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The two-dimensional cutting stock problem revisited

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Abstract. In the strip packing problem (a standard version of the two-dimensional cutting stock problem), the goal is to pack a given set of rectangles into a vertical strip of unit width so as to minimize the total height of the strip needed. The k-stage Guillotine packings form a particularly simple and attractive family of feasible solutions for strip packing.

We present a complete analysis of the quality of k-stage Guillotine strip packings versus globally optimal packings: $k = 2$ stages cannot guarantee any bounded asymptotic performance ratio. $k = 3$ stages lead to asymptotic performance ratios arbitrarily close to 1.69103; this bound is tight. Finally, $k = 4$ stages yield asymptotic performance ratios arbitrarily close to 1.

Key words. Cutting stock – Strip packing – Guillotine cuts – Packing problem – Approximation scheme – Worst case analysis

1. Introduction

We consider a version of the two-dimensional cutting stock problem that is called the *strip packing problem*: We are given a supply of raw-material in the form of a rectangular strip of width 1 and unlimited height, and a demand of n rectangular items with widths and heights from the unit interval [0, 1]. The goal is to find a packing of all items into the strip such that the total height used is minimized. It is not allowed to rotate the items, and all items have to be packed with their sides parallel to the sides of the strip. This is a natural and fundamental optimization problem that has many applications in manufacturing systems (where not allowing rotations is motivated by the structure of the raw-material, such as the grain of wood or the pattern of cloth), and in parallel scheduling systems (where the width of the strip represents the machines, and the height represents time), and in VLSI design. We refer the reader to Gilmore & Gomory [9] and to Baker, Coffman & Rivest [2] for more information on this problem.

In certain manufacturing environments machines can only perform edge-to-edge cuts that are parallel to the strip's width or height; such cuts are called *Guillotine cuts*. A Guillotine cut splits the raw material into two pieces that can be processed in parallel and that can be further subdivided by other Guillotine cuts. A strip packing that can be implemented by a sequence of Guillotine cuts is called a Guillotine strip packing. The strip packings in Figure 1(b), 1(c), 1(d) are Guillotine strip packings, whereas the

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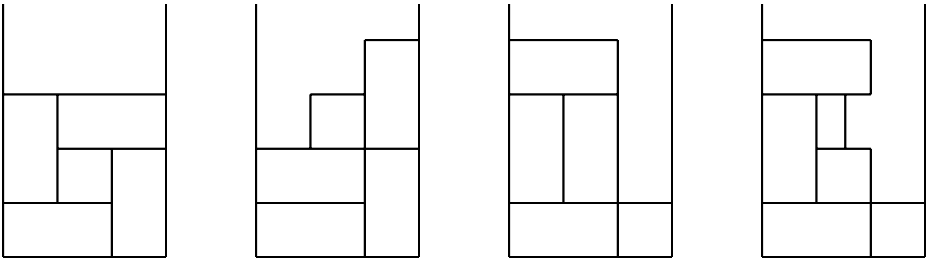

(a) **(b)** **(c)** **(d)**

Fig. 1. (a) not a Guillotine strip packing; (b) a 3-stage Guillotine strip packing; (c) a 2-stage Guillotine strip packing; (d) a 4-stage Guillotine strip packing.

packing in Figure 1(a) is not. Sometimes it is convenient to perform several Guillotine cuts in parallel; this is then called a *cutting stage*. For instance the packing in Figure 1(c) can be produced by a first stage with three horizontal Guillotine cuts, followed by a second stage of vertical Guillotine cuts through the resulting pieces. A *k-stage* Guillotine strip packing is a strip packing that can be implemented by a sequence of k stages of parallel Guillotine cuts. The cuts in the odd-numbered stages are always done horizontally, and the cuts in the even-numbered stages are always done vertically. Guillotine strip packings with a small number of stages (say, with $k = 4$ or $k = 5$ stages) are particularly simple to process and to implement, and hence they are very attractive for industrial applications.

1.1. Known results

Strip packing is an NP-hard problem: Its special case where all items are of height 1 boils down to the classical one-dimensional bin packing problem. Hence, research has turned to the design of good and efficient approximation algorithms for strip packing.

