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FULL LENGTH PAPER

On certain conditions for the existence of solutions of equilibrium problems

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Abstract The main purpose of this paper is the study of sufficient and/or necessary conditions for existence of solutions of equilibrium problems. We discuss some of the assumptions of the problem, under which the introduced conditions are sufficient and/or necessary, and also analyze the effect of these assumptions on the connection between the solution sets of the equilibrium problem and of a related convex feasibility problem.

Keywords Equilibrium problems · Convex feasibility problems

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1 Introduction

The problem of interest, called *Equilibrium Problem*, abbreviated EP, is defined as follows. Given

Honoring Alfred Auslender in his 65th birthday.

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– a real Hausdorff topological vector space X,
– a nonempty closed and convex subset K of X, and
– a function $f : K \times K \to \mathbb{R}$,

EP consists of finding $x \in K$ such that $f(x, y) \geq 0$ for all $y \in K$.

EP has been extensively studied in recent years (e.g., [1–4,9,10,12,13,18–20]). One of the reasons is that, under assumptions P1*–P3* below, it has, among its particular cases, convex optimization problems, variational inequalities (monotone or otherwise), Nash equilibrium problems, and other problems of interest in many applications. We mention parenthetically that in fact any optimization problem can be set in the framework of EP even in the absence of any additional assumptions on f, but very little can be said of EP in such a bare setting. We also comment that infinite dimensional functional spaces are a natural framework for variational inequalities (see, e.g., [15]), which leads us to present our results in as general spaces as possible.

Before formally introducing our results and their background, we will attempt a rather informal presentation. Loosely speaking, EP has been traditionally studied assuming that f is continuous in its first argument, convex in its second one and that it vanishes on the diagonal of $K \times K$. Under such assumptions, the issue of necessary and/or sufficient conditions for existence of solutions of EP was the starting point in the study of the problem. In 1972, Ky Fan proved existence of solutions assuming compactness of K (see [8]), and a short time afterward the same result was established in [6] assuming instead some form of coerciveness of f.

Recently, a necessary and sufficient condition was established in [12] for a subclass of equilibrium problems, namely those instances of EP which satisfy the basic assumptions (presented as P1*–P3* below), plus the following additional condition: positive functional values of f change sign when both arguments are interchanged (see property P4* below). In this situation, it was proved in [12] that condition P5* below is both sufficient and necessary for existence of solutions of EP.

Property P5* seems rather involved in its general statement, but it has some interesting consequences, even for particular cases of EP as thoroughly studied as convex optimization. Consider the problem CO of minimizing a convex function defined on $\mathbb{R}^n$ over a closed and convex subset $C \subset \mathbb{R}^n$, and the auxiliary problem AP consisting of finding a unitary vector $x \in \mathbb{R}^n$ such that $h(x + y) \leq h(y)$ for all $y \in C$. It follows from the relation between P5* and EP that if problem AP does not have solutions then CO does (see [12]).

Thus, condition P5* seems to be interesting enough, and it is the departure point for this paper, whose goals are threefold. In the first place, we will consider an alternative to P5*, namely condition P5 below. P5 is much simpler that P5* and more easily checkable. It also seems less demanding than P5*, but in fact it is equivalent to it, because it is also, as we prove in Sect. 4, both necessary and sufficient for existence of solutions of EP under P1*–P4*.

Next we perform some sort of sensitivity analysis on the new necessary and sufficient condition P5: we want to check whether it is robust enough to persist under some perturbations of the problem assumptions. For reasons explained in the sequel, we concentrate in the (somewhat nonstandard) property P4*. First we weaken it, obtaining P4 below, under which P5 turns out to be still sufficient, but no longer necessary, for the existence of solutions of EP. Then we introduce two slightly strengthened versions of P4, namely P4′ and P4′′ below, under which P5 becomes again necessary.

At this point we have three conditions, namely the old P4*, and the new P4′ and P4′′, which ensure that P5 is both necessary and sufficient, and P4, weaker than the previous three, under which P5 is only sufficient. Our third task consists of exploring the relations among the former three, i.e., P4*, P4′ and P4′′. We show, with appropriate examples, that they are independent, in the sense that no one of them implies any of the remaining two.

Finally, we also pursue this sensitivity analysis beyond property P4*, and we proceed to weaken also the classical properties P2* and P3*, respectively, to P2 and P3 below. This is perhaps a minor point, but anyway we show that P5 remains indeed necessary and sufficient when P2* and P3* are replaced by their weaker versions.

We introduce now formally the setting considered in [12]. In this reference, X is taken as a reflexive Banach space, and the function f of EP is required to satisfy the following assumptions:

P1*: $f(x, x) = 0$ for all $x \in K$,
P2*: $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in K$,
P3*: $f(\cdot, y)$ is upper semicontinuous for all $y \in K$,
P4*: f is *pseudomonotone*, i.e., if $f(x, y) \geq 0$ for some $x, y \in K$, then $f(y, x) \leq 0$.

