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A note on treating a second order cone program as a special case of a semidefinite program

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Abstract. It is well known that a vector is in a second order cone if and only if its "arrow" matrix is positive semidefinite. But much less well-known is about the relation between a second order cone program (SOCP) and its corresponding semidefinite program (SDP). The correspondence between the dual problem of SOCP and SDP is quite direct and the correspondence between the primal problems is much more complicated. Given a SDP primal optimal solution which is not necessarily "arrow-shaped", we can construct a SOCP primal optimal solution. The mapping from the primal optimal solution of SDP to the primal optimal solution of SOCP can be shown to be unique. Conversely, given a SOCP primal optimal solution, we can construct a SDP primal optimal solution which is not an "arrow" matrix. Indeed, in general no primal optimal solutions of the SOCP-related SDP can be an "arrow" matrix.

1. Introduction

In the literature, [1, 4, 5, 7, 8][1], when one refers to second order cone program (SOCP) as a special case of semidefinite program (SDP), one always use equivalence (1) below, which states that a vector is in a second order cone if and only if its "arrow" matrix is positive semidefinite. This can directly be applied on the dual problem of SOCP to obtain the dual problem of the corresponding SDP, as will be shown below. The relationship between the two dual problems is clear. However, when we consider the primal SOCP and try to relate it to its corresponding primal SDP, the situation is not as simple. In fact, we will show later that in general, no "arrow" matrix can be an optimal solution to the primal SDP.

Consider the primal second order cone problem $(\mathcal{P})$ given as follows:

$$(\mathcal{P}) \qquad \begin{array}{ll} \min & \sum_{i=1}^{n} c_i^T x_i \\ \text{subject to} & \sum_{i=1}^{n} A_i x_i = b \\ & \|\overline{x}_i\| \leq (x_i)_0 \quad i = 1, \ldots, n \end{array}$$

Here $x_i = ((x_i)_0, \overline{x}_i^T)^T \in \Re^{k_i+1}$ and $A_i \in \Re^{m\times(k_i+1)}$.

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[1] Note that [2, 3, 6, 9, 10] are related to second order cone programming, hence they are also included as references.

We can treat the second order cone program as a special case of semidefinite program using the following equivalence:

$$\|\bar{x}\| \le (x)_0 \iff Arw(x) := \begin{pmatrix} x_0 & \bar{x}^T \\ \bar{x} & x_0 I \end{pmatrix} \succeq 0 \tag{1}$$

where $A \succeq 0$ means that A is symmetric, positive semidefinite.

In order to write a SOCP as a SDP, it is more convenient to look at the dual problem $(\mathcal{D})$ of $(\mathcal{P})$,

$$(\mathcal{D}) \qquad \begin{array}{ll} \max \ b^T y & \\ \text{subject to } A_i^T y + s_i = c_i & i = 1, \ldots, n \\ \|\bar{s}_i\| \le (s_i)_0 & i = 1, \ldots, n \end{array}$$

where $y \in \Re^m$ and $s_i = ((s_i)_0, \bar{s}_i^T)^T \in \Re^{k_i+1}$. Now let $A_i^T = \begin{pmatrix} a_i^T \\ \tilde{A}_i \end{pmatrix}$ where $a_i \in \Re^m$ and $\tilde{A}_i \in \Re^{k_i \times m}$. By writing $\tilde{A}_i$ as $(\tilde{a}_i^1, \ldots, \tilde{a}_i^m) \in \Re^{k_i \times m}$, a_i as $(a_i^1, \ldots, a_i^m)^T \in \Re^m$ and using equivalence (1), we can easily obtain $(\mathcal{D}')$, the dual SDP problem,

$$(\mathcal{D}') \qquad \begin{array}{ll} \max \ b^T y & \\ \text{subject to } \sum_{j=1}^m y_j Arw((a_i^j \ \tilde{a}_i^{jT})^T) + S_i = Arw(c_i) & i = 1, \ldots, n \\ S_i \succeq 0 & i = 1, \ldots, n \end{array}$$

This is precisely the dual of the primal SDP $(\mathcal{P}')$ given below:

$$(\mathcal{P}') \qquad \begin{array}{ll} \min \ \sum_{i=1}^n Arw(c_i) \bullet X_i & \\ \text{subject to } \sum_{i=1}^n Arw((a_i^j \ \tilde{a}_i^{jT})^T) \bullet X_i = b_j & j = 1, \ldots, m \\ X_i \succeq 0 & i = 1, \ldots, n \end{array}$$

Here $X_i \in \Re^{(k_i+1)\times(k_i+1)}$ and symmetric.