For a strip packing instance I, we denote by $\text{OPT}(I)$ the height of its optimal packing, and by $A(I)$ the height of a packing produced by an approximation algorithm A. Then the *asymptotic performance ratio* (a.p.r.) of algorithm A (cf. Garey and Johnson [8]) is defined as the smallest real $r \geq 1$ for which there exists some positive constant const(r) such that for all instances I,

$$A(I) \leq r \cdot \text{OPT}(I) + \text{const}(r). \tag{1}$$

Because of the additive constant in (1), small instances are irrelevant for the asymptotic performance ratio. Hence, the a.p.r. mainly measures the quality of strip packings with a huge number of items, and it is robust against anomalies resulting from a small number of rectangles in the optimal packing.

Baker, Coffman & Rivest [2] designed a polynomial time approximation algorithm with asymptotic performance ratio equal to 3. Coffman, Garey, Johnson & Tarjan [4] improved the a.p.r. down to 2.7, Sleator [15] improved it to 2.5, Golan [10] to 1.33, and Baker, Brown & Katseff [1] to 1.25. Finally, Kenyon & Remila [12, 13] constructed an asymptotic fully polynomial time approximation scheme (AFPTAS) for strip packing: For every $\varepsilon > 0$ there is a polynomial time approximation algorithm with asymptotic performance ratio $1 + \varepsilon$; the running time is polynomial in the number n of items and in $1/\varepsilon$. Remarkably, this AFPTAS produces 5-stage Guillotine packings.

In the *on-line* version of strip packing, the items arrive one by one, and have to be packed immediately, without knowledge of future items. Baker & Schwarz [3] gave an on-line algorithm with $a.p.r. \approx 1.7$, and Csirik & Woeginger [5] improved this to $a.p.r. \approx 1.69103$. The on-line algorithms in [3] and [5] produce 3-stage Guillotine packings.

1.2. Our results

We perform an analysis of the quality of k-stage Guillotine strip packings for strip packing. Theorems 1.1, 1.2, and 1.3 present the complete picture how well k-stage Guillotine strip packings can approximate the optimal packing:

- $k = 2$ stages are far too weak, and cannot guarantee any bounded performance ratio.
- $k = 3$ stages are reasonable, and lead to an asymptotic performance ratio of 1.69103.
- $k = 4$ or more stages are excellent, and yield asymptotic performance ratios arbitrarily close to 1.

The result for $k = 2$ is straightforward, and half of the result for $k = 3$ has been proved in the literature. The main contribution of this paper is the other half of the result for $k = 3$, and the new result for $k = 4$. For $k \geq 2$, we denote by $G_k(I)$ the height of the best k-stage Guillotine packing for instance I.

Theorem 1.1 (Result for $k = 2$ stages)**.** *For any $r \geq 1$ and for any positive constant c, there exists an instance I such that the best 2-stage Guillotine packing for I satisfies $G_2(I) > r \cdot \mathrm{OPT}(I) + c$.*

The proof of Theorem 1.1 can be found in the first paragraph of Section 2.

Theorem 1.2 (Result for $k = 3$ stages)**.** *For any $\varepsilon > 0$, there exists a constant* $\mathrm{const}(\varepsilon)$*, such that any strip packing instance I satisfies*

$$G_3(I) \;\leq\; (h_\infty + \varepsilon) \;\cdot \mathrm{OPT}(I) + \mathrm{const}(\varepsilon)$$

where $h_\infty \approx 1.69103$ is defined as in equation (2). A 3-stage Guillotine packing of this quality can be computed in polynomial time. Moreover, the value h_∞ in the above inequality is best possible.

The positive result in Theorem 1.2 has been proved implicitly by Csirik & Woeginger [5], since the on-line strip packing algorithm in [5] produces a 3-stage Guillotine packing of the stated quality, and since it has a polynomial running time. The proof that the constant $h_\infty \approx 1.69103$ can not be improved is given in Section 2.

