Math. Program., Ser. B 98: 319–353 (2003)

Digital Object Identifier (DOI) 10.1007/s10107-003-0407-5

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The stable set problem and the lift-and-project ranks of graphs

Received: May 31, 2002 / Accepted: January 3, 2003
Published online: April 10, 2003 – © Springer-Verlag 2003

Abstract. We study the lift-and-project procedures for solving combinatorial optimization problems, as described by Lovász and Schrijver, in the context of the stable set problem on graphs. We investigate how the procedures' performances change as we apply fundamental graph operations. We show that the odd subdivision of an edge and the subdivision of a star operations (as well as their common generalization, the stretching of a vertex operation) cannot decrease the N_0-, N-, or N_+-rank of the graph. We also provide graph classes (which contain the complete graphs) where these operations do not increase the N_0- or the N-rank. Hence we obtain the ranks for these graphs, and we also present some graph-minor like characterizations for them. Despite these properties we give examples showing that in general most of these operations can increase these ranks. Finally, we provide improved bounds for N_+-ranks of graphs in terms of the number of nodes in the graph and prove that the subdivision of an edge or cloning a vertex can increase the N_+-rank of a graph.

Key words. Stable set problem – Lift-and-project – Semidefinite lifting – Semidefinite programming – Integer programming

1. Introduction

We are interested in the lift-and-project procedures for solving combinatorial optimization problems as described by Lovász and Schrijver [16] (see also Balas et al. [3] and Sherali and Adams [17]) and their performances on the stable set problem.

In Section 2 we introduce the lift-and-project procedures, set up some notations, and mention a few well-known facts about these procedures and the stable set problem. We study three lift-and-project procedures, denoted by N_0, N, and N_+ in the order of increasing strength, and define graph ranks r_0, r, and r_+ based on these procedures.

In Section 3 we observe that when a graph can be decomposed into two smaller graphs such that the intersection of the smaller graphs is a clique, then the rank of the original graph is the maximum of the ranks of the smaller graphs. Then we prove that when the graph has a cut vertex, a much stronger property holds, which can be generalized to polytopes that have a similar "cut coordinate." In both cases the behaviours of N_0, N, and N_+ can be completely described by the behaviours of these procedures on the smaller, decomposed pieces alone. These considerations naturally lead us to a

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Mathematics Subject Classification (2000): 0C10, 90C22, 90C27, 47D20

problem posed by Lovász in 1992, involving perfect graphs, which we solve here as a by-product of our work.

Section 4 is mostly concerned with procedures N_0 and N; however, many of our proofs apply also to the N_+ procedure. Among other results, we prove that r_0, r, and r_+ are monotone nondecreasing under the subdivision of a star and the odd subdivision operations (as well as under their common generalization, the stretching of a vertex operation) on the given graph, and we give an excluded-induced-subgraph characterization of odd-star subdivisions of cliques. We also prove that a subdivision of a clique is odd-star if and only if r_0 and r of the graph coincide and are both equal to two less than the order of the clique which gave rise to the subdivision. Various related technical tools which may be of independent interest are also developed in this section.

Section 5 contains some elementary facts around the similarities and differences of N_0 and N as well as the behaviour of these procedures under fundamental graph operations.

In Section 6 we study certain α-critical graphs which arise from the line graphs of nested blossoms (blossom inequalities on odd cliques are known to translate to high N_+-rank values [18] when we consider the corresponding stable set problem via taking the line graph). We prove that for such graphs, r_0 and r grow only logarithmically with the dimension.

Finally, in Section 7 we focus exclusively on r_+ and prove improved upper bounds for it.

2. Notations and basic properties

The lift-and-project procedures can be defined as follows: Let $K \in \mathbb{R}^{n+1}$ be a convex cone (the components of the vectors are indexed by $0, 1, \ldots, n$, and the elements of K are denoted by $y = (y_0, y_1, \ldots, y_n)^T =: \begin{pmatrix} y_0 \\ x \end{pmatrix}$, where $x \in \mathbb{R}^n$) such that $\left\{x \in \mathbb{R}^n : \begin{pmatrix} 1 \\ x \end{pmatrix} \in K\right\} \subseteq [0, 1]^n$. We use e_j to denote the jth unit vector and $\bar{e}$ to denote the vector of all 1's. In this paper, all vectors are column vectors. The linear operator $\operatorname{diag} : \mathbb{R}^{(n+1)\times(n+1)} \to \mathbb{R}^{n+1}$ takes $Y \in \mathbb{R}^{(n+1)\times(n+1)}$ and returns a vector whose ith component is Y_{ii}. Then

$$M_0(K) := \Big\{Y \in \mathbb{R}^{(n+1)\times(n+1)} : \; Ye_0 = Y^T e_0 = \operatorname{diag}(Y), \\ Ye_i \in K, \forall i \in 1, 2, \ldots, n, \\ Y(e_0 - e_i) \in K, \forall i \in \{1, 2, \ldots, n\}\Big\}$$

defines a lifting of the next relaxation (potentially tighter than K) of the cone of all 0-1 vectors in K. We project it back onto the space of K to get

$$N_0(K) := \{Ye_0 : Y \in M_0(K)\}.$$

Let Σ^{n+1} denote the space of $(n+1) \times (n+1)$ symmetric matrices with real entries. We can now restrict the above lifting to symmetric matrices:

$$M(K) := \Big\{Y \in \Sigma^{n+1} : \; Ye_0 = \operatorname{diag}(Y),$$

$$\left. \begin{aligned} &Ye_i \in K, \forall i \in \{1, 2, \dots, n\}, \\ &Y(e_0 - e_i) \in K, \forall i \in \{1, 2, \dots, n\} \end{aligned} \right\}.$$

Projecting this lifting back results in

$$N(K) := \{Ye_0 : Y \in M(K)\}.$$

Finally, to obtain tighter relaxations, we can restrict the matrix Y to be positive semidefinite. Let Σ_+^{n+1} denote those elements Y of Σ^{n+1} which are positive semidefinite, i.e., satisfy $h^T Yh \geq 0$ for all $h \in \mathbb{R}^{n+1}$. Defining

$$M_+(K) := \left\{ Y \in \Sigma_+^{n+1} : \begin{aligned} &Ye_0 = \text{diag}(Y), \\ &Ye_i \in K, \forall i \in \{1, 2, \dots, n\}, \\ &Y(e_0 - e_i) \in K, \forall i \in \{1, 2, \dots, n\} \end{aligned} \right\}$$

and projecting it back yields the relaxation

$$N_+(K) := \{Ye_0 : Y \in M_+(K)\}.$$

Our main interest lies in the sets

$$\left\{ x \in \mathbb{R}^n : \begin{pmatrix} 1 \\ x \end{pmatrix} \in N_0(K) \right\}$$

(and similarly for $N(K)$ and $N_+(K)$). For simplicity, when we say that we are *applying* the N_0, N, or N_+ operator to some convex set $P \subseteq [0, 1]^n$, we mean that we consider the cone corresponding to this convex set, apply the corresponding lifting-projecting procedure, then take the convex subset of $[0, 1]^n$ defined by the intersection of this new cone with $y_0 = 1$. $N_0(P)$, $N(P)$, or $N_+(P)$, resp., will denote this final subset of $[0, 1]^n$, and we will use $N_0^k(P)$, $N^k(P)$, and $N_+^k(P)$ to indicate that we applied the corresponding operator k times in succession ($k = 0$ will refer to the original polytope, so $N_0^0(P) = P$, etc.).

The following fact is well-known and some related insights also exist in Balas' work from the 1970s (see [2]).

Lemma 1. *Let F be any face of $[0, 1]^n$ and $P \subseteq [0, 1]^n$. Then*

$$N(P \cap F) = N(P) \cap F.$$

Similarly for N_+ and N_0.

Applying Lemma 1 finitely many times we get

Corollary 2. *Let F be any face of $[0, 1]^n$ and $P \subseteq [0, 1]^n$. Then for every $k \geq 0$,*

$$N^k(P \cap F) = N^k(P) \cap F.$$

Similarly for N_+ and N_0.

