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FULL LENGTH PAPER

Complete characterizations of stable Farkas' lemma and cone-convex programming duality

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Abstract We establish necessary and sufficient conditions for a stable Farkas' lemma. We then derive necessary and sufficient conditions for a stable duality of a cone-convex optimization problem, where strong duality holds for each linear perturbation of a given convex objective function. As an application, we obtain stable duality results for convex semi-definite programs and convex second-order cone programs.

Keywords Stable Farkas lemma · Stable duality · Convex optimization · Semi-definite programming · Second-order cone programming

Mathematics Subject Classification (2000) 41A65 · 41A29 · 90C30

1 Introduction

When it comes to the elegant and powerful duality theory in optimization, Farkas' lemma is the key. This Lemma states that given any vectors $a_1, a_2,\ldots,a_m$ in $\mathbb{R}^n$, the linear inequality $c^T x \geqq 0$ is a consequence of the linear system $a_i^T x \geqq 0, i = 1, 2, \ldots, m$

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if and only if $c = \sum_{i=1}^{m} \lambda_i a_i$, for some multipliers $\lambda_i \geqq 0$. Farkas' lemma underpins the linear programming duality and has also played a central role in the development of optimization theory (see e.g. [7,11,18,20]). Applications by way of extensions of the celebrated Farkas lemma range from classical nonlinear programming to modern areas of optimization such as nonsmooth optimization and semidefinite programming. For a recent look at its extensions and applications see [4,9,16].

The generalized Farkas lemma for a given cone-convex system $-g(x) \in S$ and a given real-valued convex function f states that $f(x) \geqq 0$ is a consequence of the system $-g(x) \in S$ if and only if there exists $\lambda \in S^+$ such that, for each $x \in \mathbb{R}^n$, $f(x) + (\lambda \circ g)(x) \geqq 0$. Symbolically,

$$[-g(x) \in S \ \Rightarrow f(x) \geqq 0] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in \mathbb{R}^n) f(x) + (\lambda \circ g)(x) \geqq 0, \quad (1)$$

where the set $S \subset \mathbb{R}^m$ is a closed convex cone with the dual cone S^+ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous convex function with respect to S. This equivalence in (1) holds under a regularity condition, sometimes called "closed cone condition" (see e.g. [5,8,11,14,16,19]). Motivated by the concept of a stable minimax theorem where the minimax equality holds for each linear perturbation of the function involved [12,13], we refer the generalized Farkas lemma as the *Stable Farkas Lemma* whenever, for each choice of linear functional x^* on $\mathbb{R}^n$ and each choice of scalar $\alpha \in \mathbb{R}$,

$$[-g(x) \in S \ \Rightarrow f(x) \geqq x^*(x) + \alpha] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in \mathbb{R}^n) f(x) + (\lambda \circ g)(x) \\ \geqq x^*(x) + \alpha.$$

The purpose of this paper is to establish complete characterizations of the stable Farkas lemma and to obtain stable strong duality characterizations for classes of convex optimization problems, including cone programming problems [1,2,21], which have received a great deal of attention in recent years. We show that a simple closure condition is necessary and sufficient for the stable Farkas lemma. We also derive necessary and sufficient conditions for stable "min–max" duality for convex optimization problems. For sufficient conditions for stable minimax theorems see [13].

The outline of the paper is as follows. Section 2 provides background material on convex analysis that will be used later in the paper. Section 3 presents characterizations of the stable Farkas lemma. Section 4 gives stable strong duality results for cone-convex programs. As an application, Sect. 5 establishes complete characterizations of the stable duality for cone programming problems, including semi-definite programs and second-order cone programs.

2 Preliminaries

We recall in this section some notations and basic results which will be used in this paper. Let X be a normed space with X^* its dual endowed with weak*-topology. For a subset $D \subset X^*$, the w^*-closure of D will be denoted by $\text{cl}D$ and the convex cone generated by D by $\text{cocone}D$.

Let $h : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex function. The conjugate function of h, $h^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{v(x) - h(x) \mid x \in \text{dom } h\},$$

where $\text{dom } h := \{x \in X \mid h(x) < +\infty\}$ is the effective domain of h. The function h is said to be proper if h does not take on the value $-\infty$ and $\text{dom } h \neq \emptyset$. The epigraph of h is defined by

$$\text{epi } h := \{(x, r) \in X \times \mathbb{R} \mid x \in \text{dom } h, h(x) \leqq r\}.$$

The set (possibly empty)

$$\partial h(a) := \{v \in X^* \mid h(x) - h(a) \geqq v(x - a), \forall x \in \text{dom } h\}$$

is the subdifferential of the convex function h at $a \in \text{dom } h$. For a closed convex subset D of X, the indicator function δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The support function δ^*_D is defined by $\delta^*_D(u) = \sup_{x \in D} u(x)$. Then $\partial \delta_D(x) = N_D(x)$, which is known as the normal cone of D of x. If h is proper lower semicontinuous and sublinear (i.e., convex and positively homogeneous of degree one), then $\text{epi } h^* = \partial h(0) \times \mathbb{R}_+$.

