity of solutions). *In the framework of Theorem 1.6, if neighborhoods U of \bar{p} and V of \bar{x} exist such that, for all $x \in V$, $f(\cdot, x)$ is Lipschitz continuous relative to U with constant λ , then s is Lipschitz continuous around \bar{p} with $\text{lip}(s; \bar{p}) \leq \kappa\lambda$.*

The invertibility property assumed in (b) of Theorem 1.6 is what Robinson in [14] called the “strong regularity” of the generalized equation at hand. This term has since come to refer rather to the existence of a Lipschitz continuous localization as in Corollary 1.8; cf. [6].

Differentiability of the localization s at \bar{p} cannot be deduced from the estimate in (10), not to speak of continuous differentiability around \bar{p} , and in fact differentiability may fail. Elementary one-dimensional examples of variational inequalities exhibit solution mappings that are not differentiable, usually in connection with the “solution trajectory” hitting or leaving the boundary of the set C . For such mappings, weaker concepts of differentiability are available; we will touch upon this in Sect. 3.

In the special case where the variational inequality treated by Robinson’s theorem reduces to the equation $f(p, x) = 0$ (namely $C = 0$, so $N_C \equiv 0$), the invertibility of the mapping G in (9) comes down to the invertibility of $D_x f(\bar{p}, \bar{x})$ in the classical implicit function theorem, but Robinson’s result falls short of yielding all the conclusions in that result, Theorem 1.1. It could, though, be used as an intermediate step in the proof of Theorem 1.1, when $Y = X^*$. Closing the gap would be tantamount to

supplying a corollary, beyond the two given above, which confirms that s is continuously differentiable around \bar{p} when f is continuously differentiable with respect to p and x jointly. Similarly, Theorem 1.6 is not adequate on its face for deriving the strict differentiability version of the classical implicit function theorem in Theorem 1.5.

On the other hand, Robinson's result does provide through Corollary 1.8 a property of s on a neighborhood of \bar{p} which can well be compared with the classical conclusion about continuous differentiability in Theorem 1.1. The connection can be seen through the fact that continuous differentiability of s around \bar{p} implies Lipschitz continuity around \bar{p} , whereas, when the parameter space P is finite-dimensional, Lipschitz continuity around \bar{p} implies in turn that s is differentiable almost everywhere around \bar{p} .

For stating and proving his result, Robinson was clearly motivated by the problem of how the solutions of the standard nonlinear programming problem depend on parameters, and he pursued this goal in the same paper [14] where he presented his implicit function theorem. At that time it was already known from the work of Fiacco and McCormick [8] that under linear independence of the constraint gradients and second-order sufficient conditions, together with strict complementarity slackness at the reference point, the solution mapping of the standard nonlinear programming problem has a *smooth* single-valued localization around the reference point. The proof of this result was based on the classical implicit function theorem, inasmuch under strict complementarity slackness the Kuhn-Tucker system turns into a system of equations locally. Robinson looked at the case when the strict complementarity slackness is violated, which happens, as already noted, when the "stationary point trajectory" hits or leaves the constraints. Based on his implicit function theorem, which actually reached far beyond his immediate goal, he proved, still in [14], that under a stronger form of the second-order sufficient condition, together with linear independence of the constraint gradients, the solution mapping of the standard nonlinear programming problem has a *Lipschitz continuous* single-valued localization around the reference point. This result was a stepping stone to the subsequent extensive development of stability analysis in optimization, whose maturity came with the publication of the recent books [2, 7] and [10].

Robinson's breakthrough in the stability analysis of nonlinear programming was in fact much needed for the emerging numerical analysis of variational problems more generally. In his paper [14], he noted the thesis of his Ph.D. student Josephy [9] who proved that strong regularity yields local quadratic convergence of Newton's method for solving variational inequalities, a method whose version for constrained optimization problems is well known as the sequential quadratic programming (SQP) method in nonlinear programming. In the explosion of works in this area in the 80s and 90s Robinson's contribution, if not forgotten, was sometimes taken for granted. Still, its importance was acknowledged in many papers—too many to be listed here.

About a decade after Robinson's theorem was published, it was realized that it could be used as a tool in the analysis of a variety of other contexts. To our knowledge, Alt [1] was the first to use it for optimal control problems. About the same time, a rigorous proof was obtained in [5] of the fact that everything in Theorem 1.6 actually holds for *all* generalized equations (1), without any need of restriction to $Y = X^*$ and $F = N_C$. Variational inequalities serve as an example, not a limitation.

Corresponding generalizations of Robinson’s implicit function theorem to abstract spaces came later, as well as important applications such as convergence analysis of algorithms and discrete approximations to infinite-dimensional variational problems. However, to survey these widespread developments is not within the scope of this paper.

