Math. Program., Ser. B 105, 201–213 (2006)

Digital Object Identifier (DOI) 10.1007/s10107-005-0649-5

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# Independent packings in structured graphs*

Received: April 23, 2004 / Accepted: November 30, 2004
Published online: 14 November 2005 – © Springer-Verlag 2005

**Abstract.** Packing a maximum number of disjoint triangles into a given graph $G$ is NP-hard, even for most classes of structured graphs. In contrast, we show that packing a maximum number of independent (that is, disjoint and nonadjacent) triangles is polynomial-time solvable for many classes of structured graphs, including weakly chordal graphs, asteroidal triple-free graphs, polygon-circle graphs, and interval-filament graphs. These classes contain other well-known classes such as chordal graphs, cocomparability graphs, circle graphs, circular-arc graphs, and outerplanar graphs. Our results apply more generally to independent packings by members of any family of connected graphs.

**Key words.** Independent set – Induced matching – Chordal graph – Circular-arc graph – Polygon-circle graph – Cocomparability graph – Interval-filament graph – Intersection graph – Asteroidal triple-free graph – Weakly chordal graph – Chordal bipartite graph – Dissociation set.

## 1. Introduction

Let $\mathcal{H}$ be a fixed set of connected graphs. An *$\mathcal{H}$-packing* of a given graph $G$ is a pairwise node-disjoint set of subgraphs of $G$, each isomorphic to a member of $\mathcal{H}$. An *independent $\mathcal{H}$-packing* of a given graph $G$ is an $\mathcal{H}$-packing of $G$ in which no two subgraphs of the packing are joined by an edge of $G$. When $\mathcal{H}$ is a single graph $H$, we have $H$-packings and independent $H$-packings of $G$. Thus a $K_2$-packing of $G$ is precisely a matching of $G$, and an independent $K_2$-packing of $G$ is precisely an induced matching of $G$. Similarly, an independent $K_1$-packing of $G$ is precisely an independent set of nodes in $G$. The *$\mathcal{H}$-packing problem* asks for the maximum number of nodes covered by an $\mathcal{H}$-packing, while the *independent $\mathcal{H}$-packing problem* asks for the maximum number of graphs contained in an independent $\mathcal{H}$-packing, of the input graph $G$. We can similarly study the alternate problems, of packings with maximum number of graphs, and independent packings with maximum number of covered nodes, but the more natural problems are the ones stated above. In any event, when $\mathcal{H}$ consists of one graph $H$, the two objectives coincide.

It is known [32] that the $H$-packing problem is NP-hard for any connected graph $H$ with at least three nodes. (On the other hand, polynomial-time $\mathcal{H}$-packing algorithms

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* Research of both authors is supported by the Natural Sciences and Engineering Research Council of Canada.

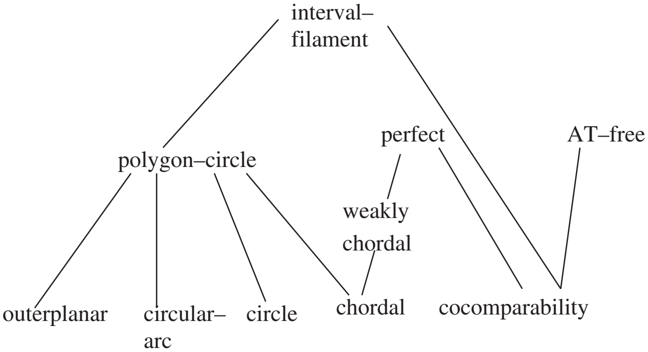
*Mathematics Subject Classification (2000):* 20E28, 20G40, 20C20 are known for nice families $\mathcal{H}$ [10, 27, 31, 33, 45].) The $H$-packing problem remains NP-hard for many graphs $H$, even when restricted to chordal graphs, planar graphs, line-graphs, and total graphs [26].

In this paper we show that independent $\mathcal{H}$-packings can be found by polynomial-time algorithms for any family $\mathcal{H}$ of connected graphs, when the input graphs have nice structure. This applies to a number of different classes of structured graphs; the reason for this seems to be related to the following construction.

Given a graph $G$ and set $\mathcal{H}$ of graphs, we define another graph, $\mathcal{H}(G)$, whose nodes are the subgraphs of $G$ isomorphic to a member of $\mathcal{H}$, and such that two nodes are adjacent exactly when the subgraphs they correspond to intersect or are joined by an edge of $G$. Then independent $\mathcal{H}$-packings in $G$ correspond precisely to independent sets of nodes in $\mathcal{H}(G)$. Often, if $G$ has nice structure, the structure is inherited by the graph $\mathcal{H}(G)$. If $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$, and if the independent set problem is polytnomial-time-solvable in $\mathcal{H}(G)$, then the independent $\mathcal{H}$-packing problem is polynomial-time-solvable in $G$.

In particular, for a family $\mathcal{H}$ of connected graphs, we show that if $G$ is a polygon-circle graph, then so is $\mathcal{H}(G)$, and the same holds for cocomparability graphs, asteroidal triple-free graphs, weakly chordal graphs and interval-filament graphs. It follows that the independent $\mathcal{H}$-packing problem is polynomial-time-solvable in these classes. Interval-filament graphs include cocomparability graphs and polygon-circle graphs, and the latter include circle graphs, circular-arc graphs, chordal graphs, and outerplanar graphs. Definitions of these graph classes are given below. Figure 1 shows inclusion relations between them.