For proper lower semicontinuous convex functions $g, h : X \to \mathbb{R} \cup \{+\infty\}$, the infimal convolution of g with h, denoted $g \Box h$, is defined by

$$(g \Box h)(x) = \inf_{x_1 + x_2 = x} \{g(x_1) + h(x_2)\}.$$

The lower semicontinuous envelope and lower semicontinuous convex hull of a function $g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ are denoted respectively by $\text{cl} g$ and $\text{clco} g$. That is, $\text{epi}(\text{cl} g) = \text{cl}(\text{epi } g)$ and $\text{epi}(\text{cl co} g) = \text{cl co}(\text{epi } g)$. For details, see [23].

Let g, h and g_i, $i \in I$ (where I is an arbitrary index set) be proper lower semicontinuous convex functions. It is well known from the dual operation (see [23]) that if $\text{dom } g \cap \text{dom } h \neq \emptyset$, then

$$(g \Box h)^* = g^* + h^*, \ (g + h)^* = \text{cl}(g^* \Box h^*)$$

and if $\sup_{i \in I} g_i$ is proper, then

$$(\sup_{i \in I} g_i)^* = \text{cl co} \left(\inf_{i \in I} g_i^*\right).$$

Thus we can check that

$$\text{epi}(g + h)^* = \text{cl}(\text{epi} g^* + \text{epi} h^*) \quad \text{and} \quad \text{epi}(\sup_{i \in I} g_i)^* = \text{cl co} \left(\bigcup_{i \in I} \text{epi} g_i^* \right).$$

The closure in the first equation is superfluous if one of g and h is continuous at some $x_0 \in \text{dom } g \cap \text{dom } h$ (see [6,23] for details).

Let Y be another normed linear space with topological dual Y^* and let S be a closed convex cone in Y. Denote by S^+ the dual cone of S, defined as

$$S^+ = \left\{y^* \in Y^* | y^*(y) \geqq 0 \text{ for any } y \in S\right\}.$$

We say that the map $g : X \to Y$ is S-convex if for any $x_1, \ x_2 \in X$ and any $\lambda \in [0, 1]$,

$$g(\lambda x_1 + (1-\lambda)x_2) \in \lambda g(x_1) + (1-\lambda)g(x_2) - S$$

and that g is S-sublinear if g is S-convex and positively homogeneous of degree 1. Note that $g^{-1}(-S) := \{x \in X | -g(x) \in S\}$.

We also need the following lemma, which is well-known, see for instance [3,22]. Let U, V be two topological spaces and let $\Phi : U \times V \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then the marginal function $h : V \to \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $h(v) := \inf_{u \in U} \Phi(u, v)$ and the projection operator, $\mathrm{Pr}_{V \times \mathbb{R}} : U \times V \times \mathbb{R} \to V \times \mathbb{R}$, is given by $\mathrm{Pr}_{V \times \mathbb{R}}(u, v, r) = (v, r)$.

Lemma 2.1 (Lemma 2.3 [3] and Theorem 2.1 [22]) *Let* $\Phi : U \times V \to \mathbb{R} \cup \{-\infty, +\infty\}$ *be a proper function. Then,* $\mathrm{Pr}_{V \times \mathbb{R}}(\mathrm{epi}\,\Phi)$ *is closed if and only if the marginal function* $h : V \to \mathbb{R} \cup \{-\infty, +\infty\}$*, defined by* $h(v) = \inf_{u \in U} \Phi(u, v)$ *is lower semicontinuous and* $\inf_{u \in U} \Phi(u, \bar{v})$ *is attained whenever* $h(\bar{v}) > -\infty$*, for* $\bar{v} \in V$.

3 Characterizations of stable Farkas lemma

In this section we present characterizations of stable Farkas lemma for cone-convex systems. Our approach makes use of the powerful convex conjugate analysis. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $g : X \to Y$ be a continuous S-convex function with $\mathrm{dom}\, f \cap g^{-1}(-S) \neq \emptyset$.