Theorem 1.3 (Result for $k \geq 4$ stages)**.** *For any $\varepsilon > 0$, there exists a constant* $\mathrm{const}(\varepsilon)$*, such that any strip packing instance I satisfies*

$$G_4(I) \;\leq\; (1 + \varepsilon) \;\cdot \mathrm{OPT}(I) + \mathrm{const}(\varepsilon)$$

A 4-stage Guillotine packing of this quality can be computed with a time complexity that is polynomial in the number n of items and in $1/\varepsilon$.

In other words, there is an AFPTAS for computing the best 4-stage Guillotine strip packing that simultaneously is an AFPTAS for computing the globally best strip packing. Theorem 1.3 is proved in Section 3.

Our proof draws many ideas from Fernandez de la Vega & Lueker [6], Karmarkar & Karp [11], Fernandez de la Vega & Zissimopoulos [7], and especially from Kenyon & Remila [12, 13]. Our result also reproves the existence of an AFPTAS for strip packing, the main result of [12, 13]. However, whereas the paper [12, 13] introduced quite a number of new ideas and new tricks, our arguments are fairly simple; the complete proof takes less than five pages. Our arguments also demonstrate that the combinatorics of strip packing is very close to the combinatorics of classical one-dimensional bin packing: If the reader is familiar with the results of Fernandez de la Vega & Lueker [6], he/she will notice that the arguments in Sections 3.1–3.3 are just simple two-dimensional generalizations of the arguments in [6] for the one-dimensional problem.

2. Two and three stages: Proofs of the negative results

In this section, we prove the negative statements in Theorems 1.1 and 1.2. The proof of Theorem 1.1 is straightforward: Let ℓ be an integer, and consider an instance I with ℓ^2 items of width $1/\ell$ and height from the interval $[1-1/\ell, 1]$ such that distinct items have distinct heights. Then the optimal packing has height $\text{OPT}(I) \le \ell$, whereas the best 2-stage Guillotine packing has height $G_2(I) \ge \ell^2 - \ell$. This yields $G_2(I)/\text{OPT}(I) \ge \ell - 1$. Letting ℓ tend to infinity completes the proof.

The rest of this section is devoted to the proof of the negative statement in Theorem 1.2. We define a sequence t_i of integers that is well-known in the on-line bin packing community. In the literature this sequence is sometimes called the *Salzer* sequence, since it was introduced in 1947 by Salzer [14].

$$\begin{aligned} t_1 &= 2 \\ t_{i+1} &= t_i(t_i - 1) + 1 \qquad\qquad \text{for } i \ge 1 \end{aligned}$$

The Salzer sequence starts with the numbers $\langle 2, 3, 7, 43, 1807, 3263443, \ldots \rangle$, and its growth is doubly exponential. Now the number h_∞ (that was already used in the statement of Theorem 1.2) can be defined precisely.

$$h_\infty \;=\; \sum_{i=1}^{\infty} \frac{1}{t_i - 1} \;=\; 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{42} + \frac{1}{1806} + \cdots\cdots \approx 1.69103. \tag{2}$$

We will now define an instance I for which the height of the best 3-stage Guillotine packing is far away from the height of the globally optimal packing. A crucial property of instance I will be that no two of its items have the same width.

Let ε with $0 < \varepsilon < 1$ be a (small) real number, and let α be a (huge) integer that satisfies $\alpha > 100/\varepsilon$. We determine the smallest index d that fulfills the inequality

$$h_\infty - \frac{\varepsilon}{2} \;\le\; \sum_{i=1}^{d} \frac{1}{t_i - 1}, \tag{3}$$

and we define a real number δ by

$$\delta = \frac{1}{d}\left(1 - \sum_{i=1}^{d} \frac{1}{t_i}\right) = \frac{1}{t_d(t_d - 1)d} > 0. \tag{4}$$

The instance I consists of d item lists $I_1, \ldots, I_d$ where list I_i consists of n_i items that all have the same height H_i, and that all have width between $1/t_i$ and $1/t_i + \delta$. Hence, these lists I_i are 'almost' homogeneous.