Note that we have obtained $(\mathcal{D}')$, starting from $(\mathcal{D})$, by merely using equivalence (1). One can see that there is a natural correspondence between the feasible sets of $(\mathcal{D})$ and $(\mathcal{D}')$, and the objective functions of both problems are exactly identical. Hence, it is clear that given an optimal solution $(s_1^*, \ldots, s_n^*, y^*)$ for $(\mathcal{D})$, one can easily obtain an optimal solution $(S_1^*, \ldots, S_n^*, y^*)$ for $(\mathcal{D}')$ and vice versa, because of equivalence (1).

Does the relation $x \leftrightarrow Arw(x)$, given by relation (1), also build the equivalence between $(\mathcal{P})$ and $(\mathcal{P}')$? The answer is "no". One can easily verify that x and $X = Arw(x)$ may not simultaneously satisfy both constraints $a_i^T x = b_i$ and $Arw(a_i) \bullet X = b_i$.

2. Equivalence between $(\mathcal{P})$ and $(\mathcal{P}')$

Having observed the above, we may ask, given an optimal solution $(x_1^*, \ldots, x_n^*)$ of $(\mathcal{P})$, what is an optimal solution for $(\mathcal{P}')$? Also, given an optimal solution $(X_1^*, \ldots, X_n^*)$ of $(\mathcal{P}')$, which is not necessarily of "arrow-shaped", what is an optimal solution of $(\mathcal{P})$? In this section, we will answer these questions.

Wlog, we consider $(\mathcal{P})$ and $(\mathcal{P}')$ for $n = 1$; our results are easily extended to $n \ge 2$.

We denote by S^{k+1} the space of all symmetric $(k+1)\times(k+1)$ matrices, $S^{k+1}_+ := \{X \in S^{k+1} : X \succeq 0\}$, and $C^{k+1} := \{x = (x_0, \bar{x}^T)^T \in \Re^{k+1} : \|\bar{x}\| \le x_0\}$.

In order to build the equivalence between $(\mathcal{P})$ and $(\mathcal{P}')$, we should find a correspondence between x and X such that $a_i^T x = b_i \iff Arw(a_i) \bullet X = b_i$ for any pair of corresponding constraints in $(\mathcal{P})$ and $(\mathcal{P}')$ respectively. Thus the following lemma is crucial:

Lemma 1. *$X = (X_{ij})_{0\le i\le k, 0\le j\le k} \in S^{k+1}$ and $x = (x_0, \dots, x_k)^T \in \Re^{k+1}$.*

$$Arw(u) \bullet X = u^T x \text{ holds for all } u \in \Re^{k+1} \qquad (\star)$$

if and only if

$$\sum_{i=0}^{k} X_{ii} = x_0 \;\text{ and }\; 2X_{i0} = x_i, \;\; i = 1, \dots, k \tag{2}$$

The above lemma leads to the following two maps.

Definition 1. *Define a point-to-set map $\tilde{M}$ which maps a point $x = (x_0, x_1, \dots, x_k)^T \in C^{k+1}$ to a subset of S^{k+1}*

$$\tilde{M}(x) := \Big\{X = (X_{ij})_{0\le i\le k, 0\le j\le k} \in S^{k+1} \mid \textstyle\sum_{i=0}^{k} X_{ii} = x_0 \text{ and } X_{i0} = \frac{1}{2}x_i, i = 1, \dots, k\Big\},$$

and a map $M' : S^{k+1} \longmapsto R^{k+1}$ by $M'(X) = (\sum_{i=0}^{k} X_{ii},\ 2X_{10}, \dots,\ 2X_{k0})^T$ for $X = (X_{ij})_{0\le i\le k, 0\le j\le k} \in S^{k+1}$.