Define a function $\Phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$\Phi(x, y) := \begin{cases} f(x) & \text{if} \quad y - g(x) \in S \\ +\infty & \text{otherwise} \end{cases}$$

and a function $\eta : X^* \to \mathbb{R} \cup \{-\infty, +\infty\}$ by

$$\eta(x^*) := \inf_{y^* \in Y^*} \Phi^*(x^*, y^*).$$

Then, for each $(x^*, y^*) \in X^* \times Y^*$,

$$\Phi^*(x^*, y^*) = \begin{cases} \sup\limits_{x \in X} \left\{x^*(x) - f(x) - (y^* \circ g)(x)\right\} & \text{if} \quad y^* \in -S^+ \\ +\infty \quad \text{otherwise} \end{cases}$$

and $\eta^*(x) := (\Phi^*)^*(x, 0) = \Phi(x, 0), \quad \forall x \in X$ and, for each $x^* \in X^*$,

$$\eta(x^*) \geqq \eta^{**}(x^*) = \sup_{x \in X} \left\{x^*(x) - \Phi(x, 0)\right\} = -\inf_{x \in g^{-1}(-S)} \left\{f(x) - x^*(x)\right\} > -\infty.$$

Theorem 3.1 (Stable Farkas Lemma) *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function, and let* $g : X \to Y$ *be a continuous and* S*-convex function with* $\operatorname{dom} f \cap g^{-1}(-S) \neq \emptyset$. *Then the following statements are equivalent*:

(i) *For each* $x^* \in X^*$ *and each* $\alpha \in \mathbb{R}$,

$$[-g(x) \in S \Rightarrow f(x) \geqq x^*(x) + \alpha] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in X)\, f(x) + (\lambda \circ g)(x) \geqq x^*(x) + \alpha.$$

(ii) $\operatorname{epi} f^* + \bigcup_{y^* \in S^+} \operatorname{epi} (y^* \circ g)^*$ *is* w^**-closed.*

Proof [(ii) $\Rightarrow$ (i)] Suppose that (ii) holds. Let $x^* \in X^*$ and $\alpha \in \mathbb{R}$. Assume that $f(x) \geqq x^*(x) + \alpha$, for each $x \in g^{-1}(-S)$. Define

$$\Phi(x, y) = \begin{cases} f(x) & \text{if } \; y - g(x) \in S \\ +\infty & \text{otherwise.} \end{cases}$$

Then one can check that $\operatorname{Pr}_{X^* \times \mathbb{R}}(\operatorname{epi}\Phi^*) = \operatorname{epi} f^* + \bigcup_{y^* \in S^+} \operatorname{epi}\, (y^* \circ g)^*$. By (ii), $\operatorname{Pr}_{X^* \times \mathbb{R}}(\operatorname{epi}\Phi^*)$ is closed, and hence from Lemma 2.1, η is lower semicontinuous and all values $\eta(x^*)$ are attained. So,

$$\begin{aligned} \eta(x^*) &= \eta^{**}(x^*) \\ &= -\inf_{x \in g^{-1}(-S)} \left\{ f(x) - x^*(x) \right\}, \end{aligned}$$

and hence there exists $\lambda \in S^+$ such that $\inf_{x \in X} \{-x^*(x) + f(x) + (\lambda \circ g)(x)\} = \inf_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\}$. By assumption, $\inf_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\} \geqq \alpha$, and so, $\inf_{x \in X} \{-x^*(x) + f(x) + (\lambda \circ g)(x)\} \geqq \alpha$ and hence (i) holds.

[(i) $\Rightarrow$ (ii)] Suppose that (i) holds. Let $(u, r) \in \operatorname{cl}\, [\operatorname{epi} f^* + \bigcup_{y^* \in S^+} \operatorname{epi}\, (y^* \circ g)^*]$. Then there exist nets $\{\lambda_\alpha\} \subset S^+$, $\{(v_\alpha, p_\alpha)\} \subset \operatorname{epi} f^*$ and $\{(w_\alpha, q_\alpha)\} \subset \operatorname{epi}(\lambda_\alpha \circ g)^*$ such that

$$(v_\alpha + w_\alpha, p_\alpha + q_\alpha) \to (u, r).$$

Now, $w_\alpha(x) \leqq q_\alpha$, for each $x \in g^{-1}(-S)$ and $v_\alpha(x) - f(x) \leqq p_\alpha$, for each $x \in X$ and so, we have

$$(v_\alpha + w_\alpha)(x) - f(x) \leqq p_\alpha + q_\alpha, \quad \forall x \in g^{-1}(-S).$$

Letting $\alpha \to \infty$, $u(x) - f(x) \leqq r$, for each $x \in g^{-1}(-S)$. Thus it follows from (i) that there exists $\lambda \in S^+$ such that

$$f(x) + (\lambda \circ g)(x) \geqq u(x) - r, \quad \forall x \in X.$$

This gives us that $(f + \lambda \circ g)^*(u) \leqq r$ and then $(u, r) \in \operatorname{epi}\, (f + \lambda \circ g)^* = \operatorname{epi} f^* + \operatorname{epi}\, (\lambda \circ g)^*$. Hence, (ii) holds. □