- The first list I_1 consists of $n_1 = \alpha$ items of height $H_1 = 1$. The width of every item in I_1 is taken from the open interval $(1/2, 1/2 + \delta)$ in such a way that no two items in I have the same width.
- Now assume that the item lists $I_1, \ldots, I_{i-1}$ have already been defined. Let $N_{i-1} = n_1 + n_2 + \cdots + n_{i-1}$ denote the overall number of items in $I_1, \ldots, I_{i-1}$. Then list I_i consists of $n_i = \alpha \cdot (t_i - 1) \cdot N_{i-1}$ items of height $H_i = 1/((t_i - 1)N_{i-1})$. The width of every item in I_i is taken from the open interval $(1/t_i, 1/t_i + \delta)$ in such a way that no two items in I have the same width.

This completes the description of instance I.

Lemma 2.1. *There exists a strip packing of instance I of height α.*

Proof. In every list I_i, we pack the $n_i = \alpha \cdot (t_i - 1) \cdot N_{i-1}$ items into α so-called *crates*, that is, rectangular objects of height 1 and width $1/t_i + \delta$. Every crate receives $(t_i - 1)N_{i-1}$ items with total height $(t_i - 1)N_{i-1} \cdot H_i = 1$, and there indeed is sufficient space to accomodate all items from I_i by stacking them on top of each other. In the strip packing, we create α so-called *shelves* of height 1 and width 1. Into every shelf we put exactly one crate from every list I_i. These crates fit together into one shelf, since their total width is at most

$$\sum_{i=1}^{d}\left(\frac{1}{t_i} + \delta\right) = d \cdot \delta + \sum_{i=1}^{d} \frac{1}{t_i} = 1.$$

Here we used the definition of δ in (4). □

Lemma 2.2. *Any 3-stage Guillotine strip packing for I has height at least* $(\alpha - 1) \sum_{i=1}^{d} \frac{1}{t_i - 1}$.

Proof. Consider an arbitrary 3-stage Guillotine strip packing for I: Recall from the definition of a k-stage Guillotine strip packing that the cuts in the first stage are done horizontally. The cuts in the first stage subdivide the strip into so-called *shelves*, and the cuts in the second stage then subdivide the shelves into so-called *crates*. If some crate contained two or more items, then the cuts in the third stage are not sufficient to cut out these items, since no two items in I have the same width. Hence, every crate contains at most one item, and the structure of any 3-stage Guillotine strip packing for I must be fairly primitive.

A shelf that contains at least one item from list I_i, but that does not contain any item from the lists $I_1, \ldots, I_{i-1}$ is said to be a shelf of type i. We denote by x_i the number of shelves of type i, and by X_{i-1} the number of shelves of type $1, 2, \dots, i-1$. Clearly, $x_i \le n_i$ and $X_{i-1} \le N_{i-1}$. Altogether, there are n_i items in the list I_i, and all these items are in shelves of types $1, \dots, i$. Since every shelf can accomodate at most $t_i - 1$ items from list I_i, we get that

$$n_i \;\le\; (x_i + X_{i-1}) \cdot (t_i - 1) \;\le\; (x_i + N_{i-1}) \cdot (t_i - 1).$$

Together with $n_i = \alpha \cdot (t_i - 1) \cdot N_{i-1}$ this yields

$$x_i \;\ge\; (\alpha - 1) \cdot N_{i-1}. \tag{5}$$

Since the height of a shelf of type i is at least $H_i = 1/((t_i - 1)N_{i-1})$, (5) yields that the height of the 3-stage Guillotine strip packing is at least

$$\sum_{i=1}^{d} x_i \cdot H_i \;\ge\; \sum_{i=1}^{d} (\alpha - 1) \cdot N_{i-1} \cdot \frac{1}{(t_i - 1)N_{i-1}} \;=\; (\alpha - 1) \sum_{i=1}^{d} \frac{1}{t_i - 1}.$$

This completes the proof of the lemma. □

The statements in Lemma 2.1 and 2.2, the inequality (3), and the inequality $\alpha > 100/\varepsilon$ together imply that the ratio between the height of the best 3-stage Guillotine strip packing and the height of the globally best strip packing is at least

$$\frac{\alpha - 1}{\alpha + 1} \sum_{i=1}^{d} \frac{1}{t_i - 1} \;\ge\; \frac{\alpha - 1}{\alpha + 1} \,(h_\infty - \frac{\varepsilon}{2}) \;>\; h_\infty - \varepsilon.$$

Since in our construction the value ε can be made arbitrarily close to 0, this completes the proof of Theorem 1.2.

