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FULL LENGTH PAPER

On the choice of parameters for the weighting method in vector optimization

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Abstract We present a geometrical interpretation of the weighting method for constrained (finite dimensional) vector optimization. This approach is based on rigid movements which separate the image set from the negative of the ordering cone. We study conditions on the existence of such translations in terms of the boundedness of the scalar problems produced by the weighting method. Finally, using recession cones, we obtain the main result of our work: a sufficient condition under which weighting vectors yield solvable scalar problems.

Keywords Vector optimization · Weak efficiency · Scalarization · Weighting method · Recession cone

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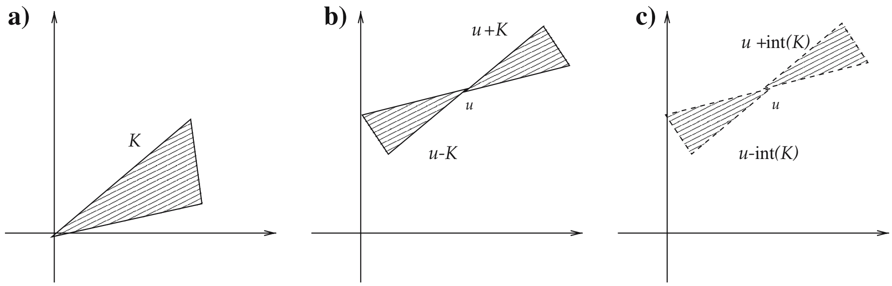

Fig. 1 In (**a**) we have the cone K. In (**b**) the vectors in the above cone are $\succeq u$, while those in the other cone are $\preceq u$. In (**c**) we have vectors which are $\succ u$ and $\prec u$ in the above and the below cone, respectively

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1 Introduction

In this work we consider the problem of finding weakly efficient points (or weak Pareto minimal elements) of a constrained vector optimization problem. Our setting will be a finite dimensional linear space, say $\mathbb{R}^m$, with the canonical inner product $\langle \cdot, \cdot \rangle$, and a preference order induced by $K \subset \mathbb{R}^m$, a closed convex and pointed (i.e., $-K \cap K = \{0\}$) cone with nonempty interior $\mathrm{int}(K)$ (see Figure 1(a)). Our objective function, defined on a subset of another finite dimensional space, will take its values in $\mathbb{R}^m$. To be more precise, the space $\mathbb{R}^m$ is endowed with the following partial order

$$u \preceq v \, (v \succeq u) \text{ for } u, v \in \mathbb{R}^m \quad \text{iff} \quad v - u \in K \text{ (see Fig. 1b)},$$

and the following stronger relation

$$u \prec v \, (v \succ u) \text{ for } u, v \in \mathbb{R}^m \quad \text{iff} \quad v - u \in \mathrm{int}(K) \text{ (see Fig. 1c)}.$$

Among the advantages of the notation "$0 \preceq w$" over "$w \in K$", we mention that K-inequalities can be handled as regular ones, e.g., two of such inequalities can be added up, or multiplied by nonnegative numbers, etc.

Regarding the importance of considering general ordering cones, we point out that even though the vast majority of real life problems formulated as vector-valued ones deals with the component-wise partial order, i.e., the one which arises from the Paretian cone, there are many others which require preference orders induced by closed convex cones other than the nonnegative orthant. Such cones have been analyzed in [1] (based on the theoretical results of [2]), where problems of portfolio selection in security markets require finding weak Pareto minimal points with respect to feasible portfolio cones, which are nonlattice, that is to say cones with more extreme rays than the ambiance space.

Now we define our problem. Given a subset Ω of $\mathbb{R}^n$ and a mapping $F : \Omega \to \mathbb{R}^m$, the vector optimization problem, understood in the weak Pareto sense ([10,11]),

$$(P) \quad \min_{x\in\Omega} F(x),$$

consists of finding a *feasible* point x^* ($x^* \in \Omega$) such that $F(x^*)$ is *weakly efficient* (or a *weak Pareto minimal element*) for $F(\Omega)$, i.e., such that

$$F(x) \prec F(x^*)$$

does not hold for any feasible x. We recall that $x^* \in \Omega$ is *efficient* (or *Pareto minimal element*) for $F(\Omega)$ if there does not exist $x \in \Omega$ such that $F(x) \preceq F(x^*)$, with $F(x) \neq F(x^*)$ ([10,11]). Trivially, efficient points are also weakly efficient.

Scalarization techniques [8,9,11,12] for solving problem (P) substitute the original vector problem by a suitable scalar one, in such a way that the optimal solutions of the new problem are also optimal for the original one. The main advantage of this approach, from a practical point of view, is that we can use a large number of fast and reliable methods developed for single-valued optimization in order to solve vector problems.