Example 3.1 Let $f(x) = \delta_{(-\infty,0]}(x),\ g(x) = [\max\{0, x\}]^2$ and $S = \mathbb{R}_+$. Then f is a proper lower semicontinuous convex function, g is a continuous convex function and $\operatorname{dom} f \cap g^{-1}(-S) = (-\infty, 0]$. Moreover, epi $f^* = [0, \infty) \times \mathbb{R}_+$ and

$$\bigcup_{y^* \in S^+} \text{epi } (y^* \circ g)^* = (\{0\} \times \mathbb{R}_+) \cup \left\{(x, r) \in \mathbb{R}^2 \mid x > 0,\ r > 0\right\}$$

and epi $f^* + \bigcup_{y^* \in S^+}$ epi $(y^* \circ g)^*$ is closed. In fact, epi $f^* + \bigcup_{y^* \in S^+}$ epi $(y^* \circ g)^* = \mathbb{R}^2_+$. For each $x^* \in \mathbb{R}$ and each $\alpha \in \mathbb{R}$,

$$\begin{aligned} &\left[-g(x) \in S \ \Rightarrow \ f(x) \geqq x^* x + \alpha\right] \\ &\Rightarrow x^* \in [0, \infty) \quad \text{and} \quad \alpha \leqq 0 \\ &\Rightarrow \text{for any } \lambda \in S^+ \text{ and any } x \in \mathbb{R}, \quad f(x) + \lambda g(x) \geqq x^* x + \alpha. \end{aligned}$$

So, Theorem 3.1 holds as the converse implication always holds. □

In the case where $f = 0$, Theorem 3.1 yields a characterization of a closed cone condition in terms of a version of the stable Farkas lemma.

Corollary 3.1 *Let $g : X \to Y$ be a continuous and S-convex function with $g^{-1}(-S) \neq \emptyset$. Then the following statements are equivalent*:

(i) *For each $x^* \in X^*$ and each $\alpha \in \mathbb{R}$,*

$$[-g(x) \in S \ \Rightarrow x^*(x) \geqq \alpha] \Leftrightarrow (\exists \lambda \in S^+)(\forall x \in X) x^*(x) + (\lambda \circ g)(x) \geqq \alpha.$$

(ii) $\bigcup_{y^* \in S^+}$ epi $(y^* \circ g)^*$ *is w^*-closed.*

Proof The conclusion follows from Theorem 3.1 by taking $f = 0$ and replacing x^* by $-x^*$ as epi $f^* = \{0\} \times \mathbb{R}_+$ and

$$\text{epi } f^* + \bigcup_{y^* \in S^+} \text{epi } (y^* \circ g)^* = \bigcup_{y^* \in S^+} \text{epi } (y^* \circ g)^*.$$

□

A characterization of condition (ii) of Corollary 3.1 in terms of strong duality can be found in [4].

The generalized Farkas lemma for cone-sublinear systems has been well known under the closed cone condition that $\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0)$ is w^*-closed whenever the function f is continuous and sublinear. For details, see [10,16] and other references therein. We now show that a special case of our closure condition completely characterizes the stable Farkas lemma for the cone-sublinear systems.

Corollary 3.2 *Suppose that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous sublinear function and that $g : X \to Y$ is a continuous S-sublinear function with $\operatorname{dom} f \cap g^{-1}(-S) \neq \emptyset$. Then the following statements are equivalent*:

(i) *For each* $x^* \in X^*$,

$$[-g(x) \in S \;\Rightarrow\; f(x) \geqq x^*(x)] \quad \Leftrightarrow \quad (\exists \lambda \in S^+)\; x^* \in \partial f(0) + \partial(\lambda \circ g)(0).$$

(ii) $\partial f(0) + \bigcup_{y^* \in S^+} \partial(y^* \circ g)(0)$ *is* w^**-closed.*

Proof The conclusion follows if we show that the statements (i) and (ii) are equivalent to the statements (i) and (ii) of Theorem 3.1 Indeed, (ii) is equivalent to the condition that $\text{epi} f^* + \bigcup_{y^* \in S^+} \text{epi}(y^* \circ g)^*$ is w^*-closed because

$$\text{epi} f^* + \bigcup_{y^* \in S^+} \text{epi}(y^* \circ g)^* = [\partial f(0) + \bigcup_{y^* \in S^+} \partial(y^* \circ g)(0)] \times \mathbb{R}_+.$$

On the other hand, (i) of Theorem 3.1 clearly implies (i). To show (i) implies (i) of Theorem 3.1, let $x^* \in X^*$ and $\alpha \in \mathbb{R}$. If $-g(x) \in S \;\Rightarrow\; f(x) \geqq x^*(x) + \alpha$, then by sublinearity of the functions, we see that

$$-g(x) \in S \;\Rightarrow\; f(x) \geq x^*(x) \quad \text{and} \quad \alpha \leq 0.$$