3. Four stages: The approximation scheme

In this section, we will prove Theorem 1.3. Hence, consider an arbitrary strip packing instance I with n items. Let ε be some small real number with $0 < \varepsilon < 1$. We define two integers $s = \lceil 6/\varepsilon \rceil + 1$ and $t = s^2$, and the real number $\delta = 1/s$. Note that these choices imply $(1 + 1/s)^2 \le 1 + \varepsilon$, and $s(s+1)/(s-1)^2 \le 1 + \varepsilon$, and that $\delta\, t = s$.

3.1. The two rounded instances

We classify the items into *big* items with widths greater than δ and into small items with widths less than or equal to δ; this classification is taken from Kenyon & Remila [12, 13]. By H we denote the total height of all big items. Throughout, we will assume that

$$H \;\ge\; 2t. \tag{6}$$

In the complementary case where $H < 2t$, the additive constant in the asymptotic worst case ratio in (1) can be used to swallow all the big items. Furthermore, we observe that the total area A_{items} of all items in instance I satifies

$$A_{items} \geq \delta \cdot H, \tag{7}$$

since every big item has width at least δ.

We subdivide the set of big items into t subsets $B_1, \ldots, B_t$ with the following properties: The items in B_i have width greater or equal to the width of items in $B_{i+1} (i = 1, \ldots, t-1)$. The total height H_i of the items in B_i is between $H/t - 1$ and $H/t + 1$ $(i = 1, \ldots, t)$. Such a partition indeed exists, since the height of every single item is bounded by 1. By W_i $(i = 1, \ldots, t)$ we denote the maximum width over all items in the subset B_i. Furthermore, we define W_{t+1} to be the minimum width over all items in the subset B_t. Note that $W_1 \geq W_2 \geq \cdots \geq W_{t+1} > \delta$ holds.

Next, we define two rounded instances I^- and I^+ from the original instance I. Both rounded instances have the same set of small items as I. Both rounded instances have one corresponding big item for every big item in I: In the plus instance I^+, the width of every big item in set B_i $(i = 1, \ldots, t)$ is rounded up to W_i; this yields a corresponding set B_i^+ of big items. In the minus instance I^-, the width of every big item in set B_i $(i = 1, \ldots, t)$ is rounded down to W_{i+1}; this yields a corresponding set B_i^- of big items. Clearly,

$$\text{OPT}(I^-) \leq \text{OPT}(I) \leq \text{OPT}(I^+). \tag{8}$$

3.2. *The two linear programming relaxations*

A *pattern* is a non-negative integer vector $q = \langle q_1, \ldots, q_t \rangle$ such that $\sum_{i=1}^{t} q_i W_i \leq 1$. Note that $W_i > \delta$ implies $q_i < 1/\delta$, and that consequently there exist at most $(1/\delta)^t$ distinct patterns. Let $\mathcal{Q}$ denote the set of all patterns. The motivation for considering these patterns is the following: Consider an arbitrary strip packing for instance I^- or I^+, and make a horizontal cut at an arbitrary place through this packing. For $i = 1, \ldots, t$ count the number q_i of items along this cut that are of width W_i; then these numbers q_i exactly form a pattern.

We introduce a linear programming relaxation (LP^+) for instance I^+, in which the variable $\phi(q)$ measures the total height of all horizontal cuts that yield the pattern q. (LP^+) completely ignores all the small items.

$$\begin{aligned} (\text{LP}^+) : \text{minimize } & \textstyle\sum_{q \in \mathcal{Q}} \phi(q) \\ \text{subject to } & \textstyle\sum_{q \in \mathcal{Q}} q_i \phi(q) \geq H_i \quad \text{for all } 1 \leq i \leq t \\ & \phi(q) \geq 0 \qquad\qquad \text{for all } q \in \mathcal{Q} \end{aligned}$$

By the above discussion, the optimal objective value $\text{VAL}(\text{LP}^+)$ of this linear program (LP^+) is a lower bound on $\text{OPT}(I^+)$.