Let $G := (V, E) := (V(G), E(G))$ denote a finite, undirected, simple graph with *vertex* or *node* set V and *edge* set E. In what follows, K_n denotes the complete graph on n vertices. We define the *fractional stable set polytope* as

$$FRAC(G) := \left\{ x \in [0,1]^V : x_i + x_j \leq 1 \text{ for all } \{i, j\} \in E \right\}.$$

This polytope is used as the initial approximation to the convex hull of incidence vectors of the *stable sets* of G (sets of vertices such that no two of them are joined by an edge), which is called the *stable set polytope*:

$$STAB(G) := \text{conv}\left(FRAC(G) \cap \{0,1\}^V \right).$$

For all $v \in V(G)$ let $G - v$ denote the graph defined by $V(G - v) := V(G) \setminus \{v\}$ and $E(G - v) := E(G) \setminus \{\{u, v\} \in E(G) : u \in V(G)\}$, and let $\Gamma_G(v) := \Gamma(v)$ denote the *neighbourhood* of v in G:

$$\Gamma(v) := \{u \in V : \{u, v\} \in E\}.$$

Let $G \ominus v$ be defined by

$$V(G \ominus v) := V \setminus (\Gamma(v) \cup \{v\})$$

and

$$E(G \ominus v) := \{\{u, w\} \in E(G) : u, w \notin (\Gamma_G(v) \cup \{v\})\}.$$

This operation was called the *contraction of* v in [16]; here we call it the *destruction of* v. For any edge $e \in E$ let $G - e$ denote the graph obtained from G by the deletion of the edge e. If the inequality $a^T x = \sum_{u \in V(G)} a(u) x_u \leq b$ is valid for $STAB(G)$, so are $\sum_{u \in V(G-v)} a(u) x_u \leq b$ and $\sum_{u \in V(G \ominus v)} a(u) x_u \leq b - a(v)$, obtained by the deletion and the destruction of the vertex v, respectively.

For a given graph $G = (V, E)$, its N_0-*rank* (and similarly its N-*rank* and N_+-*rank*) is defined as the smallest nonnegative integer k for which the application of the N_0 (N or N_+) operator k times to $FRAC(G)$ gives $STAB(G)$. Alternatively, this rank is the largest rank of a facet of $STAB(G)$ (the N_0-, N-, and N_+-rank of an inequality valid for $STAB(G)$ is defined as the minimum k for which the inequality is valid for $N_0^k(FRAC(G))$, $N^k(FRAC(G))$, and $N_+^k(FRAC(G))$ resp.). We denote these ranks by $r_0(G)$, $r(G)$, and $r_+(G)$, respectively. To simplify the notation we write $N_0^k(G)$, $N^k(G)$, and $N_+^k(G)$ for $N_0^k(FRAC(G))$, $N^k(FRAC(G))$, and $N_+^k(FRAC(G))$ respectively. The following two lemmas are due to Lovász and Schrijver [16]:

Lemma 3. *For all graphs G, we have*

$$r(G) \leq r_0(G) \leq \min_{v \in V} \{r_0(G - v)\} + 1 \quad \textit{and} \quad r(G) \leq \min_{v \in V} \{r(G - v)\} + 1.$$

Moreover, if the inequalities obtained from $a^T x \leq b$ by the deletion and destruction of $v \in V$ are valid for $N_0^k(G)$, $N^k(G)$, resp., then $a^T x \leq b$ is valid for $N_0^{k+1}(G)$, $N^{k+1}(G)$, respectively.

Lemma 4. *For all graphs G, we have*

$$r_+(G) \leq \max_{v \in V} \{r_+(G \ominus v)\} + 1.$$

So node deletion can only decrease any of the ranks, and at most by one (for N_+ this will follow later from Theorem 36).

Another fact that makes the procedures N_0, N, and N_+ very interesting is that we can optimize a linear function over any of $N_0^k(G)$, $N^k(G)$, and $N_+^k(G)$ in polynomial time, provided $k = O(1)$. (See [16, 7].)

So far, the abovementioned ranks and some of their relatives have been studied from many points of view (see [3, 5, 6, 10, 12, 16, 18]). However, many important open questions remain. Our goal here is to improve some of the known bounds on these ranks for the stable set problem and to deal with some of those open problems related to the behaviour of these ranks under fundamental graph operations.

The area of geometric representations of graphs (see Lovász [13], Grötschel, Lovász and Schrijver [7], Kotlov, Lovász and Vempala [9] and Lovász [15]) is closely connected to the subject of this paper. Even though there has been a lot of progress in understanding geometric embeddings of graphs and invariants like the Colin de Verdière number of a graph, due to the fact that we can optimize any linear function over any of $N_0^k(G)$, $N^k(G)$, and $N_+^k(G)$ in polynomial time, provided $k = O(1)$, investigating N_0-, N- and N_+-ranks of graphs and further understanding of graphs of small rank remain very interesting.

3. An elementary decomposition

Chvátal [4] has shown that if the graph G can be decomposed into two parts, G_1 and G_2, so that their intersection is a complete graph, then the facets of $STAB(G)$ are just the union of the facets of $STAB(G_1)$ and $STAB(G_2)$. Hence we get a similar property for the N_0-rank, N-rank and the N_+-rank:

Lemma 5. *If $G = G_1 \cup G_2$ such that $G_1 \cap G_2$ is a complete graph, then*

$$r_0(G) = \max\{r_0(G_1), r_0(G_2)\},$$
$$r(G) = \max\{r(G_1), r(G_2)\}$$

and

$$r_+(G) = \max\{r_+(G_1), r_+(G_2)\}.$$

Proof. In Corollary 2 let $F := \left\{x \in [0,1]^{V(G)} : x_v = 0 \text{ for all } v \in V(G_2) \backslash V(G_1)\right\}$. Then Corollary 2 and Chvátal's result imply $r_0(G) \geq r_0(G_1)$, $r(G) \geq r(G_1)$, and $r_+(G) \geq r_+(G_1)$. Analogously we get $r_0(G) \geq r_0(G_2)$, $r(G) \geq r(G_2)$, and $r_+(G) \geq r_+(G_2)$. To establish the reverse inequalities we utilize Chvátal's result again and conclude that to derive all facets of $STAB(G)$, it suffices to derive all facets of $STAB(G_1)$ and $STAB(G_2)$. By Corollary 2, the latter can be achieved in at most max $\{r_0(G_1), r_0(G_2)\}$, max $\{r(G_1), r(G_2)\}$, and max $\{r_+(G_1), r_+(G_2)\}$ iterations of the N_0, N, and N_+ operators respectively. □

The usual application of Lemma 5 occurs when there is a cut vertex v in G. In this case, we can prove the following stronger result:

Theorem 6. *If $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$, then $N_0^k(G)$ ($N^k(G)$) is defined by the facets of the polytopes $N_0^k(G_1)$ and $N_0^k(G_2)$ ($N^k(G_1)$ and $N^k(G_2)$) for every $k \geq 0$.*

Proof. We prove the claim by induction on k. For $k = 0$ the claim is easy (see the proof of Lemma 10), so we assume that the statement is true for $k - 1 \geq 0$ and prove it for k. First consider the N_0 operator only.