So, by (i), $x^* \in \partial f(0) + \partial(\lambda \circ g)(0)$, for some $\lambda \in S^+$. Thus, for each $x \in X$, $f(x) + (\lambda \circ g)(x) \geq x^*(x)$. Hence, for each $x \in X$, $f(x) + (\lambda \circ g)(x) \geq x^*(x) + \alpha$. □

Corollary 3.3 *Let* $A : X \to Y$ *be continuous and linear. Then the following statements are equivalent*:

(i) $\forall c \in X^*, \big[-Ax \in S \;\Rightarrow c(x) \geqq 0\big] \Leftrightarrow (\exists \lambda \in S^+)\; c + A^T\lambda \;=\; 0.$

(ii) $A^T(S^+)$ *is* w^**-closed.*

Proof Let $g(x) = Ax$. Then $\bigcup_{\lambda \in S^+} \partial(\lambda \circ g)(0) = A^T(S^+)$. Then, the conclusion follows from Corollary 3.1. □

Remark 3.1 If $Y = \mathbb{R}^m$ and S is a polyhedral convex cone in Y, then $A^T(S^+)$ is a finitely generated cone and hence $A^T(S^+)$ is closed. So, from Corollary 3.3, the original Farkas lemma follows (see [7]).

4 Stable Lagrangian duality

Using stable Farkas lemma, we derive necessary and sufficient conditions for a stable duality result for a cone convex optimization problem which holds for each linear perturbation of the given convex objective function.

Theorem 4.1 (Stable Duality) *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function, and let* $g : X \to Y$ *be a continuous and* S*-convex function with* $\text{dom}\, f \cap g^{-1}(-S) \neq \emptyset$. *Then the following statements are equivalent*:

(i) $\inf_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\} = \max_{y^* \in S^+} \inf_{x \in X} \{f(x) + (y^* \circ g)(x) - x^*(x)\}, \;\; \forall x^* \in X^*.$

(ii) $\operatorname{epi} f^* + \bigcup_{y^* \in S^+} \operatorname{epi}(y^* \circ g)^*$ *is* w^**-closed.*

Proof We only need to show that (i) is equivalent to (i) of Theorem 3.1. Suppose that (i) holds. Let $x^* \in X^*$ and $\alpha \in \mathbb{R}$, and assume that $f(x) \geqq x^*(x)+\alpha, \ \forall x \in g^{-1}(-S)$. Then $\inf_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\} \geqq \alpha$. From (i), there exists $\lambda \in S^+$ such that

$$\inf_{x \in X} \left\{f(x) + (\lambda \circ g)(x) - x^*(x)\right\} \geqq \alpha,$$

and so, (i) of Theorem 3.1 holds.

Conversely, assume that (i) of Theorem 3.1 holds. To show (i), let $x^* \in X^*$. If $\inf_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\} = -\infty$, then (i) trivially holds as

$$\begin{aligned} \inf_{x \in X} \left\{f(x) + (\lambda \circ g)(x) - x^*(x)\right\} &\leqq \inf_{x \in g^{-1}(-S)} \left\{f(x) + (\lambda \circ g)(x) - x^*(x)\right\} \\ &\leqq \inf_{x \in g^{-1}(-S)} \left\{f(x) - x^*(x)\right\} = -\infty. \end{aligned}$$

So, we may assume that $r := \inf_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\}$ is finite because $\operatorname{dom} f \cap g^{-1}(-S) \neq \emptyset$. Then, $f(x) - x^*(x) \geqq r$, for each $x \in g^{-1}(-S)$. Now, it follows from (i) of Theorem 3.1 that there exists $\lambda \in S^+$ such that for each $x \in X$, $f(x) + (\lambda \circ g)(x) \geqq x^*(x) + r$. Thus we have,

$$\inf_{x \in X} \left\{f(x) + (\lambda \circ g)(x) - x^*(x)\right\} \geqq \inf_{x \in g^{-1}(-S)} \left\{f(x) - x^*(x)\right\}.$$

Since $f(x) \geqq f(x) + (\lambda \circ g)(x)$, for each $x \in g^{-1}(-S)$,

$$\inf_{x \in g^{-1}(-S)} \left\{f(x) - x^*(x)\right\} \geqq \inf_{x \in X} \left\{f(x) + (\lambda \circ g)(x) - x^*(x)\right\}.$$

Thus (i) holds. □

We now present a new necessary and sufficient condition for the stable min–max duality of a given convex optimization problem. It states that

$$\min_{x \in g^{-1}(-S)} \left\{f(x) - x^*(x)\right\} = \max_{y^* \in S^+} \inf_{x \in X} \left\{f(x) + (y^* \circ g)(x) - x^*(x)\right\}, \quad \forall\, x^* \in X^*.$$

To derive such a condition, define, for each $x \in X$,

$$Ng(x)_0 := \left\{u \in X^* \mid (u, u(x)) \in \bigcup_{y^* \in S^+} \operatorname{epi}\,(y^* \circ g)^*\right\}.$$

