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Variational convergence of bivariate functions: lopsided convergence

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Abstract We explore convergence notions for bivariate functions that yield convergence and stability results for their maxinf (or minsup) points. This lays the foundations for the study of the stability of solutions to variational inequalities, the solutions of inclusions, of Nash equilibrium points of non-cooperative games and Walras economic equilibrium points, of fixed points, of solutions to inclusions, the primal and dual solutions of convex optimization problems and of zero-sum games. These applications will be dealt with in a couple of accompanying papers.

Keywords Lopsided convergence · Maxinf-points · Ky Fan functions · Variational inequalities · Epi-convergence

Mathematics Subject Classification (2000) 65K10 · 90C31 · 91A10 · 47J20 · 47J30 · 49J45

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1 Variational convergence of bivariate functions

A fundamental component of 'Variational Analysis' is the analysis of the properties of bivariate functions. For example: the analysis of the Lagrangians associated with an optimization problem, of the Hamiltonians associated with Calculus of Variations and Optimal Control problems, the reward functions associated with cooperative or non-cooperative games, and so on. In a series of articles, we deal with the stability of the solutions of a wide collection of problems that can be re-cast as finding the maxinf-points of such bivariate functions.

So, more explicitly: given a bivariate function $F : C \times D \to \mathbb{R}$, we are interested in finding a point, say $\bar{x} \in C$, that maximizes with respect to the first variable x, the infimum of F, $\inf_{y \in D} F(\cdot, y)$, with respect to the second variable y. We refer to such a point $\bar{x}$ as a *maxinf-point*. In some particular situations, for example when the bivariate function is concave–convex, such a point can be a saddle point, but in many other situation its just a maxinf-point, or a minsup-point when minimizing with respect to the first variable the supremum of F with respect to the second variable. To study the stability, and the existence, of such points, and the sensitivity of their associated values, one is lead to introduce and analyze convergence notion(s) for bivariate functions that in turn will guarantee the convergence either of their saddle points or of just their maxinf-points.

This paper is devoted to the foundations. Two accompanying papers deal with the motivating examples [10,11]: variational inequalities, fixed points, Nash equilibrium points of non-cooperative games, equilibrium points of zero-sum games, etc. We make a distinction between the situations when the bivariate function is generated from a single-valued mapping [11] or when the mapping can also be set-valued [10].

The major tool is the notion of *lopsided convergence*, that was introduced in [2], but is modified here so that a wider class of applications can be handled. The major adjustment is that bivariate functions are no longer as in [2], defined on all of $\mathbb{R}^n \times \mathbb{R}^m$ with values in the extended reals, but are now only finite-valued on a specific product $C \times D$ with C, D subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$. Dealing with 'general' bivariate functions defined on the full product space was in keeping with the elegant work of Rockafellar [13] on duality relations for convex–concave bivariate functions and the subsequent work [3] on the epi/hypo-convergence of saddle functions. However, our present analysis actually shows that notwithstanding its esthetic allurement one should not cast bivariate functions, even in the convex–concave case, in the general extended-real valued framework. In some way, this is in contradiction with the univariate case where the extension, by allowing for the values $\pm\infty$, of functions defined on a (constrained) set to all of $\mathbb{R}^n$ has been so effectively exploited to derive a 'unified' convergence and differentiation theory [5,14]. We shall show that some of this can be recovered, but one must first make a clear distinction between max-inf problems and min-sup ones, and only then one can generate the appropriate extensions; after all, also in the univariate case one makes a clear distinction when extending a function in a minimization setting or a maximization setting.

In order to be consistent in our presentation, and to set up the results required later on, we begin by a presentation of the theory of epi-convergence for real-valued univariate functions that are *only* defined on a subset of $\mathbb{R}^n$. No 'new' results are actually derived although a revised formulation is required. We make the connection with the standard approach, i.e., when these (univariate) functions are extended real-valued. We then turn to lopsided convergence and point out the shortcomings of an 'extended real-valued' approach. Finally, we exploit our convergence result to obtain a extension of Ky Fan inequality [7] to situations when the domain of definition of the bivariate function is not necessarily compact.

2 Epi-convergence

One can always represent an optimization problem, involving constraints or not, as one of minimizing an extended real-valued function. In the case of a constrained-minimization problem, simply redefine the objective as taking on the value ∞ outside the feasible region, the set determined by the constraints. In this framework, the canonical problem can be formulated as one of minimizing on all of $\mathbb{R}^n$ an extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. Approximation issues can consequently be studied in terms of the convergence of such functions. This has lead to the notion of *epi-convergence*[1] that plays a key role in 'Variational Analysis' [1,5,14]; when dealing with a maximization problem, it is hypo-convergence, the convergence of the hypographs, that is the appropriate convergence notion.

Henceforth, we restrict our development to the 'minimization setting' but, at the end of this section, we translate results and observations to the 'maximization' case.

As already indicated, in Variational Analysis, one usually deals with

$$\text{fcn}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \overline{\mathbb{R}} \right\}$$

the space of extended real-valued functions that are defined on *all* of $\mathbb{R}^n$, even allowing for the possibility that they are nowhere finite-valued. Definitions, properties, limits, etc., generally do not refer to the domain on which they are finite. For reasons that will become clearer when we deal with the convergence of bivariate functions, we need to depart from this simple, and very convenient, paradigm. Our focus will be on

$$fv\text{-fcn}(\mathbb{R}^n) = \left\{ f : D \to \mathbb{R} \,\middle|\, \text{ for some } \emptyset \neq D \subset \mathbb{R}^n \right\},$$

the class of all *finite-valued functions with non-empty domain* $D \subset \mathbb{R}^n$. It must be understood that in this notation, $\mathbb{R}^n$ does not refer to the domain of definition, but to the underlying space that contains the domains on which the functions are defined.

The *epigraph* of a function f is *always* the set of all points in $\mathbb{R}^{n+1}$ that lie on or above the graph of f, irrespective of f belonging to fv-fcn$(\mathbb{R}^n)$ or fcn$(\mathbb{R}^n)$. If $f : D \to \mathbb{R}$ belongs to fv-fcn$(\mathbb{R}^n)$, then

$$\text{epi} f = \left\{ (x, \alpha) \in D \times \mathbb{R} \,\middle|\, \alpha \geq f(x) \right\} \subset \mathbb{R}^{n+1},$$

[1] For extensive references and a survey of the field one can consult [1,5], and in particular, the Commentary section that concludes [14, Chap. 7].

and if f belongs to $\mathrm{fcn}(\mathbb{R}^n)$ then

$$\mathrm{epi}\, f = \big\{(x, \alpha) \in \mathbb{R}^{n+1} \,\big|\, \alpha \geq f(x)\big\}.$$

A function f is *lsc* (=*lower semicontinuous*) if its epigraph is closed as a subset of $\mathbb{R}^{n+1}$, i.e., $\mathrm{epi}\, f = \mathrm{cl}(\mathrm{epi}\, f)$ with cl denoting closure [14, Theorem 1.6].[2]
So, when $f \in f\upsilon\text{-fcn}(\mathbb{R}^n)$, lsc implies[3] that for all $x^\nu \in D \to x$:

- if $x \in D$: $\liminf_\nu f(x^\nu) \geq f(x)$, and
- if $x \in \mathrm{cl}\, D \setminus D$: $f(x^\nu) \to \infty$.

In our 'minimization' framework: $\mathrm{cl}\, f$ denotes the function whose epigraph is the closure relative to $\mathbb{R}^{n+1}$ of the epigraph of f, i.e., the lsc-regularization of f. Its possible that when $f \in f\upsilon\text{-fcn}$, $\mathrm{cl}\, f$ might be defined on a set thats strictly larger than D but always contained in $\mathrm{cl}\, D$.

Lets now turn to convergence issues. Recall that *set-convergence*, in the Painlevé–Kuratowski sense [14, Sect. 4.B], is defined as follows: $C^\nu \to C \subset \mathbb{R}^n$ if

- (a-set) all cluster points of a sequence $\{x^\nu \in C^\nu\}_{\nu \in I\!N}$ belong to C,
- (b-set) for each $x \in C$, one can find a sequence $x^\nu \in C^\nu \to x$.

When just condition (a-set) holds, then C is then the *outer limit* of the sequence $\{C^\nu\}_{\nu \in I\!N}$, and when its just (b-set) that holds, then C is the *inner limit* [14, Chap. 4, Sect. 2]. Note, that whenever C is the limit, the outer- or the inner-limit, its *closed* [14, Proposition 4.4] and that $C = \emptyset$ if and only if the sequence C^ν eventually 'escapes' from any bounded set [14, Corollary 4.11]. Moreover, if the sequence $\{C^\nu\}_{\nu \in I\!N}$ consists of convex sets, its inner limit, and its limit if it exists, are also convex [14, Proposition 4.15].

Definition 1 (epi-convergence) A sequence of functions $\{f^\nu,\ \nu \in I\!N\}$, whose domains lie in $\mathbb{R}^n$, *epi-converges* to a function f when $\mathrm{epi}\, f^\nu \to \mathrm{epi}\, f$ as subsets of $\mathbb{R}^{n+1}$; again irrespective of f belonging to $f\upsilon\text{-fcn}(\mathbb{R}^n)$ or $\mathrm{fcn}(\mathbb{R}^n)$. One then writes $f^\nu \xrightarrow{e} f$.