Lemma 1 implies that any $x \in N_0^k(G)$ must also satisfy the inequalities defining $N_0^k(G_1)$ and $N_0^k(G_2)$, hence it is enough to show the other way around. Assume that

$$x = \begin{pmatrix} x^{(1)} \\ x_v \\ x^{(2)} \end{pmatrix}$$

satisfies the inequalities defining $N_0^k(G_1)$ and $N_0^k(G_2)$ with $x^{(\ell)} \in \mathbb{R}^{V(G_\ell)\setminus\{v\}}$ for $\ell = 1, 2$. Then there are matrices $X^{(\ell)} = (X_{ij}^{(\ell)}) \in \mathbb{R}^{(V(G_\ell)\setminus\{v\})\times(V(G_\ell)\setminus\{v\})}$ and vectors $y^{(1,\ell)}, y^{(2,\ell)} \in \mathbb{R}^{V(G_\ell)\setminus\{v\}}$ for $\ell = 1, 2$ such that

$$Y^{(\ell)} = \left(\begin{array}{c|c|c} 1 & (x^{(\ell)})^T & x_v \\ \hline x^{(\ell)} & X^{(\ell)} & y^{(1,\ell)} \\ \hline x_v & (y^{(2,\ell)})^T & x_v \end{array}\right) \in M_0^k(G_\ell),$$

thus showing that $\begin{pmatrix} x^{(\ell)} \\ x_v \end{pmatrix} \in N_0^k(G_\ell)$. By definition this means that for all $u \in V(G_\ell)$ we have $X_{uu}^{(\ell)} = x_u^{(\ell)}$ and that the vectors $Y^{(\ell)} e_u$ and $Y^{(\ell)}(e_0 - e_u)$ are all in the cone induced by $N_0^{k-1}(G_\ell)$. Define now the following matrix that will show that $x \in N_0^k(G)$:

$$\overline{Y} := \left(\begin{array}{c|c|c|c} 1 & (x^{(1)})^T & x_v & (x^{(2)})^T \\ \hline x^{(1)} & X^{(1)} & y^{(1,1)} & \overline{X}^{(1)} \\ \hline x_v & (y^{(2,1)})^T & x_v & (y^{(2,2)})^T \\ \hline x^{(2)} & \overline{X}^{(2)} & y^{(1,2)} & X^{(2)} \end{array}\right),$$

where the matrices $\overline{X}^{(1)}$ and $\overline{X}^{(2)}$ are defined as follows:

$$\overline{X}^{(1)} := \frac{1}{1 - x_v}(x^{(1)} - y^{(1,1)})(x^{(2)} - y^{(2,2)})^T + \frac{1}{x_v} y^{(1,1)}(y^{(2,2)})^T,$$
$$\overline{X}^{(2)} := \frac{1}{1 - x_v}(x^{(2)} - y^{(1,2)})(x^{(1)} - y^{(2,1)})^T + \frac{1}{x_v} y^{(1,2)}(y^{(2,1)})^T.$$

If $x_v = 0$ or 1, we keep only the terms that make sense. The idea of this definition is that now we can write

$$\begin{pmatrix} (x^{(1)})^T \\ \overline{X}^{(2)} \\ (y^{(2,1)})^T \end{pmatrix} = \frac{1}{1 - x_v}\left(\begin{pmatrix} 1 \\ x^{(2)} \\ x_v \end{pmatrix} - \begin{pmatrix} x_v \\ y^{(1,2)} \\ x_v \end{pmatrix}\right)(x^{(1)} - y^{(2,1)})^T + \frac{1}{x_v}\begin{pmatrix} x_v \\ y^{(1,2)} \\ x_v \end{pmatrix}(y^{(2,1)})^T \tag{1}$$

and

$$\begin{pmatrix} (x^{(2)})^T \\ \overline{X}^{(1)} \\ (y^{(2,2)})^T \end{pmatrix} = \frac{1}{1-x_v}\left(\begin{pmatrix} 1 \\ x^{(1)} \\ x_v \end{pmatrix} - \begin{pmatrix} x_v \\ y^{(1,1)} \\ x_v \end{pmatrix}\right)(x^{(2)} - y^{(2,2)})^T$$
$$+\frac{1}{x_v}\begin{pmatrix} x_v \\ y^{(1,1)} \\ x_v \end{pmatrix}(y^{(2,2)})^T. \tag{2}$$

Now, to show that $\overline{Y} \in M_0^k(G)$, we need to check that for all $u \in V(G)$ we have $\overline{Y}_{uu} = x_u$ and that $\overline{Y}e_u$ and $\overline{Y}(e_0 - e_u)$ are all in the cone induced by $N_0^{k-1}(G)$. The first property trivially holds, for the remaining two we use the induction hypothesis, which says that in order to be in $N_0^{k-1}(G)$ these vectors must satisfy all valid inequalities for $N_0^{k-1}(G_\ell)$ for $\ell = 1, 2$.

Assume first that $u \in G_1 \setminus \{v\}$ and consider the vector $\overline{Y}e_u$. This satisfies the inequalities valid for $N_0^{k-1}(G_1)$, since so does $Y^{(1)}e_u$. To satisfy the valid inequalities for $N_0^{k-1}(G_2)$, we only need that

$$\begin{pmatrix} (x^{(1)})^T \\ \overline{X}^{(2)} \\ (y^{(2,1)})^T \end{pmatrix} e_u \tag{3}$$

is in the cone induced by $N_0^{k-1}(G_2)$, which follows immediately from (1), since (3) is a nonnegative linear combination of $Y^{(2)}(e_0 - e_u)$ and $Y^{(2)}e_u$.

We can similarly check that $\overline{Y}(e_0 - e_u)$ satisfies the inequalities valid both for $N_0^{k-1}(G_1)$ and for $N_0^{k-1}(G_2)$, since

$$\begin{pmatrix} 1 \\ x^{(2)} \\ x_v \end{pmatrix} - \begin{pmatrix} (x^{(1)})^T \\ \overline{X}^{(2)} \\ (y^{(2,1)})^T \end{pmatrix} e_u$$

is again a nonnegative linear combination of $Y^{(1)}(e_0 - e_u)$ and $Y^{(1)}e_u$ with coefficients

$$\left(1 - \frac{x_u^{(1)} - y_u^{(2,1)}}{1 - x_v}\right) \text{ and } \left(1 - \frac{y_u^{(2,1)}}{x_v}\right),$$

which are nonnegative since $Y^{(1)}(e_0 - e_u)$ and $Y^{(1)}e_u$ are in $N_0^{k-1}(G_1)$.

The case $u \in V(G_2) \setminus \{v\}$ is analogous, while in the case of $u = v$ it is enough to use the induction hypothesis, finishing the induction.

For the N-operator, it is easy to check that if the matrices $Y^{(1)}$ and $Y^{(2)}$ are symmetric, then so is $\overline{Y}$. □

Theorem 6 is basically valid for the N_+-operator as well; however, since $N_+(G)$ is usually nonpolyhedral, we need to slightly rephrase it:

Theorem 7. *Assume $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$. Then for every $k \geq 0$, the convex set $N_+^k(G)$ is defined by the union of all valid inequalities for $N_+^k(G_1)$ and all valid inequalities for $N_+^k(G_2)$.*

Proof. We only need to show that in the proof of the previous theorem whenever the matrices $Y^{(\ell)}$ are positive semidefinite for $\ell = 1, 2$, so is $\overline{Y}$. We use the property that the matrix $X \in \Sigma^n$ is positive semidefinite if and only if SXS^T is, for every nonsingular $n \times n$ matrix S. Consider the following matrix:

$$S := \left(\begin{array}{c|c|c|c} 1 & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & I_{V(G_1)\setminus\{v\}} & \mathbf{0} & \mathbf{0} \\ \hline 0 & \mathbf{0} & I_{\{v\}} & \mathbf{0} \\ \hline s & \mathbf{0} & t & I_{V(G_2)\setminus\{v\}} \end{array}\right),$$

where I_V is the identity matrix with rows and columns indexed by the elements of the set V, $\mathbf{0}$ indicates a matrix of 0's of the appropriate size, and the vectors $s = (s_u)$, $t = (t_u)$ are defined for $u \in V(G_2) \setminus \{v\}$ by

$$s_u := -\frac{x_u^{(2)} - y_u^{(2,2)}}{1 - x_v} \qquad \text{and} \qquad t_u := \left(\frac{x_u^{(2)} - y_u^{(2,2)}}{1 - x_v} - \frac{y_u^{(2,2)}}{x_v}\right).$$