It is easy to verify that, for each $x \in g^{-1}(-S)$, $Ng(x)_0$ is a convex cone in X^* and

$$Ng(x)_0 = \left\{u \in X^* \mid y^* \in S^+,\ u \in \partial(y^* \circ g)(x),\ (y^* \circ g)(x) = 0\right\}.$$

Note that, for each $x \in g^{-1}(-S)$, $Ng(x_0) \subset N_{g^{-1}(-S)}(x)$ and

$$N_{g^{-1}(-S)}(x) = \left\{ u \in X^* \mid (u, u(x)) \in \text{cl} \left(\bigcup_{y^* \in S^+} \text{epi } (y^* \circ g)^* \} \right) \right\},$$

where $N_{g^{-1}(-S)}(x)$ is the normal cone to $g^{-1}(-S)$ at x.

Theorem 4.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $g : X \to Y$ be a continuous S-convex function. Suppose that f is continuous at a point in dom $f \cap g^{-1}(-S)$ and that, for each $x^* \in X^*$, $f(.) - x^*(.)$ attains its minimizer over $g^{-1}(-S)$. Then the following statements are equivalent.*

(i) $\min\limits_{x \in g^{-1}(-S)} \{f(x) - x^*(x)\} = \max\limits_{y^* \in S^+} \inf\limits_{x \in X} \{f(x) + (y^* \circ g)(x) - x^*(x)\}, \ \forall x^* \in X^*.$

(ii) $\partial f(x) + N_{g^{-1}(-S)}(x) = \partial f(x) + Ng(x)_0, \ \ \forall x \in \text{dom } f \cap g^{-1}(-S).$

Proof [(ii) ⇒ (i)] Suppose that (ii) holds. Let $x^* \in X^*$. Assume that $x \in \text{dom } f \cap g^{-1}(-S)$ and $f(x) - x^*(x) = \min\limits_{y \in g^{-1}(-S)} \{f(y) - x^*(y)\}$. Then, by optimality condition (see Theorem 4.1 [6]), we have

$$x^* \in \partial f(x) + N_{g^{-1}(-S)}(x).$$

Using (ii), we have,

$$\begin{aligned} x^* &\in \partial f(x) + Ng(x)_0 \\ &= \partial f(x) + \left\{ u \in X^* \mid y^* \in S^+, \ u \in \partial(y^* \circ g)(x), \ (y^* \circ g)(x) = 0 \right\}. \end{aligned}$$

Thus, there exists $\lambda \in S^+$ such that

$$x^* \in \partial f(x) + \partial(\lambda \circ g)(x) \quad \text{and} \quad (\lambda \circ g)(x) = 0.$$

This gives us that

$$\inf_{y \in X} \left\{ f(y) + (\lambda \circ g)(y) - x^*(y) \right\} \geqq f(x) - x^*(x).$$

On the other hand, for each $y^* \in S^+$,

$$\inf_{y \in X} \left\{ f(y) + (y^* \circ g)(y) - x^*(y) \right\} \leqq \inf_{y \in g^{-1}(-S)} \left\{ f(y) - x^*(y) \right\}.$$

Hence (i) holds.

[(i) ⇒ (ii)] Suppose that (i) holds. Since, for each $x \in g^{-1}(-S)$, $Ng(x)_0 \subset N_{g^{-1}(-S)}(x)$, $\partial f(x) + Ng(x)_0 \subset \partial f(x) + N_{g^{-1}(-S)}(x), \ \ \forall x \in \text{dom } f \cap g^{-1}(-S).$

Let $x \in \mathrm{dom}\, f \cap g^{-1}(-S)$ and $u \in \partial f(x) + N_{g^{-1}(-S)}(x)$. Then there exist $v \in \partial f(x)$ and $w \in N_{g^{-1}(-S)}(x)$ such that $u = v + w$, and hence for each $y \in g^{-1}(-S)$,

$$f(y) - (v + w)(y) \geqq f(x) - (v + w)(x).$$

Now, it follows from (i) that there exists $\lambda \in S^+$ such that

$$f(x) - u(x) = \inf_{y \in X} \left\{ f(y) + (\lambda \circ g)(y) - u(y) \right\}.$$

So, we have,

$$(\lambda \circ g)(x) = 0 \quad \text{and} \quad u \in \partial f(x) + \partial(\lambda \circ g)(x).$$

This means that

$$u \in \partial f(x) + Ng(x)_0.$$

Hence (ii) holds. □

The link between the condition that characterizes the stable duality (Theorem 4.1) and the condition that characterizes the stable min–max duality (Theorem 4.2) is illustrated by the following Proposition for which a direct proof is given.