Theorem 1. *(a) For every $x \in C^{k+1}$, there exists an $X \in \tilde{M}(x)$ which is positive semidefinite. Indeed, $0 \in \tilde{M}(0)$, and for $x \ne 0$*

$$M(x) := \begin{pmatrix} \frac{1}{4}\theta & \frac{1}{2}\bar{x}^T \\ \frac{1}{2}\bar{x} & \frac{x_0 - \|\bar{x}\|}{2k} I + \frac{\bar{x}\bar{x}^T}{\theta} \end{pmatrix} \in \tilde{M}(x) \cap S^{k+1}_+, \tag{3}$$

where $\theta = x_0 + \|\bar{x}\| + \sqrt{(x_0 + \|\bar{x}\|)^2 - 4\|\bar{x}\|^2}$. Moreover, $x \in \mathrm{int}C^{k+1} \Rightarrow M(x) \in \mathrm{int}S^{k+1}_+$.

(b) For any $X \in S^{k+1}_+$, $M'(X) \in C^{k+1}$. Moreover, $X \in \mathrm{int}S^{k+1}_+ \Rightarrow M'(X) \in \mathrm{int}C^{k+1}$.

(c) Let $x \ne 0$. Then $x \in \partial C^{k+1}$, i.e., $x_0 = \|\bar{x}\|$, if and only if $\tilde{M}(x) \cap S^{k+1}_+$ is a singleton whose unique element is $\begin{pmatrix} \frac{1}{2}x_0 & \frac{1}{2}\bar{x}^T \\ \frac{1}{2}\bar{x} & \frac{\bar{x}\bar{x}^T}{2x_0} \end{pmatrix}$.

Proof. (a) For any $0 \ne x \in C^{k+1}$, an element X in $\tilde{M}(x) \cap S^{k+1}_+$ must satisfy condition (2) as well as positive semidefiniteness, thus it must have the form

$$\begin{pmatrix} X_{00} & \frac{1}{2}\bar{x}^T \\ \frac{1}{2}\bar{x} & B \end{pmatrix} \tag{4}$$

satisfying

$$B \succeq \frac{\overline{x}\overline{x}^T}{4X_{00}}, \qquad Tr(B) + X_{00} = x_0. \tag{5}$$

A simple form of matrix B that satisfy the first condition in (5) is $B = \beta I + \frac{\overline{x}\overline{x}^T}{4X_{00}}$. With this B, the second condition in (5) becomes $k\beta + \frac{\|\bar{x}\|^2}{4X_{00}} + X_{00} = x_0$. A necessary condition for the last equation to have real solution X_{00} is $\beta \le \frac{x_0 - \|\bar{x}\|}{k}$. Choose $\beta = \frac{x_0 - \|\bar{x}\|}{2k}$. One can easily verify that $X_{00} = \frac{1}{4}\theta$ (θ is given in the theorem) satisfies the equation. Moreover, if $x \in \text{int}C^{k+1}$, $\beta > 0$ and $\theta > 0$ implies that $M(x) \in \text{int}S_+^{k+1}$.

(b) To show that indeed given $X \in S_+^{k+1}$, $M'(X) \in C^{k+1}$, we need only show that $(\sum_{i=0}^k X_{ii})^2 \ge 4\sum_{i=1}^k X_{i0}^2$. Assume $X_{00} > 0$. Denote $\overline{x} = 2(X_{10}, \ldots, X_{k0})^T$. Now X (in the form of (4)) is positive semidefinite implies that the submatrix $B - \frac{\overline{x}\overline{x}^T}{4X_{00}}$ is also positive semidefinite. Hence the trace of the submatrix

$$tr(B - \frac{\overline{x}\overline{x}^T}{4X_{00}}) = \sum_{i=1}^{k} X_{ii} - \frac{\|\bar{x}\|^2}{4X_{00}} \ge 0. \tag{6}$$

This implies that $\sum_{i=1}^k X_{i0}^2 \le X_{00}\sum_{i=1}^k X_{ii}$. Using the basic result, $ab \le (a+b)^2/4$, we show that $4\sum_{i=1}^k X_{i0}^2 \le (\sum_{i=0}^k X_{ii})^2$, hence $M'(X) \in C^{k+1}$. By changing inequalities to strict inequalities, this argument can be used to show that $X \in \text{int}S_+^{k+1} \Rightarrow M'(X) \in \text{int}C^{k+1}$.

