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The strong conical hull intersection property for convex programming

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Abstract. The strong conical hull intersection property (CHIP) is a geometric property of a collection of finitely many closed convex intersecting sets. This basic property, which was introduced by Deutsch et al. in 1997, is one of the central ingredients in the study of constrained interpolation and best approximation. In this paper we establish that the strong CHIP of intersecting sets of constraints is the key characterizing property for optimality and strong duality of convex programming problems. We first show that a sharpened strong CHIP is necessary and sufficient for a complete Lagrange multiplier characterization of optimality for the convex programming model problem

$$(P_f) \quad \min\{f(x)|x \in C, -g(x) \in S\},$$

where C is a closed convex subset of a Banach space X, S is a closed convex cone which does not necessarily have non-empty interior, Y is a Banach space, $f : X \to \mathbb{R}$ is a continuous convex function and $g : X \to Y$ is a continuous S-convex function. We also show that the strong CHIP completely characterizes the strong duality for partially finite convex programs, where Y is finite dimensional and $g(x) = -Ax + b$ and S is a polyhedral convex cone. Global sufficient conditions which are strictly weaker than the Slater type conditions are given for the strong CHIP and for the sharpened strong CHIP.

Key words. Strong conical hull intersection property – global constraint qualification – strong duality – optimality conditions – constrained approximation

1. Introduction

In this paper, we consider the cone-convex programming model problem

$$(P_f) \quad \min\{f(x)|x \in C, -g(x) \in S\},$$

where C is a closed convex subset of a Banach space X, S is a closed convex cone which does not necessarily have non-empty interior, Y is a Banach space and $g : X \to Y$ is a continuous S-convex function and $f : X \to \mathbb{R}$ is a continuous convex function. This model covers a much wider spectrum of applications that can not be captured by the standard inequality constrained convex programming problems. It includes, in particular, semidefinite programming problems [3, 23] and many classes of constrained interpolation and approximation problems [6, 9].

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The strong conical hull intersection property (CHIP), which played a central role in constrained interpolation and approximation [1, 4, 6, 8, 9, 21], turns out to be the key geometric characterizing property for optimality and strong duality in cone-convex programming. We show that a sharpened strong conical hull intersection property is necessary and sufficient for a complete Lagrange multiplier characterization of optimality for (P_f) in the sense that the sharpened strong CHIP holds if and only if for each continuous convex function $f : X \to \mathbb{R}$ and for each minimizer of (P_f), there exists a Lagrange multiplier satisfying the Kuhn-Tucker type subgradient optimality condition. Furthermore, we establish that the strong CHIP completely characterizes the strong duality of partially finite convex programming problems, where Y is finite dimensional and $g(x) = -Ax + b$ and S is a polyhedral convex cone. As a consequence, we obtain a complete characterization of the cone-constrained best approximation in Hilbert spaces in terms of the strong CHIP. We present new global sufficient conditions, which are strictly weaker than Slater's condition, for the strong CHIP and for the sharpened strong CHIP.

The Lagrange multiplier characterizations of optimality and strong duality characterizations are central to most studies of convex programming. Over the years, a great deal of attention has been focused on finding conditions, known as constraint qualifications, which ensure the existence of Lagrange multipliers, characterizing a minimizer, and which gives strong duality. Such studies of constraint qualifications are abundance in the literature, e.g. see [13, 19] and other references therein. On the other hand, research on constraint qualifications, which are, in some sense, also necessary for the existence of Lagrange multipliers for convex programming problems, has so far been limited mainly to problems with inequality constraints [13, 25]. More recently, the strong CHIP has been shown to be the characterizing property for a strong duality relationship between two special pair of optimization problems in [8, 6]. Our results provide the weakest constraint qualifications for optimality and for strong duality of broad classes of convex programming problems.

The layout of the paper is as follows. In section 2 we collect definitions, notations and preliminary results that will be used later in the paper. In section 3, we establish necessary and sufficient conditions in terms of strong CHIP for the existence of Lagrange multipliers, characterizing a minimizer of (P_f). We also give global sufficient conditions for the strong CHIP and the sharpened strong CHIP. In section 4, we present a necessary and sufficient condition for the strong duality of partially finite convex programming problems, and then obtain a simple characterization of the cone-constrained best approximation in Hilbert spaces.