Proposition 4.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let $g : X \to Y$ be a continuous S-convex function. Assume that f is continuous at a point in dom $f \cap g^{-1}(-S)$. If* $\mathrm{epi}\, f^* + \bigcup_{y^* \in S^+} \mathrm{epi}\,(y^* \circ g)^*$ *is w^*-closed, then the equality*

$$\partial f(x) + N_{g^{-1}(-S)}(x) = \partial f(x) + Ng(x)_0, \quad \forall x \in \mathrm{dom}\, f \cap g^{-1}(-S)$$

holds.

Proof Clearly, $\partial f(x) + Ng(x)_0 \subset \partial f(x) + N_{g^{-1}(-S)}(x),\ \forall x \in \mathrm{dom}\, f \cap g^{-1}(-S)$. To establish the converse inclusion, let $x \in \mathrm{dom}\, f \cap g^{-1}(-S)$ and $u \in \partial f(x) + N_{g^{-1}(-S)}(x)$. Then, there exist $v \in \partial f(x)$ and $w \in N_{g^{-1}(-S)}(x)$ such that $u = v + w$, and hence we have

$$f^*(v) + f(x) = v(x)$$

and there exist nets $\{\lambda_\alpha\} \subset S^+, \{w_\alpha\} \subset X^*$ and $\{r_\alpha\} \subset \mathbb{R}$ such that $(w_\alpha, r_\alpha) \in \mathrm{epi}(\lambda_\alpha \circ g)^*$ and $(w_\alpha, r_\alpha) \to (w, w(x))$. Thus $(v, v(x) - f(x)) + (w_\alpha, r_\alpha) \in \mathrm{epi}\, f^* + \cup_{y^* \in S^+} \mathrm{epi}(y^* \circ g)^*$. Now, by the closedness assumption, there exists $\lambda \in S^+$ such that

$$\begin{aligned}(v, v(x) - f(x)) + (w, w(x)) &\in \mathrm{epi}\, f^* + \mathrm{epi}(\lambda \circ g)^* \\ &= \mathrm{epi}(f + \lambda \circ g)^*.\end{aligned}$$

So, for each $y \in X$,

$$u(y) - (f + \lambda \circ g)(y) \leqq u(x) - f(x).$$

In particular, by taking $y = x$, we get $-(\lambda \circ g)(x) \leqq 0$. As $(\lambda \circ g)(x) \leqq 0$, $(\lambda \circ g)(x) = 0$ Thus, for each $y \in X$,

$$(f + \lambda \circ g)(y) \geqq (f + \lambda \circ g)(x) + u(y - x).$$

This gives us that $u \in \partial f(x) + \partial(\lambda \circ g)(x)$ and $(\lambda \circ g)(x) = 0$, and hence $u \in \partial f(x) + Ng(x)_0$. □

Remark 4.1 (1) In general, (i) in Theorem 4.2 does not imply that $N_{g^{-1}(-S)}(x) = Ng(x)_0 \ \forall x \in \text{dom } f \cap g^{-1}(-S)$. Indeed, let f, g and S be as in Example 3.1. Then (i) in Theorem 4.2 holds. But even though $0 \in \text{dom } f \cap g^{-1}(-S)$, $N_{g^{-1}(-S)}(0) = [0, \infty)$ and $Ng(0)_0 = \{0\}$.

(2) In general, $\partial f(x_0) + N_{g^{-1}(-S)}(x_0) = \partial f(x_0) + Ng(x_0)_0$ for some $x_0 \in \text{dom } f \cap g^{-1}(-S)$ does not imply that (i) in Theorem 4.2 holds. Indeed, let $f(x) = -x$, $g(x) = [\max\{0, x\}]^2$ and $S = \mathbb{R}_+$. Let $x_0 = -1$. Then $x_0 \in \text{dom } f \cap g^{-1}(-S)$,

$$\partial f(x_0) + N_{g^{-1}(-S)}(x_0) = \partial f(x_0) + Ng(x_0)_0 = \{-1\}.$$

Even though $\min_{x \in g^{-1}(-S)} f(x) = 0$, $\max_{\lambda \geqq 0} \inf_{x \in \mathbb{R}} \{f(x) + \lambda g(x)\}$ can not be attained, and hence (i) in Theorem 4.2 does not hold.

Theorem 4.3 *Let $g : X \to Y$ be a continuous S-convex function with $g^{-1}(-S) \neq \emptyset$. Suppose that, for each $x^* \in X^*$, $x^*(.)$ attains its minimizer over $g^{-1}(-S)$. Then the following statements are equivalent*:

(i) $\min_{x \in g^{-1}(-S)} x^*(x) = \max_{y^* \in S^+} \inf_{x \in X} \{x^*(x) + (y^* \circ g)(x)\}, \quad \forall x^* \in X^*.$

(ii) $N_{g^{-1}(-S)}(x) = Ng(x)_0, \quad \forall x \in g^{-1}(-S).$

Proof The conclusion follows from Theorem 4.2 by taking $f = 0$. □

5 Stable duality in cone programming

In this section, we apply the results of the previous sections to derive stable duality results for convex semi-definite programs and convex second-order cone programs.