In a very similar manner, we introduce a linear programming relaxation (LP^-) for instance I^-, in which the variable $\psi(q)$ measures the total height of all horizontal cuts that yield the pattern q. (LP$^-$) ignores all the small items as well as all the big items in the set B_t^-.

$$\begin{aligned}(\mathrm{LP}^-): \text{minimize } & \textstyle\sum_{q\in\mathcal{Q}} \psi(q) \\ \text{subject to } & \textstyle\sum_{q\in\mathcal{Q}} q_i\psi(q) \geq H_{i-1} \quad \text{for all } 2 \leq i \leq t \\ & \psi(q) \geq 0 \qquad\qquad\qquad \text{for all } q \in \mathcal{Q}\end{aligned}$$

Then the optimal objective value $\mathrm{VAL}(\mathrm{LP}^-)$ is a lower bound on $\mathrm{OPT}(I^-)$. Note that (LP$^+$) and (LP$^-$) are actually one and the same formulation, but for two different instances. Our next goal is to prove the following inequality.

$$\mathrm{VAL}(\mathrm{LP}^+) \;\leq\; \mathrm{OPT}(I) + 2t + A_{items}/s \tag{9}$$

To this end, consider an optimal basic feasible solution ψ^* of the linear program (LP$^-$). Since ψ^* is a vertex of the underlying polytope, at most t of the values $\psi^*(q)$ are non-zero. Moreover, for $p = \langle 1, 0, 0, \ldots, 0\rangle$ the optimality of ψ^* implies $\psi^*(p) = 0$. From ψ^*, we define a solution ϕ' for (LP$^+$):

- For $p = \langle 1, 0, 0, \ldots, 0\rangle$, we let $\phi'(p) = H/t + 1$.
- For all $q \neq p$ with $\psi^*(q) = 0$, we let $\phi'(q) = 0$.
- For all q with $\psi^*(q) > 0$, we let $\phi'(q) = \psi^*(q) + 2$.

Claim 3.1. *The defined ϕ' constitutes a feasible solution for* (LP$^+$).

Proof. Indeed, by the definition of $\phi'(p)$ we have

$$\sum_{q\in\mathcal{Q}} q_1\phi'(q) \;\geq\; H/t + 1 \;\geq\; H_1.$$

Here the final inequality follows since H_1 lies between $H/t - 1$ and $H/t + 1$. Next, consider some fixed index i with $2 \leq i \leq t$. Since H_{i-1} is non-zero and since ψ^* is feasible for (LP$^-$), there exists some $q \in \mathcal{Q}$ such that q_i and $\psi^*(q)$ both are non-zero. For this particular q, we have $q_i \geq 1$ and $\phi'(q) \geq \psi^*(q) + 2$. We conclude that for $i \geq 2$

$$\sum_{q\in\mathcal{Q}} q_i\phi'(q) \;\geq\; \Big(\sum_{q\in\mathcal{Q}} q_i\psi^*(q)\Big) + 2q_i \;\geq\; H_{i-1} + 2 \;\geq\; H_i.$$

Here the second inequality follows since ψ^* is feasible for (LP^-), and the final inequality follows since H_i and H_{i-1} both lie between $H/t - 1$ and $H/t + 1$. The displayed inequalities now yield the feasibility of ϕ' for (LP$^+$). □

Hence, ϕ' is feasible for (LP$^+$) and its objective value is

$$\sum_{q\in\mathcal{Q}} \phi'(q) \;\leq\; \sum_{q\in\mathcal{Q}} \psi^*(q) + 2t + H/t \;\leq\; \mathrm{VAL}(LP^-) + 2t + A_{items}/(t\cdot\delta)$$

Here we used the inequality in (7). Since $\mathrm{VAL}(LP^-) \leq \mathrm{OPT}(I^-) \leq \mathrm{OPT}(I)$ and since $t \cdot \delta = s$, the above inequality implies the correctness of (9).

3.3. *How to approximate the plus instance*

In this subsection, we will construct a near-optimal packing for instance I^+ in two construction steps. We start from an optimal basic feasible solution ϕ^* of (LP$^+$), and we note that in any basic feasible solution at most t of the values $\phi^*(q)$ can be non-zero.