Figure 1 provides an example of two functions f and f^ν that are close to each other in terms of the distance between their epigraphs—i.e., the distance between the location of the two jumps—but are pretty far from each other pointwise or with respect to the ℓ^∞-norm,—i.e., the size of the jumps.

Let $\{f^\nu\}_{\nu \in I\!N}$ be a sequence of functions with domains in $\mathbb{R}^n$. When,

- $\mathrm{epi}\, f$ is the outer limit of $\{\mathrm{epi}\, f^\nu\}_{\nu \in I\!N}$, one refers to f as the *lower epi-limit* of the functions f^ν,
- $\mathrm{epi}\, f$ is the inner limit of the $\mathrm{epi}\, f^\nu$, one refers to f as the *upper epi-limit* of the functions f^ν.

Of course, f is the epi-limit of the sequence if its both the lower and upper epi-limit.

[2] Throughout its implicitly assumed that $\mathbb{R}^n$ is equipped with its usual Euclidean topology.

[3] Indeed, if $\liminf_\nu f(x^\nu) < \infty$, then for some subsequence $\{\nu_k\}$, $f(x^{\nu_k}) \to \alpha \in \mathbb{R}$ and because $\mathrm{epi}\, f$ is closed, it implies that $(x, \alpha) \in \mathrm{epi}\, f$ which would place x in the domain of f, contradicting $x \notin D$.

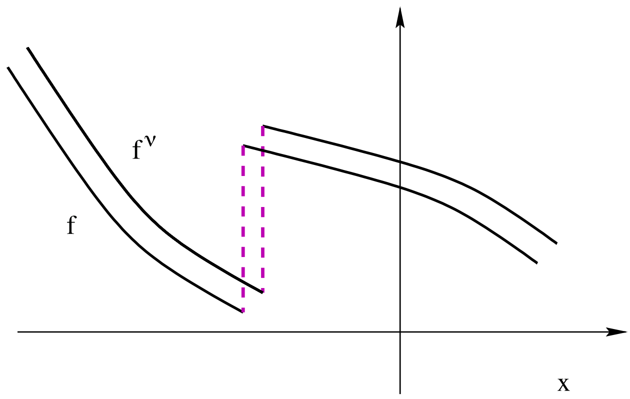

Fig. 1 f and f^ν epigraphically close to each other

Proposition 1 (properties of epi-limits) *Let* $\{f^\nu\}_{\nu\in\mathbb{N}}$ *be a sequence of functions with domains in* $\mathbb{R}^n$*. Then, the lower and upper epi-limits and the epi-limit, if it exists, are all lsc. Moreover, if the functions* f^ν *are convex, so is the upper epi-limit, and the epi-limit, if it exists.*

Proof Follows immediately from the properties of set-limits. □

The last proposition implies in particular that the *family of lsc functions is closed under epi-convergence*.

The definition of epi-convergence for families of functions in $\text{fcn}(\mathbb{R}^n)$ is the usual one [14, Chap. 7, Sect. B] with all the implications concerning the convergence of the minimizers and infimal values [14, Chap. 7, Sect. E]. But, in a certain sense, the definition is 'new' when the focus is on epi-convergent families in $f\upsilon\text{-fcn}(\mathbb{R}^n)$, and its for this class of functions that we need to know the conditions under which one can claim convergence of the minimizers and infimums. *We chose to make the presentation self-contained*, although as will be shown later, one could also embed $f\upsilon\text{-fcn}(\mathbb{R}^n)$ in a subclass of $\text{fcn}(\mathbb{R}^n)$ and then appeal to the 'standard' results, but unfortunately this requires that the non-initiated reader plows through a substantial amount of material.

When f is *an epi-limit its necessarily a lsc function* since its epigraph is the set-limit of a collection of sets in $\mathbb{R}^{n+1}$. Its epigraph is closed but its domain D is not necessarily closed. Simply think of the collection of functions $f^\nu = f$ for all ν with $D = (0, \infty)$ and $f(x) = 1/x$ on D. This collection clearly epi-converges to the lsc function f on D with closed epigraph but not with closed domain.

Lemma 1 (epi-limit value at boundary points). *Suppose* $f : D \to \mathbb{R}$ *is the epi-limit of a sequence* $\left\{f^\nu : D^\nu \to \mathbb{R}\right\}_{\nu\in\mathbb{N}}$ *with all functions in* $f\upsilon\text{-fcn}(\mathbb{R}^n)$*. Then, for any sequence* $x^\nu \in D^\nu \to x$*:* $\liminf_\nu f^\nu(x^\nu) > -\infty$.

Proof We proceed by contradiction. Suppose that $x^\nu \in D^\nu \to x$ and $\liminf_\nu f^\nu(x^\nu) = -\infty$. By assumption $f > -\infty$ on D, thus the x^ν cannot converge to a point in D, i.e., necessarily $x \notin D$. If thats the case and since $\text{epi}\, f^\nu \to \text{epi}\, f$, the line $\{x\} \times \mathbb{R}$ would have to lie in $\text{epi}\, f$ contradicting the assumption that f, the epi-limit of the f^ν, belongs to $f\upsilon\text{-fcn}(\mathbb{R}^n)$. □

Example 1 [an epi-limit thats not in $fv\text{-fcn}(\mathbb{R}^n)$] Consider the sequence of functions $\{f^\nu : [0,\infty) \to \mathbb{R}\}_{\nu \in I\!N}$ with

$$f^\nu(x) = \begin{cases} -\nu^2 x & \text{if } 0 \le x \le \nu^{-1}, \\ \nu^2 x - 2\nu & \text{if } \nu^{-1} \le x \le 2\nu^{-1}, \\ 0 & \text{for } x \ge 2\nu^{-1}. \end{cases}$$

Detail The functions $f^\nu \in fv\text{-fcn}(\mathbb{R})$ and for the sequence $x^\nu = \nu^{-1}$, $f^\nu(x^\nu) \to -\infty$ and $f^\nu \xrightarrow{e} f$ where $f : [0,\infty) \to \mathbb{R}$ with $f \equiv 0$ on $(0,\infty)$ and $f(0) = -\infty$. Thus, the functions f^ν epi-converge to f as functions in $\text{fcn}(\mathbb{R})$, provided they are appropriately extended, i.e., taking on the value ∞ on $(-\infty, 0)$. But they do not epi-converge to a function in $fv\text{-fcn}(\mathbb{R})$. □

In addition to the 'geometric' definition, the next proposition provides an 'analytic' characterization of epi-converging sequences in $fv\text{-fcn}(\mathbb{R}^n)$.

Proposition 2 [epi-convergence in $fv\text{-fcn}(\mathbb{R}^n)$] *Let* $\{f : D \to \mathbb{R}, f^\nu : D^\nu \to \mathbb{R},\ \nu \in I\!N\}$ *be a collection of functions in* $fv\text{-fcn}(\mathbb{R}^n)$. *Then,* $f^\nu \xrightarrow{e} f$ *if and only the following conditions are satisfied:*

(a) $\forall\, x^\nu \in D^\nu \to x$ *in* D, $\liminf_\nu f^\nu(x^\nu) \ge f(x)$,

(a$^\infty$) *for all* $x^\nu \in D^\nu \to x \notin D$, $f^\nu(x^\nu) \nearrow \infty$,[4]

(b) $\forall x \in D$, $\exists\, x^\nu \in D^\nu \to x$ *such that* $\limsup_\nu f^\nu(x^\nu) \le f(x)$.

Proof If epi $f^\nu \to$ epi f and $x^\nu \in D^\nu \to x$ either $\liminf_\nu f^\nu(x^\nu) = \alpha < \infty$ or not; Lemma 1 reminds us that $\alpha = -\infty$ is not a possibility. In the first instance, (x, α) is a cluster point of $\{(x^\nu, f^\nu(x^\nu)) \in \text{epi}\, f^\nu\}_{\nu \in I\!N}$ and thus belongs to epi f, i.e., $f(x) \le \alpha$ and hence (a) holds; $\alpha > -\infty$ since otherwise f would not be finite valued on D. If $\alpha = \infty$ that means that $f^\nu(x^\nu) \nearrow \infty$ and x cannot belong to D, and thus (a$^\infty$) holds. On the other hand, if $x \in D$ and thus $f(x)$ is finite, there is a $\{(x^\nu, \alpha^\nu) \in \text{epi}\, f^\nu\}_{\nu \in I\!N}$ such that $x^\nu \in D^\nu \to x \in D$ and $\alpha^\nu \to f(x)$ with $\alpha^\nu \ge f^\nu(x^\nu)$, i.e., $\limsup_\nu f^\nu(x^\nu) \le f(x)$, i.e., (b) is also satisfied.