One of the most widely used scalarization techniques in multicriteria (i.e., in the Paretian cone case or, in other words, the point-wise partial order) is the weighting method, which consists of minimizing a weighted sum of the different objectives. The weights, which are critical for the method, in general are not known in advance, so computational implementations of this technique are not always straightforward. In fact, unproper choices of weighting vetors may lead to scalar problems without optimal solutions.

In this paper we study an extension of the weighting method for vector optimization. In Sect. 2, we show that the method can be derived from the following procedure: if possible, first, by adding an adequate vector, separate the image of the objective from the negative of the ordering cone and then find the closest point at the image set from the separating hyperplane. We devote this section to study existence conditions of translations that separate the image and the negative of the cone. An interpretation of such conditions in terms of the weighting method is presented. Roughly speaking, it turns out that the existence of suitable (separating) rigid movements is deeply connected with the boundedness of the scalar problems produced by the weighting method.

In the last section we take a step further and study the existence of optima of the scalarized problems. This analysis yelds the main result of this article: a new condition, under which a vector of weights determines a solvable scalar problem (whose optima are weakly efficient). We end our work by pointing out that cone-boundedness, a usually required assumption in classical results on existence of efficient points, is not assumed on our analysis. By means of an example, we show that our condition can be satisfied even in the absence of cone-boundedness of the image set. Furthermore, we prove that this new condition is implied by cone-boundedness and, therefore, is a more general assumption.

2 The weighting method scalarization and the existence of suitable rigid movements

In this work, we will say that a scalar minimization problem is a *scalarization* of (P) if its optimal solutions are weakly efficient for (P) (see, e.g., [10] and [11]).

The scalarization procedure which consists of minimizing a weighted sum of the different objectives is known as the *weighting method*, and can be traced up to the mid fifties and early sixties [6,15].

Roughly speaking, for a given vector optimization problem, as (P), the weighting method consists of minimizing the scalar-valued problem:

$$\min_{x\in\Omega} \langle w, F(x)\rangle, \tag{1}$$

where w is the *weighting vector* and $w \in K^* := \{y \in \mathbb{R}^m \mid \langle v, y\rangle \geq 0 \; \forall\, v \in K\}$, the *positive dual* (or *polar*) of K..

Despite the simplicity and notoriety of the weighting method -practically no multiobjective or vector optimization book fails to discuss it (e.g., [11,14])-, its implementation aspects are not fully satisfactory. Indeed, (1) may be an unbounded problem and, even if it is bounded, it may lack minimizers. Boundedness of (1) will be discussed in this section, and existence of minimizers for such a bounded problem will be analyzed in Sect. 3. Of course, there are some simple cases for which this is not an issue; for instance, if $F(\Omega)$ is compact, all scalarizations as (1) will be useful, since this kind of problems always has optimal solutions.

Even though proving that an optimal solution of (1) is weakly efficient for (P) is an easy task (by a simple "ad absurdum" reasoning), we sketch not just a proof of this fact, but a whole different presentation of the method.

For this geometrical approach, we will focus our attention on problems for which the convex hull of $F(\Omega)$, after a suitable rigid movement, does not touch $-K$, i.e.,

$$[\mathrm{conv}(F(\Omega)) + u_0] \cap (-K) = \emptyset, \quad \text{for some } u_0 \in \mathbb{R}^m. \tag{2}$$

It is easy to see that we do not loose generality on assuming $u_0 = 0$. Indeed, if $F_u(x) := F(x) + u$, for $u \in \mathbb{R}^m$, and (P_u) is the vector problem $\min_{x\in\Omega} F_u(x)$, then, (P_u) satisfies $\mathrm{conv}(F_u(\Omega)) \cap (-K) = \emptyset$, for all $u \succeq u_0$, where u_0 is given by (2). Moreover, these problems ((P) and (P_u)) have the same optima.

Assuming condition (2), with $u_0 = 0$, let us now consider the problem of how to find weakly efficient solutions for problem (P). Observe that the optimality condition $F(x) \not\prec F(x^*) \;\forall\, x \in \Omega$ can be written as

$$[F(x^*) - \mathrm{int}(K)] \cap F(\Omega) = \emptyset. \tag{3}$$

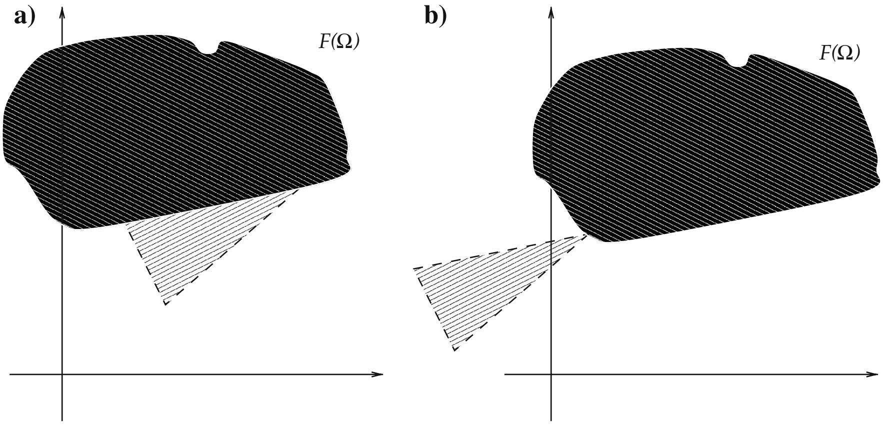

Fig. 2 The vertex of the shifted cone K in (**a**) is weak Pareto minimal for $F(\Omega)$, as well as all points in the line supported by the cone. In (**b**) the shifted cone touches $F(\Omega)$ just at its vertex, which is (strong) Pareto optimal