These results have been anticipated, more or less independently, by two earlier developments. First of all, the problem of finding a largest induced matching, i.e., the independent $K_2$-packing problem, has been widely studied [5–9, 11–13, 22, 23, 38, 41, 46]. It is known to be NP-hard for bipartite graphs [5, 50], and for planar graphs [40]; polynomial-time algorithms for various classes of structured graphs were given in [5–7, 9, 22, 23]. The techniques of those papers are closely related to the present paper. Secondly, the problem of maximum sized independent $r$-clique packings, i.e., the independent $K_r$-packing problem, has been solved in the class of chordal graphs in [34–37]. (Note




Upward lines indicate inclusion

**Fig. 1.**

that in (independent) $K_r$-packings, the size $r$ of the cliques is fixed.) In addition to algorithms, independent $K_r$-packings were related to generalizations of split graphs. In fact, it is shown in [34, 36] that, for chordal graphs $G$, the maximum number of independent $K_r$'s in $G$ is equal to the minimum number of cliques that meet all $K_r$'s of $G$. These results also include an efficient algorithm to find a minimum $s$ such that a chordal graph $G$ can be partitioned into $s$ cliques and $r-1$ independent sets. A weighted version of the min-max theorem was given in [37]. While these papers also employ the general technique of the auxiliary graph $\mathcal{H}(G)$, they also offer direct algorithms using the perfect elimination ordering of $G$, resulting in linear time performance.

We remark that the definition of $\mathcal{H}(G)$ can be extended so that we are allowed to specify which subgraphs of $G$ isomorphic to a member of $\mathcal{H}$ are to be included. For instance one may want to only allow induced subgraphs, or some other pre-determined family of subgraphs. (This point of view for ordinary packings has been taken in [10].)

For the independent $\mathcal{H}$-packing problem in $G$, each node of $\mathcal{H}(G)$ has weight 1. Different weightings of $\mathcal{H}(G)$ can be used to solve other problems. For example, if we want to cover as many nodes as possible by an independent $\mathcal{H}$-packing in $G$, we would give the node of $\mathcal{H}(G)$ corresponding to a subgraph $G'$ of $G$ weight equal to the number of nodes of $G'$, and then find a maximum weight independent set of nodes in $\mathcal{H}(G)$. In particular, a *dissociation set* is a set of nodes which induce a subgraph with maximum degree 1. The problem of finding a largest dissociation set is NP-hard for bipartite graphs [51], but polynomial-time-solvable for bipartite graphs without an induced "skew star" [2]. Dissociation sets are precisely independent $\{K_1, K_2\}$-packings, so the problem of finding a largest dissociation set in graph $G$ can be solved by finding a maximum weight independent set of nodes in $\mathcal{H}(G)$, where the weight of nodes corresponding to $K_1$'s in $G$ is 1 and the weight of nodes corresponding to $K_2$'s in $G$ is 2. Our results imply that dissociation set problem can be solved in polynomial time for chordal graphs, weakly chordal graphs, asteroidal triple-free graphs and interval-filament graphs since maximum weight independent sets of nodes can be found in these graphs in polynomial time [4, 16, 18, 29, 42, 48].

We now give a simple proof that packing independent triangles in general graphs is NP-hard. We give a reduction from induced matching, which is known to be NP-hard [5, 40, 50]. In fact, since induced matching is NP-hard for planar graphs [40], and our reduction preserves planarity, it follows that packing independent triangles is NP-hard in planar graphs. Let $G$ be a graph; form a new graph $G'$ by adjoining to $G$ one new node $v_e$ for each edge $e$ of $G$, which is adjacent to both endpoints of $e$. Then $G$ has an induced matching of $k$ edges if and only if $G'$ has an independent packing of $k$ triangles: Each $k$ independent edges of $G$ give rise to $k$ independent triangles of $G'$. Conversely, $k$ independent triangles of $G'$ yields $k$ independent edges of $G$ as follows: If the triangle contains one of the added nodes $v_e$, remove it to get the edge $e$. Otherwise, remove any one node of the triangle to obtain an edge.

Given a family $\mathcal{F}$ of non-empty sets, the *intersection graph* $\mathcal{I}(\mathcal{F})$ of $\mathcal{F}$ has node-set $\mathcal{F}$ and an edge between $u$ and $v$ exactly when $u \cap v \neq \emptyset$. There are many well-studied classes of intersection graphs including the following. *Interval graphs* are the intersection graphs of a set of intervals on a line; *chordal graphs* are the intersection graphs of a set of subtrees of a tree; *circular-arc graphs* are the intersection graphs of a set of arcs of a circle; *circle graphs* are the intersection graphs of a set of chords of a circle;

*polygon-circle graphs* are the intersection graphs of a set of convex polygons inscribed on a circle. The class of polygon-circle graphs includes the classes of chordal graphs [39], circular-arc graphs [39], circle graphs, and outerplanar graphs (see [43]).

*Cocomparability graphs* are the complements of comparability graphs, which are graphs which have a transitive acyclic orientation. Cocomparability graphs can be characterized as the intersection graphs of a set of curves between two parallel lines, $L_1$ and $L_2$, in the plane, each curve having one endpoint on $L_1$ and the other on $L_2$ [24].

Recently, Gavril [18] has introduced a new class of graphs which he calls interval-filament graphs. A graph is an *interval-filament* graph if it is the intersection graph of a set of curves $C$ (called interval-filaments) in the $xy-$plane with left endpoint, $l(C)$, and right endpoint, $r(C)$, lying on the $x-$axis such that $C$ lies in the plane above and within the interval $[l(C), r(C)]$. The curves can be self-intersecting. Interval-filament graphs include polygon-circle graphs and cocomparability graphs [18].