The existence condition analyzed in [12] is the following:
P5*: For all sequences $\{x^n\} \subset K \setminus \{0\}$ such that

(i) $\lim_{n\to\infty} \|x^n\| = \infty$,
(ii) the sequence $\{\|x^n\|^{-1} x^n\}$ converges weakly to a point $\bar{x} \in K$ such that $f(y, y+\bar{x}) \leq 0$ for all $y \in K$,

there exists another sequence $\{u^n\} \subset K$ such that, for large enough n,

(a) $\|u^n\| < \|x^n\|$,
(b) $f(x^n, u^n) \leq 0$.

Theorem 3.12 in [12] establishes that, assuming that P1*, P2*, P3* and P4* hold, P5* is necessary and sufficient for the existence of solutions of EP.

We will concentrate our sensitivity analysis on variations of property P4*, due to the following reasons:

(i) Differently from P1*–P3*, which can be considered "classical", since they appear (perhaps with slight variations), in practically all the literature on the subject, P4* is to some extent a "newcomer", and thus its incorporation to the set of assumptions deserves some careful analysis.
(ii) P4* is rather demanding, since it excludes certain relevant problems which are particular cases of EP under P1*–P3*, like most non-monotone variational inequalities. Thus, it is interesting to consider alternative options which encompass some of these excluded instances.
(iii) P4* is a condition on f as a joint function of its two arguments, while P2* and P3* deal with f as a function of each of its arguments separately.

Parenthetically, we comment that P4* was introduced in [3] as a form of pseudomonotonicity. It extends the notion of pseudomonotonicity introduced in [14]. It holds for complementarity problems or variational inequality problems (seen as particular cases of EP) when the operator of interest is monotone (or more generally, pseudomonotone in the sense of Karamardian). A fortiori, it also holds for convex optimization problems.

A recurrent theme in the analysis of the aforementioned conditions for the existence of solutions of EP is the connection between them and the solutions of the following convex feasibility problem (CFP in the sequel), often called the "dual equilibrium problem" find $\bar{x} \in \cap_{y \in K} L_f(y)$ where $L_f(y) = \{x \in K : f(y, x) \leq 0\}$. Note that $\bar{x}$ solves CFP iff $f(y, \bar{x}) \leq 0$ for all $y \in K$. It was proved in [12] that under P1*–P3* every solution of CFP solves EP, and then both solution sets trivially coincide under P4*. We also prove in this paper that they still coincide under either P4′ or P4″, but not under P4.

The paper is organized as follows: in Sect. 2 we present the new assumptions under consideration: P1–P5, P4′ and P4″, and establish the relations between P4*, P4, P4′ and P4″. In Sect. 3 we discuss the connections between EP and CFP under different sets of assumptions. In Sect. 4 we prove that P5 is sufficient for the existence of solutions of EP under P1–P4, and necessary and sufficient when P4 in replaced either by P4′ or by P4″.

2 The new assumptions and the connection between the variants of P4*

We start with the statement of our new set of assumptions, which will replace P1*–P4*. We recall the following two definitions:

Definition 1 A function $h : K \to \mathbb{R}$ is said to be *pseudoconvex* if for all $x, y \in K$ and all $t \in (0, 1)$ it holds that

$$h(z) \geq h(x) \quad \text{implies} \quad h(y) \geq h(z), \tag{1}$$

where $z := tx + (1 - t)y$.

Definition 2 A function $h : K \to \mathbb{R}$ is said to be *upper hemicontinuous* if it is upper semicontinuous on any segment contained on K.

In connection with Definition 1, we mention that it is implied by the usual notion of pseudoconvexity, in the differentiable case. It is easy to verify that convexity implies pseudoconvexity, which implies quasiconvexity. It follows that the sublevel sets of a pseudoconvex function are convex. Also, the maximum of a finite family of pseudoconvex functions is pseudoconvex.

Regarding Definition 2, it is worthwhile to comment that hemicontinuous functions which are not continuous appear in many significant applications in infinite dimensional spaces.

In the sequel we assume that the function $f : K \times K \to \mathbb{R}$ satisfies the following conditions:

P1 $f(x, x) = 0$ for each $x \in K$,
P2 $f(x, \cdot) : K \to \mathbb{R}$ is pseudoconvex and lower semicontinuous for all $x \in K$,
P3 $f(\cdot, y) : K \to \mathbb{R}$ is upper hemicontinuous for all $y \in K$.

Note that P1 coincides with P1*, and that P2 and P3 are weaker versions of P2* and P3*, which were the ones used in [12], as explained in Sect. 1. Observe also that, due to the lower semicontinuity requirement included in P2, it does not make any difference if pseudoconvexity is replaced by semistrict quasiconvexity in condition P2. Now we discuss our announced three alternatives for the pseudomonotonicity assumption P4*. We begin with a definition.

Definition 3 The function $f : K \times K \to \mathbb{R}$ is said to be *properly quasimonotone* if for all $x_1, \ldots, x_n \in K$ and all $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ it holds that

$$\min_{1 \leq i \leq n} f\left(x_i, \sum_{j=1}^{n} \lambda_j x_j\right) \leq 0. \tag{2}$$

Proper quasimonotonicity was introduced by Zhou and Chen in [21]. In the case of variational inequalities, it is stronger than quasimonotonicity, as defined, e.g., in [14]. In the general case, as considered here, it neither implies quasimonotonicity nor is implied by it (appropriate examples are given in [1]).