Using (1) and (2) we can easily check (using the symmetry of $\overline{Y}$) that

$$S\overline{Y}S^T = \left(\begin{array}{c|c|c|c} 1 & (x^{(1)})^T & x_v & \mathbf{0} \\ \hline x^{(1)} & X^{(1)} & y^{(1,1)} & \mathbf{0} \\ \hline x_v & (y^{(2,1)})^T & x_v & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{X}^{(2)} \end{array}\right),$$

where $\overline{X}^{(2)}$ is given by

$$\left(\begin{array}{c|c|c} 1 & \mathbf{0} & x_v \\ \hline \mathbf{0} & \overline{X}^{(2)} & \mathbf{0} \\ \hline x_v & \mathbf{0} & x_v \end{array}\right) := \overline{S}Y^{(2)}\overline{S}^T,$$

with

$$\overline{S} := \left(\begin{array}{c|c|c} 1 & \mathbf{0} & 0 \\ \hline s & I_{V(G_2)\setminus\{v\}} & t \\ \hline 0 & \mathbf{0} & 1 \end{array}\right).$$

Since S is nonsingular, $\overline{Y}$ is positive semidefinite if and only if $S\overline{Y}S^T$ is positive semidefinite. Clearly $S\overline{Y}S^T$ is positive semidefinite if and only if the two square submatrices it can be decomposed are positive semidefinite, i.e. if and only if $Y^{(1)}$ and $\overline{X}^{(2)}$ are positive semidefinite. Since $\overline{S}$ is also nonsingular, $\overline{S}Y^{(2)}\overline{S}^T$ is positive semidefinite if and only if $Y^{(2)}$ is positive semidefinite, thus its symmetric minor, $\overline{X}^{(2)}$, is also positive semidefinite, and the theorem is proved. □

A by-product of the above technique is the following positive semidefinite extension fact:

Proposition 8. *Let* $Y \in \Sigma^{n+1}$*, and suppose that* $Y_{0j} = Y_{jj}$ *for some* j*. Define the matrix*

$$\overline{Y} := \begin{pmatrix} Y & Y(e_0 - e_j) \\ (e_0 - e_j)^T Y & (Y_{00} - Y_{0j}) \end{pmatrix}.$$

Then

$$\overline{Y} \in \Sigma_+^{n+2} \text{ if and only if } Y \in \Sigma_+^{n+1}.$$

Moreover, the (linear algebraic) ranks of the matrices Y *and* $\overline{Y}$ *are the same.*

Theorems 6 and 7 can be generalized to polytopes. We will assume that the coordinates of the polytope are split into two sets I and J such that $I \cap J = \{v\}$, i.e. there is one common coordinate, and let I' and J' denote the nonempty sets $I \setminus \{v\}$ and $J \setminus \{v\}$. We will use the following notations: If $x = (x_i)_{i \in I} \in \mathbb{R}^I$, then $x' = (x_i)_{i \in I'} \in \mathbb{R}^{I'}$ and similarly for J. Thus, if $x \in \mathbb{R}^I$ and $y \in \mathbb{R}^J$, then both vectors (x, y') and (x', y) are in $\mathbb{R}^{I \cup J}$, but they are the same only if $x_v = y_v$.

Now given a polytope $P \subset [0, 1]^{I \cup J}$, let its projections to coordinates I, J, resp., be P_I, P_J, respectively. So, e.g.,

$$P_I = \left\{ x \in \mathbb{R}^I : \text{there is a } y' \in \mathbb{R}^{J'} \text{ such that } (x, y') \in P \right\}.$$

We will say that v is a *cut coordinate* for the polytope P, if P has the property that whenever $x \in P_I$ and $y \in P_J$ with $x_v = y_v$, then $(x, y') = (x', y) \in P$. Thus v is a cut coordinate for P if and only if the inequalities defining its projections P_I and P_J suffice to define P itself.

From Corollary 2 it easily follows that $\left[N^k(P)\right]_I = N^k(P_I)$ and $\left[N^k(P)\right]_J = N^k(P_J)$ for any $k \geq 1$, and similarly for N_0 and N_+. Hence, Theorems 6 and 7 can be phrased for polytopes as follows:

Theorem 9. *If* v *is a cut coordinate for the polytope* P*, then it is a cut coordinate for* $N_0^k(P)$*,* $N^k(P)$*, and* $N_+^k(P)$ *as well for every* $k \geq 0$*.*

Proof. The proof is essentially the same as that of Theorem 6 with I playing the role of $V(G_1)$ and J the role of $V(G_2)$. □

To see that Theorems 6 and 7 are special cases of Theorem 9, one just needs to check the following:

Lemma 10. *If* G *is a graph with a vertex* v*, then* v *is a cut coordinate for* $FRAC(G)$ *(and for* $STAB(G)$ *as well) if and only if* v *is a cut vertex of* G*.*

Proof. If v is a cut vertex in G, then $G = G_1 \cup G_2$ such that $V(G_1) \cap V(G_2) = \{v\}$, and then it is easy to check that v is a cut coordinate for $FRAC(G)$ with $I = V(G_1)$ and $J = V(G_2)$, since if $x \in FRAC(G_1)$ and $y \in FRAC(G_2)$ with $x_v = y_v$, then (x, y') will also satisfy all edge inequalities.

On the other hand, if v is not a cut vertex in G, then for any split $V(G) = I \cup J$ of the vertices with $I \cap J = \{v\}$ and $I' \neq \emptyset$, $J' \neq \emptyset$, there will be an edge $\{u, w\} \in E(G)$ with $u \in I'$ and $w \in J'$, and then the vectors x and y defined by

$$x_i := \begin{cases} 1 \text{ if } i = u, \\ 0 \text{ if } i \in I \setminus \{u\}; \end{cases}$$

and

$$y_i := \begin{cases} 1 \text{ if } i = w, \\ 0 \text{ if } i \in J \setminus \{w\}; \end{cases}$$

will be in $FRAC(G_1)$, $FRAC(G_2)$, respectively, but (x, y') is not in $FRAC(G)$ (or $STAB(G)$). □

Theorem 6 does not generalize to the case when $G_1 \cap G_2$ is a larger clique (even if only an edge). An example is presented in Figure 1.

Claim 11. *For the graph G given in Figure* 1*, the inequality*

$$2x_1 + 3x_2 + 3x_6 + x_3 + x_4 + x_5 \leq 3 \tag{4}$$

has N_0-rank 2*. Moreover, it defines a facet of $N^2(G)$.*

Proof. Theorem 2.3 of [16] shows that for all graphs, one application of N and N_0 are the same and both are defined by the trivial, edge, and odd cycle inequalities. Therefore in our example $N_0(G) = N(G)$, and they are both defined by the nonnegativity and the triangle constraints.

It is clear that (4) is a valid inequality for $STAB(G)$ (K_5 inequality plus twice the triangle inequality of {1, 2, 6}). Since the deletion and destruction of vertex 1 both result in an inequality of N_0-rank 1 (deletion of 1 gives the sum of three triangles, destruction of 1 leaves just a triangle inequality), (4) has N_0-rank at most 2.

The point $\frac{1}{3}\bar{e} \in [0, 1]^6$ satisfies all triangle inequalities, but it violates (4). Thus the N-rank of the inequality is at least 2, hence the N_0- and N-ranks of (4) are both equal to 2.

To see that (4) defines a facet of $N^2(G)$, consider the characteristic vectors of the stable sets {1, 3}, {1, 4}, {1, 5}, {2}, {6} and the vector $\left(\frac{3}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$. The first five vectors are in $STAB(G)$, so they also lie in $N^2(G)$. The last one is in $N^2(G)$ because the following matrix is in $M^2(G)$:

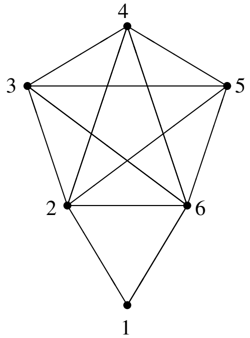

Fig. 1. $G = K_5 \cup K_3$ with $K_5 \cap K_3 = K_2$

$$\frac{1}{8}\begin{pmatrix} 8 & 3 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

It is easy to see that the six vectors mentioned above are affinely independent and they all satisfy (4) with equality. Since the dimensions of $STAB(G)$ and $N^2(G)$ are equal to 6 in this example, the claim follows. □

In fact, for this example, $N_0^2(G) = N^2(G)$ and both are defined by all K_4 inequalities, the triangle inequality on $\{1, 2, 6\}$, and (4). However, $STAB(G)$ has just two nontrivial facets, the two maximal cliques (hence $r_0(G) = 3 = r_0(K_5)$), and we need both of them to get (4) as a linear combination of facets of $STAB(G)$.