Conversely, if (a) and (a$^\infty$) hold, and $(x^\nu, \alpha^\nu) \in \text{epi}\, f^\nu \to (x, \alpha)$ then either $x \in D$ or not; recall also, that in view of Lemma 1, α cannot be $-\infty$ since we are dealing with epi-convergence in $fv\text{-fcn}(\mathbb{R}^n)$. In the latter instance, by (a$^\infty$) $\alpha = \infty$, so we are not dealing with a converging sequence of points (in R^{n+1}) and there is no need to consider this situation any further. When $x \in D$, since then $\liminf_\nu f^\nu(x^\nu) \ge f(x)$ and $\alpha^\nu \ge f^\nu(x^\nu)$, one has $\alpha \ge f(x)$ and consequently (x, α) belongs to epi f; this means that condition (a-set) is satisfied. If $(x, \alpha) \in \text{epi}\, f$, from (b) follows the existence of a sequence $x^\nu \in D^\nu \to x$ such that $\limsup_\nu f^\nu(x^\nu) \le f(x) \le \alpha$. We can then choose the $\alpha^\nu \ge f^\nu(x^\nu)$ so that $\alpha^\nu \to \alpha$ that yields (b-set), the second condition for the set-convergence of epi $f^\nu \to$ epi f. □

Theorem 1 (epi-convergence: basic properties) *Consider a sequence* $\{f^\nu : D^\nu \to \mathbb{R},\ \nu \in I\!N\} \subset fv\text{-fcn}(\mathbb{R}^n)$ *epi-converging to* $f : D \to \mathbb{R}$, *also in* $fv\text{-fcn}(\mathbb{R}^n)$. *Then*

[4] $\nearrow$ means non-decreasing and converging to, i.e., not necessarily monotonically.

$$\limsup_{\nu\to\infty} (\inf f^\nu) \;\le\; \inf f.$$

Moreover, if $x^k \in \operatorname{argmin}_{D^{\nu_k}} f^{\nu_k}$ *for some subsequence* $\{\nu_k\}$ *and* $x^k \to \bar{x}$*, then* $\bar{x} \in \operatorname{argmin}_D f$ *and* $\min_{D^{\nu_k}} f^{\nu_k} \to \min_D f$.

If $\operatorname{argmin}_D f$ *is a singleton, then every convergent subsequence of minimizers converges to* $\operatorname{argmin}_D f$.

Proof Let $\{x^l\}_{l=1}^\infty$ be a sequence in D such that $f(x^l) \to \inf f$. By 2(b), for each l one can find a sequence $x^{\nu,l} \in D^\nu \to x^l$ such that $\limsup_\nu f^\nu(x^{\nu,l}) \le f(x^l)$. Since for all ν, $\inf f^\nu \le f^\nu(x^{\nu,l})$, it follows that for all l,

$$\limsup_\nu (\inf f^\nu) \le \limsup_\nu f^\nu(x^{\nu,l}) \le f(x^l),$$

and one has, $\limsup_\nu (\inf f^\nu) \le \inf f$ since $f(x^l) \to \inf f$.

For the sequence $x^k \in D^{\nu_k} \to \bar{x}$, from the above and 2(a),

$$\inf f \ge \limsup_k f^{\nu_k}(x^k) \ge \liminf_k f^{\nu_k}(x^k) \ge f(\bar{x}),$$

i.e., $\bar{x}$ minimizes f on D and $f^{\nu_k}(x^k) = \min_{D^{\nu_k}} f^{\nu_k} \to \min_D f$.

Finally, since every convergent subsequence of minimizers of the functions f^ν converges to a minimizer of f, it follows that it must converge to the unique minimizer when $\operatorname{argmin}_D f$ is a singleton. □

In most of the applications, we shall rely on a somewhat more restrictive notion than 'simple' epi-convergence to guarantee the convergence of the infimums.

Definition 2 (tight epi-convergence) The sequence $\{f^\nu : D^\nu \to \mathbb{R}\}_{\nu\in\mathbb{N}} \subset fv\text{-fcn}(\mathbb{R}^n)$ *epi-converges tightly* to $f : D \to \mathbb{R} \in fv\text{-fcn}(\mathbb{R}^n)$, if $f^\nu \xrightarrow{e} f$ and for all $\varepsilon > 0$, there exist a compact set B_ε and an index ν_ε such that

$$\forall\, \nu \ge \nu_\varepsilon : \quad \inf_{B_\varepsilon\cap D^\nu} f^\nu \le \inf_{D^\nu} f^\nu + \varepsilon.$$

Theorem 2 (convergence of the infimums). *Let* $\{f^\nu : D^\nu \to \mathbb{R}\}_{\nu\in\mathbb{N}} \subset fv\text{-fcn}(\mathbb{R}^n)$ *be a sequence of functions that epi-converges to the function* $f : D \to \mathbb{R}$ *also in* $fv\text{-fcn}(\mathbb{R}^n)$*, with* $\inf_D f$ *finite. Then, they epi-converge tightly*

(a) *if and only if* $\inf_{D^\nu} f^\nu \to \inf_D f$.

(b) *if and only if there exists a sequence* $\varepsilon^\nu \searrow 0$ *such that* $\varepsilon^\nu\text{-}\operatorname{argmin} f^\nu \to \operatorname{argmin} f$.

Proof Lets start with necessity in (a). For given $\varepsilon > 0$, the assumptions and Theorem 1 imply

$$\liminf_\nu (\inf_{D^\nu\cap B_\varepsilon} f^\nu) \le \liminf_\nu (\inf_{D^\nu} f^\nu) + \varepsilon \le \limsup_\nu (\inf_{D^\nu} f^\nu) + \varepsilon \le \inf_D f + \varepsilon < \infty.$$

If there is a subsequence $\{\nu_k\}$ such that $f(x^k) < \kappa$ for some $x^k \in D^{\nu_k} \cap B_\varepsilon$, it would follow that $\inf_D f < \kappa$. Indeed, since B_ε is compact, the sequence $\{x^k\}$ has a cluster point, say $\bar{x}$, and then conditions (a$^\infty$) and (a) of Proposition 2 guarantee $f(\bar{x}) < \kappa$ with $\bar{x} \in D$, and consequently, also $\inf_D f < \kappa$. Since its assumed that $\inf_D f$ is finite, it follows that there is no such sequences with κ arbitrarily negative. In other words, excluding possibly a finite number of indexes, the $\inf_{D^\nu \cap B_\varepsilon} f^\nu$ stay bounded away from $-\infty$ and one can find $x^\nu \in \varepsilon\text{-}\operatorname{argmin}_{D^\nu \cap B_\varepsilon} f^\nu$. The sequence $\{x^\nu\}_{\nu \in \mathbb{N}}$ admits a cluster point, say $\bar{x}$, that lies in B_ε and again by 2(a$^\infty$,a), $f(\bar{x}) \le \operatorname{liminf}_\nu f^\nu(x^\nu)$. Hence,

$$\inf_D f - \varepsilon \le f(\bar{x}) - \varepsilon \le \operatorname{liminf}_\nu f^\nu(x^\nu) - \varepsilon \le \operatorname{liminf}_\nu(\inf_{D^\nu} f^\nu).$$

In combination with our first string of inequalities and the fact that $\varepsilon > 0$ can be chosen arbitrarily small, it follows that indeed $\inf_{D^\nu} f^\nu \to \inf_D f$.

Next, we turn to sufficiency in (a). Since $\inf f^\nu \to \inf f \in \mathbb{R}$ by assumption, its enough, given any $\delta > 0$, to exhibit a compact set B such that $\limsup_\nu \left(\inf_{B \cap D^\nu} f^\nu\right) \le \inf_D f + \delta$. Choose any point x such that $f(x) \le \inf_D f + \delta$. Because $f^\nu \xrightarrow{e} f$ in $fv\text{-fcn}(\mathbb{R}^n)$, there exists a sequence, 2(a), $x^\nu \to x$ such that $\limsup_\nu f^\nu(x^\nu) \le f(x)$. Let B be any compact set large enough to contain all the points x^ν. Then $\inf_B f^\nu \le f^\nu(x^\nu)$ for all ν, so B has the desired property.

We derive (b) from (a). Let $\bar{\alpha}^\nu = \inf f^\nu \to \inf f = \bar{\alpha}$ that is finite by assumption, and consequently for ν large enough, also $\bar{\alpha}^\nu$ is finite. Since convergence of the epigraphs implies the convergence of the level sets [14, Proposition 7.7], one can find a sequence of $\alpha^\nu \searrow \bar{\alpha}$ such that $\operatorname{lev}_{\alpha^\nu} f^\nu \to \operatorname{lev}_{\bar{\alpha}} f = \operatorname{argmin} f$. Simply set $\varepsilon^\nu := \alpha^\nu - \bar{\alpha}^\nu$.

For the converse, suppose there is a sequence $\varepsilon^\nu \searrow 0$ with ε^ν-argmin $f^\nu \to$ argmin $f \ne \emptyset$. For any $x \in \operatorname{argmin} f$ one can select $x^\nu \in \varepsilon^\nu$-argmin f^ν with $x^\nu \to x$. Then because $f^\nu \xrightarrow{e} f$, one obtains

$$\begin{aligned} \inf f = f(x) &\le \liminf_\nu f^\nu(x^\nu) \le \liminf_\nu(\inf f^\nu + \varepsilon^\nu) \\ &\le \liminf_\nu(\inf f^\nu) \le \operatorname{limsup}_\nu(\inf f^\nu) \le \inf f, \end{aligned}$$

where the last inequality comes from Theorem 1. □

Remark 1 (convergence of domains) Although, epi-convergence essentially implies convergence of the level sets [14, Proposition 7.7], it does not follow that it implies the convergence of their (effective) domains. Indeed, consider the following sequence $f^\nu : \mathbb{R} \to \mathbb{R}$ with $f^\nu \equiv \nu$ except for $f^\nu(0) = 0$ that epi-converges to $\delta_{\{0\}}$ the indicator function of $\{0\}$. We definitely do not have $\operatorname{dom} f^\nu = \mathbb{R}$ converging to $\operatorname{dom} \delta_{\{0\}} = \{0\}$. This vigorously argues against the temptation of involving the convergence of their domains in the definition of epi-convergence, *even* for functions in $fv\text{-fcn}(\mathbb{R}^n)$.