Note that $F(x^*) - \text{int}(K)$ is the interior of the cone obtained by translating $-K$ till the moment $F(x^*)$ becomes its vertex (see Fig. 2). Note too that other points on the intersection of the boundaries of $F(\Omega)$ and shifted cones of the form $F(x^*) - \text{int}(K)$ are also weakly efficient (Fig. 2a. Whenever the vertex is the only point in that intersection, it is Pareto optimal for $F(\Omega)$ (Fig. 2b).

We will devote our efforts to locate those weakly efficient points x^*, which are "F's pre-images of shifted cones' vertices satisfying (3). But how can we effectively locate such vertices?

Let us now give an answer to the above question. Since we are assuming that $\text{conv}(F(\Omega)) \cap (-K) = \emptyset$, by virtue of the Convex Separation Theorem (see [3], Theorem 4.7 or [4], Proposition B13), there exists a hyperplane H_w that separates $-K$ and $\text{conv}(F(\Omega))$. Since K is a closed cone, we do not loose generality if we assume that H_w is a subspace and $||w|| = 1$. We therefore have that $\langle w, y\rangle \leq 0 \leq \langle w, z\rangle \; \forall\, y \in -K$ and $\forall\, z \in \text{conv}(F(\Omega))$. In particular,

$$\langle w, y\rangle \leq 0 \quad \text{for all } y \in -K \tag{4}$$

and

$$\langle w, F(x)\rangle \geq 0 \quad \text{for all } x \in \Omega\,. \tag{5}$$

Note that (4) is telling us that w is an element of K^*.

According to Fig. 3, good candidates for vertices $F(x^*)$ of shifted cones of the above mentioned form, which touch $F(\Omega)$ without overlapping image's points other than the boundary ones, are the closest points from H_w at $F(\Omega)$.

At this point, it's not difficult to prove that such points are indeed weakly efficient for $F(\Omega)$; in other words, that x^* is a weak Pareto minimal element of problem (P) if $F(x^*)$ realizes the minimal distance between $F(\Omega)$ and H_w.

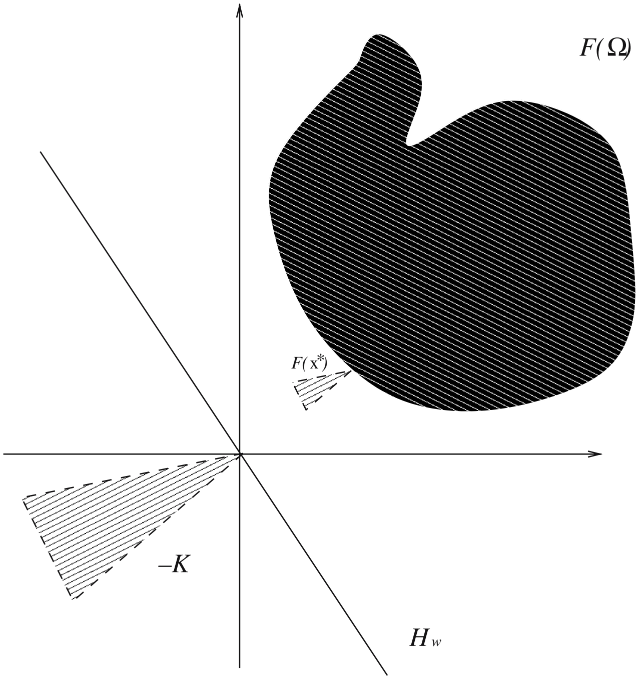

Fig. 3 The closest points in $F(\Omega)$ to H_w are weakly efficient

Calling P_w the orthogonal projector onto the separating hyperplane H_w, i.e.,

$$P_w(y) := y - \langle w, y \rangle w \,, \tag{6}$$

the distance between a point $y \in \mathbb{R}^m$ and H_w is clearly given by $||y - P_w(y)||$.

Therefore, we have the following result:

If the unitary m-vector w satisfies (4)–(5)*, then every optimal solution of the scalar problem*

$$(S) \qquad \min_{x \in \Omega} ||F(x) - P_w(F(x))||$$

is weakly efficient for the vector problem (P).