We introduce next the announced variations of property P4*.

P4 f is properly quasimonotone.

It is easy to check that P4 is equivalent to condition C1 in Lemma 3.1, with $L_f(y)$ instead of $C(y)$. We will consider also the following variants of P4:

P4′: f satisfies P4, with strict inequality in (2) if the x_is are pairwise distinct and the λ_is are all strictly positive.
P4′: For every $x_1, \ldots, x_n \in K$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, it holds that

$$\sum_{i=1}^{n} \lambda_i f\left(x_i, \sum_{j=1}^{n} \lambda_j x_j\right) \leq 0. \tag{3}$$

We start with the following elementary result.

Proposition 2.1 *Under P1 and P2, anyone among P4*, P4′ and P4′′ implies P4.*

Proof For P4*, we remark that, as commented above, pseudoconvexity of $f(x, \cdot)$ implies semistrict quasiconvexity of the same function, and then we invoke Proposition 1.1 of [1], where it is proved that under P1 and semistrict quasiconvexity of $f(x, \cdot)$, P4* implies P4. The result is trivial for P4′. For P4′′, use the fact that any convex combination of $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ is greater than or equal to $\min_{1 \leq i \leq n} \alpha_i$.

Now we proceed to prove that P4*, P4′ and P4″ are mutually independent, meaning that under P1–P3 none of them implies any of the remaining ones. For this we construct three examples, for all of which we take $K = [0, 1]$.

Example 1 Define $f_1 : K \times K \to \mathbb{R}$ as

$$f_1(x, y) = \begin{cases} (y + 1 - x)(y - x) & \text{if } y \le x \\ y(y - x) & \text{if } x \le y \end{cases} \tag{4}$$

Note that $f_1(x, y) \ge 0$ for $x \le y$, $f_1(x, y) \le 0$ for $y \le x$.

Example 2 Define $f_2 : K \times K \to \mathbb{R}$ as

$$f_2(x, y) = 0. \tag{5}$$

Example 3 Define $f_3 : K \times K \to \mathbb{R}$ as

$$f_3(x, y) = \begin{cases} \frac{1}{2}(y - x) & \text{if } y \le x \\ y - x & \text{if } x \le y \end{cases} \tag{6}$$

These functions provide all the needed counterexamples. The analysis of f_1 is more delicate, and we encapsulate it in the following proposition.

Proposition 2.2 *f_1 satisfies P1, P2, P3, P4′, P4″ but not P4*.*

Proof (i) P1 and P3 hold trivially (note that f_1 is indeed continuous)

(ii) For P2, we claim that if a function $h : [0, 1] \to \mathbb{R}$ is unimodal in the following way: there exists $t^* \in [0, 1]$ such that h is strictly decreasing in $[0, t^*]$ and strictly increasing in $[t^*, 1]$, then h is pseudomonotone. For the sake of completeness, we proceed to prove this elementary claim (which is very likely to be already known), for which we must verify that (1) holds for any choice of $x, y, z \in [0, 1]$. Without loss of generality, we may assume that $x < z < y$. If both x and y belong to $[0, t^*]$, then we have $h(x) > h(z) > h(y)$ and (1) is vacuous. If they both belong to $[t^*, 1]$, then $h(x) < h(z) < h(y)$ and (1) holds. If $x \le t^* \le y$ and $h(z) \ge h(x)$, then unimodality implies that z belongs to $[t^*, y]$, and hence that $h(z) \le h(y)$, establishing (1) and consequently the claim. Next, observe that $f_1(x, \cdot)$ is a continuous function with two pieces: a convex quadratic function continuously followed by an increasing quadratic one, i.e., a unimodal function with the shape prescribed in the claim above. It follows that f_1 satisfies P2.

(iii) Note that $f_1(1, 0) = 0$, $f_1(0, 1) = 1$, so that P4* does not hold.

(iv) We prove that P4′ holds. Take $t_1, \dots, t_m \in (0, 1)$, $0 \le x_1 < x_2 < \cdots < x_m \le 1$, and let $\bar{x} = \sum_{i=1}^{m} t_i x_i$. It follows that $0 < \bar{x} < x_m$, so that

$$f_1(x_m, \bar{x}) = (\bar{x} + 1 - x_m)(\bar{x} - x_m) < 0,$$

because $1 - x_m \ge 0$. Thus $\min_{1 \le i \le m} f_1(x_i, \bar{x}) < 0$, and P4′ holds.