2. Preliminaries

We begin this section by fixing the notations, definitions and preliminaries that will be used later in the paper. Let X and Y be Banach spaces. The continuous dual space of X will be denoted by X^*. For a set $W \subset X^*$, the weak*closure of W will be denoted by $\operatorname{cl} W$. For a subset A of X, we shall denote the interior of A by $\operatorname{int} A$, and the indicator function δ_A is defined by $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ if $x \notin A$. The support function σ_A is defined by

$$\sigma_A(u) = \sup_{x \in A} u(x) \quad (u \in X^*).$$

Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$. Then, the *conjugate function* of f, $f^* : X' \to \mathbb{R} \cup \{-\infty, +\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \mathrm{X}\}.$$

The function f is said to be *proper* if it does not take on the value $-\infty$ and $dom\ f \neq \emptyset$, where the *domain* of f, $dom\ f$, is given by $dom\ f = \{x \in X \mid f(x) < +\infty\}$. The *epigraph of* f, $Epif$, is defined by

$$Epif = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}.$$

For details see [26]. The function f is lower semicontinuous if and only if $Epif$ is a closed subset of $X \times \mathbb{R}$. The *lower semicontinuous regularization*, $\mathrm{cl}\, f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, is the function whose epigraph is equal to the closure of the epigraph of f in $X \times \mathbb{R}$:

$$Epi(\mathrm{cl}\, f) := \mathrm{cl}(Epif).$$

For the functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$, the *infimal convolution* of f with g, denoted by $f \oplus g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, is defined by

$$f \oplus g(x) := \inf_{x_1 + x_2 = x} \{f(x_1) + g(x_2)\}.$$

The infimal convolution of f with g is said to be *exact* provided the infimum above is achieved. Note that if $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous convex functions, and if $\mathrm{cl}(f^* \oplus g^*)$ is proper, then $(f + g)^* = \mathrm{cl}(f^* \oplus g^*)$. Moreover, if $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous convex functions with $dom\ f \cap dom\ g \neq \emptyset$ then $cl(f^* \oplus g^*)$ is proper. These facts can be found in, or can easily be derived from, [24, 26].

For $\varepsilon > 0$, the ε-subdifferential of a proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $a \in dom\ f$ is defined as the non-empty weak* closed convex set

$$\partial_\varepsilon f(a) = \{v \in X^* \mid f(x) - f(a) \geq v(x - a) - \varepsilon,\ \forall x \in \mathrm{dom}\ f\}.$$

The elements of $\partial_\varepsilon f(a)$ are called ε-subgradients of f at a. For $\varepsilon = 0$, $\partial_0 f(a)$ is the usual subdifferential of f at a and is often denoted by $\partial f(a)$. See [13, 14] for a discussion of this set and its properties. Note that $\bigcap_{\varepsilon > 0} \partial_\varepsilon f(a) = \partial f(a)$. It follows from the definitions of $Epif^*$and the ε-subdifferential of f that if $a \in domf$, then

$$Epif^* = \bigcup_{\epsilon \geq 0} \{(v, v(a) + \epsilon - f(a)) \mid v \in \partial_\epsilon f(a)\}.$$

For details see [16].

It is also worth noting that, for the proper lower semicontinuous convex functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ with $dom\ f \cap dom\ g \neq \emptyset$, if $(f + g)^* = f^* \oplus g^*$ and the infimal convolution is exact, then $\partial(f + g)(x) = \partial f(x) + \partial g(x)$, for each $x \in dom\ f \cap dom\ g$.

For the details, see [12]. If $Epif^* + Epig^*$ is weak*closed then $(f+g)^* = f^* \oplus g^*$ and the infimal convolution is exact. For details see [5].

For convenience, we denote the composite mapping $\lambda \circ g$ by λg, where $\lambda \in Y^*$ and $g : X \longrightarrow Y$ is a function. For a non-empty subset A of X, define the polar cone of A by

$$A^o := \{v \in X^* : v(w) \leq 0 \quad \forall\, w \in A\},$$

and the dual cone of A by $A^+ := -A^o$. Also, A^o is called the normal cone of A at 0 whenever $0 \in A$. For the non-empty subset A of X, the conical hull of A is denoted by $cone(A)$. A function $g : X \longrightarrow Y$ is called S-convex if, for each $x, y \in X$ and $0 \leq \alpha \leq 1$,

$$\alpha g(x) + (1-\alpha)g(y) - g(\alpha x + (1-\alpha)y) \in S,$$

where S is a closed convex cone in Y. Let C be a non-empty closed convex subset of X and let

$$g^{-1}(-S) := \{x \in X : -g(x) \in S\}, \tag{2.1}$$

where $g : X \longrightarrow Y$ is a continuous S-convex function. It is easy to check that $g^{-1}(-S)$ is a closed convex subset of X. Let $K := C \cap g^{-1}(-S)$. If K is non-empty, then we easily obtain that

$$Epi\sigma_K = cl(Epi\sigma_{g^{-1}(-S)} + Epi\sigma_C). \tag{2.2}$$

Moreover, if $g^{-1}(-S)$ is non-empty then

$$Epi\sigma_{g^{-1}(-S)} = cl(\cup_{\lambda \in S^+} Epi(\lambda g)^*). \tag{2.3}$$

A proof of the result is given in [4, 16].