In this paper, we will also look at two other classes of graphs: asteroidal triple-free graphs and weakly chordal graphs, which we now define. Neither class is known to be the intersection graphs of some nice family. An independent set of three nodes is called an *asteroidal triple (AT)* if between each pair in the triple there exists a path that avoids the neighborhood of the third. A graph is *asteroidal triple-free (AT-free)* if it contains no asteroidal triple. Interval graphs [44] and cocomparability graphs [14] are AT-free.

A graph is *weakly chordal* if neither the graph nor the complement of the graph has an induced circuit on five or more nodes. Weakly chordal graphs were introduced by Hayward in [28] and are a well-studied class of perfect graphs. The class of weakly chordal graphs includes the classes of interval graphs, chordal graphs, permutation graphs, and trapezoid graphs, and the complements of these. For more on specially structured classes of graphs, see [3].

Finding a largest independent set of nodes in a graph, and in particular, in a perfect graph, is a well-studied problem in mathematical programming. Chordal, weakly chordal, and cocomparability graphs are perfect, but other types of graphs discussed in this paper - circular-arc graphs, circle graphs, polygon-circle graphs, interval-filament graphs, and AT-free graphs - need not be perfect. There is a polynomial-time algorithm for largest independent set in a perfect graph [25], but this algorithm is not combinatorial. For the classes of graphs studied in this paper, there are combinatorial polynomial-time algorithms for largest independent set, and these algorithms also have better complexity. We sometimes study certain subclasses of a class of graphs, since often better algorithms exist for the subclass.

For the rest of the paper, we shall always assume that $\mathcal{H}$ is a family of connected graphs. We will show that for the classes of interval graphs, chordal graphs, circular-arc graphs, polygon-circle graphs, cocomparability graphs, interval-filament graphs, AT-free graphs, and weakly chordal graphs, if $G$ is in the class, then so is $\mathcal{H}(G)$. Since the independent set problem is polynomial-time-solvable in each of these classes, so is the independent $\mathcal{H}$-packing problem, as long as $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$.

## 2. Intersection graphs

The following idea was used in [5] to prove that if $G$ is chordal, then $K_2(G)$ is chordal.

**Proposition 1.** *Let $G = \mathcal{I}(\mathcal{F})$ be the intersection graph of a family $\mathcal{F}$. Let $\mathcal{H}$ be a set of graphs. Then $\mathcal{H}(G)$ is the intersection graph of the family $\mathcal{F}'$*
*$=\{\cup S$ : $S$ is a subfamily of $\mathcal{F}$ corresponding to an $\mathcal{H}$-subgraph of $G\}$*
*$= \{S_{i_1} \cup \ldots \cup S_{i_r} : \{i_1, \ldots, i_r\}$ is the node-set of a subgraph of $G$ isomorphic to a graph in $\mathcal{H}$, $S_{i_k} \in \mathcal{F}\}$.*
*That is, $\mathcal{H}(G) = \mathcal{I}(\mathcal{F}')$.*

*Proof.* Let $\{S_{i_1}, \ldots, S_{i_m}\} \subseteq \mathcal{F}$ and $\{S_{j_1}, \ldots, S_{j_n}\} \subseteq \mathcal{F}$ correspond to the node-sets of subgraphs $C_i$ and $C_j$ of $G$, each isomorphic to a graph in $\mathcal{H}$.
$(S_{i_1} \cup \ldots \cup S_{i_m}) \cap (S_{j_1} \cup \ldots \cup S_{j_n}) \neq \emptyset$
if and only if there exist $p$ and $q$ such that $S_{i_p} \cap S_{j_q} \neq \emptyset$
if and only if there exist $p$ and $q$ such that $i_p$ is a node of $C_i$, $j_q$ is a node of $C_j$ and either $i_p$ and $j_q$ are the same node or are joined by an edge in $G$
if and only if $C_i$ and $C_j$ have a node in common or are joined by an edge. □

Note that if $S$ is a set of subtrees of a tree $T$ whose intersection graph is connected, then $\cup S$ is also a subtree of $T$. Similarly, if $S$ is a set of intervals on a line $L$ whose intersection graph is connected, then $\cup S$ is also an interval on $L$, and, if $S$ is a set of arcs of a circle $C$ whose intersection graph is connected, then $\cup S$ is also an arc of $C$, or can be replaced by an arc of $C$ without changing the graph (in case $\cup S$ is the entire circle). These observations, together with Proposition 1 give the following corollaries.

**Corollary 1.** *If $G$ is a chordal graph, then $\mathcal{H}(G)$ is also a chordal graph.*

**Corollary 2.** *If $G$ is an interval graph, then $\mathcal{H}(G)$ is also an interval graph.*

**Corollary 3.** *If $G$ is a circular-arc graph, then $\mathcal{H}(G)$ is also a circular-arc graph.*

We now prove that for the classes of polygon-circle graphs, cocomparability graphs, and interval-filament graphs, if $G$ is in the class, then so is $\mathcal{H}(G)$. These results are more difficult than for interval, chordal, and circular-arc graphs, because the union of a set of pairwise intersecting polygons inscribed on a circle is not generally another polygon inscribed on a circle, the union of a set of pairwise intersecting curves between two parallel lines is not generally another such curve, and the union of a set of pairwise intersecting interval-filaments is not generally another interval-filament. Thus we can't apply Proposition 1 directly. Rather, we show that in each case, if $G$ is the intersection graph of a family $\mathcal{F}$, then $\mathcal{H}(G)$ is the intersection graph of a family $\mathcal{F}''$
$= \{M(Z) : Z \in \mathcal{F}'\}$
$= \{M(\cup S)$ : $S$ is a subfamily of $\mathcal{F}$ corresponding to an $\mathcal{H}$-subgraph of $G\}$
$= \{M(S_{i_1} \cup \ldots \cup S_{i_m}) : \{i_1, \ldots, i_m\}$ is the node-set of a subgraph of $G$ isomorphic to a graph in $\mathcal{H}$, $S_{i_k} \in \mathcal{F}\}$,
where $M$ has the properties that for $\cup\{S_{i_1}, \ldots, S_{i_m}\}, \cup\{S_{j_1}, \ldots, S_{j_n}\} \in \mathcal{F}'$,