In Sect. 2 we will present a result (Theorem 2.1) which carries Robinson’s theorem to a setting of generalized equations (1) without any, even rudimentary, form of differentiability required of f .

Robinson, in a different paper [15] written long after [14], introduced a concept of first-order approximation as a replacement of derivative, and employed it for obtaining an implicit function theorem for equations (not generalized equations) with non-smooth functions f . By means of that approximation idea, a version of Theorem 1.6 for generalized equations (1) was obtained in [4] in which differentiability properties of the localized solution mapping s at \bar{p} were shown to follow from differentiability properties of f , and differentiability could even be replaced by semidifferentiability. In Sect. 3 (Theorem 3.1), that result of [4] will be extended further as well. A first-order approximation formula at \bar{p} will be furnished for the localized solution mapping s whose existence is guaranteed by Theorem 2.1. We will have arrived then at an implicit function theorem for generalized equations (1) which, despite not relying in its statement on any form of derivatives, fully covers the sharpened version of Theorem 1.1 given in Theorem 1.5.

The main contributions of this paper, Theorems 2.1 and 3.1, can easily be adapted to the framework of X being a complete metric space, Y being a linear metric space with shift-invariant metric, and P being any metric space. The statements and the proofs require hardly more than changes in notation.

2 Robinson’s theorem: extended

We start with defining a concept introduced by Robinson in [15] which we will be able to use as a replacement for the linearization in condition (b) of Theorem 1.6.

First-order approximations. For a function $f : X \rightarrow Y$ and a point $\bar{x} \in \text{int dom } f$, a function $h : X \rightarrow Y$ is said to be a first-order approximation of f at \bar{x} when

$$f(\bar{x}) = h(\bar{x}) \quad \text{and} \quad \text{clm}(f - h; \bar{x}) = 0. \tag{11}$$

This approximation is said to be strict if actually

$$f(\bar{x}) = h(\bar{x}) \quad \text{and} \quad \text{lip}(f - h; \bar{x}) = 0. \tag{12}$$

Traditional approximations through linearization fit as the case where $h = f(\bar{x}) + A(x - \bar{x})$ for some $A \in \mathcal{L}(X, Y)$. In (11), f is (Fréchet) differentiable with $A = Df(\bar{x})$, whereas in (12) we have not just differentiability but strict differentiability.

To meet the challenges ahead, we will have to distinguish between different behaviors of $f(p, x)$ with respect to its two arguments, and also to capture uniformities

in these behaviors. Moduli for “partial” calmness and Lipschitz continuity will be required. They will serve in defining partial first-order approximations.

Partial calmness and partial Lipschitz continuity. For a function $f : P \times X \rightarrow Y$ and a point $(\bar{p}, \bar{x}) \in \text{dom } f$ that is not an isolated point of $\text{dom } f$, the partial calmness modulus of f with respect to x at (\bar{p}, \bar{x}) is defined by

$$\widehat{\text{clm}}_x(f; (\bar{p}, \bar{x})) = \limsup_{\substack{x \rightarrow \bar{x}, p \rightarrow \bar{p}, \\ (p,x) \in \text{dom } f, x \neq \bar{x}}} \frac{\|f(p, x) - f(p, \bar{x})\|}{\|x - \bar{x}\|}.$$

Analogously, for a point $(\bar{p}, \bar{x}) \in \text{int dom } f$ the partial Lipschitz modulus is defined by

$$\widehat{\text{lip}}_x(f; (\bar{p}, \bar{x})) = \limsup_{\substack{x, x' \rightarrow \bar{x}, p \rightarrow \bar{p}, \\ x \neq x'}} \frac{\|f(p, x') - f(p, x)\|}{\|x' - x\|}.$$

Partial first-order approximations. For $f : P \times X \rightarrow Y$ and a point $(\bar{p}, \bar{x}) \in \text{int dom } f$, a function $h : X \rightarrow Y$ is said to be a first-order approximation of f with respect to x uniformly in p at (\bar{p}, \bar{x}) when

$$f(\bar{p}, \bar{x}) = h(\bar{x}) \quad \text{and} \quad \widehat{\text{clm}}_x(f - h; (\bar{p}, \bar{x})) = 0.$$

This approximation is said to be strict if

$$f(\bar{p}, \bar{x}) = h(\bar{x}) \quad \text{and} \quad \widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x})) = 0.$$

For the first of the two main results we feature in this paper, which will be stated next, we continue in the setting of a generalized equation (1) and its solution mapping S in (2). Surprisingly, perhaps, no assumptions at all are imposed directly on the set-valued mapping $F : X \rightrightarrows Y$ in (1), although certain properties of F are obviously implicit in the “invertibility” condition imposed jointly on F and an auxiliary function h . The function h could be a linearization as in Theorem 2.1, or a nonlinear first-order approximation, but not necessarily.