3. Strong CHIP and Lagrange Multipliers

In this section we first establish that the strong CHIP of $\{C, g^{-1}(-S)\}$ is necessary and sufficient for an asymptotic Lagrange multiplier conditions, characterizing a minimizer. We then show that a sharpened strong CHIP is necessary and sufficient for the standarad Lagrange multiplier conditions [13, 26], characterizing a minimizer. We begin by recalling the notion of strong CHIP (see, e.g. [1, 4, 6, 8, 9]).

Definition 3.1. *Let C_1, and C_2 be two closed convex subsets in X and let $x \in C_1 \cap C_2$. Then $\{C_1, C_2\}$ is said to have the strong CHIP at x, if*

$$(C_1 \cap C_2 - x)^o = (C_1 - x)^o + (C_2 - x)^o.$$

The pair $\{C_1, C_2\}$ is said to have the strong CHIP if it has the strong CHIP at each $x \in C_1 \cap C_2$.

Note that if $C_1 \cap C_2 \neq \emptyset$, then we always have

$$(C_1 - x)^o + (C_2 - x)^o \subset (C_1 \cap C_2 - x)^o. \tag{3.1}$$

If $(\delta_{C_1} + \delta_{C_2})^* = \delta^*_{C_1} \oplus \delta^*_{C_2}$ and the infimal convolution is exact then the pair $\{C_1, C_2\}$ has the strong CHIP at each $x \in C_1 \cap C_2$. In particular, if $Epi\sigma_{C_1} + Epi\sigma_{C_2}$ is weak* -closed then the pair $\{C_1, C_2\}$ has the strong CHIP. For details see [4, 5]. Now, for each $x \in X$,

$$Ng(x) := \left\{u \in X^* | (u, u(x)) \in \mathrm{cl}\,(\cup_{\lambda \in S^+} Epi(\lambda g)^*)\right\}. \tag{3.2}$$

Clearly, $Ng(x)$ is a weak*closed convex cone in X^* as $\mathrm{cl}(\cup_{\lambda \in S^+} Epi(\lambda g)^*)$ is a weak* closed convex cone. Let us first deduce from the definitions that $Ng(x)$ is, in fact, the normal cone of $g^{-1}(-S)$ at x. Recall that the dual space X^* is equipped with its weak* topology, and that "lim" means limit in the appropriate topology.

Proposition 3.1. *For each $x \in X$, $(g^{-1}(-S) - x)^o = Ng(x)$. Moreover, if $x \in g^{-1}(-S)$, then $u \in Ng(x)$ if and only if there exist nets $\{\epsilon_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^+$ and $\{u_\alpha\} \subset X^*$ with $u_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ such that* $\lim_\alpha u_\alpha = u$, $\lim_\alpha (\lambda_\alpha g)(x) = 0$ *and* $\lim_\alpha \epsilon_\alpha = 0$.

Proof. The point $u \in (g^{-1}(-S) - x)^o$ if and only if $\sigma_{g^{-1}(-S)}(u) \leq u(x)$, which, in turn, is equivalent to $(u, u(x)) \in Epi\sigma_{g^{-1}(-S)}$. Since

$$Epi\sigma_{g^{-1}(-S)} = \mathrm{cl}\,(\cup_{\lambda \in S^+} Epi(\lambda g)^*),$$

it follows that $u \in (g^{-1}(-S) - x)^o$ if and only if $(u, u(x)) \in \mathrm{cl}\,(\cup_{\lambda \in S^+} Epi(\lambda g)^*)$, which, by definition, is equivalent to the condition that $u \in Ng(x)$.

Let $x \in g^{-1}(-S)$. Now, by the definition $u \in Ng(x)$ if and only if

$$(u, u(x)) \in \mathrm{cl}\,(\cup_{\lambda \in S^+} \cup_{\varepsilon \geq 0} \{(v, \varepsilon + v(x) - (\lambda g)(x)) : v \in \partial_\varepsilon(\lambda g)(x)\}),$$

which means that there exist nets $\{\epsilon_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^+$ and $\{u_\alpha\} \subset X^*$ with $u_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ such that $\lim_\alpha u_\alpha = u$ and $\lim_\alpha \epsilon_\alpha + u_\alpha(x) - (\lambda_\alpha g)(x) = u(x)$. Thus, $u \in Ng(x)$ if and only if there exist nets $\{\epsilon_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^+$ and $\{u_\alpha\} \subset X^*$ with $u_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$, such that $\lim_\alpha u_\alpha = u$, $\lim_\alpha (\lambda_\alpha g)(x) = 0$ and $\lim_\alpha \epsilon_\alpha = 0$. □

In the following Theorem we derive an optimality condition using the strong CHIP.