1. If $(\cup\{S_{i_1}, \ldots, S_{i_m}\}) \cap (\cup\{S_{j_1}, \ldots, S_{j_n}\}) \neq \emptyset$,
   then $M(\cup\{S_{i_1}, \ldots, S_{i_m}\}) \cap M(\cup\{S_{j_1}, \ldots, S_{j_n}\}) \neq \emptyset$.
2. If $M(\cup\{S_{i_1}, \ldots, S_{i_m}\}) \cap M(\cup\{S_{j_1}, \ldots, S_{j_n}\}) \neq \emptyset$,
   then $(\cup\{S_{i_1}, \ldots, S_{i_m}\}) \cap (\cup\{S_{j_1}, \ldots, S_{j_n}\}) \neq \emptyset$.

Then, since $\mathcal{H}(G)$ is the intersection graph of $\mathcal{F}'$, it is also the intersection graph of $\mathcal{F}''$.

**Theorem 1.** *If $G$ is a polygon-circle graph, then $\mathcal{H}(G)$ is also a polygon-circle graph.*

*Proof.* Let $CH(Z)$ denote the convex hull of $Z$. For $M = CH$, property (1) above clearly holds since a set is contained in its convex hull. To prove (2), we first note the following.

*Claim 1.* If $\{S_{i_1}, \ldots, S_{i_m}\}$ is a set of polygons inscribed on a circle $C$, whose intersection graph is connected, and if $P$ is another polygon inscribed on $C$, then

$CH(\cup\{S_{i_1}, \ldots, S_{i_m}\}) \cap P \neq \emptyset$ implies there is a $k$ such that $S_{i_k} \cap P \neq \emptyset$.

Then (2) follows by applying Claim 1 twice, once with $P = CH(\cup\{S_{j_1}, \ldots, S_{j_n}\})$, and then a second time, replacing $CH(\cup\{S_{i_1}, \ldots, S_{i_m}\})$ with $CH(\cup\{S_{j_1}, \ldots, S_{j_n}\})$ and replacing $P$ by the $S_{i_k}$ which intersects $CH(\cup\{S_{j_1}, \ldots, S_{j_n}\})$. For $m = 2$, Claim 1 was proved in [6].

*Proof of Claim.* Assume $CH(\cup\{S_{i_1}, \ldots, S_{i_m}\}) \cap P \neq \emptyset$. Polygons $CH(\cup\{S_{i_1}, \ldots, S_{i_m}\})$ and $P$ are convex, so assuming they intersect, they intersect in a boundary point $B$ of each. Now $B$ is either a vertex of $P$ or lies on an edge $E$ of $P$. In the first case, $B$ lies on the circle $C$, and it follows that $B$ must be a vertex of some $S_{i_k}$. In the second case, edge $E$ is a chord of the circle. If $E$ does not intersect any $S_{i_k}$, then each $S_{i_k}$ lies completely in the left half $L$ of $C - E$ or the right half $R$ of $C - E$. If all of the $S_{i_k}$'s lie completely on one side of the line segment $E$, then so does $CH(\cup\{S_{i_1}, \ldots, S_{i_m}\})$, a contradiction. Suppose polygon $S_{i_p}$ lies in $L$ and $S_{i_q}$ lies in $R$. Since the intersection graph of $\{S_{i_1}, \ldots, S_{i_m}\}$ is connected, there is a path between the nodes corresponding to $S_{i_p}$ and $S_{i_q}$; that is, there exists a sequence of $S_{i_k}$'s starting with $S_{i_p} \in L$ and ending with $S_{i_q} \in R$, such that adjacent members of the sequence intersect. But then one of these must intersect the line segment $E$, a contradiction. □

Note that not all nice subclasses of polygon-circle graphs have the property that for $G$ in the subclass, $K_r(G)$ is also in the subclass. This is not true for the outerplanar graph $C_5$, the circuit on five nodes, for instance, since $K_2(C_5)$ is the complete graph on five nodes, which is not outerplanar. In [6], an example is given of a circle graph $G$ for which $K_2(G)$ is not a circle graph.

**Theorem 2.** *If $G$ is an interval-filament graph, then $\mathcal{H}(G)$ is also an interval-filament graph.*

*Proof.* Let $G$ be the intersection graph of a set $\mathcal{C}$ of interval-filaments on a horizontal line $L$. We now describe the construction of $M(Z)$ satisfying (1) and (2). Without loss of generality, we may assume that no two filaments have a common endpoint. Consider a set of members of $\mathcal{C}$, say $C_1, \ldots, C_r$, whose intersection graph is connected. Consider a plane (multi)graph $H$, whose nodes are the endpoints and intersection points of the $C_i$'s and whose edges are the pieces of the $C_i$'s between these points. Clearly, $H$ is connected. All nodes of $H$ except the endpoints of the $C_i$'s have even degree. Let $l^*$ be the leftmost of the left endpoints of the $C_i$'s and let $r^*$ be the rightmost of the right endpoints of the