Consider now the following convex semi-definite program.

$$\begin{array}{lll} \text{(CSP)} & \text{Minimize} & f(x) \\ & \text{subject to} & F_0 + \Sigma_{i=1}^m x_i F_i \succeq 0, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a convex function.

Let S_n be a vector space of $n \times n$ symmetric matrices with the trace inner product $(M, N) := \mathrm{Tr}[MN]$ which is partially ordered by Löwner partial order $\succeq$ of S_n; that is, for $M,\ N \in S_n,\ M \succeq N$ if and only if $(M - N)$ is positive semidefinite. Let $K = \{M \in S_n \mid M \succeq 0\}$. Then K is self-dual, that is, $K^+ = K$. Let $F(x) = F_0 + \Sigma_{i=1}^m x_i F_i$, where $F_i \in S_n,\ i = 0, 1, \ldots, m$ and $\hat{F}(x) = \Sigma_{i=1}^m x_i F_i$. Clearly, $\hat{F}$ is a continuous linear operator from $\mathbb{R}^m$ to S_n. The adjoint operator $\hat{F}^* : S_n \to \mathbb{R}^m$ is given as $(\hat{F}^*(Z))_i = \mathrm{Tr}[ZF_i]$, $i = 1, \ldots, m$. We denote the feasible set of (CSP) by $F^{-1}(K)$.

Theorem 5.1 *Let $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Suppose that dom $f \cap F^{-1}(K) \neq \emptyset$. Then the following statements are equivalent*:

(i) *For each $a \in \mathbb{R}^m$ and each $\alpha \in \mathbb{R}$*

$$\left[F(x) \in K \Rightarrow f(x) \geqq a^T x + \alpha\right] \iff (\exists Z \in K)(\forall x \in \mathbb{R}^m)\ f(x) - \mathrm{Tr}[ZF(x)] \geqq a^T x + \alpha.$$

(ii) $$\inf_{x \in F^{-1}(K)} \left\{f(x) - a^T x\right\} = \max_{Z \in K} \inf_{x \in \mathbb{R}^m} \left\{f(x) - \mathrm{Tr}(ZF(x)) - a^T x\right\}, \quad \forall a \in \mathbb{R}^m.$$

(iii) epi $f^* + \bigcup_{Z \in K, r \geqq 0} \left\{(-\hat{F}^*(Z), \mathrm{Tr}(ZF_0) + r)\right\}$ *is closed.*

Proof Let $g : \mathbb{R}^m \to S_n$ be defined by $g(x) = -F(x)$ and $S = K$. Then we have,

$$\bigcup_{y^* \in S^+} \mathrm{epi}\ (y^* \circ g)^* = \bigcup_{Z \in K,\ r \geqq 0} \left\{(-\hat{F}^*(Z), \mathrm{Tr}(ZF_0) + r)\right\}.$$

Then, the conclusions follow from Theorems 3.1 and 4.1. □

Consider the following convex second-order cone program:

(SOCP) Minimize $f(x)$
subject to $x \in M := \{x \in \mathbb{R}^n \mid \|(h_1(x), \ldots, h_{m-1}(x))^T\| \leqq h_m(x)\}$,

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are convex functions and $\|z\| = \sqrt{z^T z},\ z \in \mathbb{R}^{m-1}$. Let $C = \{(y, t) \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|y\| \leq t\}$, that is, C is a second-order cone in $\mathbb{R}^m$. Then C is self-dual, that is, $C = C^+$.

Theorem 5.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $h_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \ldots, m$, be convex functions with dom $f \cap M \neq \emptyset$. Then the following statements are equivalent*:

(i) *For each $a \in \mathbb{R}^n$ and each $\alpha \in \mathbb{R}$*

$$[x \in M] \Rightarrow \left[f(x) \geqq a^T x + \alpha\right] \iff (\exists \lambda \in C)\ (\forall x \in \mathbb{R}^n)\ f(x) - \sum_{i=1}^m \lambda_i h_i(x) \geqq a^T x + \alpha.$$

(ii) $\inf_{x \in M} \left\{ f(x) - a^T x \right\} = \max_{\lambda \in C} \inf_{x \in \mathbb{R}^n} \left\{ f(x) - \sum_{i=1}^m \lambda_i h_i(x) - a^T x \right\}, \quad \forall a \in \mathbb{R}^n.$

(iii) $\text{epi } f^* + \bigcup_{\lambda \in C} \sum_{i=1}^m \text{epi } (\lambda_i h_i)^*$ *is closed.*

Proof Let $g(x) = -(h_1(x)), \ldots, h_m(x))^T$ and $S = C$. Then the conclusions follow from Theorems 3.1 and 4.1. □

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