Theorem 3.1. *Let $x \in K$ be a feasible point of (P). Assume that $\{C, g^{-1}(-S)\}$ has the strong CHIP at x. Then $f(x) = \min(P_f)$ if and only if $0 \in \partial f(x) + Ng(x) + (C - x)^o$.*

Proof. Suppose that $x \in K$ is a minimizer of (P). Then, there exists $p \in \partial f(x)$ such that $-p \in (K - x)^o$. Since $\{C, g^{-1}(-S)\}$ has the strong CHIP at x, $(K - x)^o = (C - x)^o + (g^{-1}(-S) - x)^o$, and so, there exist $q \in (g^{-1}(-S) - x)^o$ and $r \in (C - x)^o$ such that $-p = q + r$. As $(g^{-1}(-S) - x)^o = Ng(x)$, we see that

$$0 = p + q + r \in \partial f(x) + Ng(x) + (C - x)^o.$$

Conversely, suppose that $0 \in \partial f(x) + Ng(x) + (C - x)^o$. Then there exist $p \in \partial f(x)$, $q \in Ng(x)$ and $r \in (C - x)^o$ such that $p + q + r = 0$. So,

$$q + r \in Ng(x) + (C - x)^o = (g^{-1}(-S) - x)^o + (C - x)^o \subset (K - x)^o.$$

Hence, $-p \in (K - x)^o$; thus, $0 \in \partial f(x) + (K - x)^o$. This gives us that x is a minimizer of (P). □

We will now see how the optimality condition can be expressed in terms of a limiting Lagrange multiplier condition.

Theorem 3.2. *Let $x \in K$. Assume that $\{C, g^{-1}(-S)\}$ has the strong CHIP at x. Then $f(x) = \min\ (P_f)$ if and only if there exist $p \in \partial f(x)$, $q \in (C - x)^o$, nets $\{\lambda_\alpha\} \subset S^+$ $\{\varepsilon_\alpha\} \subset \mathbb{R}_+$ and $\{u_\alpha\} \subset X^*$with $u_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ such that*

$$p + q + \lim_\alpha u_\alpha = 0,\ \lim_\alpha(\lambda_\alpha g)(x) = 0 \,\textit{and}\, \lim_\alpha \varepsilon_\alpha = 0.$$

Proof. Suppose that $\{C, g^{-1}(-S)\}$ has the strong CHIP at x. If $x \in K$ is a minimizer of (P_f), then by Theorem 3.1 there exist $p \in \partial f(x)$, $q \in (C - x_0)^o$ and $u \in Ng(x)$ such that $-p = q + u$. Since $u \in Ng(x)$, it follows from Proposition 3.1 that there exist nets $\{\epsilon_\alpha\} \subset \mathbb{R}_+$, $\{\lambda_\alpha\} \subset S^+$ and $\{u_\alpha\} \subset X^*$ with $u_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$, such that $\lim_\alpha u_\alpha = u$, $\lim_\alpha(\lambda_\alpha g)(x) = 0$ and $\lim_\alpha \epsilon_\alpha = 0$. Hence, $p + q + \lim_\alpha u_\alpha = 0$, $\lim_\alpha(\lambda_\alpha g)(x) = 0$, $\lim_\alpha \varepsilon_\alpha = 0$.

Conversely, suppose that there exist $p \in \partial f(x)$, $q \in (C - x)^o$, nets $\{\lambda_\alpha\} \subset S^+$ $\{\varepsilon_\alpha\} \subset \mathbb{R}_+$ and $\{u_\alpha\} \subset X^*$ with $u_\alpha \in \partial_{\varepsilon_\alpha}(\lambda_\alpha g)(x)$ such that

$$p + q + \lim_\alpha u_\alpha = 0,\ \lim_\alpha(\lambda_\alpha g)(x) = 0 \,\text{and}\, \lim_\alpha \varepsilon_\alpha = 0.$$

Then, by Proposition 3.1, $-p - q \in Ng(x)$. Thus, $0 \in p + q + Ng(x) \subset \partial f(x) + (C - x)^o + Ng(x)$. Hence by Theorem 3.1, x is a minimizer of (P_f). □

For each $x \in X$,

$$Ng(x)_o := \left\{u \in X^* \mid (u, u(x)) \in \cup_{\lambda \in S^+} Epi(\lambda g)^*\right\}.$$

It is easy to verify that $Ng(x)_o$ is a convex cone of X^* and that $Ng(x)_o \subset (g^{-1}(-S) - x)^o$. Moreover, $Ng(x)_o = \cup_{\lambda \in S^+}\{\partial(\lambda g)(x) : \lambda g(x) = 0\}$.