Note that although for perfect graphs (by the results of Lovász and Schrijver [16]), $r_+(G) = 1$ and

$$r_0(G) = r(G) = (\text{size of the largest clique in } G) - 2,$$

it is not true in general that $N_0^k(G)$ (or $N^k(G)$) is equal to the $(k+2)$-clique polytope of G (the polytope defined by the clique inequalities for every clique of G of size at most $(k+2)$). This obviously holds for $k = 0$ and $k = r(G)$, and also for $k = 1$, and it is an interesting question for what perfect graphs it will hold for some other values of k.

Our example above and Claim 11 solve Problem 5 of Lovász [14]. Lovász asked (paraphrased here) whether $N^k(G)$ is defined only by the clique inequalities (up to K_{k+2}) and nonnegativity constraints when G is perfect. Our example and claim prove that the answer to this question is "no."

4. The N_0-rank and the N-rank of a graph

Lovász and Schrijver [16] proved that for any graph G the polytopes $N_0(G)$ and $N(G)$ are the same and both are the odd-cycle polytope of G. This motivates the following conjecture:

Conjecture 12. *$N_0^k(G) = N^k(G)$ for all graphs G and all $k \geq 0$.*

By the abovementioned results, Conjecture 12 is true for $k = 1$ for all graphs and for $k = 2$ when $r_0(G) = 2$. This conjecture is also true for every clique. It is easy to check this directly (since the stronger condition $M_0^k(G) = M^k(G)$ holds for every $k \geq 0$); also, this fact can be seen as a consequence of a general geometric condition given in Theorem 6.3 of [6]. As we prove in Theorem 21, a weaker version of this conjecture (namely that the N_0- and N-ranks of all graphs are equal) holds for a wide variety of subdivisions of cliques. Upcoming Proposition 14 provides similar additional evidence. First let us point out that for some very special polytopes P we do have $N_0^k(P) = N^k(P)$.

Theorem 13. *(Theorem* 3.1 *of Cook and Dash [5]) If the convex polytope* $P \subset [0,1]^d$ *contains all vertices of the unit cube except one, then* $N_0^k(P) = N^k(P) = N_+^k(P)$ *for every* $k \geq 0$.

However, $N_0(P) \neq N(P)$ in general, even if P is lower-comprehensive, i.e. for any $y \in [0,1]^d$ if $y \leq x$, and $x \in P$, then $y \in P$ as well (it was observed in [5] and [6] that N_0, N, and N_+ preserve lower-comprehensiveness of the argument—of course, $FRAC(G)$ is lower-comprehensive). An example is easy to find when two adjacent vertices of the unit cube are cut.

The question of the equivalence of the N_0 and the N-operators in our setting and Lemma 3 motivate the examination of such graphs that have a vertex whose deletion decreases its N_0-rank or N-rank. Hence, we define the following two classes of graphs, $\mathcal{B}_0$ and $\mathcal{B}$:

– Bipartite graphs belong to $\mathcal{B}_0$.
– If G has a vertex v such that its deletion decreases its N_0-rank and $G - v$ is in $\mathcal{B}_0$, then G is in $\mathcal{B}_0$.

The definition is similar for $\mathcal{B}$, we just use the N-rank instead.

Proposition 14. *We have* $\mathcal{B} \subseteq \mathcal{B}_0$. *Moreover, for all* $G \in \mathcal{B}$ *there exists a bipartite subgraph* (V_B, E_B) *of* G *such that*

$$r(G) = r_0(G) = |V| - |V_B|.$$

Proof. If $G \in \mathcal{B}$, there exists a sequence of graphs

$$G = G_{|V|-|V_B|} \supset G_{|V|-|V_B|-1} \supset \cdots \supset G_1 \supset G_0 = G_B$$

such that G_{i+1} is obtained from G_i by adding a new node v to G_i and some edges incident to v such that $r(G_{i+1}) = r(G_i) + 1$. Initially, $r(G_B) = r_0(G_B) = 0$ (since G_B is bipartite). We proceed by induction to show that $r(G_{k+1}) = r_0(G_{k+1}) = k + 1$ for all k. Assume that we have $r(G_i) = r_0(G_i)$ for all $i \leq k$. Clearly

$$r_0(G_{k+1}) \geq r(G_{k+1}) = r(G_k) + 1.$$

On the other hand, the deletion of the special node in G_{k+1} gives G_k and $r_0(G_k) = k$. Thus $r_0(G_{k+1}) \leq r_0(G_k)+1 = r(G_k)+1$, and we proved $r(G_{k+1}) = r_0(G_{k+1}) = k+1$. Since every $G \in \mathcal{B}$ can be constructed in this way, this also proves $G \in \mathcal{B}_0$. □

We do not know whether $\mathcal{B} = \mathcal{B}_0$, and finding a description of the graphs belonging to these classes would also be interesting. Clearly $K_n \in \mathcal{B}$, but there are other examples, see Figure 2. We are interested in properties of graphs in $\mathcal{B}_0$ and in $\mathcal{B}$. However, since not all graphs of N_0-rank k contain an induced subgraph of N_0-rank (or N-rank) k that is also in $\mathcal{B}_0$ (or $\mathcal{B}$), we introduce another pair of graph classes, $\mathcal{C}_0$ and $\mathcal{C}$:

– If $r_0(G) = k$, and for every vertex $v \in V$ the graph $G - v$ has N_0-rank $k - 1$, then G is in $\mathcal{C}_0$.

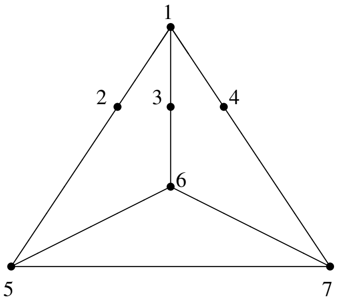

Fig. 2. A graph $G \in \mathcal{B}$

The definition is similar for $\mathcal{C}$ with the N-rank. Then we have the following fact: If the N_0-rank (or the N-rank) of the graph G is $k \geq 1$, then G has an induced subgraph G' of N_0-rank (or N-rank) k such that $G' \in \mathcal{C}_0$ (or $\mathcal{C}$).

A *subdivision* of a graph G is obtained by replacing every edge by a path of length at least 1 (the new vertices, if any, on these paths should be all different). A vertex of degree at least 3 in the subdivision must be also a vertex of the original graph; it will be referred to as a vertex of G. The path that replaced the edge $\{v, w\} \in E(G)$ in the subdivision is called the *path induced* by v and w (or by the edge $\{v, w\}$); these paths are the *induced paths* of the subdivision. An *odd subdivision of an edge* replaces the edge by a path of odd length (again, all new vertices on the path are different). The *subdivision of a star* operation picks a vertex v in G, and introduces a new vertex on every edge incident to v. The graph in Figure 2 is obtained from K_4 by applying the subdivision of a star operation at vertex 1.

Theorem 15. *If $G \in \mathcal{B}_0 \cup \mathcal{C}_0$ (or $G \in \mathcal{B} \cup \mathcal{C}$), then deletion or odd subdivision of an edge or subdivision of a star does not increase the N_0-rank (the N-rank). Moreover, for $G \in \mathcal{B}_0$, $\mathcal{B}$, resp., the new graph obtained by the odd subdivision of an edge or the subdivision of a star operation is also an element of $\mathcal{B}_0$, $\mathcal{B}$, respectively.*

Proof. We prove the first claim for the N_0-rank, the case of the N-rank is identical. Assume first that $G \in \mathcal{C}_0$. Let $\{v, w\} = e$ be the deleted edge. Since deleting v decreases the N_0-rank of G, and $G - v = (G - e) - v$, we get that $r_0(G-e) \leq r_0(G-e-v)+1 = r_0(G - v) + 1 = r_0(G)$. The proof is similar for the odd subdivision of e or for the subdivision of a star of v, since then the deletion of v gives a graph that is $G - v$ plus a path (paths) of length 2 (1). By Lemma 5 this graph has the same N_0-rank as $G - v$.