In order to prove the above statement, suppose that x^*, an optimal solution of (S), is not weakly efficient for (P); this means that the optimality condition (3) does not hold, so there exist $\bar{y} \in \text{int}(K)$ and $\bar{x} \in \Omega$ such that $F(x^*) = \bar{y} + F(\bar{x})$. Now, it is easy to see that $||F(x^*) - P_w(F(x^*))||^2 > ||F(\bar{x}) - P_w(F(\bar{x}))||^2$, in contradiction with the optimality of x^*.

Finally, observe that from the definition of the orthogonal projector P_w given in (6), from (5) and the fact that $||w|| = 1$, it follows that $||F(x) - P_w(F(x))|| = \langle w, F(x) \rangle$ for all $x \in \Omega$. Hence, we obtain, the following well-known result (see, for example, [11, Theorem 3.1.1]):

If $w \in K^*\backslash\{0\}$, then all optimal solutions, if any, of

$$\min_{x\in\Omega} \langle w, F(x)\rangle, \tag{7}$$

are weakly efficient for (P).

Indeed, if $x^* \in \Omega$ is a minimizer of (7), the optimal set of (7) (i.e., of (S)) coincides with the optimal set of $\min_{x\in\Omega} \langle w, F_{u^*}(x)\rangle$, where $u^* = -F(x^*)$. Moreover, F_{u^*} satisfies (5), and (4) trivially holds. The result follows, since, under conditions (4)–(5), as we have just seen, all optima of (S) are weakly efficient for (P).

Up to now, we know that by performing a rigid movement, if necessary, we can try to compute (at least theoretically) a weakly efficient solution of problem (P), by minimizing the distance between $F(\Omega)$ (or $F(\Omega) + u$, for some $u \in \mathbb{R}^m$) and a hyperplane that separates this set from $-K$. So at this point we go back to the very beginning of our discussion and study when we can perform a rigid movement which prevents the convex hull of F's image from touching $-K$. That is to say, we will see in which cases there exists u_0 such that (2) holds.

The following fact will be needed.

$$K \cap \left(K^* \setminus \{0\}\right) \neq \emptyset\,. \tag{8}$$

In order to verify it, note that if (8) does not hold, then, by the nonstrict Convex Separation Theorem, there exists $0 \neq w \in \mathbb{R}^m$ such that

$$\langle w, v\rangle \geq 0 \quad \forall\, v \in K, \tag{9}$$

$$\langle w, v\rangle \leq 0 \quad \forall\, v \in\ K^*\backslash\{0\}. \tag{10}$$

By (9), $w \in K^*$. Since $w \neq 0$, by virtue of (10), $||w||^2 = \langle w, w\rangle \leq 0$, in contradiction with $w \neq 0$.

Recall now that the recession cone of a convex set $C \subset \mathbb{R}^m$ (see, e.g., [13]) is the set

$$0^+C := \{v \in \mathbb{R}^m \ \mid C + tv\ \subset C \quad \forall\, t \geq 0\}.$$

We begin the analysis with a sufficient condition for the existence of a separating translation. We will first study the general case of a convex set $C \subset \mathbb{R}^m$. As usual, $\mathrm{cl}(C)$ stands for the closure of C.

Lemma 1 *Let $C \subset \mathbb{R}^m$ be convex. If $0^+\mathrm{cl}(C) \cap (-K) = \{0\}$, then there exists $u \in \mathbb{R}^m$ such that*

$$[C + u] \cap (-K) = \emptyset.$$

Proof Suppose that

$$[C+u]\cap(-K)\neq\emptyset,\quad \forall\, u\in\mathbb{R}^m. \tag{11}$$

It is enough to show that, under this assumption, $0^+\mathrm{cl}(C)\cap(-K)\neq\{0\}$.

By (8), we can take $e\in K\cap(K^*\setminus\{0\})$ and for each $k\in\mathbb{N}$, we use (11) to obtain

$$x^k\in[C+ke]\cap(-K).$$

Therefore,

$$x^k=c^k+ke,\quad c^k\in C,\ \ x^k\in -K.$$

Since $e\in K^*$ and $x^k\in -K$, we have $\langle e,x^k\rangle\leq 0$, i.e., $\langle e,c^k+ke\rangle\leq 0$, and so, using the Cauchy-Schwartz inequality,

$$k\|e\|^2\leq\langle -e,c^k\rangle\leq\|e\|\,\|c^k\|\,,$$

which implies $\|c^k\|\geq k\|e\|$. So

$$\lim_{k\to\infty}\|c^k\|=\infty\,.$$

We claim that the accumulation points of the normalized sequence $\{c^k/\|c^k\|\}$ belong to $0^+\mathrm{cl}(C)$. Indeed, let $c=\lim_{j\to\infty}c^{k_j}/\|c^{k_j}\|$, $x\in\mathrm{cl}(C)$ and $t\geq 0$. Since $\{c^k\}$ diverges, we have that

$$x+tc=\lim_{j\to\infty}(1-t/\|c^{k_j}\|)x+(t/\|c^{k_j}\|)c^{k_j}\,.$$

Therefore, since x and c^k belong to the convex set $\mathrm{cl}(C)$, the above sequence of their convex combinations is in $\mathrm{cl}(C)$, and so $x+tc\in\mathrm{cl}(C)$. Hence, $c\in 0^+\mathrm{cl}(C)$, as we claimed. Observe that the sequence $\{c^k/\|c^k\|\}$ has at least one accumulation point because it is bounded.