Definition 3.2. *The pair $\{C, g^{-1}(-S)\}$ is said to have the sharpened strong CHIP at x if*

$$(K - x)^o = (C - x)^o + Ng(x)_o.$$

The pair $\{C, g^{-1}(-S)\}$ is said to have the *sharpened strong CHIP* if it has the *sharpened strong CHIP* at each $x \in C \cap g^{-1}(-S)$.

Proposition 3.2. *If $\{C, g^{-1}(-S)\}$ has the sharpened strong CHIP then it has the strong CHIP.*

Proof. Let $x \in K$. The conclusion will easily follow from the fact that

$$(C - x)^o + Ng(x)_o \subset (C - x)^o + (g^{-1}(-S) - x)^o \subset (K - x)^o.$$

□

We will now show that a strong CHIP (resp. sharpened strong CHIP) is necessary and sufficient for a dual optimality condition, characterizing a minimizer.

Theorem 3.3. *The following assertions are equivalent:*

(1) $\{C, g^{-1}(-S)\}$ *has the sharpened strong CHIP (resp. the strong CHIP).*
(2) For each continuous convex function $f : X \to \mathbb{R}$, *and for each minimizer* x *of* (P_f),

$$0 \in \partial f(x) + Ng(x)_o + (C - x)^o \ \ \textit{(resp. } 0 \in \partial f(x) + Ng(x) + (C - x)^o\textit{)}.$$

Proof. [(1) $\Longrightarrow$ (2)]. Assume that (1) holds. Let f be a continuous convex function. Suppose that $x \in K$ is a minimizer of (P_f). Then, there exists $p \in \partial f(x)$ such that $-p \in (K - x)^o$. So, (2) holds, since $(K - x)^o = (C - x)^o + Ng(x)_o$ (resp. $(K - x)^o = (C - x)^o + Ng(x)$).

[(2) $\Longrightarrow$ (1)]. Assume that (2) holds. Let $x_0 \in K$ be arbitrary. If $u \in (K - x_0)^o$ then $-u(x_0) = \min (P_{-u})$. Now, by (2), $0 \in \{-u\} + (C - x_0)^o + Ng(x_0)_o$ (resp. $0 \in \{-u\} + (C - x_0)^o + Ng(x_0)$). So, $u \in (C - x_0)^o + Ng(x_0)_o$ (resp. $u \in (C - x_0)^o + Ng(x_0)$). Hence,

$$\begin{aligned}(K - x_0)^o &\subset (C - x_0)^o + Ng(x_0)_o (\text{resp.} \subset (C - x_0)^o + Ng(x_0))\\ &\subset (C - x_0)^o + (g^{-1}(-S) - x_0)^o (\text{resp.} = (C - x_0)^o + (g^{-1}(-S) - x_0)^o)\\ &\subset (K - x_0)^o (\text{resp.} \subset (K - x_0)^o).\end{aligned}$$

□

We will show that if $\cup_{\lambda \in S^+} Epi(\lambda g)^*$ weak*closed then the sharpened strong CHIP of $\{C, g^{-1}(-S)\}$ is equivalent to the strong CHIP of $\{C, g^{-1}(-S)\}$.

Proposition 3.3. *If* $(\cup_{\lambda \in S^+} Epi(\lambda g)^*)$ *is weak* closed, then* $\{C, g^{-1}(-S)\}$ *has the sharpened strong CHIP if and only if* $\{C, g^{-1}(-S)\}$ *has the strong CHIP.*

Proof. Clearly, if $\{C, g^{-1}(-S)\}$ has the sharpened strong CHIP then, by Proposition 3.2, $\{C, g^{-1}(-S)\}$ has the strong CHIP. The converse implication will follow from the fact that, for each, $x \in g^{-1}(-S)$, $Ng(x)_o = Ng(x) = (g^{-1}(-S) - x)^o$, since $(\cup_{\lambda \in S^+} Epi(\lambda g)^*)$ is weak*closed. □

The following simple example illustrates that without the closure condition the strong CHIP does not necessarily imply the sharpened strong CHIP.