To save on words, we henceforth refer to a “Lipschitz continuous single-valued graphical localization” simply as a *Lipschitz localization*.

Theorem 2.1 (extension of Robinson’s implicit function theorem). *In the generalized equation (1) with its solution mapping S in (2), let \bar{p} and \bar{x} be such that $\bar{x} \in S(\bar{p})$. Assume that*

- (a) $f(\cdot, \bar{x})$ is continuous at \bar{p} and $h : X \rightarrow Y$ is a function having

$$h(\bar{x}) = f(\bar{p}, \bar{x}) \quad \text{and} \quad \widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x})) < \infty. \tag{13}$$

(b) For the set-valued mapping $G = h + F$, with $G(\bar{x}) \ni 0$, the inverse G^{-1} has a Lipschitz localization σ at 0 for \bar{x} with

$$\text{lip}(\sigma; 0) \cdot \widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x})) < 1. \tag{14}$$

Then S has a single-valued graphical localization s at \bar{p} for \bar{x} which is continuous at \bar{p} and such that for every

$$\kappa > \left(\text{lip}(\sigma; 0)^{-1} - \widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x})) \right)^{-1} \tag{15}$$

the estimate holds that for all p', p in a neighborhood of \bar{p} ,

$$\|s(p') - s(p)\| \leq \kappa \|f(p', s(p)) - f(p, s(p))\|. \tag{16}$$

Theorem 1.6 follows at once from Theorem 2.1 by taking F to be N_C and h to be the linearization of f given by $h(x) = f(\bar{p}, \bar{x}) + D_x f(\bar{p}, \bar{x})(x - \bar{x})$. In contrast to Robinson’s result in Theorem 1.6, however, h no longer needs to be a linearization. It does not even have to be a first-order approximation, although the conditions in (13) are certainly satisfied when h is a strict first-order approximation of f with respect to x uniformly in p at (\bar{p}, \bar{x}) , as defined above. That case, which has further implications, will be taken up in Sect. 3. However, Theorem 2.1 is able to extract information from relationships between f and h that are much weaker than first-order approximation but still can have important consequences for the behavior of solutions to a generalized equation.

As an illustration of what we just said, observe that Theorem 2.1 specialized to linear mappings yields a classical result sometimes called the Banach lemma. Namely, for $f(p, x) = (A + B)x + p$ and $h(x) = Ax$ for linear bounded mappings A and B acting from X to Y , $F \equiv 0$ and $\bar{x} = \bar{p} = 0$, condition (b) means that A is invertible. Moreover, since $\text{lip}(\sigma; 0) = \|A^{-1}\|$ and also $\widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x})) = \|B\|$, (14) becomes $\|B\| \cdot \|A^{-1}\| < 1$. Then Theorem 2.1 says that the mapping $A + B$ is invertible, and $\|(A + B)^{-1}\| \leq \kappa$ for any $\kappa > (\|A^{-1}\| - \|B\|)^{-1}$. This leads directly to the following: if $A \in \mathcal{L}(X, Y)$ is invertible, then for any $B \in \mathcal{L}(X, Y)$ with $\|B\| < \|A^{-1}\|^{-1}$ one has

$$\|(A + B)^{-1}\| \leq (\|A^{-1}\|^{-1} - \|B\|)^{-1}.$$

Note that this result cannot be obtained from Theorem 1.6.

Corollaries 1.7 and 1.8 of Theorem 1.6 translate fully to the framework of Theorem 2.1, of course, since they only depend on the estimate available for $\|s(p') - s(p)\|$, which has the same character in (16) as in (10).

Our proof of Theorem 2.1 will proceed through an intermediate step which we isolate as a lemma with a somewhat lengthy statement. Proving the lemma requires an appeal to the Banach fixed point theorem, as stated next in its Banach space version.

Here we use the notation that

$$\mathbb{B}_a(\bar{x}) = \{x \mid \|x - \bar{x}\| \leq a\}.$$

Banach fixed point theorem. Consider a point $\bar{x} \in X$ and a function $\Phi : X \rightarrow X$ for which there exist scalars $a > 0$ and $\theta, 0 \leq \theta < 1$ such that

- (a) $\|\Phi(\bar{x}) - \bar{x}\| \leq a(1 - \theta);$
- (b) Φ is Lipschitz continuous on $\mathbb{B}_a(\bar{x})$ with Lipschitz constant θ .

Then Φ has a unique fixed point in $\mathbb{B}_a(\bar{x})$; in other words, there exists one, and only one, $x \in \mathbb{B}_a(\bar{x})$ satisfying $x = \Phi(x)$.