Next let $G \in \mathcal{B}_0$. We prove the claim by induction on the N_0-rank. For bipartite graphs the claim is true, since the deletion or odd subdivision of an edge e or the subdivision of a star gives another bipartite graph. Now, we assume that $r_0(G) = k > 0$ and that we have shown the claim for any graph in $\mathcal{B}_0$ with N_0-rank $k - 1$. Then by definition there is a vertex v such that $G - v$ has N_0-rank $k - 1$ and $G - v \in \mathcal{B}_0$. We have two cases:

(1) If v is incident to e to be deleted or subdivided or the subdivision of a star is applied to v, then the proof is the same as in the case $G \in \mathcal{C}_0$.

(2) Otherwise $G - v \in \mathcal{B}_0$ has N_0-rank $r_0(G - v) = k - 1$, hence by the induction hypothesis the deletion or subdivision of an edge or the subdivision of a star does not increase its N_0-rank. Thus, by Lemma 3, the same holds for G.

To prove the second claim ($G' \in \mathcal{B}_0$) for N_0 (the proof for N is identical), it is enough to show that $r_0(G') = r_0(G)$ and $G' - v \in \mathcal{B}_0$ with $r_0(G' - v) = r_0(G) - 1$, where v is a vertex in G with $G - v \in \mathcal{B}_0$ and $r_0(G - v) = r_0(G) - 1$.

To show $r_0(G') = r_0(G)$ we already have $r_0(G') \leq r_0(G)$, and we prove the other direction separately, in Theorem 16 below.

Finally we show by induction on $r_0(G)$ that $G' - v$ is also in $\mathcal{B}_0$, and $r_0(G' - v) = r_0(G) - 1$. This is clear for $r_0(G) = 1$, since then $G - v$ and hence $G' - v$ are both bipartite. Now let $r_0(G) \geq 2$. In case (2), the claims follow from the induction hypothesis, while in case (1) they follow from Lemma 5. □

Remark 1. From the proof of Theorem 15 it is clear that the first claim holds for *any* subdivision of G. Deletion of an edge can increase the rank for $G \notin \mathcal{B}_0$, an example is given later in Figure 4 (p. 339).

Theorem 16. *If the graph G' is obtained from G using the subdivision of a star or odd subdivision of an edge operations, then $r_0(G') \geq r_0(G)$ (and similarly for the N- and the N_+-rank).*

Proof. Let us consider the odd subdivision operation first. To prove $r_0(G') \geq r_0(G)$, it is enough to show the property when we replace edge $\{v, w\} \in E(G)$ by a path of length 3, say $vv'w'w$. If $r_0(G) = k$, then there is a point $x \in \mathbb{R}^{V(G)}$ such that $x \in N_0^{k-1}(G)$ but $x \notin STAB(G)$. Define $x' \in \mathbb{R}^{V(G')}$ as follows:

$$x'_u := \begin{cases} 1 - x_v & \text{if } u = v', \\ x_v & \text{if } u = w', \\ x_u & \text{otherwise.} \end{cases}$$

Now the claim will follow from the following lemma:

Lemma 17. *If $x \notin STAB(G)$, then $x' \notin STAB(G')$, and if $x \in N_0^k(G)$, then $x' \in N_0^k(G')$ (similarly for N and N_+).*

Proof. To prove the first claim we use the following fact from Wolsey [20]: If $a^T x \leq b$ is a valid inequality for $STAB(G)$, then $(a')^T x \leq b'$ is valid for $STAB(G')$, where $b' := b + a(v)$ and

$$a'(u) := \begin{cases} a(v) \text{ if } u = v' \text{ or } u = w', \\ a(u) \text{ otherwise.} \end{cases}$$

(Sometimes $a'(v')$, $a'(w')$, and b' can be chosen smaller, but this is not important for us.) Now if $x \notin STAB(G)$, then there is an inequality $a^T x \leq b$ valid for $STAB(G)$ that is violated by x. It is very easy to check that x' will also violate $(a')^T x \leq b'$, which is valid for $STAB(G')$, thus $x' \notin STAB(G')$.

We prove the second statement by induction on k. For $k = 0$ the statement is trivial, since if x satisfies the edge inequalities, so will x' (and $x \in [0, 1]^V$ implies

$x' \in [0,1]^{V(G')}$). Now, we assume that the claim holds for $k \geq 0$ and prove it for $k+1$. If $x \in N_0^{k+1}(G)$, then there is a matrix $X = (X_{ij}) \in \mathbb{R}^{V(G)\times V(G)}$ such that

$$Y := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in M_0^{k+1}(G),$$

hence $X_{uu} = x_u$ and $Ye_u, Y(e_0 - e_u) \in N_0^k(G)$ for any $u \in V(G)$. We define the following matrix that will show $x' \in N_0^{k+1}(G')$:

$$Y' := \begin{pmatrix} 1 & (x')^T \\ x' & X' \end{pmatrix},$$

where the matrix $X' = (X'_{ij}) \in \mathbb{R}^{V(G')\times V(G')}$ is defined as follows:

$$X'_{v'u} := \begin{cases} 1 - x_v & \text{if } u = v', \\ 0 & \text{if } u = w', \\ x_u - X_{vu} & \text{otherwise.} \end{cases} \qquad X'_{w'u} := \begin{cases} 0 & \text{if } u = v', \\ x_v & \text{if } u = w', \\ X_{vu} & \text{otherwise.} \end{cases}$$

$$X'_{uv'} := \begin{cases} 1 - x_v & \text{if } u = v', \\ 0 & \text{if } u = w', \\ x_u - X_{uv} & \text{otherwise.} \end{cases} \qquad X'_{uw'} := \begin{cases} 0 & \text{if } u = v', \\ x_v & \text{if } u = w', \\ X_{uv} & \text{otherwise.} \end{cases}$$

$$X'_{uu'} := X_{uu'} \qquad \text{if } u, u' \in V(G).$$

Thus in Y' the row corresponding to w' is the same as the row corresponding to v while the row corresponding to v' is the first row minus the row corresponding to v, and similarly for the columns. Because of this, it is easy though somewhat tedious to check that $Y' \in M_0^{k+1}(G')$ using the induction hypothesis. We consider one example:

To show that $Y'(e_0 - e_{v'}) \in M_0^k(G')$ just notice that

$$Y'(e_0 - e_{v'}) = \begin{pmatrix} x_v \\ y \end{pmatrix},$$

where $y = (y_u) \in \mathbb{R}^{V(G')}$ is given by

$$y_u := \begin{cases} 0 & \text{if } u = v', \\ x_v & \text{if } u = w', \\ X_{uv} & \text{otherwise.} \end{cases}$$

Using that $Ye_v \in N_0^k(G)$ and the induction hypothesis it now follows that $Y'(e_0 - e_{v'}) \in N_0^k(G')$. The other cases are analogous. Since the matrix Y' is symmetric if Y is symmetric, the statement is also valid for the N-operator.