This concludes the presentation of the results that will be used in the sequel. As indicated earlier, its also possible to derive these results from those for extended real-valued functions. To do so, one identifies $fv\text{-fcn}(\mathbb{R}^n)$ with

$$pr\text{-fcn}(\mathbb{R}^n) := \left\{ f \in \text{fcn}(\mathbb{R}^n) \,\middle|\, -\infty < f \not\equiv \infty \right\},$$

the subset of proper functions in $\text{fcn}(\mathbb{R}^n)$; in a minimization context, a function f is said to be *proper* if $f > -\infty$ and $f \not\equiv \infty$, in which case, its finite on its (*effective*) *domain*

$$\text{dom } f = \{x \in \mathbb{R}^n \mid f(x) < \infty\}.$$

There is an one-to-one correspondence, a *bijection*[5] denoted η, between the elements of $fv\text{-fcn}(\mathbb{R}^n)$ and those of $pr\text{-fcn}(\mathbb{R}^n)$: If $f \in fv\text{-fcn}(\mathbb{R}^n)$, its extension to all of $\mathbb{R}^n$ by setting $\eta f = f$ on its domain and $\eta f \equiv \infty$ on the complement of its domain, uniquely identifies a function in $pr\text{-fcn}(\mathbb{R}^n)$. And, if $f \in pr\text{-fcn}(\mathbb{R}^n)$, the restriction of f to dom f, uniquely identifies a function $\eta^{-1} f$ in $fv\text{-fcn}(\mathbb{R}^n)$. Its important to observe that under this bijection, any function, either in $pr\text{-fcn}(\mathbb{R}^n)$ or $fv\text{-fcn}(\mathbb{R}^n)$, and the corresponding one in $fv\text{-fcn}(\mathbb{R}^n)$ or $pr\text{-fcn}(\mathbb{R}^n)$, have the *same* epigraphs!

Since, epi-convergence for sequences in $fv\text{-fcn}(\mathbb{R}^n)$ or in in $\text{fcn}(\mathbb{R}^n)$ is always defined in terms of the convergence of the epigraphs, there is really no need to verify that the analytic versions (Proposition 2 and [14, Proposition 7.2]) also coincide. However, for completeness sake and to highlight the connections, we go through the details of an argument.

Proposition 3 (epi-convergence in $fv\text{-fcn}(\mathbb{R}^n)$ and $\text{fcn}(\mathbb{R}^n)$) *Let* $\{f : D \to \mathbb{R},\, f^\nu : D^\nu \to \mathbb{R},\, \nu \in I\!N\}$ *be a collection of functions in* $fv\text{-fcn}(\mathbb{R}^n)$. *Then,* $f^\nu \xrightarrow{e} f$ *if and only* $\eta f^\nu \xrightarrow{e} \eta f$ *where* η *is the bijection defined above.*

Proof Now, $\eta f^\nu \xrightarrow{e} \eta f$ ([14, Proposition 7.2]) if and only if for all $x \in \mathbb{R}^n$:

(aη) $\liminf_\nu \eta f^\nu(x^\nu) \geq \eta f(x)$ for every sequence $x^\nu \to x$,

(bη) $\limsup_\nu \eta f^\nu(x^\nu) \leq \eta f(x)$ for some sequence $x^\nu \to x$.

Since for $x \notin D$, $\eta f(x) = \infty$, (aη) clearly implies (a) & (a$^\infty$). Conversely, if (a) and (a$^\infty$) hold, $x \in D$ and $x^\nu \to x$, when computing the $\liminf_\nu \eta f^\nu(x^\nu)$ one can ignore elements $x^\nu \notin D^\nu$ since then $\eta f^\nu(x^\nu) = \infty$. Hence, for $x \in D$, actually (a) implies (aη). If $x \notin D$ and $x^\nu \to x$, (a$^\infty$) and, again, the fact that $\eta f^\nu(x^\nu) = \infty$ when $x \notin D^\nu$, yield (aη).

If (bη) hold and $x \in D$, then the sequence $x^\nu \to x$ must, at least eventually, have $x^\nu \in D^\nu$ since otherwise the $\limsup_\nu \eta f^\nu(x^\nu)$ would be ∞ whereas $f(x) = \eta f(x)$ is finite. Thus, (bη) implies (b). Conversely, (b) certainly yields (bη) if $x \in D$. If $x \notin D$, $\eta f(x) = \infty$ and so the inequality in (bη) is also trivially satisfied in that case. □

As long as we restrict our attention to $pr\text{-fcn}(\mathbb{R}^n)$, in view of the preceding observations, *all the basic results*, cf. [14, Chap. 7, Sect. E] of the theory of epi-convergence related to the convergence of infimums and minimizers apply equally well to functions in $fv\text{-fcn}(\mathbb{R}^n)$ and *not just those featured here*. In particular, if one takes into account the bijection between $fv\text{-fcn}(\mathbb{R}^n)$ and $pr\text{-fcn}(\mathbb{R}^n)$, then Theorem 1 is simply an adaptation of the standard results for epi-converging sequences in $\text{fcn}(\mathbb{R}^n)$, cf. [14, Proposition 7.30, Theorem 7.31]. Similarly, again by relying on the bijection

[5] In fact, this bijection is a homeomorphism when we restrict our attention to lsc functions. The continuity of this correspondence is immediate if both of these function-spaces are equipped with the topology induced by the convergence of the epigraphs, see below.

η to translate the statement of Theorem 2 into an equivalent one for functions ηf^ν, ηf that belong to $\mathrm{fcn}(\mathbb{R}^n)$, one comes up with [14, Theorem 7.31] about the convergence of the infimal values.

Finally, in a maximization setting, one can simply pass from f to $-f$, or one can repeat the previous arguments with the following changes in the terminology: min to max (inf to sup), ∞ to $-\infty$, epi to hypo, $\leq$ to $\geq$ (and vice-versa), lim inf to lim sup (and vice-versa), and lsc to usc. The *hypograph* of f is the set of all points in $\mathbb{R}^{n+1}$ that lie on or below the graph of f, f is *usc* (=*upper semicontinuous*) if its hypograph is closed, and its *proper*, in the maximization framework, if $-\infty \not\equiv f < \infty$; in the maximization setting cl f denotes the function whose hypograph is the closure, relative to $\mathbb{R}^{n+1}$ of hypo f, its also called its *usc regularization*.

A sequence is said to *hypo-converge*, written $f^\nu \xrightarrow{h} f$, when $-f^\nu \xrightarrow{e} -f$, or equivalently if hypo $f^\nu \to$ hypo f, and it *hypo-converge tightly* if $-f^\nu$ epi-converge tightly to $-f$. And consequently, if the sequence hypo-converges tightly to f with $\sup_D f$ finite, then $\sup_{D^\nu} f^\nu \to \sup_D f$.

When hypo f is the inner set-limit of the hypo f^ν, then f is the *lower hypo-limit* of the functions f^ν and if its the outer set-limit then its their *upper hypo-limit*. It then follows from Proposition 1 that the lower and upper hypo-limits, and the hypo-limit, if it exists, are all usc. Moreover, if the functions f^ν are concave, so is the lower hypo-limit, and the hypo-limit, if it exists. Hence, one also has that the *family of usc functions is closed under hypo-convergence*.

3 Lopsided convergence

Lopsided convergence for bivariate functions was introduced in [2]; we already relied on this notion to formalize the convergence of pure exchange economies and to study the stability of their Walras equilibrium points [9]. Its aimed at the convergence of maxinf-points, or minsup-points but not at both; therefore the name *lopsided*, or *lop-convergence*. However, our present, more comprehensive, analysis has lead us to adjust the definition since otherwise some 'natural' classes of bivariate functions with domain and values like those depicted in Fig. 2, would essentially be excluded, i.e., could not be included in (lopsided or) lop-convergent families. And these are precisely the class of functions that needs to be dealt with in many applications. Moreover, like in Sect. 2, the main focus will not be on extended real-valued functions but on finite-valued bivariate functions that are only defined on a product of non-empty sets rather than on extended real-valued functions defined on the full product space. The motivation for proceeding in this manner, again, coming from the applications. But this time, its not just one possible approach, its in fact mandated by the underlying structure of the class of bivariates that are of interest in the applications. We shall, however, like in the previous section, provide the bridge with the 'extended real-valued' framework that was used in [2].