(c) "$\Rightarrow$" Suppose $0 \ne x \in \partial C^{k+1}$. Let X be any element in $\tilde{M}(x) \cap S_+^{k+1}$. Similar to the proof of (b), we have inequality (6). Because $\sum_{i=0}^k X_{ii} = x_0$ and $x_0 = \|\bar{x}\|$, (6) is equivalent to

$$tr(B - \frac{\overline{x}\overline{x}^T}{4X_{00}}) = -\frac{(X_{00} - x_0/2)^2}{X_{00}} \ge 0$$

This only holds if $X_{00} = x_0/2$. Thus, the trace of the submatrix $B - \frac{\overline{x}\overline{x}^T}{4X_{00}}$ is zero. This implies that the submatrix itself is zero since the submatrix is positive semidefinite. Therefore, $B = \frac{\overline{x}\overline{x}^T}{4X_{00}}$ and $X_{00} = x_0/2$. This shows that $X = \begin{pmatrix} \frac{1}{2}x_0 & \frac{1}{2}\bar{x}^T \\ \frac{1}{2}\bar{x} & \frac{\bar{x}\bar{x}^T}{2x_0} \end{pmatrix}$.

"$\Leftarrow$" Follows from the definition of $\tilde{M}(x)$ and $x_0 \ge 0$. □

Now, the equivalence between $(\mathcal{P})$ and $(\mathcal{P}')$ follows immediately from the above theorem, noticing that Condition ($\star$) also implies that the objective values for $(\mathcal{P})$ and $(\mathcal{P}')$ are equal.

Corollary 1. *Given the data set* (A, c, b) *of a SOCP from which we obtained* $(\mathcal{P})$ *and* $(\mathcal{P}')$. $(\mathcal{P})$ *and* $(\mathcal{P}')$ *are equivalent in the sense that the point-to-set map* $\tilde{M}$ *has the property that any element of* $\tilde{M}(x^*) \cap S_+^{k+1}$ *is an optimal solution of* $(\mathcal{P}')$ *if* x^* *is an optimal solution of* $(\mathcal{P})$ *and the map* M' *is such that* $M'(X^*)$ *is an optimal solution of* $(\mathcal{P})$ *whenever* X^* *is an optimal solution of* $(\mathcal{P}')$.

We state below some properties of the maps $\tilde{M}$ and M'.

(i) $\tilde{M}(x) = (M')^{-1}(\{x\})$ for every $x \in C^{k+1}$,
(ii) $\tilde{M}(x) \cap \tilde{M}(x') = \emptyset$ for $x \neq x' \in C^{k+1}$, $\bigcup_{x \in C^{k+1}} \tilde{M}(x) \cap S_+^{k+1} = S_+^{k+1}$.
(iii) M' is the unique map such that $X \in S_+^{k+1}$ and $M'(X)$ satisfy Condition $(\star)$.

Our final remark is that most instances of primal SDP $(\mathcal{P}')$ have no "arrow-shaped" optimal solution because for most instances, all its optimal solutions are on the boundary of the cone. In this case, we must have that, by Theorem 1(c), every nonzero primal optimal solution is of the form $\frac{1}{2}\begin{pmatrix} x_0^* & (\bar{x}^*)^T \\ \bar{x}^* & \frac{\bar{x}^*(\bar{x}^*)^T}{x_0^*} \end{pmatrix}$, for some $x^* \in \partial C^{k+1}$, which is not arrow-shaped. An example of when the above occurs is if the strong duality holds for the SDP pair $(\mathcal{P}') - (\mathcal{D}')$ and there exists a (dual) optimal solution (S^*, y^*) such that $S^* \neq 0$.

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