For the N_+-operator one needs to check that whenever Y is positive semidefinite, so is Y'. But this follows immediately from our earlier observation that the new rows (and columns) are linear combinations of the first row (column) and the row (column) corresponding to v, hence by simple row and column operations we can eliminate them, showing that Y' is also positive semidefinite. □

The rest of the proof for the subdivision of a star operation is similar. Assume that we get G' by applying the subdivision of a star operation to vertex $v \in V(G)$, and for any $w \in \Gamma(v)$ replace the edge $\{v, w\} \in E(G)$ with the path $vw'w$. Given $x \in \mathbb{R}^{V(G)}$ define $x'' \in \mathbb{R}^{V(G')}$ by

$$x''_u := \begin{cases} 1 - x_v & \text{if } u = v, \\ x_v & \text{if } u = w' \text{ and } \{v, w\} \in E(G), \\ x_u & \text{otherwise.} \end{cases}$$

Again the claim will follow from the following lemma, analogous to Lemma 17:

Lemma 18. *If $x \notin STAB(G)$, then $x'' \notin STAB(G')$, and if $x \in N_0^k(G)$, then $x'' \in N_0^k(G')$ (similarly for N and N_+).*

Proof. For the first part, we use the following theorem from [12]: Let $a^T x \leq b$ be a valid inequality for $STAB(G)$, and define $a(\Gamma_G(v)) := \sum_{w \in \Gamma_G(v)} a(w)$. Then $(a'')^T x \leq b''$ is valid for $STAB(G')$, where $b'' := b + a(\Gamma_G(v)) - a(v)$ and

$$a''(u) := \begin{cases} a(\Gamma_G(v)) - a(v) & \text{if } u = v, \\ a(w) & \text{if } u = w' \text{ and } w \in \Gamma_G(v), \\ a(u) & \text{otherwise.} \end{cases}$$

(Again, sometimes some of the new weights $a''(w')$, $a''(v)$, and b'' can be chosen smaller.) Now if $x \notin STAB(G)$, then there is an inequality $a^T x \leq b$ valid for $STAB(G)$ that is violated by x. It is very easy to check that x'' will also violate $(a'')^T x \leq b''$, which is valid for $STAB(G')$, thus $x'' \notin STAB(G')$.

The proof of the second statement is similar to that of Lemma 17, so we only give a sketch of the induction. For $k = 0$ the statement is trivial, since if x satisfies the edge inequalities, so will x'' (and $x \in [0, 1]^{V(G)}$ implies $x'' \in [0, 1]^{V(G')}$). If $x \in N_0^{k+1}(G)$, then there is a matrix $X = (X_{ij}) \in \mathbb{R}^{V(G) \times V(G)}$ such that

$$Y := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in M_0^{k+1}(G).$$

We define the corresponding matrix Y''

$$Y'' := \begin{pmatrix} 1 & (x'')^T \\ x'' & X'' \end{pmatrix},$$

where the matrix $X'' = (X''_{ij}) \in \mathbb{R}^{V(G') \times V(G')}$ is defined as follows:

$$X''_{vu} := \begin{cases} 1 - x_v & \text{if } u = v, \\ 0 & \text{if } u = w' \text{ and } w \in \Gamma_G(v), \\ x_u - X_{vu} & \text{otherwise;} \end{cases}$$

$$X''_{uv} := \begin{cases} 1 - x_v & \text{if } u = v, \\ 0 & \text{if } u = w' \text{ and } w \in \Gamma_G(v), \\ x_u - X_{uv} & \text{otherwise;} \end{cases}$$

for any $w \in \Gamma_G(v)$

$$X''_{w'u} := \begin{cases} 0 & \text{if } u = v, \\ x_v & \text{if } u = w' \text{ and } w \in \Gamma_G(v), \\ X_{vu} & \text{otherwise;} \end{cases} \quad X''_{uw'} := \begin{cases} 0 & \text{if } u = v \\ x_v & \text{if } u = w' \text{ and } w \in \Gamma_G(v), \\ X_{uv} & \text{otherwise;} \end{cases}$$

and finally $X''_{uu'} := X_{uu'}$ if $u, u' \in V(G) \setminus \{v\}$. The rest of the proof, showing $Y'' \in M_0^{k+1}(G')$, is the same. Note that Y'' is symmetric if and only if Y is, therefore the proof also applies to the N-operator. For the N_+-operator one just needs to notice that the new rows (columns) corresponding to the new vertices $w \in \Gamma_G(v)$ are simple extensions of the rows (columns) corresponding to v in Y, while the row (column) corresponding to v in Y'' is just the first row (column) minus the row (column) corresponding to any $w \in \Gamma_G(v)$. Thus by appropriate row and column operations we can eliminate these extra rows and columns to get that Y'' is positive semidefinite if and only if Y is, finishing the proof. □

This finishes the proof of Theorem 16. □

Remark 2. The rank can increase in Theorem 16 when applying these operations. For the N_0- and N-rank, an example can be obtained from the graph in Figure 3 (use odd subdivision for edge $\{1, 4\}$ or subdivision of a star for vertex 1), while for the N_+-rank an example is mentioned later in Proposition 37 for odd subdivision.

Theorem 15 also immediately implies the following:

Corollary 19. *If $G \in \mathcal{B}_0$ (resp. $\mathcal{B}$) and G' is obtained from G using a sequence of the subdivision of a star and the odd subdivision of an edge operations, then $G' \in \mathcal{B}_0$ (resp. $\mathcal{B}$) and $r_0(G') = r_0(G)$ (resp. $r(G') = r(G)$).*

A similar statement for the N_0-rank (or N-rank) is true for graphs in $\mathcal{C}_0$ (or $\mathcal{C}$) if every edge in the graph G is rank-critical, i.e. $r_0(G-e) < r_0(G)$ (or $r(G-e) < r(G)$) for all $e \in E(G)$ (though G' will usually not be in $\mathcal{C}_0$ or $\mathcal{C}$).

Combining Theorem 15 with Lemma 14 gives a lot of graphs G with the property that $r_0(G) = r(G)$. In particular, every graph obtained from K_n by a sequence of odd subdivisions of edges and/or subdivisions of stars belongs to $\mathcal{B}$ and $\mathcal{B}_0$ and has N_0- and N-rank $n-2$.

When G can be obtained from K_n using the subdivision of a star and the odd subdivision of an edge operations, we say that G is an *odd-star subdivision* of K_n. We can recognize when a subdivision of K_n is odd-star:

Lemma 20. *Let G be a subdivision of K_n ($n \geq 3$) where every induced path of G contains 0, 1, or 2 new vertices. The graph G is an odd-star subdivision of K_n if and only if the following two properties are satisfied:*

(a) *Every cycle formed by three induced paths of G is odd;*
(b) *G does not contain the graph shown in Figure 3 as an induced subgraph.*

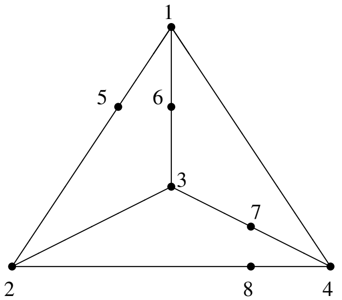

Fig. 3. A non-odd-star subdivision of K_4

Remark 3. It is easy to see that if a subdivided edge contains at least three additional vertices, then having two fewer vertices on the path does not change whether the graph is an odd-star subdivision of K_n or not. Thus, it is enough to examine the graphs specified in the lemma.

Proof. (of Lemma 20) Assume first that G is an odd-star subdivision of K_n. Property (a) holds for K_n, and it clearly remains valid after the application of these operations. If property (b) is not true, then G contains the induced subgraph shown on Figure 3. Clearly the vertices 1, 2, 3, and 4 belonged to the original K_n. Since the path from vertex 1 to vertex 2 contains exactly one additional vertex, to obtain G from K_n we must have applied the subdivision of a star operation to either vertex 1 or 2. However, in the first case we should have at least one additional vertex on the path joining vertex 1 to vertex 4, while in the second case we should have at least one additional vertex on the path joining vertex 2 to vertex 3. Since none of these paths is subdivided, we get a contradiction, so property (b) is valid for G.