$C_i$'s. Pair up the endpoints of the $C_i$'s other than $l^*$ and $r^*$ in their left-to-right order. For each pair, find a chordless path in $H$ between the members of the pair. Create a plane graph $H'$ from $H$ by doubling the edges of each of these paths in $H$, (that is, for each edge of one of the paths, add a new edge parallel to it), keeping each new edge "close" to the original, so that a copy of a piece of $C_i$ intersects only filaments that $C_i$ does. Then $H'$ is a plane graph with exactly two nodes of degree 1, $l^*$ and $r^*$, and the rest of even degree, and thus corresponds to a single (self-intersecting) filament $W$ with endpoints $l^*$ and $r^*$. Let $M(\cup\{S_{i_1}, \ldots, S_{i_r}\}) = W$. Then it is clear that (1) and (2) hold, and thus $\mathcal{H}(G)$ is also an interval-filament graph. □

This idea can also be used to prove:

**Theorem 3.** *If $G$ is a cocomparability graph, then $\mathcal{H}(G)$ is also a cocomparability graph.*

*Proof of Theorem 3.* Let cocomparability graph $G$ be the intersection graph of a set $\mathcal{C}$ of curves between two parallel vertical lines, $L_1$ and $L_2$, in the plane, each curve having one endpoint on $L_1$ and the other on $L_2$. Without loss of generality, we may assume that no two curves have a common endpoint. Consider a set of members of $\mathcal{C}$, say $C_1, \ldots, C_r$, whose intersection graph is connected. Consider a plane (multi)graph $H$, whose nodes are the endpoints and intersection points of the $C_i$'s and whose edges are the pieces of the $C_i$'s between these points. All nodes of $H$ except the endpoints of the $C_i$'s have even degree. Let $l^*$ be one of the left endpoints of the $C_i$'s, say the topmost one, and let $r^*$ be one of the right endpoints of the $C_i$'s, say the topmost one. Pair up the endpoints of the $C_i$'s other than $l^*$ and $r^*$ in their top-to-bottom order. For each pair, find a chordless path in $H$ between the members of the pair. Create a plane graph $H'$ from $H$ by doubling the edges of each path in $H$, keeping each new edge "close" to the original, so that a copy of a piece of $C_i$ intersects only curves that $C_i$ does. Then $H'$ is a plane graph with exactly two nodes of degree 1, $l^*$ and $r^*$, and the rest of even degree, and thus corresponds to a single (self-intersecting) curve $W$ with one endpoint ($l^*$) on $L_1$ and the other ($r^*$) on $L_2$. Let $M(\cup\{S_{i_1}, \ldots, S_{i_r}\}) = W$. Then it is clear that (1) and (2) hold, and thus $\mathcal{H}(G)$ is also a cocomparability graph. □

Polynomial-time algorithms have been given for finding a largest independent set of nodes in chordal graphs [16], in circular-arc graphs [17, 21], and in interval-filament graphs [18], and one is well-known for cocomparability graphs (find a transitive acyclic orientation of the complement [19, 20] and then find a largest clique). By Corollary 1, Corollary 3, Theorem 2 and Theorem 3, these provide polynomial-time algorithms for finding a largest independent $\mathcal{H}$-packing in these classes, as long $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$. This is for instance the case for the independent $K_r$-packing problem.

Gavril's algorithm for finding a largest independent set of nodes in an interval-filament graph requires as input the interval-filament representation. Where $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$, if we are given the interval-filament representation of $G$, an interval-filament representation of $\mathcal{H}(G)$ can be constructed in polynomial time as described in the proof of Theorem 2. Then Gavril's algorithm can be applied to find a largest independent set of nodes in $\mathcal{H}(G)$, giving a largest independent

$\mathcal{H}$-packing in $G$ assuming every graph in $\mathcal{H}$ is connected. In particular, this works for the independent $K_r$-packing problem.

So far, the only polynomial-time algorithm known for the independent set problem in polygon-circle graphs is Gavril's algorithm for the more general class of interval-filament graphs. For the other classes of graphs mentioned, namely, chordal graphs, cocomparability graphs, and circular-arc graphs, polynomial-time algorithms exist for the independent set problem, and thus for the independent $K_r$-packing problem and certain independent $\mathcal{H}$-packing problems, which do not require any representation but simply the adjacency information. (However, we note that more efficient algorithms can sometimes be obtained by directly describing a representation of $\mathcal{H}(G)$ from that for $G$. For instance, in [35], this is done for independent $K_r$-packings of chordal graphs, represented by simplicial orderings.)

An NC-algorithm is one which uses polynomially many parallel processors and whose running time is polynomial in the logarithm of the length of the input. For fixed $r$, it is straightforward to design an NC algorithm to find $K_r(G)$ for graph $G$. Thus the NC algorithm in [47] for finding a largest independent set of nodes in chordal graphs provides an NC algorithm for finding a maximum independent $K_r$-packing in chordal graphs. Similarly, NC algorithms for maximum independent set of nodes in circular-arc graphs [1, 49] provide NC algorithms for maximum independent $K_r$-packing in this class.

## 3. Asteroidal triple-free and weakly chordal graphs

An independent set of three nodes is called an *asteroidal triple (AT)* if between each pair in the triple there exists a path (which can be assumed to be chordless) that avoids the neighborhood of the third. A graph is *asteroidal triple-free (AT-free)* if it contains no asteroidal triple. AT-free graphs are not yet known to be the intersection graphs of some nice family, however they contain several classes of intersection graphs including interval graphs [44] and cocomparability graphs [14]. Chordal graphs may have asteroidal triples.