Lemma 2.2 Consider a function $\varphi : P \times X \rightarrow Y$ and a point $(\bar{p}, \bar{x}) \in \text{int dom } \varphi$ and let the scalars $\mu \geq 0, b \geq 0, a > 0$ and the set $U \subset P$ be such that $\bar{p} \in U$ and

$$\begin{cases} \|\varphi(p, x) - \varphi(p, x')\| \leq \mu\|x - x'\| & \text{for all } x, x' \in \mathbb{B}_a(\bar{x}) \text{ and } p \in U, \\ \|\varphi(p, \bar{x}) - \varphi(\bar{p}, \bar{x})\| \leq b & \text{for all } p \in U. \end{cases} \tag{17}$$

Consider also a set-valued mapping $M : Y \rightrightarrows X$ with $(\bar{y}, \bar{x}) \in \text{gph } M$ where $\bar{y} = \varphi(\bar{p}, \bar{x})$, such that for each $y \in \mathbb{B}_{\mu a + b}(\bar{y})$ the set $M(y) \cap \mathbb{B}_a(\bar{x})$ consists of exactly one point, denoted by $r(y)$, and suppose that the function

$$r : y \mapsto M(y) \cap \mathbb{B}_a(\bar{x}) \quad \text{for } y \in \mathbb{B}_{\mu a + b}(\bar{y}) \tag{18}$$

is Lipschitz continuous in its domain with Lipschitz constant γ . In addition, suppose that

- (a) $\gamma\mu < 1;$
- (b) $\gamma\mu a + \gamma b \leq a$.

Then for each $p \in U$ the set $\{x \in \mathbb{B}_a(\bar{x}) \mid x \in M(\varphi(p, x))\}$ consists of exactly one point, denoted by $s(p)$, and the associated function

$$s : p \mapsto \{x \in \mathbb{B}_a(\bar{x}) \mid x \in M(\varphi(p, x))\} \quad \text{for } p \in U \tag{19}$$

satisfies

$$\|s(p') - s(p)\| \leq (\gamma^{-1} - \mu)^{-1} \|\varphi(p', s(p)) - \varphi(p, s(p))\| \quad \text{for all } p', p \in U. \tag{20}$$

Proof Fix $p \in U$ and consider the function $\Phi_p : X \rightarrow X$ defined by

$$\Phi_p : x \mapsto r(\varphi(p, x)) \quad \text{for } x \in \mathbb{B}_a(\bar{x}).$$

Then, if $x \in \mathbb{B}_a(\bar{x})$, we have from the Lipschitz continuity of r , (17) along with condition (b) and the identity $\bar{x} = r(\varphi(\bar{p}, \bar{x}))$, that

$$\begin{aligned} \|\Phi_p(\bar{x}) - \bar{x}\| &= \|r(\varphi(p, \bar{x})) - r(\varphi(\bar{p}, \bar{x}))\| \\ &\leq \gamma \|\varphi(p, \bar{x}) - \varphi(\bar{p}, \bar{x})\| \leq \gamma b \leq a(1 - \gamma\mu) \end{aligned}$$

and for any $x, x' \in \mathbb{B}_a(\bar{x})$, $x \neq x'$, that

$$\begin{aligned} \|\Phi_p(x) - \Phi_p(x')\| &= \|r(\varphi(p, x)) - r(\varphi(p, x'))\| \\ &\leq \gamma \|\varphi(p, x) - \varphi(p, x')\| \leq \gamma\mu \|x - x'\| < \|x - x'\|. \end{aligned}$$

We are in position then to apply the Banach fixed point theorem and to conclude from it that Φ_p has a unique fixed point in $\mathbb{B}_a(\bar{x})$.

Denoting that fixed point by $s(p)$, and doing this for every $p \in U$, we get a function $s : U \rightarrow \mathbb{B}_a(\bar{x})$. But having $x = \Phi_p(x)$ is equivalent to having $x = r(\varphi(p, x)) = M(\varphi(p, x)) \cap \mathbb{B}_a(\bar{x})$. Hence s is the function in (19). Moreover, since $s(p) = r(\varphi(p, s(p)))$, we have from the Lipschitz continuity of r and (17) that, for any $p, p' \in U$,

$$\begin{aligned} \|s(p') - s(p)\| &= \|r(\varphi(p', s(p'))) - r(\varphi(p, s(p)))\| \\ &\leq \|r(\varphi(p', s(p'))) - r(\varphi(p', s(p)))\| \\ &\quad + \|r(\varphi(p', s(p))) - r(\varphi(p, s(p)))\| \\ &\leq \gamma \|\varphi(p', s(p')) - \varphi(p', s(p))\| + \gamma \|\varphi(p', s(p)) - \varphi(p, s(p))\| \\ &\leq \gamma\mu \|s(p') - s(p)\| + \gamma \|\varphi(p', s(p)) - \varphi(p, s(p))\|. \end{aligned}$$