The definition of lop-convergence is necessarily one-sided. One is either interested in the convergence of maxinf-points or minsup-points but not both. In general, the maxinf-points are not minsup-points, and vice-versa. When, they identify the same points, such points are *saddle-points*. In this article, our concern is with the

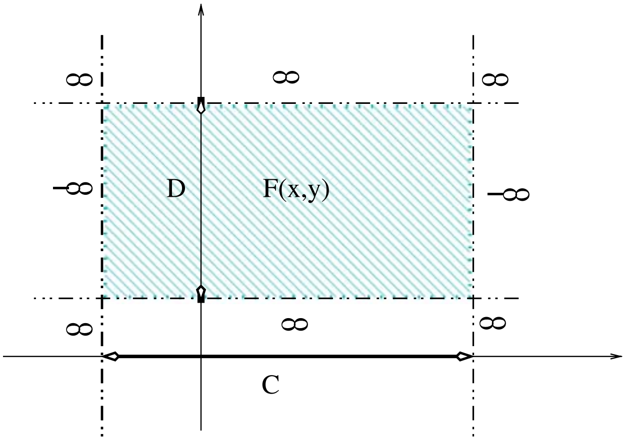

Fig. 2 Partition of the domain of a *proper* bivariate function: maxinf framework

'lopsided'-situation, and will deal with the 'saddle-point'-situation in the last section of the article.

Definitions and results can be stated either in terms of the convergence of maxinf-points or minsup-points with some obvious adjustments for signs and terminology. However, *its important to know if we are working in a 'maxinf' or a 'minsup' framework*, and this is in keeping with the (plain) univariate case where one has to focus on either minimization or maximization. Because most of the applications we are interested in, are more naturally formulated in terms of maxinf-problems, thats the version that will be dealt with in this section. We provide, at the end of the section, the necessary translations required to deal with minsup-problems.

Here, the term *bivariate function* always refers to functions defined on the product of two non-empty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.[6] We write

$$\mathrm{biv}(\mathbb{R}^{n+m}) = \left\{F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}\right\}$$

for the class of bivariate functions that are extended real-valued and defined on all of $\mathbb{R}^n \times \mathbb{R}^m$, and

$$fv\text{-}\mathrm{biv}(\mathbb{R}^{n+m}) = \left\{F : C \times D \to \mathbb{R} \,\middle|\, \emptyset \neq C \subset \mathbb{R}^n,\ \emptyset \neq D \subset \mathbb{R}^m\right\}$$

for the class of bivariate functions that are real-valued and defined on the product $C \times D$ of non-empty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively; here, its understood that $\mathbb{R}^{n+m}$ does not refer to the domain of definition but to the (operational) product space that includes $C \times D$.

[6] In a follow-up paper, we deal with bivariate functions defined on the product of non-empty subsets of two topological spaces potentially equipped with different topologies.

For a bivariate function in $\text{biv}(\mathbb{R}^{n+m})$ or $fv\text{-biv}(\mathbb{R}^{n+m})$, one refers to $\bar{x}$ as a *maxinf-point* if

$$\bar{x} \in \operatorname*{argmax}_{x \in C} \big[\inf_{y \in D} F(x, y) \big],$$

its a *minsup-point* if

$$\bar{x} \in \operatorname*{argmin}_{x \in C} \big[\sup_{y \in D} F(x, y) \big];$$

$C = \mathbb{R}^n$ and $D = \mathbb{R}^m$ are not excluded.

Thus, for now, lets focus on $fv\text{-biv}(\mathbb{R}^{n+m})$, keeping in mind that we are dealing with the maxinf case.

Definition 3 (lop-convergence, fv-biv) A sequence in $fv\text{-biv}(\mathbb{R}^{n+m})$, $\{F^\nu : C^\nu \times D^\nu \to \mathbb{R}\}_{\nu \in \mathbb{N}}$ *lop-converges*, or *converges lopsided*, to a function $F : C \times D \to \mathbb{R}$, also in $fv\text{-biv}(\mathbb{R}^{n+m})$, if

(a) For all $x^\nu \to x$ with $x^\nu \in C^\nu$, $x \in C$ and for all $y \in D$, there exists $y^\nu \to y$ with $y^\nu \in D^\nu$ such that $\limsup_\nu F^\nu(x^\nu, y^\nu) \leq F(x, y)$,

(a$^\infty$) For all $x^\nu \to x$ with $x^\nu \in C^\nu$ and $x \notin C$ and for all $y \in D$, there exists $y^\nu \to y$ with $y^\nu \in D^\nu$ such that $F^\nu(x^\nu, y^\nu) \to -\infty$.

(b) For all $x \in C$, there exists $x^\nu \to x$ with $x^\nu \in C^\nu$ such that for any sequence $\{y^\nu \in D^\nu\}_{\nu \in \mathbb{N}}$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(x, y)$ when the sequence converges to a point $y \in D$, and $F^\nu(x^\nu, y^\nu) \to \infty$ when the sequence converges to a point $y \notin D$.

Although a number of properties can be immediately derived from this convergence notion, cf. Theorem 8 for example, to obtain the convergence of the maxinf-points, however, we need to require (partial) 'ancillary-tightness', cf. Theorem 3, condition (b–t).

This (*partial*) *ancillary-tightness* condition is new; it was inspired by the work of Bagh [6] on approximation for optimal control problems. A more conventional condition that implies ancillary-tightness would be the following: (b) *holds and there is a compact set* $B \subset \mathbb{R}^m$ *such that*

$$\forall x \in \mathbb{R}^n : \quad B \supset \big\{ y \,\big|\, F^\nu(x, y) < \infty \big\}.$$

This last condition, suggested in [2], is too restrictive in many applications. Moreover, the use of ancillary-tightness allows for a generalization of Ky Fan's inequality, see the next section, that can be exploited in situations when the domain of definition of the bivariate function is not compact.

Now, lets turn to the convergence of the marginal functions

$$g^\nu = \inf\nolimits_{y \in D^\nu} F^\nu(\cdot, y) \qquad \text{to} \qquad g = \inf\nolimits_{y \in D} F(\cdot, y);$$

in the extended real-valued framework, one can find a number of related results in the literature, see in particular [12].

Theorem 3 (hypo-convergence of the inf-projections) *Suppose the sequence* $\{F^\nu\}_{\nu\in\mathbb{N}} \subset fv\text{-biv}(\mathbb{R}^{n+m})$ *lop-converges to* F *with condition 3*(b) *strengthened as follows:*

(b–t) *not only, for all* $x \in C$, $\exists x^\nu \in C^\nu \to x$ *such that* $\forall y^\nu \in D^\nu \to y$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(x, y)$ *or* $F(x^\nu, y^\nu) \to \infty$ *depending on* y *belonging or not to* D*, but also, for any* $\varepsilon > 0$ *one can find a compact set* B_ε*, possibly depending on the sequence* $\{x^\nu \to x\}$*, such that for all* ν *larger than some* ν_ε*,*

$$\inf_{D^\nu \cap B_\varepsilon} F^\nu(x^\nu, \cdot) \leq \inf_{D^\nu} F^\nu(x^\nu, \cdot) + \varepsilon.$$

Let $g^\nu = \inf_{y\in D^\nu} F^\nu(\cdot, y)$*,* $g = \inf_{y\in D} F(\cdot, y)$*. Then* $g^\nu \xrightarrow{h} g$ *in* $fv\text{-fcn}(\mathbb{R}^n)$ *assuming that their domains are non-empty, i.e.,* $C^\nu_g = \{x \in C^\nu \,|\, g^\nu(x) > -\infty\}$ *and* $C_g = \{x \in C \,|\, g(x) > -\infty\}$ *are non-empty sets, except possibly for a finite number of indexes* ν*.*

Proof The functions g^ν and g never take on the value ∞, so the proof does not have to deal with that possibility. This means that g^ν and g belong to $fv\text{-fcn}(\mathbb{R}^n)$ whenever they are defined on non-empty sets. Note, however, that in general, the function g^ν and g are not necessarily finite-valued on all of C^ν and C, since they can take on the value $-\infty$ implying that $C^\nu_g = \{x \,|\, g^\nu(x) > -\infty\}$ and $C_g = \{x \,|\, g(x) > -\infty\}$ could be strictly contained in C^ν and C, even potentially empty, this later instance, however, has been excluded by the hypotheses.

We need to verify the conditions of Proposition 2. Lets begin with (a) and (a$^\infty$). Suppose $x^\nu \in C^\nu_g \to x \in C_g$. So, $g(x) \in \mathbb{R}$ and $y_\varepsilon \in \varepsilon\text{-}\operatorname{argmin}_D F(x, \cdot)$ for $\varepsilon > 0$. By 3(a), one can find $y^\nu \in D^\nu \to y_\varepsilon$ such that $\limsup_\nu F^\nu(x^\nu, y^\nu) \leq F(x, y_\varepsilon)$. Hence,

$$\limsup_\nu g^\nu(x^\nu) \leq \limsup_\nu F^\nu(x^\nu, y^\nu) \leq F(x, y_\varepsilon) \leq g(x) + \varepsilon.$$

Since, this holds for arbitrary $\varepsilon > 0$, it follows that $\limsup_\nu g^\nu(x^\nu) \leq g(x)$. When $g(x) = -\infty$ which means that $x \notin C_g$. If $x \in C$, for any $\kappa < 0$ there is a $y_\kappa \in D$ such that $F(x, y_\kappa) < \kappa$, and 2(a) then yields a sequence $y^\nu \in D^\nu \to y_\kappa$ such that

$$\limsup_\nu g^\nu(x^\nu) \leq \limsup_\nu F^\nu(x^\nu, y^\nu) \leq F(x, y_\kappa) < \kappa.$$

Since this holds for κ arbitrarily negative, it follows that $\limsup_\nu g^\nu(x^\nu) = -\infty$. When $x \notin C$, one appeals directly to 3(a$^\infty$) to arrive at the same implication.