(v) We prove that P4$''$ holds. Take $0 \le x_1 \le x_2 \le \cdots \le x_m \le 1$, $t_1, \ldots, t_m \in [0, 1]$, and let $\bar{x} = \sum_{i=1}^{m} t_i x_i$. Note that $x_1 \le \bar{x} \le x_m$. Take k such that $x_k \le \bar{x} \le x_{k+1}$, so that $x_i \le \bar{x}$ for $1 \le i \le k$, and $\bar{x} \le x_i$ for $k+1 \le i \le m$. It follows from (4) that

$$f_1(x_i, \bar{x}) = \begin{cases} \bar{x}(\bar{x} - x_i) & \text{if } \quad 1 \le i \le k \\ (\bar{x} + 1 - x_i)(\bar{x} - x_i) & \text{if } k+1 \le i \le m \end{cases} \tag{7}$$

Thus

$$\begin{aligned} \sum_{i=1}^{m} t_i f_1(x_i, \bar{x}) &= \sum_{i=1}^{k} t_i f_1(x_i, \bar{x}) + \sum_{i=k+1}^{m} t_i f_1(x_i, \bar{x}) \\ &= \bar{x} \sum_{i=1}^{k} t_i(\bar{x} - x_i) + \sum_{i=k+1}^{m} t_i(\bar{x} + 1 - x_i)(\bar{x} - x_i) \\ &\le \bar{x} \sum_{i=1}^{k} t_i(\bar{x} - x_i) + (\bar{x} + 1 - x_m) \sum_{i=k+1}^{m} t_i(\bar{x} - x_i) \\ &= \bar{x} \sum_{i=1}^{k} t_i(\bar{x} - x_i) - (\bar{x} + 1 - x_m) \sum_{i=k+1}^{m} t_i(x_i - \bar{x}) \end{aligned} \tag{8}$$

using in the inequality the facts that $x_i \le x_m$ for all i, and that $\bar{x} - x_i \le 0$ for $k+1 \le i \le m$. We claim now that

$$\sum_{i=1}^{k} t_i(\bar{x} - x_i) = \sum_{i=k+1}^{m} t_i(x_i - \bar{x}). \tag{9}$$

Indeed, (9) is equivalent to

$$\sum_{i=1}^{m} t_i \bar{x} = \sum_{i=1}^{m} t_i x_i,$$

which holds because both sides are clearly equal to $\bar{x}$. Replacing (9) in (8) we get

$$\begin{aligned} \sum_{i=1}^{m} t_i f_1(x_i, \bar{x}) &\le \left[\sum_{i=1}^{k} t_i(\bar{x} - x_i)\right] [\bar{x} - \bar{x} - (1 - x_m)] \\ &= -\left[\sum_{i=1}^{k} t_i(\bar{x} - x_i)\right] (1 - x_m) \le 0 \end{aligned} \tag{10}$$

which holds because $\bar{x} - x_i \ge 0$, $t_i \ge 0$ $(1 \le i \le k)$ and $1 - x_m \ge 0$. In view of (10), P4$''$ holds.

Next we establish the announced independence result.

Proposition 2.3 *Assuming P1–P3, none among P4′, P4″ and P4* implies any of the remaining ones.*

Proof The function f_1 of Example 1 shows that neither P4′ nor P4″ imply P4*. Clearly, f_2 of Example 2 satisfies P1, P2, P3, P4* and P4″, but not P4′, so that neither P4* nor P4″ imply P4′. Finally we look at f_3 in Example 3. It satisfies P1, it is continuous in $K \times K$ and $f_3(x, \cdot)$ is convex, being the maximum of two affine functions. Hence, it satisfies P2 and P3. It satisfies P4*, because $f_3(x, y) \geq 0$ iff $x \leq y$, in which case $f_3(y, x) \leq 0$. It is easy to check that it satisfies P4′, but it does not satisfy P4″: take $x_1 = 1/3$, $x_2 = 2/3$, $t_1 = t_2 = 1/2$. Then $\bar{x} = t_1 x_1 + t_2 x_2 = 1/2$ and

$$t_1 f_3(x_1, \bar{x}) + t_2 f_3(x_2, \bar{x}) = \frac{1}{12} - \frac{1}{24} = \frac{1}{24} > 0.$$

It follows that neither P4* nor P4′ imply P4″.

In order to complete the analysis of the relations among P4, P4′, P4″ and P4*, it remains to check whether P4 implies any of the others. It does not, but we leave this result for Sect. 4.

3 Equilibrium problems and convex feasibility problems

We are interested now in the following convex feasibility problem (to be denoted CFP), and its relation with EP.

CFP consists of finding $\bar{x} \in \cap_{y \in K} L_f(y)$, where $L_f(y) = \{x \in K : f(y, x) \leq 0\}$.

Note that for each $y \in K$, $L_f(y)$ is a nonempty, closed and convex subset of K, because $f(y, y) = 0$ for all $y \in K$, K is closed and convex, and f is lower semicontinuous and pseudoconvex in the second argument, so that it has convex sublevel sets.

Note also that $\bar{x}$ solves CFP iff

$$f(y, \bar{x}) \leq 0 \quad \forall\ y \in K. \tag{11}$$

The following lemma was proved by Ky Fan in 1961, and gives a sufficient condition for existence of solutions of CFP.

Lemma 3.1 ([7], Lemma 1) *Let Y be a nonempty subset of a real Hausdorff topological vector space X. For each $y \in Y$, consider a closed subset $C(y)$ of X. If the following two conditions hold:*

C1. *the convex hull of any finite subset $\{x_1, \ldots, x_n\}$ of Y, denoted as co$\{x_1, \ldots x_n\}$, is contained in $\bigcup_{i=1}^{n} C(x_i)$,*

C2. *$C(x)$ is compact for at least some $x \in Y$,*

then $\bigcap_{y \in Y} C(y) \neq \emptyset$.