Since $\gamma\mu < 1$, we see that s satisfies (20), as needed. □

Its worth noting that, although the Banach fixed point theorem was utilized in proving Lemma 2.2, it can in turn be derived from Lemma 2.2. For that, the data in the lemma need to be specified as follows: $P = Y = X$, $\mu = \theta$, a is unchanged, $b = \|\Phi(\bar{x}) - \bar{x}\|$, $\bar{p} = 0$, $U = \mathbb{B}_b(0)$, $\varphi(p, x) = \Phi(x) + p$, $\bar{y} = \Phi(\bar{x})$, $M(y) = y + \bar{x} - \Phi(\bar{x})$, and consequently $\gamma = 1$. All the conditions of Lemma 2.2 hold for such data under assumptions (a)(b) of the Banach fixed point theorem, hence for $p = \Phi(\bar{x}) - \bar{x} \in U$ the set $\{x \in \mathbb{B}_a(\bar{x}) \mid x = M(\varphi(p, x)) = \Phi(x)\}$ consists of exactly one point; that is, Φ has a unique fixed point in $\mathbb{B}_a(\bar{x})$. Thus, Lemma 2.2 is actually equivalent³ to the Banach fixed point theorem. There is no point of course, in giving a fairly complicated equivalent formulation of a classical result unless, as it is in our case, this formulation would bring some insights and dramatically simplify the proofs of later results.

Proof of Theorem 2.1. Let $\mu > \text{lip}(\sigma; 0)$ and $\gamma > \widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x}))$ be such that $\mu\gamma < 1$, as is possible under (14). Choose $\kappa \geq (\gamma^{-1} - \mu)^{-1}$; then κ satisfies (15). Let a, b and c be positive numbers such that

$$\begin{aligned} \|\sigma(y) - \sigma(y')\| &\leq \mu \|y - y'\| \text{ for } y, y' \in \mathbb{B}_{\mu a + b}(0), \\ \|-f(p, x) + h(x) + f(p, x') - h(x')\| &\leq \gamma \|x - x'\| \text{ for } x, x' \in \mathbb{B}_a(\bar{x}), p \in \mathbb{B}_c(\bar{p}), \end{aligned}$$

³ As mentioned at the end of Sect. 1, Theorem 2.1 and, consequently, Lemma 2.2 can be stated in a complete metric space X . Such a formulation would be equivalent to the standard statement of the Banach fixed point theorem in a complete metric space.

and

$$\|f(p, \bar{x}) - f(\bar{p}, \bar{x})\| \leq b \quad \text{for } p \in \mathbb{B}_c(\bar{p}). \tag{21}$$

Take b smaller if necessary so that $b\gamma \leq a(1 - \gamma\mu)$, and accordingly adjust c to ensure having (21). Now apply Lemma 2.2 with $r = \sigma$, $M = (h + F)^{-1}$ and $\varphi = -f + h$, keeping the rest of the notation the same. Its straightforward to check that the estimates in (17) and the conditions (a) and (b) hold for the function in (18). Then, through the conclusion of Lemma 2.2 and the observation that

$$x \in (h + F)^{-1}(-f(p, x) + h(x)) \iff x \in S(p)$$

we obtain that the solution mapping S in (2) has a single-valued graphical localization s at \bar{p} for \bar{x} . Since $(\gamma^{-1} - \mu)^{-1} \leq \kappa$, the estimate in (16) is confirmed for $U = \mathbb{B}_c(\bar{p})$. That estimate implies the continuity of s at \bar{p} , in particular. \square

3 Solution approximation and semidifferentiability

The featured result of this section, the theorem coming next, demonstrates that if we add some relatively mild assumptions about the function f (but still allow F to be arbitrary!), we can develop a first-order approximation of the localized solution mapping s in Theorem 2.1. In this way differentiability properties of s can be obtained, for example.

Theorem 3.1 (extended implicit function theorem with solution approximations). *Specialize the assumptions in Theorem 2.1 to the case where, in (a), h is a strict first-order approximation of f with respect to x uniformly in p at (\bar{p}, \bar{x}) ; in other words, suppose (13) holds with*

$$\widehat{\text{lip}}_x(f - h; (\bar{p}, \bar{x})) = 0.$$

Add the further assumptions that $\text{clm}(f(\cdot, \bar{x}); \bar{p}) < \infty$ and that $f(\cdot, \bar{x})$ has a first-order approximation r at \bar{p} . Let U be as in (16).