Lets now turn to the second condition 2(b) for hypo-convergence: for all $x \in C_g$ there exists $x^\nu \in C^\nu_g \to x$ such that $\liminf_\nu g^\nu(x^\nu) \geq g(x)$. The inequality would clearly be satisfied if $g(x) = -\infty$ but then $x \notin C_g$ and that case does not need to concern us. So, when $g(x) \in \mathbb{R}$, let $x^\nu \in C^\nu \to x$ be a sequence predicated by condition (b–t) for ancillary-tight lop-convergence. It follows that the functions $F^\nu(x^\nu, \cdot)\colon D^\nu \to \mathbb{R}$ epi-converge to $F(x, \cdot)\colon D \to \mathbb{R}$. Hence, one can apply Theorem 2, since $g(x) = \inf_D F(x, \cdot)$ is finite and the condition on tight epi-convergence is satisfied as immediate consequence of (partial) ancillary-tight lop-convergence. Hence, $g^\nu(x^\nu) \to g(x)$. □

Theorem 4 (convergence of maxinf-points, fv-biv) *Let* $\{F^\nu\}_{\nu\in I\!N}$ *and* F *be a family of bivariate functions that satisfy the assumptions of Theorem 3, so, in particular the* F^ν *lop-converge to* F *and the condition* (b–t) *on ancillary-tightness is satisfied. For all* ν *large enough, let* x^ν *be a maxinf-point of* F^ν *and* $\bar{x}$ *any cluster point of the sequence* $\{x^\nu,\ \nu \in I\!N\}$*, then* $\bar{x}$ *is a maxinf-point of the limit function* F*. Moreover, with* $\{x^\nu,\ \nu \in N \subset I\!N\}$ *the (sub)sequence converging to* $\bar{x}$*,*

$$\lim_{\nu \underset{N}{\rightarrow} \infty} \Big[\inf_{y\in D^\nu} F^\nu(x^\nu, y) \Big] = \inf_{y\in D} F(\bar{x}, y)\,],$$

i.e., there is also convergence of the 'values' of these maxinf-points.

Proof Theorem 3 tells us that with

$$g^\nu(x) = \inf\nolimits_{y\in D^\nu} F^\nu(x, y), \quad g(x) = \inf\nolimits_{y\in D} F(x, y),$$

the functions g^ν hypo-converge to g. Maxinf-points of F^ν and F are then maximizers of the corresponding functions g^ν and g. The assertions now follow immediately from the convergence of the argmax of hypo-converging sequences, cf. Theorem 1 translated to the 'maximization' framework. □

However, a number of approximation results require 'full tightness' of the converging sequence, not just ancillary-tightness.

Definition 4 (tight lopsided convergence, fv-biv) A sequence of bivariate functions $\big\{F^\nu : C^\nu \times D^\nu \to \mathbb{R}\big\}_{\nu\in I\!N}$ in fv-biv$(\mathbb{R}^{n+m})$ *lop-converges tightly* to a function $F : C \times D \to \mathbb{R}$, also in fv-biv$(\mathbb{R}^{n+m})$, if they lop-converge, and in addition the following conditions are satisfied:

(a–t) for all $\varepsilon > 0$ there is a compact set A_ε such that for all ν large enough,

$$\sup\nolimits_{x\in C^\nu\cap A_\varepsilon} \inf\nolimits_{y\in D^\nu} F^\nu(x, y) \geq \sup\nolimits_{x\in C^\nu} \inf\nolimits_{y\in D^\nu} F^\nu(x, y) - \varepsilon,$$

(b–t) for $x \in C$ and the corresponding sequence $x^\nu \in C^\nu \to x$ identified in condition 3(b), for any $\varepsilon > 0$ one can find a compact set B_ε, possibly depending on the sequence $\{x^\nu \to x\}$, such that for all ν larger enough,

$$\inf\nolimits_{D^\nu\cap B_\varepsilon} F^\nu(x^\nu, \cdot) \leq \inf\nolimits_{D^\nu} F^\nu(x^\nu, \cdot) + \varepsilon.$$

Theorem 5 (approximating maxinf-points) *Suppose the sequence of bivariate functions* $\big\{F^\nu : C^\nu \times D^\nu \to \mathbb{R}\big\}_{\nu\in I\!N}$ *in* fv-biv$(\mathbb{R}^{n+m})$ *lop-converges tightly to a function* $F : C \times D \to \mathbb{R}$*, also in* fv-biv$(\mathbb{R}^{n+m})$*. Moreover, suppose the inf-projections* $g^\nu = \inf_{y\in D^\nu} F^\nu(\cdot, y)$ *and* $g = \inf_{y\in D} F(\cdot, y)$ *are finite-valued on* C^ν *and* C*, respectively, with* $\sup g = \sup_x \inf_y F(x, y)$ *finite. Then*

$$\sup_x \inf_y F^\nu(x, y) \to \sup_x \inf_y F(x, y)$$

and if $\bar{x}$ is a maxinf point of F one can always find sequences $\{\varepsilon^\nu \searrow 0,\ x^\nu \in \varepsilon^\nu\text{-argmax}_x(\inf_y F^\nu)\}_{\nu\in I\!N}$ *such that* $x^\nu \underset{N}{\to} \bar{x}$. *Conversely, if such sequences exist, then* $\sup_x(\inf_y F^\nu) \underset{N}{\to} \inf_y F(\bar{x}, \cdot)$.

Proof Tightness, in particular condition (b–t), implies that $g^\nu \xrightarrow{h} g$, see Theorem 3. From (a–t), it then follows that they hypo-converge tightly. The assertions then proceed directly from Theorem 2 □

Lets now turn to the situation when our bivariate functions are extended real-valued and defined on all of $\mathbb{R}^n \times \mathbb{R}^m$, keeping in mind that we remain in the maxinf setting. To define convergence, we cannot proceed as in Sect. 2, where we tied the convergence of functions with that of their epigraphs. Here, there is no easily identifiable (unique) geometric object that can be associated with a bivariate function.

Recall that $\text{biv}(\mathbb{R}^{n+m})$ is the family of all extended-real valued functions defined on $\mathbb{R}^n \times \mathbb{R}^m$. In our maxinf case, as in [13], the *effective domain* dom F of a bivariate function $F : \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is

$$\text{dom}\, F = \text{dom}_x\, F \times \text{dom}_y\, F,$$

where

$$\text{dom}_x\, F = \{x \mid F(x, y) < \infty,\ \forall\, y \in \mathbb{R}^m\},$$
$$\text{dom}_y\, F = \{y \mid F(x, y) > -\infty,\ \forall\, x \in \mathbb{R}^n\}.$$

Thus, F is finite-valued on dom F; it does not exclude the possibility that F might be finite-valued at some points that do not belong to dom F.

In the 'maxinf' framework, the term *proper* is reserved for bivariate functions with non-empty domain and such that

$$F(x, y) = \infty \ \text{ when } \ x \notin \text{dom}_x\, F$$
$$F(x, y) = -\infty \ \text{ when } \ x \in \text{dom}_x\, F \ \text{ but } \ y \notin \text{dom}_y\, F,$$

see Fig. 2. If F is proper, we write $F \in pr\text{-biv}(\mathbb{R}^{n+m})$, a sub-collection of $\text{biv}(\mathbb{R}^{n+m})$.

Definition 5 (lopsided convergence, biv) A sequence of bivariate functions $\{F^\nu,\ \nu \in I\!N\} \subset \text{biv}(\mathbb{R}^{n+m})$ *lop-converges* to a function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ if
(a) $\forall\,(x, y) \in \mathbb{R}^{n+m}$, $x^\nu \to x$, $\exists\, y^\nu \to y$: $\limsup_\nu F^\nu(x^\nu, y^\nu) \le F(x, y)$,
(b) $\forall x \in \text{dom}_x\, F$, $\exists\, x^\nu \to x$: $\liminf_\nu F^\nu(x^\nu, y^\nu) \ge F(x, y)\ \forall\, y^\nu \to y \in \mathbb{R}^m$.

Observe that when the functions F^ν and F do not depend on x, they lop-converge if and only if they epi-converge, and that if they do not depend on y, they converge lop-sided if and only if they hypo-converge. This later assertion follows from Proposition 3. Moreover, if for all (x, y), the functions $F^\nu(x, \cdot) \xrightarrow{e} F(x, \cdot)$ *and* $F^\nu(\cdot, y) \xrightarrow{h} F(\cdot, y)$, then the functions F^ν lop-converge to F; however, one should keep in mind that this is a sufficient condition but by no means a necessary one.

Remark 2 ('83 versus new definition). The definition of lop-convergence, in [2], required condition 5(b) to hold not just for all $x \in \mathrm{dom}_x F$ but for all $x \in \mathbb{R}^n$. The implication is that then lop-convergent families must be restricted to those converging to a function F with $\mathrm{dom}_x F = \mathbb{R}^n$.