**Theorem 4.** *If $G$ is AT-free, then $\mathcal{H}(G)$ is also AT-free.*

*Proof.* We will prove the contrapositive. Where $C$ is a subgraph of $G$ isomorphic to a graph in $\mathcal{H}$, let $v(C)$ denote the corresponding node of $\mathcal{H}(G)$. Suppose $\{v(X), v(Y), v(Z)\}$ is an AT in $\mathcal{H}(G)$. Let $v(X) = v(C_1), v(C_2), \ldots v(C_k) = v(Y)$ be the nodes of a chordless path $P$ in $\mathcal{H}(G)$, which avoids the neighbourhood of $v(Z)$. Since $v(C_i)v(C_{i+1})$ is an edge of $\mathcal{H}(G)$, $C_i$ and $C_{i+1}$ either contain a common node of $G$ or are joined by an edge of $G$. If three of the $C_i$'s contained a common node of $G$, then the corresponding $v(C_i)$'s would induce a triangle in $\mathcal{H}(G)$, so $P$ would not be a chordless path in $\mathcal{H}(G)$. Also, $C_i$ can only intersect $C_{i-1}$ and $C_{i+1}$, since if $C_i$ intersects $C_j$, where $j \notin \{i-1, i+1\}$, then $P$ would have a chord between $v(C_i)$ and $v(C_j)$. Where $C_q, C_{q+1}, \ldots, C_r$ is a maximal sequence of $C_i$'s where each $C_i$ intersects the one(s) next to it in the sequence, we'll call $C_q \cup C_{q+1} \cup \ldots \cup C_r$ an *$\mathcal{H}$-string*. Where $C_j$ does not intersect any other $C_i$'s, we'll call $C_j$ an *$\mathcal{H}$-graph*. It follows that $\{C_i : 1 \leq i \leq k\}$ is a set of pairwise-disjoint $\mathcal{H}$-graphs and $\mathcal{H}$-strings. Let's call these $\mathcal{H}$-graphs and $\mathcal{H}$-strings $D_1, D_2, \dots, D_m$, where $D_1$ is either $C_1$ or the $\mathcal{H}$-string (of $P$) containing $C_1$, and the others follow in order as in $P$.

The edge of $P$ between the nodes of $\mathcal{H}(G)$ representing $D_j$ and $D_{j+1}$ corresponds to a set of edges in $G$; choose one such edge of $G$ for each $j$ and call these the *suppressed edges.* (If $m = 1$, the set of suppressed edges is empty.) Note that each suppressed edge has one end in some $C_i$ and the other end in $C_{i+1}$. Let $x'$ be any node of $X$ and let $y'$ be any node of $Y$. Extend the suppressed edges to a $x'y'$-path in $G$ by choosing a path from each $D_j$. These paths exist because each $C_i$ is isomorphic to a graph in $\mathcal{H}$, and thus is connected, and thus so is $D_j$. Call the $x'y'$-path, $P'$.

Note that every node of $P'$ is in some $C_i$.

Let $z'$ be any node of $Z$. We claim that $\{x', y', z'\}$ is an AT in $G$. Since $\{v(X), v(Y), v(Z)\}$ is an independent set of nodes in $\mathcal{H}(G)$, $\{x', y', z'\}$ is an independent set of nodes in $G$. Path $P'$ does not contain any neighbours of $z'$ in $G$ because if node $u$ of $P'$ is a neighbour of $z'$ in $G$, and $u$ is in $C_i$, then $v(C_i)$ and $v(Z)$ are joined in $\mathcal{H}(G)$, contradicting the choice of $P$. It follows that $\{x', y', z'\}$ is an AT in $G$. □

In [4] and [42], polynomial-time algorithms are given for finding a largest independent set of nodes in AT-free graphs. Thus these algorithms provide polynomial-time algorithms for finding a largest independent $\mathcal{H}$-packing, as long as $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$.

A *hole* in a graph $G$ is an induced circuit on four or more nodes. An *antihole* is the complement of a hole. A graph is *weakly chordal* if it has no hole or antihole on five or more nodes. Interval graphs, chordal graphs, permutation graphs, and trapezoid graphs, and the complements of these are all weakly chordal.

We will show that if $G$ is weakly chordal, then $\mathcal{H}(G)$ is also weakly chordal. Polynomial-time algorithms for finding a largest independent set of nodes in weakly chordal graphs are known [29, 48, 30]. This then provides a polynomial-time algorithm for the largest independent $K_r$-packing problem in weakly chordal graphs, and for the largest independent $\mathcal{H}$-packing problem in weakly chordal graphs in cases where $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$.

A graph is *chordal bipartite* if it is both weakly chordal and bipartite; equivalently, a bipartite graph is chordal bipartite if it has no holes on six or more nodes. Theorem 5 below implies that the largest independent $\mathcal{H}$-packing problem can be solved in polynomial time for chordal bipartite graphs in cases where $\mathcal{H}(G)$ can be constructed in time polynomial in the size of $G$, whereas the largest independent $\mathcal{H}$-packing problem is NP-hard for general bipartite graphs since the induced matching problem is [5, 50, 46].

**Theorem 5.** *If $G$ is a weakly chordal graph, then $\mathcal{H}(G)$ is also a weakly chordal graph.*

Our proof extends the proof in [7]. We first prove two lemmas.

**Lemma 1.** *If $G$ has no holes on at least $k$ nodes, where $k \geq 4$, then $\mathcal{H}(G)$ has no holes on at least $k$ nodes.*

*Proof.* Where $C$ is a subgraph of $G$ isomorphic to a graph in $\mathcal{H}$, let $v(C)$ denote the corresponding node of $\mathcal{H}(G)$. Let $P = v(C_1), v(C_2), \dots, v(C_k)$ be the nodes of a hole in $\mathcal{H}(G)$ where $k \geq 4$. We will show that $G$ has a hole on at least $k$ nodes. In what follows, arithmetic is either mod $k$ or mod $m$, as appropriate.