Example 3.1. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$g(x, y) = (x^2 + y^2)^{\frac{1}{2}} - y.$$

Let $S = \mathbb{R}_+$ and

$$C = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ 0 \leq y \leq 1\}.$$

Then C is a closed convex subset of $\mathbb{R}^2$ and

$$g^{-1}(-S) = \{(x, y) \in \mathbb{R}^2 : -g(x, y) \in S\} = \{(x, y) \in \mathbb{R}^2 : x = 0,\ y \geq 0\}.$$

Let $x = (0, 1) \in K := C \cap g^{-1}(-S) = \left\{(x, y) \in \mathbb{R}^2 : x = 0, 0 \leq y \leq 1\right\}$. A direct calculation shows that $(C - x)^o = -\mathbb{R}_+ \times \mathbb{R}$, $(g^{-1}(-S) - x)^o = \mathbb{R} \times \{0\}$, $(K - x)^o = \mathbb{R} \times \mathbb{R}_+$ and $Ng(x)_o = \{0\} \times \{0\}$. It is now easy to verify that $\{C, g^{-1}(-S)\}$ has the strong CHIP at x; however, $\{C, g^{-1}(-S)\}$ does not have the sharpened strong CHIP at x as $\mathbb{R} \times \mathbb{R}_+ = (K - x)^o \neq (C - x)^o + Ng(x)_o = (-\mathbb{R}_+ \times \mathbb{R}) + (\{0\} \times \{0\})$. Note that $(-1, 0, 0) \in cl(\cup_{\lambda \in S^+} Epi(\lambda g)^*)$, but $(-1, 0, 0) \notin \cup_{\lambda \in S^+} Epi(\lambda g)^*$, and so $\cup_{\lambda \in S^+} Epi(\lambda g)^*$ is not closed.

We will now look at global conditions which ensure the strong CHIP and the sharpened strong CHIP of $\{C, g^{-1}(-S)\}$.

Proposition 3.4. *If $(\cup_{\lambda \in S^+} Epi(\lambda g)^* + Epi\sigma_C)$ is weak*closed, then $\{C, g^{-1}(-S)\}$ has the sharpened strong CHIP.*

Proof. Let $x \in K$ and let $u \in (K - x)^o$. Then, $(u, u(x)) \in Epi\sigma_K$. Now, by the hypothesis, $(u, u(x)) \in \cup_{\lambda \in S^+} Epi(\lambda g)^* + Epi\sigma_C$. So, by the definition of conjugate functions, we can find $v_1, v_2 \in X^*$ such that $(v_1, v_1(x)) \in \cup_{\lambda \in S^+} Epi(\lambda g)^*$ and $(v_2, v_2(x)) \in Epi\sigma_C$ such that $u = v_1 + v_2$. Hence, by the definitions of $Ng(x)_o$ and $(C - x)^o$, we get $v_1 \in Ng(x)_o$ and $v_2 \in (C - x)^o$. Thus, $u = v_1 + v_2 \in Ng(x)_o + (C - x)^o$. This means that $(K - x)^o \subset (C - x)^o + Ng(x)_o$. As the opposite inclusion always holds, we get $(K - x)^o = (C - x)^o + Ng(x)_o$. □

Proposition 3.5. *If $(cl(\cup_{\lambda \in S^+} Epi(\lambda g)^*) + Epi\sigma_C)$ is weak*closed, then $\{C, g^{-1}(-S)\}$ has the strong CHIP.*

Proof. Since $Epi\sigma_{g^{-1}(-S)} = cl(\cup_{\lambda \in S^+} Epi(\lambda g)^*$, it follows from the hypothesis that

$$cl(\cup_{\lambda \in S^+} Epi(\lambda g)^*) + Epi\sigma_C = Epi\sigma_{g^{-1}(-S)} + Epi\sigma_C$$

is weak*closed, and so, $\{C, g^{-1}(-S)\}$ has the strong CHIP. □

Note that $(\cup_{\lambda \in S^+} Epi(\lambda g)^* + Epi\sigma_C)$ is weak*-closed if, in particular, $int(S)$ is non-empty and $-g(x_0) \in int(S)$ for some $x_0 \in C$, which is known as the Slater condition. The closure condition is strictly weaker than the Slater condition. To see this, let $X = Y = \mathbb{R}$, $C = [-1, 1]$ and $S = \mathbb{R}_+$. Let $g(x) = \max\{0, x\}$. Then it is easy to check that $(\cup_{\lambda \in S^+} Epi(\lambda g)^* + Epi\sigma_C)$ is closed, whereas the Slater condition does not hold. For various other generalized interior point or Slater type conditions which ensure that $(\cup_{\lambda \in S^+} Epi(\lambda g)^* + Epi\sigma_C)$ is weak*-closed, see [17].