Then, added to the conclusions in Theorem 2.1 in relation to the Lipschitz localization σ in (b) of that theorem is the fact that the function $\eta : U \rightarrow X$ defined by

$$\eta(p) = \sigma(r(\bar{p}) - r(p)) \quad \text{for } p \in U, \quad \text{where } \sigma(0) = \bar{x}, \quad r(\bar{p}) = f(\bar{p}, \bar{x}), \tag{22}$$

is a first-order approximation at \bar{x} to the localized solution mapping s in Theorem 2.1.

If in addition σ is affine, i.e., $\sigma(y) = \bar{x} + Ay$ for some $A \in \mathcal{L}(Y, X)$, and furthermore $\widehat{\text{lip}}_p(f; (\bar{p}, \bar{x})) < \infty$ and the first-order approximation r is strict with respect to p uniformly in x at (\bar{p}, \bar{x}) , then η is a strict first-order approximation of s at \bar{p} in the form

$$\eta(p) = \bar{x} + Ar(\bar{p}) - Ar(p) \quad \text{for } p \in U.$$

Proof Let the constants a and c be as in the proof of Theorem 2.1; then $U = \mathbb{B}_c(\bar{p})$. Let $V = \mathbb{B}_a(\bar{x})$. For $p \in U$ we have

$$s(p) = \sigma(-f(p, s(p)) + h(s(p))) \tag{23}$$

along with $\bar{x} = s(\bar{p}) = \sigma(0)$. Let $\mu > \text{lip}(\sigma; 0)$; then we can take $\kappa = \mu$ in (16). Let $p \in U$, $p \neq \bar{p}$ and $p' = \bar{p}$ in (16), and divide both sides of (16) by $\|p - \bar{p}\|$. Take the limit as $p \rightarrow \bar{p}$ and $\mu \rightarrow \text{lip}(\sigma; 0)$. This gives

$$\text{clm}(s; \bar{p}) \leq \text{lip}(\sigma; 0) \cdot \text{clm}(f(\cdot, \bar{x}); \bar{p}). \tag{24}$$

Then, by our assumptions, s is calm at \bar{p} .

Consider any $\lambda > \text{clm}(s; \bar{p})$ and $\varepsilon > 0$. Make the neighborhoods U and V smaller if necessary so that, for all $p \in U$ and $x, x' \in V$,

$$\|s(p) - s(\bar{p})\| \leq \lambda \|p - \bar{p}\| \tag{25}$$

and

$$\begin{cases} \| -f(p, x) + h(x) + f(p, \bar{x}) - h(\bar{x}) \| \leq \varepsilon \|x - \bar{x}\|, \\ \|f(p, \bar{x}) - r(p)\| \leq \varepsilon \|p - \bar{p}\|. \end{cases} \tag{26}$$

Then, for $p \in U$ we get by way of (23), the Lipschitz continuity of σ , the inequality in (25), plus (26) and the fact that $h(\bar{x}) = f(\bar{p}, \bar{x})$, the estimate that

$$\begin{aligned} & \|s(p) - \sigma(-r(p) + f(\bar{p}, \bar{x}))\| \\ &= \|\sigma(-f(p, s(p)) + h(s(p))) - \sigma(-r(p) + f(\bar{p}, \bar{x}))\| \\ &\leq \mu (\| -f(p, s(p)) + h(s(p)) + f(p, \bar{x}) - h(\bar{x}) \| + \|f(p, \bar{x}) - r(p)\|) \\ &\leq \mu \varepsilon \|s(p) - \bar{x}\| + \mu \varepsilon \|p - \bar{p}\| \leq \varepsilon \mu (\lambda + 1) \|p - \bar{p}\|. \end{aligned}$$

Since ε can be arbitrarily small, the function η defined in (22) is a first-order approximation of s at \bar{p} .

Moving on to the second part of the theorem, suppose $\sigma(y) = \bar{x} + Ay$. Again, choose any $\varepsilon > 0$ and adjust the neighborhoods U of \bar{p} and V of \bar{x} so that (25) holds and

$$\begin{cases} \| -f(p, x) + h(x) + f(p, x') - h(x') \| \leq \varepsilon \|x - x'\| & \text{for } x, x' \in V, p \in U, \\ \|f(p', x) - r(p') - f(p, x) + r(p)\| \leq \varepsilon \|p' - p\| & \text{for } p', p \in U, x \in V. \end{cases} \tag{27}$$