Note that

$$c^k=x^k-ke.$$

Since $-K$ is a closed convex cone, $c^k\in -K$ and the accumulation points of $\{c^k/\|c^k\|\}$ also belong to $-K$. Therefore, $0^+\mathrm{cl}(C)\cap(-K)\neq\{0\}$. □

Now we discuss a necessary condition for the existence of a separating translation.

Lemma 2 *Let $C \subset \mathbb{R}^m$ be convex. If there exists $u \in \mathbb{R}^m$ such that $[C + u] \cap (-K) = \emptyset$, then*

$$0^+C \cap \mathrm{int}(-K) = \emptyset.$$

Proof We will show that, if

$$0^+C \cap \mathrm{int}(-K) \neq \emptyset\,, \tag{12}$$

then for any $u \in \mathbb{R}^m$,

$$[C + u] \cap (-K) \neq \emptyset\,. \tag{13}$$

Assume that (12) holds and take $v \in 0^+C \cap \mathrm{int}(-K)$. For any $u \in \mathbb{R}^m$ and $c \in C$, there exists some (large enough) $t > 0$ such that

$$(1/t)(c + u) + v \in -K\,.$$

Therefore $c + u + tv \in -K$. On the other hand, since $0^+C = 0^+[C + u]$, $v \in 0^+[C + u]$ and so we have $c + u + tv \in C + u$. Altogether, we deduce that $c + u + tv \in [C + u] \cap (-K)$ and (13) holds for any u. □

We end this section by giving a straightforward interpretation of Lemmas 1 and 2 from the weighting method's point of view.

Proposition 1 *If* $0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \cap (-K) = \{0\}$*, then there exists a vector* $w \in K^*\backslash\{0\}$ *such that the problem*

$$\min_{x\in\Omega}\langle w, F(x)\rangle$$

is bounded (*from below*)*. Conversely, if the above problem is bounded for some* $w \in K^*\backslash\{0\}$*, then* $0^+\mathrm{conv}(F(\Omega)) \cap \mathrm{int}(-K) = \emptyset$*.*

Proof First assume that $0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \cap (-K) = \{0\}$. By Lemma 1, with $C = \mathrm{conv}(F(\Omega))$, there exists $u_0 \in \mathbb{R}^m$ such that $\mathrm{conv}(F(\Omega) + u_0) \cap (-K) = \emptyset$. Then, using again the nonstrict Convex Separation Theorem and the fact that $-K$ is a closed cone, we have that

$$\langle w, y\rangle \leq 0 \leq \langle w, F_{u_0}(x)\rangle \quad \forall\ x \in \Omega,\ \ y \in -K\,,$$

for some $w \in K^*\backslash\{0\}$. Hence,

$$\langle w, -u_0\rangle \leq \langle w, F(x)\rangle \quad \forall x \in \Omega\,.$$

Let us now see the converse. Assume that, for some $\alpha \in \mathbb{R}$ and $w \in K^*\backslash\{0\}$,

$$\langle w, F(x)\rangle \geq \alpha \quad \forall x \in \Omega\,. \tag{14}$$

Suppose that $0^+\text{conv}(F(\Omega)) \cap \text{int}(-K) \neq \emptyset$, and let v be a nonzero m-vector in that intersection. Then, for any $z \in \text{conv}(F(\Omega))$ and $t \geq 0$, $z + tv \in \text{conv}(F(\Omega))$. So, by Caratheodory's Theorem ([5], Proposition 1.3.1),

$$z + tv = \sum_{i=1}^{m+1} \lambda_i F(x^i)\,, \tag{15}$$

where $x^i \in \Omega$, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots m+1$ and $\sum_{i=1}^{m+1} \lambda_i = 1$. From (15) and (14), we get

$$\langle w, z + tv \rangle \geq \alpha > -\infty \quad \forall\, t \geq 0\,. \tag{16}$$

But, since $v \in \text{int}(-K)$ and $w \in K^* \backslash \{0\}$, we have

$$\lim_{t \to +\infty} \langle w, z + tv \rangle = -\infty\,,$$

in contradiction with (16). So such v cannot exist and the conclusion follows. □

The condition $0^+\text{cl}(\text{conv}(F(\Omega))) \cap (-K) = \{0\}$, which appears in the fisrt part of the previous corollary, has already been used for ensuring existence of optima in vector optimization problems (see [7]). On the other hand, Theorem 3.2.12 [14], when applied to our problem, shows that $\text{cl}(\text{conv}(F(\Omega))$ has efficient points.