Since $v(C_i)v(C_{i+1})$ is an edge of $\mathcal{H}(G)$, $C_i$ and $C_{i+1}$ either contain a common node of $G$ or are joined by an edge of $G$. If three of the $C_i$'s contained a common node of $G$, then the corresponding $v(C_i)$'s would induce a triangle in $\mathcal{H}(G)$, so $P$ would not be chordless. Also, $C_i$ can only intersect $C_{i-1}$ and $C_{i+1}$, since if $C_i$ intersects $C_j$, where $j \notin \{i-1, i+1\}$, then $P$ would have a chord between $v(C_i)$ and $v(C_j)$. Where $C_q, C_{q+1}, \ldots, C_r$ is a maximal sequence of $C_i$'s where each $C_i$ intersects the one(s) next to it in the sequence, we'll call $C_q \cup C_{q+1} \cup \ldots \cup C_r$ an *$\mathcal{H}$-string.* Where $C_j$ does not intersect any other $C_i$'s, we'll call $C_j$ an *$\mathcal{H}$-graph.* It follows that $\{C_i : 1 \leq i \leq k\}$ is a set of pairwise-disjoint $\mathcal{H}$-graphs and $\mathcal{H}$-strings. Let's call these $\mathcal{H}$-graphs and $\mathcal{H}$-strings $D_1, D_2, \ldots, D_m$, where $D_1$ is either $C_1$ or the $\mathcal{H}$-string containing $C_1$, and the others follow in order as in $P$. We will first assume that $m > 1$. Then, without loss of generality, assume that $C_1$ is either an $\mathcal{H}$-graph or the beginning of an $\mathcal{H}$-string.

The edge of $P$ between the nodes of $\mathcal{H}(G)$ representing $D_j$ and $D_{j+1}$ corresponds to a set of edges in $G$; choose one such edge of $G$ for each $j$ and call these the *suppressed edges.* Note that each suppressed edge has one end in some $C_i$ and the other end in $C_{i+1}$; in fact, one end is in $C_i - C_{i-1}$ and the other end is in $C_{i+1} - C_{i+2}$. Extend the suppressed edges to a circuit in $G$ by choosing a chordless path from each $D_j$. These paths exist because each $C_i$ is isomorphic to a graph in $\mathcal{H}$, and thus is connected, and thus so is $D_j$. Call the circuit $P'$.

Note that every node of $P'$ is in some $C_i$. In fact, the nodes of $P'$ are nodes of $C_1$, followed by nodes of $C_2$, and so on to $C_k$. We now show that circuit $P'$ contains at least one node from each $C_i$. Since $C_i$ and $C_{i+1}$ either contain a common node of $G$ or are joined by an edge of $G$, and since $C_i$ and $C_j$, $j \notin \{i-1, i+1\}$, neither contain a common node of $G$ nor are joined by an edge of $G$, and since suppressed edges have one end in $C_i - C_{i-1}$ and the other end is in $C_{i+1} - C_{i+2}$ for some $i$, it follows that circuit $P'$ intersects $\mathcal{H}$-string $D_j = C_q \cup C_{q+1} \cup \ldots \cup C_r$ at a node of $C_q - C_{q+1}$, then a node of $C_i$, $i \in q+1, \ldots, r-1$, and finally a node of $C_r - C_{r-1}$. It follows that $P'$ has at least as many nodes as $P$.

Any chord of $P'$ must have its ends in two consecutive $C_i$'s; in fact, in $C_i - C_{i-1}$ and $C_{i+1} - C_{i+2}$, for some $i$. A hole in $G$ with at least as many nodes as $P$ has can be found as follows.

Starting with $i = 1$, for each $i$, if there is a chord of $P'$ with ends in $C_i$ and $C_{i+1}$, choose a longest such chord, say $e_i$ (that is, a chord such that the part of $P'$ between its ends has the largest number of edges), and replace the part of $P'$ between the ends of $e_i$ by $e_i$. The circuit obtained has no chord between nodes of $C_i$ and $C_{i+1}$, however it still has a node from each $C_i$, and thus has at least as many nodes as $P$.

We now look at the case when $m = 1$; that is, when the union of the $C_i$'s is a single "closed" $\mathcal{H}$-string. Choose a node $u$ in $C_1 \cap C_2$ and a node $v$ in $C_1 \cap C_k$. Note that $u$ and $v$ are distinct. Find a chordless path $P_1$ between $u$ and $v$ in $C_1$ and a chordless path $P_2$ between $u$ and $v$ in $C_2 \cup C_3 \cup \ldots \cup C_k$. Considering $P_2$ to be a path from $u$ to $v$, let $u'$ be the last node of $P_2$ which is also a node of $P_1$. Considering $P_2$ to be a path from $v$ to $u$, let $v'$ be the last node of $P_2$ which is also a node of $P_1$. The union of the parts of the paths between $u'$ and $v'$ is a circuit $Q$. Any chord of $Q$ must have one end in $C_1$ and the other in $C_2$ or $C_k$. As in the $m > 1$ case, by first choosing a longest such chord between $C_k$ and $C_1$ and replacing the part of $Q$ between the ends of the chord by the chord, and then doing the same with a longest remaining chord between $C_1$ and $C_2$ , we get a hole $Q'$. By an argument similar to that in the $m > 1$ case, it follows that $Q'$ has a node from each $C_i$. □