Proof See Lemma 1 in [7].

The connection between EP and CFP is established in the following result.

Lemma 3.2 *Under P1–P3, the solution set of CFP is contained in the solution set of EP.*

Proof Let $\bar{x}$ be a solution of CFP, and take any $y \in K$. For each $t \in (0, 1)$, define $w_t := ty + (1 - t)\bar{x}$. Since K is convex, w_t belongs to K. Using P1 and the fact that $\bar{x}$ solves CFP, we get $f(w_t, \bar{x}) \leq 0 = f(w_t, w_t)$. By P2, $f(w_t, \cdot)$ is pseudoconvex and thus, using P1 again, $0 = f(w_t, w_t) \leq f(w_t, y)$ for all $t \in (0, 1)$. Taking limits as $t \to 0$ we conclude, using P3, that $0 \leq f(\bar{x}, y)$, i.e., $\bar{x}$ solves EP.

We comment that Lemma 3.2 has been proved in [1] under another generalized convexity assumption, called semistrict quasiconvexity, first introduced in 1993 in [11], and used later on in, e.g., [2,9] and [10]. The function $h : K \to \mathbb{R}$ is said to be *semistrictly quasiconvex*, if for all $x, y \in K$ it holds that

$$h(x) < h(y) \text{ implies } h(x + t(y - x)) < h(y), \tag{12}$$

for all $t \in (0, 1)$. It is easy to check that pseudoconvexity implies semistrict quasiconvexity, and that both notions coincide when h is lower semicontinuous, as in our case. In order to avoid further dealing with semistrict quasiconvexity, we chose the option of proving Lemma 3.2 from scratch.

We remark that the converse of the previous lemma does not hold (a counterexample can be found in [12]).

Next we prove that under any of the variants of P4 under consideration, both solution sets coincide.

Proposition 3.3 *If f satisfies P1–P3, and either P4*, P4′ or P4″, then the solution sets of EP and CFP coincide.*

Proof The solution set of CFP is included in the solution set of EP by Lemma 3.2. We analyze the reciprocal inclusion under each of the three variants of P4.

(i) Under P4*, the reciprocal inclusion is immediate, in view of (11).
(ii) Assume that P4′ holds. Let $\bar{x}$ be a solution of EP, and take any $y \in K, y \neq \bar{x}$. For each $t \in (0, 1)$, define $w_t := ty + (1-t)\bar{x}$. By P4′, $\min\{f(\bar{x}, w_t), f(y, w_t)\} < 0$. Since $f(\bar{x}, w_t) \geq 0$ because $\bar{x}$ solves EP, we get that $f(y, w_t) < 0$ for all $t \in (0, 1)$. Taking the limit as $t \to 0$, we conclude, using P2, that $f(y, \bar{x}) \leq 0$, i.e., in view of (11), that $\bar{x}$ solves CFP.
(iii) Assume that P4″ holds, and take $\bar{x}$, y and w_t as in item (ii). By P4″, $(1 - t)f(\bar{x}, w_t) + tf(y, w_t) \leq 0$. Again, $f(\bar{x}, w_t) \geq 0$ because $\bar{x}$ solves EP, so that $f(y, w_t) \leq 0$. As in item (ii) we get the result by taking the limit as $t \to 0$, invoking P2 and (11).

In [12], a sort of reciprocal of Proposition 3.3 was conjectured, namely, that if f satisfies P1*–P3* and the solution sets of EP and CFP coincide, then f satisfies P4*. Assuming P2 instead of P2*, the function f_1 of Example 1 disproves the conjecture: by Proposition 2.2, it does not satisfy P4*, but it does satisfy P4′, so that the solution sets of CFP and EP coincide, in view of Proposition 3.3. However, f_1 does not satisfy P2*, because for $x \in (0, 1)$, $f_1(x, \cdot)$, consisting of a convex quadratic piece followed by an increasing quadratic one, is pseudoconvex but not convex. We construct next another counterexample for which $f(x, \cdot)$ is indeed convex.

Example 4 Let X be $\mathbb{R}^2$ with the Euclidean inner product, and consider the set

$$K := co\{(0, 0), (1, 0), (0, -1)\}, \tag{13}$$

where "*co*" denotes the convex hull. Let K_0 be the set $K \setminus \{(x_1, 0) : 0 \leq x_1 \leq 1\}$, i.e., we take off the upper side of the triangle. Let $f_4 : K \times K \to \mathbb{R}$ be defined by

$$f_4(x, y) := \langle T(x), y - x\rangle, \tag{14}$$

where the operator $T : K \to \mathbb{R}^2$ is defined as follows:

1. If $x = (x_1, 0) \in K \setminus K_0$, we take $T(x) = (-\sin(x_1\pi/4), \cos(x_1\pi/4))$ (rotation of the vector (0, 1) with an angle equal to $x_1\pi/4$).
2. If $x = (x_1, x_2) \in K$ with $x_2 < 0$ (i.e., $x \in K_0$), then we construct $T(x)$ by making use of the following property: There exists a unique vector $u = (u_1, 0) \in K \setminus K_0$ with $x_1 < u_1 \leq 1$ such that the vector $x - u$ is orthogonal to $T(u)$ ($T(u)$ has been defined in step 1). Indeed, denote by $t = (t_1, 0)$ an arbitrary vector belonging to $K \setminus K_0$ and let $\varphi(t)$ be the cosine of the angle between the vectors $x - t$ and $T(t)$ (defined also in step 1). Then $\varphi((x_1, 0)) < 0$, $\varphi((1, 0)) \geq 0$ and φ is continuous and strictly increasing on the line segment joining $(x_1, 0)$ and $(1, 0)$. Thus, there exists a unique u_1 with $x_1 < u_1 \leq 1$ such that $\varphi((u_1, 0)) = 0$, i.e., denoting by u the vector $(u_1, 0)$ we have that $x - u$ and $T(u)$ are orthogonal. Now define $T(x) := T(u) = T((u_1, 0))$, where u is the unique vector attached to x using the procedure above.

The operator T which appears in the definition of f_4 in Example 4 was introduced in [17] for different purposes. We comment also that EP with a function f defined as f_4 in (14), for an arbitrary $T : K \to X$, where X is a Hilbert space, is equivalent to the variational inequality problem with operator T and feasible set K.

Proposition 3.4 *The function f_4 of Example* 4 *satisfies P1*–P3* and the solution sets of the instances of EP and CFP associated to it coincide, but it does not satisfy P4*.*

Proof It is immediate from (14) that $f_4(x, x) = 0$ for all $x \in K$. It is easy to check that T is well-defined and continuous on K. Therefore f_4 is continuous with respect to each of its variables. Since $f(x, \cdot)$ is affine, it is certainly convex. Thus P1*–P3* are satisfied.

Observe also that the triangle K has been decomposed into infinitely many line segments on which our operator T is constant and that these line segments (level lines) are disjoint (each two lines have empty intersection). By this property it is easy to show that the solution sets of CFP and EP are equal, namely, they coincide with the closed line segment $[(1, 0), (0, -1)]$.

Finally, f_4 does not satisfy P4*. Indeed, for $x = (0, 0)$ and $y = (1, 0)$ we have that $f_4(x, y) = \langle T(x), y - x\rangle = 0$ while $f_4(y, x) = \langle T(y), x - y\rangle > 0$.

4 Existence results

In this section we assume that X is a reflexive Banach space. For each $n \in \mathbb{N}$ let $K_n := \{x \in K : \|x\| \leq n\}$. Since K_n is nonempty for sufficiently large n, in what follows, for the sake of simplicity, we suppose without loss of generality that K_n is nonempty for all $n \in \mathbb{N}$. Define, for each $y \in K$, the set $L_f(n, y) := \{x \in K_n : f(y, x) \leq 0\}$. Note that, applying Lemma 3.2 with K_n instead of K, we have that $\cap_{y \in K_n} L_f(n, y) \subset \{x \in K_n : f(x, y) \geq 0, \ \ \forall y \in K_n\}$, i.e., each solution of the convex feasibility problem restricted to K_n is a solution of the equilibrium problem restricted to K_n. We will also use in the sequel the sets $K_n^o \subset K$, defined as the intersection of K with the open ball of radius n around the origin, i.e., $K_n^o := \{x \in K : \|x\| < n\}$. We need the following technical lemma for our existence result.

Lemma 4.1 *Suppose that P1-P3 hold. If for some $n \in \mathbb{N}$ and some $\bar{x} \in \cap_{y \in K_n} L_f(n, y)$ there exists $\bar{y} \in K_n^o$ such that $f(\bar{x}, \bar{y}) \leq 0$, then $f(\bar{x}, y) \geq 0$ for all $y \in K$, i.e., $\bar{x}$ solves the Equilibrium Problem associated to K.*

Proof If $\bar{x}$ belongs to $\cap_{y \in K_n} L_f(n, y)$, then, by the above observation, we have that $f(\bar{x}, w) \geq 0$ for all $w \in K_n$. Thus, we only need to show that $f(\bar{x}, w) \geq 0$ for all $w \in K \setminus K_n$. Take any $w \in K \setminus K_n$. Since $\bar{y}$ belongs to K_n^o, there exists $t \in (0, 1)$ such that $z := tw + (1-t)\bar{y}$ belongs to K_n. It follows that $0 \leq f(\bar{x}, z)$. Since $f(\bar{x}, \bar{y}) \leq 0$ by assumption, we conclude from the pseudoconvexity of f that $0 \leq f(\bar{x}, w)$.

Next we present the condition which will be sufficient (or necessary and sufficient, depending on the variant of P4 used as assumption), for the existence of solutions of EP.

- P5: For any sequence $\{x^n\} \subset K$ satisfying $\lim_{n \to \infty} \|x^n\| = +\infty$, there exists $u \in K$ and $n_0 \in \mathbb{N}$ such that $f(x^n, u) \leq 0$ for all $n \geq n_0$.