3 On the choice of the weighting vector

In the previous section we obtained, via a geometrical approach, necessary and sufficient conditions for the boundedness of the (scalar) minimization problems of the weighting method. In this section, our goal is to give a criterion for the proper choice of the weighting vectors. First we study existence of optimal solutions of the associated scalar problems.

Theorem 1 *Take $w \neq 0$ and let $H_w = \{y \in \mathbb{R}^m \mid \langle w, y \rangle = 0\}$. Suppose that*

1. *$F(\Omega)$ is closed,*
2. *the scalar problem*

$$\min_{x \in \Omega} \langle w, F(x) \rangle \tag{17}$$

is bounded from below,

3. $0^+\text{cl}(\text{conv}(F(\Omega))) \cap H_w = \{0\}$.

Then, problem (17) has an optimal solution.

Proof Let

$$\nu := \inf_{x \in \Omega} \langle w, F(x) \rangle\,.$$

There exists a minimizing sequence $\{x^k\} \subset \Omega$, i.e.,

$$\lim_{k\to\infty} \langle w, F(x^k)\rangle = \nu\,.$$

If $\{F(x^k)\}$ has a bounded subsequence, as $F(\Omega)$ is closed, there exists $\bar{x} \in \Omega$ such that $F(\bar{x})$ is an accumulation point of $\{F(x^k)\}$. So, $\bar{x}$ is an optimal solution of (17).

If $\{F(x^k)\}$ has no bounded subsequences, then

$$\lim_{k\to\infty} \|F(x^k)\| = \infty\,.$$

We will show that this condition contradicts our assumptions. Refining the sequence if necessary, we can assume that $\{F(x^k)/\|F(x^k)\|\}$ converges to some non-null m-vector $y \in \mathbb{R}^m$. Since $\{\langle w, F(x^k)\rangle\}$ converges to ν,

$$\langle w, y\rangle = \lim_{k\to\infty} \frac{1}{\|F(x^k)\|}\left\langle w, F(x^k)\right\rangle = 0,$$

and $y \in H_w$. By means of the same reasoning used in Lemma 1, we see that $y \in 0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))$. Altogether,

$$y \in 0^+\mathrm{cl}(\mathrm{conv}F(\Omega))) \cap H_w, \quad \|y\| = 1\,,$$

in contradiction with our hypotheses. □

We now give conditions which are equivalent to assumptions 2 and 3 of our last theorem. For $w \neq 0$ in $\mathbb{R}^m$, let us call S_w the half-space determined by w, that is to say,

$$S_w := \{y \in \mathbb{R}^m \mid \langle w, y\rangle \leq 0\}.$$

Lemma 3 *Conditions* 2 *and* 3 *of Theorem* 1 *are equivalent to the following one:*

$$0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \cap S_w = \{0\}\,, \tag{18}$$

which, in turn, is equivalent to

$$w \in \mathrm{int}\Big(\left[0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))\right]^*\Big). \tag{19}$$

Proof First we will show that conditions 2 and 3 of Theorem 1 are equivalent to (18). Suppose that conditions 2 and 3 hold. We will prove (18) by contradiction.

Assume that $z^0 \in 0^+\text{cl}(\text{conv}(F(\Omega))) \cap S_w$ and $z^0 \neq 0$. Since $z^0 \in S_w$, we have $\langle w, z^0\rangle \leq 0$. So, from condition 3 it follows that

$$\langle w, z^0\rangle < 0. \tag{20}$$

Now take $x^0 \in \Omega$. Due to the fact that $z^0 \in 0^+\text{cl}(\text{conv}(F(\Omega)))$, we have that $F(x^0)+kz^0 \in \text{cl}(\text{conv}(F(\Omega)))$ for all $k = 1, 2, \ldots$ Hence, there exists a sequence $\{y^k\} \subset \text{conv}(F(\Omega))$ such that

$$||y^k - (F(x^0) + kz^0)|| < 1 \quad \text{for all } k = 1, 2, \ldots$$

Combining the above inequality with Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}\langle w, y^k\rangle &= \langle w, y^k - (F(x^0) + kz^0)\rangle + \langle w, F(x^0) + kz^0\rangle \\ &\leq \|w\| + \langle w, F(x^0)\rangle + k\langle w, z^0\rangle \quad \text{for all} \quad k = 1, 2, \ldots,\end{aligned}$$

Now combining this result with (20), we obtain $\lim_{k\to\infty}\langle w, y^k\rangle = -\infty$. Whence, $\inf_{y\in\text{conv}(F(\Omega))}\langle w, y\rangle = -\infty$, which implies $\inf_{x\in\Omega}\langle w, F(x)\rangle = -\infty$, in contradiction with assumption 2. So (18) must hold.