Detail Indeed, consider the following simple example: For all $\nu \in I\!N$,

$$F^\nu(x, y) = F(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0.1] \times [0, 1], \\ -\infty & \text{if } y \in (0, 1), x \notin [0, 1], \\ \infty & \text{elsewhere.} \end{cases}$$

Then, in terms of Definition 5, the F^ν lop-converge to F, but not if we had insisted that condition 5(b) holds for all $x \in \mathbb{R}^n$. Indeed, there is no way to find a sequence $x^\nu \to -1$, for example, such that for all $y^\nu \to 0$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(-1, 0) = \infty$; simply consider $y^\nu = 1/\nu \to 0$. □

As in Sect. 2, we set up a bijection, also denoted η, between the elements of fv-fcn$(\mathbb{R}^{n+m})$ and the (max-inf) proper bivariate functions, pr-biv$(\mathbb{R}^{n+m})$. For $F \in fv$-biv$(\mathbb{R}^{n+m})$, set

$$\eta F(x, y) = \begin{cases} F(x, y) & \text{when } (x, y) \in C \times D, \\ \infty & \text{when } y \notin D, \\ -\infty & \text{when } y \in D \text{ but } x \notin C, \end{cases}$$

i.e., ηF extends F to all of $\mathbb{R}^n \times \mathbb{R}^m$. Then, for $F \in pr$-biv, $\eta^{-1} F$ will be the restriction of F to its domain of finiteness, namely $\mathrm{dom}_x F \times \mathrm{dom}_y F$.

Proposition 4 (lop-convergence in fv-biv and biv) *A sequence*

$$\left\{F^\nu : C^\nu \times D^\nu \to \mathbb{R},\ \nu \in I\!N\right\} \subset fv\text{-biv}(\mathbb{R}^{n+m})$$

converges lopsided to $F : C \times D \to \mathbb{R}$ *if and only the corresponding sequence of extended real-valued bivariate functions*

$$\left\{\eta F^\nu : \mathbb{R}^{n+m} \to \overline{\mathbb{R}},\ \nu \in I\!N\right\} \subset pr\text{-biv}(\mathbb{R}^{n+m})$$

lop-converges (*Definition* 5) *to* $\eta F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, *where* η *is the bijection between* fv-biv$(\mathbb{R}^{n+m})$ *and* pr-biv$(\mathbb{R}^{n+m})$ *defined above.*

Proof To show: conditions (a), (a$^\infty$) and (b) of Definition 3 for the sequence $\{F^\nu\}_{\nu=1}^\infty$ imply and are implied by the conditions (a) and (b) of 5 for the sequence $\{\eta F^\nu\}_{\nu=1}^\infty$ in pr-biv$(\mathbb{R}^{n+m})$; lets denote these later conditions (ηa) and (ηb).

We begin with the implications involving (ηa) and (a), (a$^\infty$) Suppose (ηa) holds, $(x, y) \in C \times D$ and $x^\nu \in C^\nu \to x$, then there exists $y^\nu \to y$ such that $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) \leq \eta F(x, y) = F(x, y)$. Necessarily, for ν sufficiently large, $y^\nu \in D^\nu$ since otherwise $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = \infty$ contradicting the possibility of having this upper limit less than or equal to $F(x, y)$ that is finite. This takes care of (a). If $x^\nu \in C^\nu \to x \notin C$, $y \in D$ this implies $\eta F(x, y) = -\infty$, and consequently there exists $y^\nu \to y$ such that $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = -\infty$. Again, this sequence $\{y^\nu\}_{\nu=1}^\infty$ cannot have a subsequence with $y^\nu \notin D^\nu$ since otherwise this upper limit would be ∞. This yields (a$^\infty$)

Now, suppose (a) and (a$^\infty$) hold. As long as $y \notin D$, $\eta F(x, y) = \infty$ the inequality in (ηa) will always be satisfied, henceforth we consider only the case when $y \in D$. If $x^\nu \in C^\nu \to x \in \mathbb{R}^n$, then (a) or (a$^\infty$) directly guarantee the existence of a sequence $\{y^\nu\}_{\nu=1}^\infty$ such that $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) \le \eta F(x, y)$. Finally, consider the case when $x^\nu \to x$, but $x^\nu \notin C^\nu$ for a subsequence $N \subset I\!N$; there is no loss of generality in actually assuming that $N = I\!N$. Pick any sequence $y^\nu \in D^\nu \to y$, hence $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = -\infty$ will certainly be less than or equal to $\eta F(x, y)$. So, (ηa) holds also, trivially, in this situation.

When (ηb) holds and $(x, y) \in C \times D$, hence $\eta F(x, y) = F(x, y) \in \mathbb{R}$. We only have to consider sequences $y^\nu \in D^\nu \to y \in D$ and for all such sequences: $\exists\, x^\nu \to x$ such that $\liminf_\nu \eta F^\nu(x^\nu, y^\nu) \ge F(x, y)$. This sequence $x^\nu \to x$ cannot have a subsequence whose elements do not belong to the corresponding sets C^ν since otherwise the lower limit of $\{\eta F^\nu(x^\nu, y^\nu)\}_{\nu=1}^\infty$ would be $-\infty < F(x, y) \in \mathbb{R}$. This means that for this sequence $x^\nu \to x$, the $x^\nu \in C^\nu$ for ν sufficiently large. Hence, (b) holds when $y \in D$. When $x \in C$, $y \notin D$, $\eta F(x, y) = \infty$. For any $y^\nu \in D^\nu \to y$ there is a sequence $x^\nu \to x$ such that $\liminf_\nu \eta F^\nu(x^\nu, y^\nu) = \infty$. Since $y^\nu \in D^\nu$, $x^\nu \notin C^\nu$ would imply $\eta F^\nu(x^\nu, y^\nu) = -\infty$, this sequence $x^\nu \to x$ must be such that $x^\nu \in C^\nu$ for ν sufficiently large, and consequently one must have $F^\nu(x^\nu, y^\nu) \to \infty$ which means that (b) is also satisfied when $y \notin D$.

In the other direction that (b) yields (ηb) is straightforward. If $y \notin D$ and $x \in C$, then $\liminf_\nu \eta F^\nu(y^\nu, x^\nu) = \infty = F(x, y)$ for the sequence $x^\nu \in C^\nu \to x$, predicated by (b), irrespective of the sequence $y^\nu \to y$. Finally, if $(x, y) \in C \times D$, then (b) foresees a sequence $x^\nu \in C^\nu \to x$ such that the inequality in (ηb) is satisfied as long as the sequence $y^\nu \to y$ is such that all ν, or at least for ν sufficiently large, the $y^\nu \in D^\nu$. But, if they do not $\eta F^\nu(x^\nu, y^\nu) = \infty$, and these terms will certainly contribute to making $\liminf_\nu \eta F^\nu(x^\nu, y^\nu) \ge F(x, y)$. □

Ancillary-tight Lop-convergence is also the key to the convergence of the maxinf-points of extended real-valued bivariate functions.

Definition 6 (ancillary-tight lop-convergence, biv). A sequence of bivariate functions in $\mathrm{biv}(\mathbb{R}^{n+m})$, *ancillary-tight lop-converges* if it converges lopsided and for all $x \in C$, the following augmented condition of 5(b) holds:

(b–t) not only $\exists\, x^\nu \to x$ such that $\forall\, y^\nu \to y$, $\operatorname{liminf}_\nu F^\nu(x^\nu, y^\nu) \ge F(x, y)$, but also, for any $\varepsilon > 0$ one can find a compact set B_ε, possibly depending on the sequence $\{x^\nu \to x\}$, such that for all ν larger than some ν_ε,

$$\inf_{B_\varepsilon} F^\nu(x^\nu, \cdot) \le \inf F^\nu(x^\nu, \cdot) + \varepsilon.$$

Proposition 5 (ancillary-tight lop-convergence in fv-biv and biv). *A sequence*

$$\left\{F^\nu : C^\nu \times D^\nu \to \mathbb{R},\ \nu \in I\!N\right\} \subset fv\text{-biv}(\mathbb{R}^n \times \mathbb{R}^m)$$

lop-converges ancillary-tightly to $F : C \times D \to \mathbb{R}$ *in* fv-biv$(\mathbb{R}^n \times \mathbb{R}^m)$ *if and only if the corresponding sequence*

$$\{\eta F^\nu : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}},\ \nu \in I\!N\} \subset pr\text{-biv}(\mathbb{R}^n \times \mathbb{R}^m)$$

lop-converges ancillary-tightly to $\eta F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$*, where* η *is the bijection from* fv-biv$(\mathbb{R}^n \times \mathbb{R}^m)$ *onto* pr-biv$(\mathbb{R}^n \times \mathbb{R}^m)$ *defined earlier.*

Proof We already showed that lop-convergence in fv-biv and pr-biv are equivalent, cf. Proposition 4, so there only remains to verify the 'tightly' condition. But thats immediate because in both cases it only involves points that belong to $C \times \mathbb{R}^m = \text{dom}_x\, \eta F \times \mathbb{R}^m$ and sequences converging to such points. □

Theorem 6 (biv: hypo-convergence of the inf-projections). *Suppose the sequence* $\{F^\nu\}_{\nu \in I\!N} \subset \text{biv}(\mathbb{R}^{n+m})$ *lop-converges ancillary-tightly to F and let* $g^\nu = \inf_{y \in D^\nu} F^\nu(\cdot, y)$, $g = \inf_{y \in D} F(\cdot, y)$. *Then, assuming that* $g < \infty$, $g^\nu \xrightarrow{h} g$ *in* fcn$(\mathbb{R}^n)$.