Let $p, p' \in U$. Using (25) and (27), we obtain

$$\begin{aligned} & \|s(p) - s(p') - \sigma(-r(p) + f(\bar{p}, \bar{x}) + \sigma(-r(p') + f(\bar{p}, \bar{x})))\| \\ &= \|\sigma(-f(p, s(p)) + h(s(p))) - \sigma(-f(p', s(p')) + h(s(p')))\| \\ &\quad - \sigma(-r(p) + f(\bar{p}, \bar{x})) + \sigma(-r(p') + f(\bar{p}, \bar{x}))\| \\ &= \|A(-f(p, s(p)) + h(s(p)) + f(p', s(p')) - h(s(p')) + r(p) - r(p'))\| \\ &\leq \|A\| \| -f(p, s(p)) + h(s(p)) + f(p, s(p')) - h(s(p')) \| \\ &\quad + \|A\| \|f(p', s(p')) - r(p') - f(p, s(p')) + r(p)\| \\ &\leq \|A\| (\varepsilon \|s(p) - s(p')\| + \varepsilon \|p' - p\|) \leq \|A\| \varepsilon (\lambda + 1) \|p - p'\|, \end{aligned}$$

where now $\lambda \geq \text{lip}(s; \bar{p})$. Since ε can be arbitrary small, we are done. □

Inequality (24) gives us a bound for the calmness modulus of the localized solution mapping s at \bar{p} . Moreover, from the conclusion of Theorem 3.1 we obtain an exact formula for this modulus, namely

$$\text{clm}(s; \bar{p}) = \text{clm}(\eta; \bar{p}).$$

(This follows from having $\text{clm}(s - \eta; \bar{p}) = 0$.) Under the additional assumptions in the second part of Theorem 3.1, an analogous formula gives us the Lipschitz modulus of s , namely

$$\text{lip}(s; \bar{p}) = \text{lip}(\eta; \bar{p}). \tag{28}$$

Note that the assumption in the second part of Theorem 3.1 that the Lipschitz localization σ of $G^{-1} = (h + F)^{-1}$ at 0 for \bar{x} is affine can be interpreted as a sort of differentiability condition on G^{-1} at 0 with A as the derivative mapping.

Corollary 3.2 (utilization of strict differentiability). *Suppose in the generalized equation (1) with solution mapping S given by (2), that $\bar{x} \in S(\bar{p})$ and f is strictly differentiable at (\bar{p}, \bar{x}) . Assume that the inverse G^{-1} of the mapping*

$$G(x) = f(\bar{p}, \bar{x}) + D_x f(\bar{p}, \bar{x})(x - \bar{x}) + F(x),$$

with $G(\bar{x}) \ni 0$, has a Lipschitz localization σ at 0 for \bar{x} .

Then not only do the conclusions of Theorem 2.1 hold for a solution localization s , but also there is a first-order approximation η to s at \bar{p} given by

$$\eta(p) = \sigma(-D_p(\bar{p}, \bar{x})(p - \bar{p})).$$

If in addition $F \equiv 0$, then this conclusion holds with η being the strict first-order approximation to s at \bar{p} given by

$$\eta(p) = \bar{x} - D_x f(\bar{p}, \bar{x})^{-1} D_p(\bar{p}, \bar{x})(p - \bar{p}), \tag{29}$$

so s is strictly differentiable at \bar{p} ; in other words, Theorem 1.5 is recovered from Theorem 3.1.

Proof In this case Theorem 3.1 is applicable with h taken to be the linearization of $f(\bar{p}, \cdot)$ at \bar{x} and r taken to be the linearization of $f(\cdot, \bar{x})$ at \bar{p} . When $F \equiv 0$, $\sigma(y)$ can be identified with $\bar{x} + D_x f(\bar{p}, \bar{x})^{-1}y$, and η is the affine function given by (29). Strict first-order approximation by an affine function means strict differentiability. \square

Corollary 3.2 confirms that we have achieved, in Theorem 3.1, an implicit function theorem which adds to our extension of Robinson’s theorem in Theorem 2.1 in such a manner as to obtain full coverage of the classical implicit function theorem, Theorem 1.1, by way of its sharpening in Theorem 1.5.

Next, though, we will present an application of Theorem 3.1 that goes beyond strict differentiability to strict semidifferentiability. Recall that a function $g : X \rightarrow Y$ is said to be *semidifferentiable* at \bar{x} if it has a first-order approximation at \bar{x} of the form $h(x) = g(\bar{x}) + H(x - \bar{x})$ in which the function $H : X \rightarrow Y$ is continuous and positively homogeneous. It is *strictly* semidifferentiable at \bar{x} if this is a strict first-order approximation. Either way, H is uniquely determined; for want of a better notation we can denote it here by $\tilde{D}g$.