Let us now prove the converse, i.e., that (18) implies conditions 2 and 3. Clearly, (18) implies condition 3. Suppose condition 2 does not hold; then $\inf_{x\in\Omega}\langle w, F(x)\rangle = -\infty$. So, there exists a sequence $\{x^k\}$ in Ω, such that

$$\lim_{k\to\infty}\langle w, F(x^k)\rangle = -\infty .$$

Trivially, $\|F(x^k)\| \to +\infty$. As $F(x^k) \neq 0$ for large enough k, for those k's we consider the normalized sequence

$$y^k = \frac{F(x^k)}{\|F(x^k)\|} .$$

Refining the sequence if necessary, we may assume that $y^k \to \bar{y}$. Clearly, $\|\bar{y}\| = 1$. Since $\langle w, F(x^k)\rangle \to -\infty$, we have

$$\langle w, \bar{y}\rangle \leq 0. \tag{21}$$

We claim that $\bar{y} \in 0^+\text{cl}(\text{conv}(F(\Omega)))$. Take $\lambda > 0$ and $z \in \text{cl}(\text{conv}(F(\Omega)))$. Note that $\lambda/\|F(x^k)\| < 1$ for large enough k. Since

$$\lim_{k\to\infty}\left(1 - \frac{\lambda}{\|F(x^k)\|}\right) z + \frac{\lambda}{\|F(x^k)\|}F(x^k) = z + \lambda\bar{y},$$

it follows that $z+\lambda\bar{y} \in \text{cl}(\text{conv}(F(\Omega)))$. So, as we claimed, $\bar{y} \in 0^+\text{cl}(\text{conv}(F(\Omega)))$.

Therefore, by (21), $\bar{y} \in 0^+\text{cl}(\text{conv}(F(\Omega))) \cap S_w$, with $\bar{y} \neq 0$, in contradiction with (18). So condition 2 must hold. Hence, conditions 2 and 3 are equivalent to (18).

Finally we prove the equivalence between (18) and (19). Assume that (18) does not hold, that is to say, there exists u such that

$$0 \neq u \in 0^+\text{cl}(\text{conv}(F(\Omega))) \cap S_w\,.$$

We claim that this implies that (19) does not hold neither. Indeed, for all $\varepsilon > 0$ we have

$$\langle w - \varepsilon u, u\rangle = \langle w, u\rangle - \varepsilon\|u\|^2 \leq -\varepsilon\|u\|^2 < 0\,,$$

where the first inequality follows from the fact that $u \in S_w$ and the last one from $u \neq 0$. Hence, $w - \varepsilon u \notin [0^+\text{cl}(\text{conv}(F(\Omega)))]^*$. Therefore, as $\varepsilon > 0$ can be arbitrarily small, (19) does not hold.

Assume now that (19) does not hold, that is to say, there exists a sequence $\{w^k\}$ such that

$$\lim_{k\to\infty} w^k = w \quad \text{and} \quad w^k \notin [0^+\text{cl}(\text{conv}(F(\Omega)))]^*\,.$$

We claim that this implies that (18) does not hold neither. Indeed, for each k, there exists u^k such that

$$u^k \in 0^+\text{cl}(\text{conv}(F(\Omega))) \quad \text{and} \quad \langle w^k, u^k\rangle < 0\,.$$

As $0^+\text{cl}(\text{conv}(F(\Omega)))$ is a cone, we may assume that $\|u^k\| = 1$. Refining the sequence, if necessary, we may assume that $\{u^k\}$ converges to some $\bar{u}$. Since $\{w^k\}$ converges to w, we have

$$\langle w, \bar{u}\rangle \leq 0\,,$$

or equivalently, $\bar{u} \in S_w$. Moreover, $\|\bar{u}\| = 1$ and, since $0^+\text{cl}(\text{conv}(F(\Omega)))$ is closed, we also have that $\bar{u} \in 0^+\text{cl}(\text{conv}(F(\Omega)))$. Therefore, (18) does not hold and the proof is complete. □

Altogether, the previous results of this section give us the following theorem regarding the proper choice of vectors for the weighting method.

Theorem 2 *Take $w \neq 0$ and suppose that $F(\Omega)$ is closed. If*

$$0^+\text{cl}(\text{conv}(F(\Omega))) \cap S_w = \{0\},$$

or, equivalently, if

$$w \in \text{int}\Big(\,[0^+\text{cl}(\text{conv}(F(\Omega)))]^*\,\Big),$$

then problem

$$\min_{x\in\Omega}\langle w, F(x)\rangle$$

has an optimum.

Proof Suppose that w satisfies the assumptions. Then, by Lemma 3, conditions 2 and 3 of Theorem 1 hold. As $F(\Omega)$ is closed, from that theorem we conclude that the scalar problem $\min_{x\in\Omega}\langle w, F(x)\rangle$ has optimal solutions. □

We are now in position to exhibit a sufficient condition for existence of weakly efficient solutions for the vector-valued problem (P), together with an original criterion for choosing adequate weighting vectors.