We remark that P5 looks stronger than P5*, because it does not include the requirement on the sequence $\{x^n\}$ in item (ii) of the definition of P5*, and demands existence of a fixed $u \in K$, instead of a sequence $\{u^n\}$, as is the case in items (a) and (b) in the definition of P5*. In fact P5 turns out to be equivalent to P5*, because it is also necessary and sufficient for the existence of solutions of EP, under P1*–P4*, and also under P1–P3 plus either P4$'$ or P4$''$, as we prove below. In any case it is apparent that P5 has a much simpler statement and is more easily checkable than P5*.

Now we proceed to prove the following result.

Theorem 4.2 *Suppose that P1-P5 hold. Then EP admits a solution.*

Proof Let $n \in \mathbb{N}$ be arbitrary. We intend to invoke Lemma 3.1 with $Y = K_n$, $C(y) = L_f(n, y)$, and thus we must check the validity of its hypotheses (it is here where the reflexiveness of X plays its role). We consider the Banach space X with its weak topology, which certainly makes it a Hausdorff topological vector space.

Next we verify C1. Take $x_1, \ldots, x_k \in K_n$ and $\lambda_1, \ldots, \lambda_k \in [0, 1]$ such that $\sum_{j=1}^k \lambda_j = 1$. Let $\bar{x} = \sum_{j=1}^k \lambda_j x_j$. We must verify that $\bar{x}$ belongs to $\cup_{i=1}^k L_f(n, x_i)$, i.e., that $\bar{x}$ belongs to K_n and that $f(x_i, \bar{x}) \leq 0$ for some i. The first of this facts follows from convexity of K_n and the second one from P4, which ensures that $\min_{1 \leq j \leq k} f(x_j, \bar{x}) \leq 0$.

Regarding C2, since $C(y) = L_f(n, y) = \{x \in K_n : f(y, x) \leq 0\}$, $C(y)$ is closed in the norm-topology by P2 (which entails the lower semicontinuity of $f(y, \cdot)$), convex because it is a sublevel set of the pseudoconvex function $f(y, \cdot)$ (invoking again P2), and bounded because it is contained in K_n, which is itself bounded, being the intersection of K with a ball. It is well known that in a reflexive Banach space any closed, convex and bounded set is compact in the weak topology (e.g., [16], vol. I, p. 248). Thus, $C(y)$ is compact in the weak topology for all $y \in K_n$, and C2 holds.

Thus, we are within the hypotheses of Lemma 3.1, and we conclude that $\cap_{y \in K_n} L_f(n, y)$ is nonempty for each $n \in \mathbb{N}$, so that for each n we may choose $x^n \in \cap_{y \in K_n} L_f(n, y)$. We distinguish two cases.

(i) There exists $n \in \mathbb{N}$ such that $\|x^n\| < n$. In this case x^n belongs to K_n^o, and it solves EP by Lemma 4.1.

(ii) $\|x^n\| = n$ for all $n \in \mathbb{N}$. In this case assumption P5 ensures the existence of $u \in K$ and n_0 such that $f(x^n, u) \leq 0$ for all $n \geq n_0$. Take $\hat{n} \geq n_0$ such that $\|u\| < \hat{n}$. Then $f(x^{\hat{n}}, u) \leq 0$ and $u \in K_{\hat{n}}^o$. Again, $x^{\hat{n}}$ turns out to be a solution of EP by Lemma 4.1.

It is natural to ask whether P5 is also necessary for the existence of solutions of EP, under P1-P4. The following example gives a negative answer to this question.

Example 5 Let $X = \mathbb{R}$, $K := [0, \infty)$. Define $f_5 : K \times K \to \mathbb{R}$ as

$$f_5(x, y) = x(x - y). \tag{15}$$

It is clear that f_5 satisfies P1, P2 and P3, and it is easy to check that it also satisfies P4. Note that $x = 0$ is a solution of EP. On the other hand, for the sequence $x^n := n$ there exists no $u \in [0, \infty)$ such that $f_5(n, u) = n(n - u) \leq 0$ for large enough n, i.e., P5 does not hold (observe also that CFP does not have solutions).

On the other hand, it was proved in Theorem 3.12 of [12], that condition P5* (presented in Sect. 1) is indeed necessary and sufficient for existence of solutions of EP, under P1*–P4*. It turns out that the same holds for P5, even if we replace P2* and P3* by its weaker versions P2 and P3. In fact, this is also the case if instead of P4* we assume any of the slightly stronger versions of P4, namely P4′ or P4″. We mention again that P5 looks more demanding than P5*, because it asks for the existence of a fixed u (instead of a variable u^n) associated to any sequence $\{x^n\}$ divergent to infinity (and not only to those whose normalizations converge to a point with specific properties). However, at least in the ambience defined by P1-P3 and either P4′, P4′ or P4*, both P5 and P5* have exactly the same strength, since both are equivalent to the existence of solutions of EP.

Theorem 4.3 *Assume that P1, P2 and P3 hold. Assume also that any one among P4*, P4′ and P4″ holds. Then EP has solutions if and only if P5 holds.*

Proof *Sufficient condition* By Proposition 2.1, any one among P4*, P4′ and P4″ implies P4. Since P4 holds, by Theorem 4.2 EP has solutions.