Sufficient conditions for the sharpened strong CHIP (resp. strong CHIP) can also be obtained in terms of metric regularity (resp. calmness) conditions, used in the study of stability analysis and Lagrange multipliers in optimization. See [10, 11] and other reference therein. For instance, in the case where $X = Y = \mathbb{R}^m$, it is shown in [17] that if there exist neighbourhoods V and U of 0 and some $x_0 \in C \cap g^{-1}(-S)$ as well as some $\gamma > 0$ such that

$$d(x, C \cap g^{-1}(y - S)) \leq \gamma d(y - g(x), S), \quad \forall y \in V, \forall x \in U,$$

then $(\cup_{\lambda \in S^+} Epi(\lambda g)^* + Epi\sigma_C)$ is closed, and so, $\{C, g^{-1}(-S)\}$ has the sharpened strong CHIP. Recall that for a non-empty subset W of X and $x \in X$, we define

$$d(x, W) := \inf_{w \in W} \|x - w\|.$$

In Corollary 4.2 [11], calmness condition is used for the strong CHIP. For other sufficient conditions involving bounded linear regularity for the strong CHIP, see [1, 4].

4. Necessary and Sufficient Conditions for Strong Duality

Consider the partially finite convex program

$$(PF_f) \quad \min\{f(x) | \in C, \ Ax - b \in S\},$$

where $C \subset X$ is a closed convex subset of X, $A : X \to \mathbb{R}^m$ is a continuous linear mapping and $S \subset \mathbb{R}^m$ is a *polyhedral* convex cone and $b \in \mathbb{R}^m$. The partially finite convex model program arises in many important constrained approximation problems and shape preserving interpolation problems (see [2] for details).

We assume that for each continuous convex function $f : X \to \mathbb{R}$, (PF_f) has a minimizer x.

Let $g(x) := -Ax + b$ and let $B : X \times \mathbb{R} \to \mathbb{R}^m$ be the linear map defined by $B(x, \beta) := -Ax - \beta b$. Then,

$$Epi(\lambda g)^* = \text{epi } [\lambda(-A(.) + b)]^* = \{-A^T \lambda\} \times [-\lambda b, +\infty).$$

So, $\bigcup_{\lambda \in S^+} Epi(\lambda g)^* = \bigcup_{\lambda \in S^+} \{-A^T \lambda\} \times [-\lambda b, +\infty)$. This can be re-expressed as

$$\bigcup_{\lambda \in S^+} Epi(\lambda g)^* = B^T(S^+) + \{0\} \times \mathbb{R}_{+,}$$

where $B^T \lambda = (-A^T \lambda, -\lambda b)$. As $B^T(S^+)$ and $\{0\} \times \mathbb{R}_+$ are finitely generated cones, $B^T(S^+) + \{0\} \times \mathbb{R}_+$ is also weak*closed. Hence, for the partially finite convex program (PF_f), $\bigcup_{\lambda \in S^+} Epi(\lambda g)^*$ is weak*closed, and hence, sharpened strong CHIP is equivalent to strong CHIP. Moreover, in this case, if $A^{-1}(b + S) \neq \emptyset$ then

$$(A^{-1}(b + S) - x)^o = \cup_{\lambda \in S^+} \{-A^T \lambda : \lambda^T(-Ax + b) = 0\},$$

for each $x \in A^{-1}(b + S)$, where $A^{-1}(b + S) := \{x \in X \mid Ax - b \in S\}$. In the following Theorem we will show that the strong CHIP completely characterizes the strong duality of (PF_f). Let $K := C \cap A^{-1}(b + S)$.

Theorem 4.1. *The pair* $\{C, A^{-1}(b+S)\}$ *has the strong CHIP if and only if, for each continuous convex function* $f : X \to \mathbb{R}$,

$$\min(PF_f) = \max_{\lambda \in S^+} -(f+\delta_C)^*(A^T\lambda) + b^T\lambda.$$

Proof. Let $f : X \to \mathbb{R}$ be a continuous convex function and let $f(x_0) = \min(PF_f)$. Suppose that that $\{C, A^{-1}(b+S)\}$ has the strong CHIP. Then, it follows from Theorem 3.3 that

$$\exists \lambda \in S^+,\ \ 0 \in \partial f(x_0) - A^T\lambda + (C - x_0)^o,\ \lambda^T(-Ax_0 + b) = 0.$$

This gives us that

$$\exists \lambda \in S^+,\ b^T\lambda + \inf_{x \in C} f(x) - A^T\lambda x \geq f(x_0);$$

thus,

$$\max_{\lambda \in S^+} -(f+\delta_C)^*(A^T\lambda) + b^T\lambda \geq f(x_0) = \min(PF_f)\,.$$