In the first construction step, for every pattern $q \in \mathcal{Q}$ we introduce $\lceil \phi^*(q)/s \rceil$ shelves of height $s+1$. These shelves are packed on top of each other in the strip. In every shelf, we put q_i crates of width W_i and of height $s+1$ $(i = 1, \ldots, t)$. Then we fill these crates of width W_i one by one greedily with the big items in B_i^+, until we run out of such items. If some crate has remained empty in the end, it is deleted. As the item heights are less than or equal to 1, all these crates of width W_i (with the possible exception of the final crate), are filled up to a level strictly greater than s.

Claim 3.2. *There is sufficient space for packing all items from B_i^+ into these crates.*

Proof. Suppose not. Note that the total number of crates is

$$\sum_{q \in \mathcal{Q}} q_i \lceil \phi^*(q)/s \rceil \;\geq\; \sum_{q \in \mathcal{Q}} q_i \, \phi^*(q)/s \;\geq\; H_i/s.$$

Here the first inequality follows from the definition of the ceiling function, and the second inequality follows from the first restriction in the linear program (LP$^+$). Further we know that every crate is filled to a level strictly greater than s. This implies that the total height of all items in B_i^+ is strictly greater than H_i, a contradiction. □

In the second construction step, we add the small items of instance I^+ to the packing of the big items. We say that a small item has *type* k $(k \geq 0)$, if its width w satisfies $\delta(1-\delta)^{k+1} < w \leq \delta(1-\delta)^k$. Small items of type k are packed greedily into crates of type k, that is, crates of height $s+1$ and width $\delta(1-\delta)^k$. The small items are simply stacked on top of each other. Finally, the crates of type $k \geq 0$ are packed greedily into the shelves that were created during the first construction step as long as there is space. Everytime some crate does not fit into any shelf, we start a new empty shelf of height $s+1$ on top of the current packing.

Now let us analyze the height $\textsc{App}(I^+)$ of the constructed packing. Let us first assume that the second construction step does not open any new shelf. Then

$$\begin{aligned}
\textsc{App}(I^+) &= (s+1) \cdot \sum_{q \in \mathcal{Q}} \lceil \phi^*(q)/s \rceil \\
&\leq (s+1) \cdot (t + \sum_{q \in \mathcal{Q}} \phi^*(q)/s) \\
&= (s+1) \cdot t + \frac{s+1}{s} \, \textsc{Val}(\mathrm{LP}^+) \\
&\leq (s+1) \cdot t + \frac{s+1}{s} \left(\textsc{Opt}(I) + 2t + \frac{1}{s} A_{items} \right) \\
&\leq (1+\varepsilon) \, \textsc{Opt}(I) + \mathrm{const}(\varepsilon).
\end{aligned}$$

Here the first inequality follows since in a basic feasible solution at most t of the values $\phi^*(q)$ are non-zero. The second inequality follows from (9). The final inequality follows from the definitions of s and t, and from the fact that $\textsc{Opt}(I)$ is at least the total area A_{items}.

Next, let us assume that the second construction step opens at least one new shelf. Let us analyze the total area of the items in this packing: With at most one exception, every crate of width W_i and height $s+1$ ($i = 1, \dots, t$) is filled up to a level more than s, and to their full width; this yields a filling factor of $s/(s+1)$. With at most one exception, the crates of type k with width $\delta(1-\delta)^k$ and height $s+1$ is filled up to a level more than s, and to a width of at least $\delta(1-\delta)^{k+1}$; this yields a filling factor of $(1-\delta)s/(s+1)$. Let A_{crates} denote the total area of all crates in the packing. Then

$$\begin{aligned} A_{crates} &\le \frac{s+1}{s(1-\delta)} A_{items} + \sum_{i=1}^{t} W_i \cdot (s+1) + \sum_{k=0}^{\infty} \delta(1-\delta)^k \cdot (s+1) \\ &\le \frac{s+1}{s(1-\delta)} A_{items} + (t+1)(s+1). \end{aligned} \tag{10}$$

Here the first (finite) sum accounts for the exceptional crates from the first construction phase, and the second (infinite) sum accounts for the exceptional crates from the second phase. Moreover, we used $W_i \le 1$ to bound the terms in the first sum.