However, we need to adapt such terminology and notation to our framework of a function $f : P \times X \rightarrow Y$. Semidifferentiability of f at (\bar{p}, \bar{x}) comes out then in terms of a first-order approximation $f(\bar{p}, \bar{x}) + \tilde{D}f(\bar{p}, \bar{x})(p - \bar{p}, x - \bar{x})$ of f at (\bar{p}, \bar{x}) in which $\tilde{D}f(\bar{p}, \bar{x})$ is a continuous, positively homogeneous function from $P \times X$ to Y . Likewise for strict semidifferentiability. (These notions reduce to differentiability and strict differentiability when the approximation is affine.) It will be convenient to let

$$\tilde{D}_p(\bar{p}, \bar{x}) = \tilde{D}f(\bar{p}, \bar{x})(\cdot, 0) \quad \text{and} \quad \tilde{D}_x(\bar{p}, \bar{x}) = \tilde{D}f(\bar{p}, \bar{x})(0, \cdot).$$

These “partial semiderivatives” give first-order approximations of f separately in p and x at (\bar{p}, \bar{x}) .

In these terms, Theorem 3.1 can be specialized to the following result, which was foreshadowed in [4, Corollary 2.10], without proof, in a slightly weaker form but also an inadequate hypothesis. That shortcoming is hereby corrected.

Theorem 3.3 *For a generalized equation (1) with solution mapping S as in (2), and $\bar{x} \in S(\bar{p})$, suppose that f is strictly semidifferentiable at (\bar{p}, \bar{x}) . Assume that the inverse G^{-1} of the mapping*

$$G(x) = f(\bar{p}, \bar{x}) + \tilde{D}_x f(\bar{p}, \bar{x})(x - \bar{x}) + F(x),$$

for which $0 \in G(\bar{x})$, has a single-valued graphical localization σ at 0 for \bar{x} which is semidifferentiable at 0 (hence Lipschitz continuous around 0).

Then S has a Lipschitz localization s at \bar{p} which is semidifferentiable at \bar{p} , and a composition formula for the semiderivative is available:

$$\tilde{D}s(\bar{p}) = \tilde{D}\sigma(0) \circ (-\tilde{D}_p f(\bar{p}, \bar{x})).$$

Proof We proceed as in the proof of Corollary 3.2 with h and r . Then

$$\begin{aligned} & \|s(p) - s(\bar{p}) - \tilde{D}\sigma(0) \circ (-\tilde{D}_p f(\bar{p}, \bar{x}))(p - \bar{p})\| \\ & \leq \|s(p) - \sigma(-r(p) + f(\bar{p}, \bar{x})) - s(\bar{p}) - \sigma(0)\| \\ & \quad + \|\sigma(-\tilde{D}_p f(\bar{p}, \bar{x}))(p - \bar{p}) - \sigma(0) - \tilde{D}\sigma(-\tilde{D}_p f(\bar{p}, \bar{x}))(p - \bar{p})\|. \end{aligned}$$

According to Theorem 3.1 the first term on the right side of this inequality is of order $o(\|p - \bar{p}\|)$. Such a property holds also for the second term, since σ is assumed to be semidifferentiable. It remains only to observe that the composition of two continuous, positively homogeneous mappings is again continuous and positively homogeneous. □

When f has the special form in the classical inverse function theorem, we obtain from Theorems 2.1 and 3.1 an extended inverse function theorem which generalizes Theorem 1.3 to include a set-valued mapping F .

Theorem 3.4 (inverse version). *In the case of the generalized equation (1) where $f(p, x) = g(x) - p$ for a function $g : X \rightarrow Y = P$, so that $S(p) = \{x \mid p \in g(x) + F(x)\} = (g + F)^{-1}(p)$, consider any pair (\bar{p}, \bar{x}) with $\bar{x} \in S(\bar{p})$. Let h be any strict first-order approximation to g at \bar{x} .*

Then $(g + F)^{-1}$ has a Lipschitz localization s at \bar{p} for \bar{x} if and only if $(h + F)^{-1}$ has a Lipschitz localization σ at \bar{p} for \bar{x} , in which case σ is a first-order approximation of s at \bar{p} and

$$\text{lip}(s; \bar{p}) = \text{lip}(\sigma; \bar{p}).$$

If, in addition, $\sigma(y)$ is affine, $\sigma(y) = \bar{x} + Ay$, then s is strictly differentiable at \bar{p} with $Ds(\bar{p}) = A$.

Proof For the “if” part, suppose that $(h + F)^{-1}$ has a Lipschitz localization σ at \bar{p} for \bar{x} . Then from (16) we get $\text{lip}(s; \bar{p}) < \infty$. The “only if” part is completely analogous because g and h play symmetric roles in the statement. Through the observation that $r(p) = g(\bar{x}) - p + \bar{p}$, its clear that the rest follows from Theorem 3.1 and (28). □

A version of Theorem 3.4 with semidifferentiability, parallel to Theorem 3.3, could be stated as well.

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