By the weak duality, we get that

$$\max_{\lambda \in S^+} -(f+\delta_C)^*(A^T\lambda) + b^T\lambda = f(x_0) = \min(PF_f).$$

Conversely, assume that, for each continuous convex function $f : X \to \mathbb{R}$,

$$\min(PF_f) = \max_{\lambda \in S^+} -(f+\delta_C)^*(A^T\lambda) + b^T\lambda.$$

Let $x_0 \in K$ be arbitrary. If $u \in (K - x_0)^o$ then $-u(x_0) = \min(PF_{-u})$. Now, by the assumption,

$$-u(x_0) = \max_{\lambda \in S^+} -(-u+\delta_C)^*(A^T\lambda) + b^T\lambda.$$

So, there exists $\lambda \in S^+$ such that $-u(x_0) = -(-u+\delta_C)^*(A^T\lambda) + b^T\lambda$. This gives us that

$$u + A^T\lambda \in (C - x_0)^o \text{ and } \lambda^T(-Ax_0 + b) = 0\,.$$

Since $\lambda^T(-Ax_0 + b) = 0$, $-A^T\lambda \in Ng(x_0) = (A^{-1}(b+S) - x_0)^o$. Hence,

$$u = (u + A^T\lambda) - A^T\lambda \in (C - x_0)^o + (A^{-1}(b+S) - x_0)^o.$$

Thus, $\{C, A^{-1}(b+S)\}$ has the strong CHIP. □

It is worth noting that if $ri(AC) \cap (b+S) \neq \emptyset$ then $\{C, A^{-1}(b+S)\}$ has the strong CHIP. In particular, if for some $x_0 \in qri(C)$ such that $Ax_0 - b \in S$ then $\{C, A^{-1}(b+S)\}$ has the strong CHIP, where $qri(C)$ is the quasi relative interior of C and $ri(AC)$ is the relative interior of AC. For details, see [2, 22].

We shall now look at how a best constrained approximation can be characterized in terms of strong CHIP. The point $w_0 \in W$ is called a best approximation for $x \in X$ (i.e. $w_o \in P_W(x)$), if

$$d(x, W) = \|x - w_0\|.$$

If for each $x \in X$ there exists a unique best approximation $w_0 \in W$, then W is called a Chebyshev subset of X. Every closed convex set in a Hilbert space is Chebyshev. The following characterization of best approximation in Hilbert spaces is well known (see [7]).

Lemma 4.1. *Let X be a Hilbert space; and let W be a closed convex subset of X,* $x \in X$ *and* $w_0 \in W$*. Then* $w_0 = P_W(x)$ *if and only if* $x - w_0 \in (W - w_0)^o$.

Theorem 4.2. *Let X be a Hilbert space, and let* $x_0 \in K$*. Then, the following assertions are equivalent:*

(1) The pair $\{C, A^{-1}(b+S)\}$ *has the strong CHIP at* x_0.
(2) For any $x \in X$,

$x_0 = P_K(x)$ *if and only if* $x_0 = P_C(x + A^T\lambda)$ *for some* $\lambda \in S^+$ *and* $\lambda^T(-Ax_0 + b) = 0$.

Proof. [(1) $\Longleftrightarrow$ (2)]. This equivalence easily follows from Lemma 4.1 on noting that the statement (2) is equivalent to the fact that, for each $x \in X$,

$$x - x_0 \in (K - x_0)^o \Longleftrightarrow \exists\lambda \in S^+,\ x - x_0 + A^T\lambda \in (C - x_0)^o,\ \lambda^T(-Ax_0 + b) = 0,$$

which is, in turn, equivalent to the statement that

$$x \in (K - x_0)^o \Longleftrightarrow \exists\lambda \in S^+,\ x \in (C - x_0)^o - A^T\lambda,\ \lambda^T(-Ax_0 + b) = 0.$$

This means that

$$(K - x_0)^o = (C - x_0)^o + \{-A^T\lambda\ :\ \lambda \in S^+,\ \lambda^T(-Ax_0 + b) = 0\},$$

which is equivalent to the statement (1). □

A particular case of Theorem 4.2 in the case of linear inequality constraints was recently given in [15].

We conclude by noting that in the case where $Y = \mathbb{R}^n$ and $S = \mathbb{R}^n_+$, the condition, $(g^{-1}(-S) - x)^o = Ng(x)_o$, is the basic constraint qualification (BCQ), used to study convex optimization in [13]. So, our condition is an extension of (BCQ) for the general case.

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