Every shelf (except possibly the top shelf) is filled by crates to a width of at least $1-\delta$. Therefore, the total area $\textsc{App}(I^+)$ of the constructed packing satisfies

$$\begin{aligned} \textsc{App}(I^+) &\le \frac{1}{1-\delta} A_{crates} + (s+1) \\ &\le \frac{s+1}{s(1-\delta)^2} A_{items} + \frac{(t+1)(s+1)}{1-\delta} + (s+1). \\ &\le (1+\varepsilon)\,\textsc{Opt}(I) + \mathrm{const}(\varepsilon). \end{aligned}$$

Here we first used the inequality in (10) to bound A_{crates}, and then we used the fact that $\textsc{Opt}(I)$ is at least the total area A_{items}.

To summarize, in either case the height $\textsc{App}(I^+)$ of the just constructed approximate packing for instance I^+ satisfies

$$\textsc{App}(I^+) \;\le\; (1+\varepsilon)\,\textsc{Opt}(I) + \mathrm{const}(\varepsilon). \tag{11}$$

The constructed packing is a 4-stage Guillotine strip packing: The shelves can be separated from each other by the first stage of Guillotine cuts. In the second stage, we can split every shelf into crates. A third and fourth stage are sufficient to cut every crate into pieces.

3.4. The approximation scheme

We just have to put the above things together: For a given strip packing instance I with n items, we first compute the corresponding plus instance I^+. This can be done in time polynomial in n.

Then we determine an optimal basic feasible solution ϕ^* for the linear program (LP^+). Since $|\mathcal{Q}| \leq (1/\delta)^t$ and since $H_i \leq n$ for all $i = 1, \ldots, t$, this solution can be found by standard linear programming algorithms in time polynomial in $\log n$ and exponential in $1/\varepsilon$. Such a time complexity would be sufficient for an APTAS, an asymptotic polynomial time approximation scheme. By envoking the result of Karmarkar & Karp [11] on fractional one-dimensional bin packing, one can even reach a time complexity that is polynomial in $1/\varepsilon$ and $\log n$; this time complexity leads to the desired AFPTAS.

Then we compute a packing for I^+ of height $\text{APP}(I^+) \leq (1+\varepsilon)\text{OPT}(I) + \text{const}(\varepsilon)$ as described in Section 3.3 and in (11). This computation consists of simple grouping and classification steps, and clearly can be done within time polynomial in n. Finally, we replace every big item (h, w) in B_i^+ by the corresponding big item of the same height h and width $\leq w$ in B_i. This yields a packing for the original instance I of height at most $(1+\varepsilon)\text{OPT}(I) + \text{const}(\varepsilon)$, and this packing is a 4-stage Guillotine packing. This completes the proof of Theorem 1.3.

4. Discussion

In this paper, we gave a complete analysis of the quality of k-stage Guillotine strip packings versus globally optimal packings: Whereas $k = 2$ stages cannot guarantee any bounded asymptotic performance ratio, $k = 3$ stages already lead to an asymptotic performance ratio of 1.69103, and $k = 4$ stages yield asymptotic performance ratios arbitrarily close to 1. Hence, with respect to the asymptotic performance ratio there is no need to use $k = 5$ or more stages, since they can not improve on $k = 4$ stages.

The situation changes drastically when we consider the *absolute* performance ratio of k-stage Guillotine strip packings: For $k \geq 2$, let $\rho(k)$ denote the supremum of $G_k(I)/\text{OPT}(I)$ over all instances I (we recall that $G_k(I)$ denotes the height of the best k-stage Guillotine packing for instance I). Note that this time we do *not* allow any additive constant in the definition of the performance ratio. It is easy to conclude from the results in this paper that $\rho(2) = \infty$, and that for $k \geq 3$ the values $\rho(k)$ are finite. The instance depicted in Figure 1(a) illustrates that $\rho(k) \geq 4/3$ holds for all k. We have no idea about the exact values of $\rho(k)$ for $k \geq 3$. We expect that in order to determine the exact values $\rho(k)$, one will have to analyze many small instances and many special cases.

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