**Lemma 2.** *If $G$ is weakly chordal, then $\mathcal{H}(G)$ does not contain an antihole on five or more nodes.*

Our proof relies on a known structural property of weakly chordal graphs. We need to introduce some notation and definitions. Let $S$ be a set of nodes in graph $G$. We use $G[S]$ to denote the subgraph induced in $G$ by $S$. We use $\overline{G}[S]$ to denote the subgraph induced by $S$ in $\overline{G}$, the complement of $G$. The *neighbourhood, N(S),* is the set of nodes of $G - S$ that are adjacent to at least one node of $S$. We say $S$ *separates* sets $X$ and $Y$ of nodes if every path from a node in $X$ to a node in $Y$ contains some node in $S$. We say $S$ *minimally separates* $X$ and $Y$ if no proper subset of $S$ separates $X$ and $Y$. The following theorem was proved by Hayward [28]:

**Theorem 6 ([28]).** *Let $S$, $X$, and $Y$ be sets of nodes in a weakly chordal graph $G$ such that $X$ and $Y$ induce connected components of $G - S$, $S$ minimally separates $X$ and $Y$, and $\overline{G}[S]$ is connected. Then $X$ and $Y$ each contain a node that is adjacent to every node of $S$.*

We prove Lemma 2 by contradiction. We show that if graph $G$ is weakly chordal, but the graph $\mathcal{H}(G)$ contains an antihole on five or more nodes, we can find a induced subgraph $F$ of $G$ such that the weakly chordal graph $F$ has sets $X$, $Y$, and $S$ that satisfy the hypotheses of Theorem 6, but not the conclusion of Theorem 6.

*Proof of Lemma 2.* Suppose $G$ is weakly chordal, but $\mathcal{H}(G)$ contains the antihole $v(C_1), v(C_2), v(C_3), v(C_4), \ldots, v(C_k)$ as an induced subgraph, where $k \geq 5$. The order of the nodes listed corresponds to the cyclic order along the hole in the complement of $\mathcal{H}(G)$.

Each $C_i$ is a subgraph of $G$. We refer to the nodes of $C_i$ in $G$ as type $i$ nodes. Note that it is possible for a node of $G$ to be a type $i$ node as well as a type $j$ node, where $i \neq j$. Note however, that

(*) no type $i$ node is equal to or adjacent to a type $i + 1$ node, where addition is mod $k$. Let $F$ be the subgraph induced in $G$ by the union of type 1 through type $k$ nodes. Subgraph $F$ must be weakly chordal as $G$ is weakly chordal. In what follows, any adjacency mentioned is with respect to the graph $F$.

Let $S' = N(C_1)$. Note that the type of every node in $N(C_1)$ is among 3 through $k - 1$, and further, $S'$ has at least one node of type $i$, for each $i$, $3 \leq i \leq k - 1$. Also, since there is no edge between $C_1$ and $C_2$ in $F$, it must be the case that $S'$ separates $C_1$ from $C_2$ in $F$.

Let $Y$ be the component of $F - S'$ that contains $C_2$. Note that, since there are no edges of $F$ between $C_1$ and $C_2$ or between $C_1$ and $C_k$, and since the union of type 2 and type $k$ nodes induces a connected subgraph in $F$ (since $C_2$ is adjacent to $C_k$ in $\mathcal{H}(G)$, and both $C_2$ and $C_k$ are connected), all the type 2 and type $k$ nodes belong to $Y$.

Now, let $S = N(Y) \cap S'$ and let $X$ be the component of $F - S$ that contains $C_1$. Note that $S$ minimally separates $X$ from $Y$ in $F$. As $S \subseteq S'$, the type of any node in $S$

is among 3 through $k-1$. Trivially, all the type 1 nodes belong to $X$ and all the type 2 and type $k$ nodes belong to $Y$.

We now claim that $S$ has at least one node of type $i$, for each $i$, $3 \leq i \leq k-1$. First assume that $S$ had no type 3 nodes. Then, as type 1, type 3, and type $k$ nodes induce a connected subgraph in $F$, there would exist a path in $F - S$ from a type 1 node in $X$ to a type $k$ node in $Y$ contradicting the assumption that $S$ separates $X$ from $Y$ in $F$. Now suppose none of the type $i$ nodes are in $S$, for some $i$ between 4 and $k-1$ inclusive. Then a similar argument applied to type 1, type $i$, and type 2 nodes (in place of type 1, type 3, and type $k$ nodes respectively) shows that $X$ and $Y$ are not separated in $F$ by $S$, a contradiction. Finally, by *, in $\overline{F}$, every type $i$ node is adjacent to every type $i+1$ node ($3 \leq i \leq k-2$). It follows that $\overline{F}[S]$ is connected. Therefore, $F$ is a weakly chordal graph in which the sets $X$, $Y$, and $S$ satisfy the conditions of Theorem 6.

We will now show that no node in $Y$ can be adjacent to all the nodes in $S$, contradicting Theorem 6. We observe that the type of any node in $Y$ must be among 2 through $k$. By *, a type $i$ node in $Y$ can not be adjacent to a type $i+1$ node in $S$, $2 \leq i \leq k-2$. Also by *, a type $k-1$ node in $Y$ can not be adjacent to a type $k-2$ node in $S$. Finally, again by *, a type $k$ node in $Y$ can not be adjacent to a type $k-1$ node in $S$. □

*Proof of Theorem 5.* Suppose $G$ is weakly chordal. Then $G$ contains neither a hole on at least five nodes nor an antihole on at least five nodes. It then follows from Lemma 1 and Lemma 2 that $\mathcal{H}(G)$ is also weakly chordal. □

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