Proof The proof is the same as that of Theorem 3 with the obvious adjustments when the sequences do not belong to dom F^ν and the limit point does not lie in dom F. □

Theorem 7 (biv: convergence of maxinf-points) *Let* $\{F^\nu\}_{\nu \in I\!N}$ *and* F *be a family of bivariate functions that satisfy the assumptions of Theorem 6, so, in particular the* F^ν *lop-converge ancillary-tightly to* F*, Then, if for all* ν, x^ν *is a maxinf-point of* F^ν *and* $\bar{x}$ *is any cluster point of the sequence* $\{x^\nu,\ \nu \in I\!N\}$*, then* $\bar{x}$ *is a maxinf-point of the limit function* F*. Moreover, with* $\{x^\nu,\ \nu \in N \subset I\!N\}$ *the (sub)sequence converging to* $\bar{x}$,

$$\lim_{\nu \underset{N}{\to} \infty} \Big[\inf_{y \in \mathbb{R}^m} F^\nu(x^\nu, y) \Big] = \inf [\sup_{y \in \mathbb{R}^m} F(\bar{x}, y)],$$

i.e., there is also convergence of the 'values' of the maxinf-points.

Proof Theorem 6 tells us that with

$$g^\nu(x) = \inf_{y \in D^\nu} F^\nu(x, y), \quad g(x) = \inf_{y \in D} F(x, y),$$

the functions g^ν hypo-converge to g. Maxinf-points for F^ν and F are then maximizers of the corresponding functions g^ν and g. The assertions now follow immediately from the convergence of the argmax of hypo-converging sequences, cf. Theorem [14, Theorem 7.31] translated to the 'maximization' framework. □

To deal with a 'minsup' situations one can either repeat all the arguments changing inf to sup, liminf to limsup and vice-versa, or simply re-integrate the questions to the 'maxinf' framework by changing signs of the approximating and limit bivariate functions.

4 Ky Fan's Inequality extended

The class of usc functions is closed under hypo-converge [14, Theorem 7.4], and so is the class of concave usc functions [14, Theorem 7.17]. A class of functions that is closed under lopsided convergence is the class of *Ky Fan functions* (Theorem 8). We exploit this result to obtain a generalization of Ky Fan Inequality that allows us to claim existence of maxinf-points in situations when the domain of definition of the Ky Fan function is not necessarily compact.

Definition 7 A bivariate function $F : C \times D \to \mathbb{R}$ with $C$and D convex sets, in $fv\text{-biv}(\mathbb{R}^{n+m})$, is called a *Ky Fan function* if
(a) $\forall\, y \in D$: $x \mapsto F(x, y)$ is usc on C,
(b) $\forall\, x \in C$: $y \mapsto F(x, y)$ is convex on D.
Note that the sets C or D are not required to be compact.

Theorem 8 (lop-limits of Ky Fan functions) *The lopsided limit* $F : C \times D \to \mathbb{R}$ *of a sequence* $\left\{F^\nu : C^\nu \times D^\nu \to \mathbb{R}\right\}_{\nu\in\mathbb{N}}$ *of Ky Fan functions in* $fv\text{-biv}(\mathbb{R}^{n+m})$ *is also a Ky Fan function.*

Proof For the convexity of $y \mapsto F(x, y)$, let $x^\nu \in C^\nu \to x \in C$ be the sequence set forth by 3(b) and $y^0, y^\lambda, y^1 \in D$ with $y^\lambda = (1-\lambda)y^0 + \lambda y^1$ for $\lambda \in [0, 1]$. In view of 3(a), we can choose sequences $\left\{y^{0,\nu} \in D^\nu \to y^0\right\}$, $\left\{y^{1,\nu} \in D^\nu \to y^1\right\}$ such that $F^\nu(x^\nu, y^{0,\nu}) \to F(x, y^0)$ and $F^\nu(x^\nu, y^{1,\nu}) \to F(x, y^1)$. Let $y^{\lambda,\nu} = (1-\lambda)y^{0,\nu} + \lambda y^{1,\nu}$; $y^{\lambda,\nu} \in D^\nu$ since the functions $F^\nu(x, \cdot)$ are convex and the sequence $\{y^{\lambda,\nu}\}_{\nu\in\mathbb{N}}$ certainly converges to y^λ. For all ν, one has

$$F^\nu(x^\nu, y^{\lambda,\nu}) \le (1-\lambda)F^\nu(x^\nu, y^{0,\nu}) + \lambda F^\nu(x^\nu, y^{1,\nu}),$$

Taking lininf on both sides yields

$$F(x, y^\lambda) \le \operatorname{liminf}_\nu F^\nu(x^\nu, y^{\lambda,\nu}) \le (1-\lambda)F(x, y^0) + \lambda F(x, y^1),$$

that establishes the convexity of $F(x, \cdot)$.

To prove the upper semicontinuity of F with respect to x-variable, we show that for $y \in D$,

$$\text{hypo}\, F(\cdot, y) \quad \text{is the inner set-limit of the} \quad \text{hypo}\, F^\nu(\cdot, y^\nu),$$

where the limit is with respect to all sequences $\{y^\nu \in D^\nu\}_{\nu\in\mathbb{N}}$ converging to y and $\nu \to \infty$. This yields the upper semicontinuity since the inner set-limit is always closed and a function is usc if and only if its hypograph is closed. We have to show that if $(x, \alpha) \in \text{hypo}\, F(\cdot, y)$, then whenever $y^\nu \in D^\nu \to y$, one can find $(x^\nu, \alpha^\nu) \in \text{hypo}\, F^\nu(\cdot, y^\nu)$ such that $(x^\nu, \alpha^\nu) \to (x, \alpha)$. But that follows immediately from 3(a) since we can adjust the $\alpha^\nu \le F^\nu(x^\nu, y^\nu)$ so that they converge to $\alpha \le F(x, y)$. □

Given a Ky Fan function with compact domain and non-negative on the diagonal, we have the following important existence result:

Lemma 2 (Ky Fan's Inequality; [7], [4, Theorem 6.3.5]) *Suppose $F : C \times C \to \mathbb{R}$ is a Ky Fan function with C compact. Then, the set of maxinf-points of F is a nonempty subset of C. Moreover, if $F(x, x) \geq 0$ (on $C \times C$), then for every maxinf-point $\bar{x}$ of C, $F(\bar{x}, \cdot) \geq 0$ on C.*

One of the consequences of the lopsided convergence is an extension of the Ky Fan's Inequality to the case when it is not possible to apply it directly because one of the conditions is not satisfied, for example the compactness of the domain. However, we are able to approach the bivariate function F by a sequence $\{F^\nu\}_{\nu \in I\!N}$ defined on compact sets C^ν. This procedure could be useful in many situation where the original maxinf-problem is unbounded, and then the problem is approached by a family of truncated maxinf-problems. Such is the case, for example, when we consider as variables in the original problem the multipliers associated to inequality constraints, or when the original problem is a Walras equilibrium with a positive orthant as consumption set; in [8] one is precisely confronted with such situations. Another simple, illustrative example follows the statement of the theorem.

Theorem 9 (Extension of Ky Fan's Inequality) *Let $\emptyset \neq C \subset \mathbb{R}^n$ and F a finite-valued bivariate function defined on $C \times C$. Suppose one can find sequences of compact convex sets $\left\{C^\nu \subset \mathbb{R}^n\right\}$ and (finite-valued) Ky Fan functions $\{F^\nu : C^\nu \times C^\nu \to \mathbb{R}\}_{\nu \in I\!N}$ lop-converging ancillary-tightly to F, then every cluster point $\bar{x}$ of any sequence $\{x^\nu,\ \nu \in I\!N\}$ of maxinf-points of the F^ν is a maxinf-point of F*

Proof Ky Fan's Inequality 2 implies that for all ν, the set of maxinf-points of F^ν is non-empty. On the other hand, in view of Theorems 8 and 4 any cluster point of such maxinf-points will be a maxinf-point of F. □

Example 2 (Extended Ky Fan's Inequality applied) We consider a Ky Fan function $F(x, y) = \sin x + (y + 1)^{-1}$ defined on the set $[0, \infty)^2$. Although,

$$\inf{}_{y \in [0,\infty)} F(x, y) = \sin x,$$

and the set maxinf-points is not empty, we cannot apply Ky Fan Inequality because the domain of F is not compact; the function $F(\cdot, y)$ is not even sup-compact.

Detail If we consider the functions $F^\nu(x, y) = \sin x + (y + 1){-}1$ on the compact domains $[0, \nu]^2$, one can apply Ky Fan's Inequality. Indeed, in this case we have,

$$\inf{}_{y \in [0,\nu)} F(x, y) = \sin x + (\nu + 1)^{-1},$$

that converges pointwise and hypo- to $\sin x$, and

$$\operatorname{argmax}_{y \in [0,\nu]} \inf{}_{y \in [0,\nu]} F(x, y) = \left\{\pi/2 + 2k\pi \;\middle|\; k \in I\!N\right\}.$$

Thus, $x^\nu = \pi/2$ and $\tilde{x}^\nu = \pi/2 + 2\nu\pi$ are maxinf-points of the F^ν. The sequence $\{x^\nu\}_{\nu \in I\!N}$ converges to a maxinf-point of F, the second sequence $\{\tilde{x}^\nu\}_{\nu \in I\!N}$ does not. □

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