Corollary 1 *Suppose that* $F(\Omega)$ *is closed. If*

$$K^* \cap \operatorname{int}\left(\left[0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))\right]^*\right) \neq \emptyset, \tag{22}$$

then the vector-valued problem (P) *has weakly efficient solutions. Moreover, for any* $w \neq 0$ *in the above intersection, the scalar-valued problem*

$$\min_{x\in\Omega}\langle w, F(x)\rangle$$

has optimal solutions, which are weakly efficient for (P).

Proof If $0 \in K^* \cap \operatorname{int}([0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))]^*)$, then the set $[0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))]^*$ coincides with the whole space $\mathbb{R}^m$ and the above intersection is equal to K^*. So, assuming (22), there always exists some $w\neq 0$, $w\in K^*\cap \operatorname{int}([0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))]^*)$. By Theorem 2, the scalar problem $\min_{x\in\Omega}\langle w, F(x)\rangle$ has optimal solutions. As $w \in K^* \setminus \{0\}$, weak optimality of these scalar optima follows from classical results on the weighting method mentioned on Sect. 2 (see the statement made on (7)). □

Some comments are in order. First, condition (22) ensures that

$$\operatorname{int}(K^*) \cap \operatorname{int}\left(\left[0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))\right]^*\right) \neq \emptyset.$$

So, for w in the above intersection, the Ω-minimizers of $\langle w, F(x)\rangle$ will not just be weakly efficient points for (P), but, actually, efficient ones. Indeed, it is a well-known fact that whenever the weighting vector w is in the topological interior of K^*, all optima of $\min_{x\in\Omega}\langle w, F(x)\rangle$ are Pareto optimal points for (P).

Finally, we point out that condition (22) does not require K-boundedness of the image set. The K-boundedness is frequently used for ensuring existence of efficient points. Actually, there are two notions of K-boundedness. According

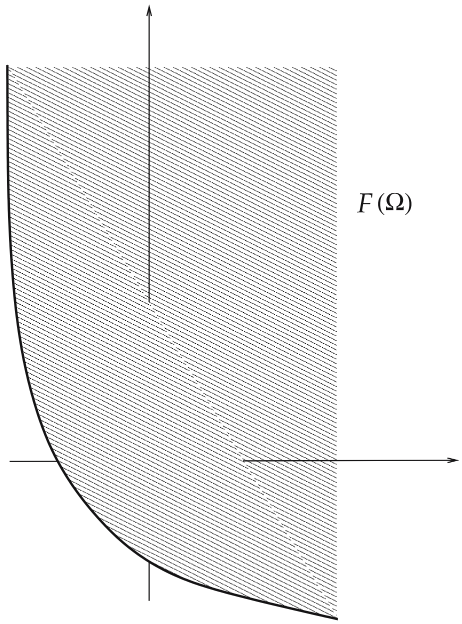

Fig. 4 $F(\Omega)$, the epigraph of the function $t \mapsto -\log(t+1)$, is not K-bounded

to [10, Definition 3.1], in our setting, $C \subset \mathbb{R}^m$ is *K-bounded* if there exist y such that

$$C \subseteq K + \{y\},$$

or, equivalently, $y \preceq z$ for all $z \in C$. On the other hand, according to [14, Definition 3.2.4], a nonempty set $C \subset \mathbb{R}^m$ is *K-bounded* if

$$C^+ \cap (-K) = \{0\},$$

where $C^+ = \left\{x : \exists \quad \{x^k\} \subset C, \quad \{\alpha_k\} \subset \mathbb{R}_{++} \quad \text{such that} \quad \alpha_k x^k \to x\right\}$ (see the definition which follows Theorem 3.1.3 in [14]).

Let us now see an example where condition (22) holds and $F(\Omega)$ is *not* K-bounded (in both senses). Take $K = \mathbb{R}^2_+$, $\Omega = (-1, +\infty) \times [0, +\infty)$ and $F : \Omega \to \mathbb{R}^2$, given by

$$F(x_1, x_2) := (x_1, x_2 - \log(x_1 + 1)).$$

Clearly, $F(\Omega)$ is the epigraph of the real-valued function $(-1, +\infty) \ni t \mapsto -\log(t+1)$ (see Fig. 4). It is easy to check that $F(\Omega)$ is not K-bounded, neither in the sense of [10] nor in the sense of [14]. On the other hand,

$$0^+\text{cl}(\text{conv}(F(\Omega))) = \mathbb{R}^2_+ .$$

Therefore, condition (22) holds. Moreover, any weighting vector $w \in \mathbb{R}^2_{++}$ defines a scalar problem with optimal solutions which are weakly efficient.

Condition (22) is implied by K-boundedness (in the sense of [10, Definition 3.1]), because, if $F(\Omega)$ is K-bounded, then

$$0^+\mathrm{cl}(\mathrm{conv}(F(\Omega))) \subset K\,,$$

and therefore $K^* \subset \left[0^+\mathrm{cl}(\mathrm{conv}(F(\Omega)))\right]^*$. Hence, under the K-boundedness assumption, condition (22) holds.

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