Necessary condition. Assume that EP has solutions and that any one among P4*, P4′ and P4″ holds. We must verify P5. Take any sequence $\{x^n\} \subset K \setminus \{0\}$ with

$\|x^n\| \to +\infty$. Let $\bar{x} \in K$ be a solution of EP. By Proposition 3.3, $\bar{x}$ is also a solution of CFP. Take $u := \bar{x}$. Since u solves CFP, we have that $f(x^n, u) \le 0$ for all $n \in \mathbb{N}$, and thus P5 holds.

As commented above, P5 and P5* are, in a certain sense, equally strong. P5 has certainly a much simpler and appealing presentation, but there are cases in which P5* might be more useful, specially as a sufficient condition for the existence of solutions of EP. For instance, Theorem 4.3. of [12] is a corollary of the existence result with P5*, for which item (ii) in the statement of P5* is essential. Thus, it is worthwhile to check that the existence result related to P5* is preserved under the new conditions P1–P4, P4′ and P4′′, because Theorem 3.12 in [12] assumes P1*–P4*. We deal with this issue in our next result.

Theorem 4.4 (i) *Under P1–P4, condition P5* is sufficient for the existence of solutions of EP.*

(ii) *Under P1–P3, and either P4′ or P4′′, condition P5* is necessary and sufficient for the existence of solutions of EP.*

Proof (a) For the sufficient condition, either in item (i) or in (ii), we follow the argument in the proof of sufficiency in Theorem 3.12 of [12], which remains valid under the new assumptions. As in Theorem 4.2, we take $x^n \in \cap_{y \in K_n} L_f(n, y)$, which exists by Lemma 3.1. If $\|x^n\| < n$ for some n, then x^n solves EP, as in item (i) of the proof of Theorem 4.2. Thus, we may assume that $\|x^n\| = n$ for all n, which ensures that $\{x^n\}$ satisfies (i) in the statement of P5*. We proceed to establish that it also satisfies (ii). Since the unit ball of X is weakly compact, without loss of generality we may assume that $\{n^{-1}x^n\}$ is weakly convergent to some $\bar{x} \in K$. Fix $y \in K$ and $m > \|y\|$. For $n \ge m$, y belongs to K_n. Since $x^n \in L_f(n, y)$, we have

$$f(y, x^n) \le 0. \tag{16}$$

Let $z^n = (1 - 1/n)y + (1/n)x^n$. We invoke now P1 and P2, claiming that $f(y, z^n) \le 0$: otherwise, $f(y, z^n) > 0 = f(y, y)$ by P1, in which case, using P2 and (16), we get $f(y, z^n) \le f(y, x^n) \le 0$, which is a contradiction. Thus,

$$0 \ge f(y, z^n) = f(y, (1 - 1/n)y + (1/n)x^n). \tag{17}$$

Clearly, lower semicontinuous functions have closed sublevel sets, and, as mentioned above, pseudoconvex functions have convex sublevel sets. Since closed and convex sets are weakly closed (see, e.g., Theorem 3.7 in [5]), we obtain that $f(y, \cdot)$ is weakly lower semicontinuous, because it satisfies P2. Hence, taking limits as $n \to \infty$ in (17), we get $0 \ge f(y, y + \bar{x})$. Since y is an arbitrary point in K, we conclude that $\{x^n\}$ satisfies (ii) in the statement of P5*. Now we invoke P5* to ensure that there exists $\{u^n\} \subset K$ such that $\|u^n\| < \|x^n\|$ and $f(x^n, u^n) \le 0$ for large enough n. Finally, Lemma 4.1 implies that any u^n satisfying these two conditions is a solution of EP. Hence, EP has solutions.

(b) For the necessary condition in item (ii), we invoke Theorem 4.3, which implies that if EP has solutions then P5 holds. The result follows after observing that P5 implies P5*: since for any sequence $\{x^n\} \subset K$ divergent to infinity there exists $u \in K$ such that $f(x^n, u) \leq 0$ for large enough n, given a sequence $\{x^n\}$ satisfying (i) and (ii) in the statement of P5*, henceforth divergent to infinity, such a u exists, and we can take $u^n = u$ for all n. It is immediate that, for large enough n, $\{u^n\}$ satisfies (a) and (b) in the statement of P5*.

Finally, we complete our analysis of the connections among P4, P4′, P4′′ and P4*, bringing together in a unique corollary all our results on the issue.

Corollary 4.5 *Assuming, P1–P3,*

(i) *any one among P4*, P4′ and P4′′ implies P4,*
(ii) *the three cases in item* (i) *are the only ones in which a property among P4*, P4, P4′ and P4′′ implies another one.*

Proof Item (i) is just Proposition 2.1. In view of Proposition 2.3, for establishing (ii) it suffices to prove that P4 implies none of the remaining three, which is indeed the case in view of Example 5 and Theorem 4.3: f_5 satisfies P4 but not P5, and EP with f_5 has solutions. Thus f_5 does not satisfy either P4*, P4′ or P4′′, because under any of them existence of solutions of EP implies P5.

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