Now assume that G satisfies properties (a) and (b). If every induced path of the subdivision contains 0 or 2 new vertices, then G can be obtained from K_n by using only the odd subdivision of an edge operation. Assume now that the path induced by the vertices $v, w \in K_n$ has exactly one new vertex. Let A be the set of those vertices of K_n that with v, induce a path having exactly 1 new vertex, and let B be the remaining vertices of K_n, i.e. those that with v induce a path having 0 or 2 new vertices. Because of property (a), induced paths between a vertex of A and a vertex of B must have exactly 1 new vertex, while induced paths within vertices of A or within vertices of B can contain 0 or 2 new vertices. If every induced path within vertices of $B \cup \{v\}$ contain 2 new vertices, then G can be obtained from K_n by applying the subdivision of a star operation to every vertex of $B \cup \{v\}$ and odd-subdividing those edges within vertices of A that induce a path having 2 new vertices in G. Similarly, if every induced path within vertices of A has exactly 2 new vertices, then G can be obtained from K_n by applying the subdivision of a star operation to every vertex of A and odd-subdividing those edges within vertices of $B \cup \{v\}$ that induce a path having 2 new vertices. If neither of these cases occur, then we have two vertices of $B \cup \{v\}$ (say 1 and 4) and two vertices of A (say 2 and 3) such that the paths induced by 1 and 4 and by 2 and 3 contain no new vertices, while the rest of the paths between these vertices contain exactly 1 new vertex. Since this is exactly a forbidden induced subgraph in property (b), this third case is impossible, thus G is an odd-star subdivision of K_4. □

Using the above lemma we can prove that only these odd-star subdivisions of K_n have the property that their N_0-ranks and N-ranks are the same as that of K_n:

Theorem 21. *Let G be a subdivision of K_n. Then the following are equivalent:*

(i) $r_0(G) = r_0(K_n) = n - 2$,
(ii) $r(G) = r(K_n) = n - 2$,
(iii) *G is an odd-star subdivision of K_n.*

Proof. First assume that G is not an odd-star subdivision of K_n. By Lemma 20 either property (a) or (b) is not satisfied. If property (a) is not satisfied, then a cycle induced by three vertices of K_n is even, hence deleting the remaining $n - 3$ vertices of K_n in G leaves a bipartite graph having rank 0, hence $r_0(G)$ (and thus $r(G)$) is at most $n - 3$ by Lemma 3. Similarly, if property (b) is not satisfied, then four vertices of K_n induce the graph shown on Figure 3 in G (or a graph having an odd number of vertices on the paths 1–2, 2–4, 4–3, and 3–1). Thus, deleting the remaining $n - 4$ vertices of K_n leaves this graph plus possibly paths. It was shown by Gerards and Schrijver [8] that the N_0-rank of such a graph is 1, thus again G has N_0-rank at most $n - 3$.

When G is an odd-star subdivision of K_n, we have already seen that its N_0- and N-rank are both $n - 2$, and the theorem is proved. □

Lovász and Schrijver [16] proved $r(K_n) = n - 2$ by showing that the point $x = (\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})^T$ belongs to $N^{k-2}(K_n)$ (they actually showed this for any graph). Using Lemmas 17 and 18 one can similarly obtain points in $N^{k-2}(G)$ for the graph G when G is obtained from K_n using odd subdivision of an edge and subdivision of a star operations. That these points belong to $N^{k-2}(G)$ also follows from the following, more general statement:

Lemma 22. *Let $S \subset V(G)$ be a stable set in the graph G. For $k \geq 3$ define the vector $x^{(S,k)} \in \mathbb{R}^{V(G)}$ as follows:*

$$x_i^{(S,k)} = \begin{cases} \frac{k-1}{k} & \text{if } i \in S, \\ \frac{1}{k} & \text{if } i \notin S. \end{cases}$$

If $x^{(S,3)} \in N(G)$, then $x^{(S,k)} \in N^m(G)$ for all $k \geq m + 2$ for any $m \geq 1$.

Remark 4. Lemma 2.7 of [16] is the statement of this lemma with $S = \emptyset$.

Proof. (of Lemma 22) First define an equivalence relation on the vertices of G depending on S: Let G' be a subgraph of G containing those edges of G that are incident to a vertex of S, and for $i, j \in V(G)$ define $i \sim j$ if and only if there is a path from i to j in G', and let $S_j = \{i \in S : i \sim j\}$.

The proof of the lemma goes by induction on m. For $m = 1$, if $x^{(S,3)} \in N(G)$, then since $N(G)$ is defined by the trivial, edge, and odd cycle inequalities (see Lovász and Schrijver [16]), the assumption $x^{(S,3)} \in N(G)$ is equivalent to requiring that S contains at most $j - 1$ vertices of any odd cycle of length $2j + 1$ (note that by the definition of

$x^{(S,3)}$ only the odd cycle inequalities have to be checked), but then $x^{(S,k)}$ also satisfies any odd cycle inequality for $k \geq 4$, thus $x^{(S,k)} \in N(G)$.

Next we assume that the statement holds for $m \geq 1$, and prove it for $m+1$. Let $k \geq m+3$, and define the matrix $X = (X_{ij}) \in \mathbb{R}^{V(G)\times V(G)}$ as follows:

If $i \sim j$, then let

$$X_{ij} = \begin{cases} \frac{1}{k} & \text{if } x_i^{(S,k)} = x_j^{(S,k)} = \frac{1}{k}, \\ \frac{k-1}{k} & \text{if } x_i^{(S,k)} = x_j^{(S,k)} = \frac{k-1}{k}, \\ 0 & \text{otherwise;} \end{cases}$$

while if $i \not\sim j$, then set

$$X_{ij} = \begin{cases} 0 & \text{if } x_i^{(S,k)} = x_j^{(S,k)} = \frac{1}{k}, \\ \frac{k-2}{k} & \text{if } x_i^{(S,k)} = x_j^{(S,k)} = \frac{k-1}{k}, \\ \frac{1}{k} & \text{otherwise.} \end{cases}$$

We claim now that the matrix

$$Y := \begin{pmatrix} 1 & (x^{(S,k)})^T \\ x^{(S,k)} & X \end{pmatrix} \in M(N^m(G)),$$

showing that $x^{(S,k)} \in N^{m+1}(G)$.

X is clearly symmetric, thus so is Y. Next we check that every column of Y is in the cone defined by $N^m(G)$. This is trivial for the first column by the induction hypothesis. If $x_j^{(S,k)} = 1/k$, then after rescaling (multiplying the vector by $1/x_j^{(S,k)}$) the jth column becomes

$$k \cdot X_{ij} = \begin{cases} 1 \text{ if } i \sim j \text{ and } x_i^{(S,k)} = \frac{1}{k}, \text{ or if } i \not\sim j \text{ and } x_i^{(S,k)} = \frac{k-1}{k}, \\ 0 \text{ otherwise.} \end{cases}$$

This is the characteristic vector of the set containing the vertices $\Gamma(S_j) \setminus S_j$ and $S \setminus S_j$. We claim that this is a stable set. Clearly, two vertices of $S \setminus S_j$ cannot be adjacent, since S is a stable set. If a vertex $v \in \Gamma(S_j) \setminus S_j$ is adjacent to a vertex $w \in S \setminus S_j$, then $w \in S_j$, contradiction. Finally, if $v, w \in \Gamma(S_j) \setminus S_j$ are adjacent, then $v \sim w$ implies that there is a path from v to w such that every second vertex on the path belongs to S (but neither v nor w), thus together with the edge $\{v, w\}$ this forms an odd cycle, and $x^{(S,k)}$ violates the corresponding odd cycle inequality, again a contradiction. Since this vector is the characteristic vector of a stable set, it is in $STAB(G) \subseteq N^m(G)$.

If $x_j^{(S,k)} = (k-1)/k$, then similarly after rescaling (multiplying by $k/(k-1)$) the jth column becomes

$$\frac{k}{k-1} \cdot X_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ and } x_i^{(S,k)} = \frac{k-1}{k}, \\ 0 & \text{if } i \sim j \text{ and } x_i^{(S,k)} = \frac{1}{k}, \\ \frac{k-2}{k-1} & \text{if } i \not\sim j \text{ and } x_i^{(S,k)} = \frac{k-1}{k}, \\ \frac{1}{k-1} & \text{if } i \not\sim j \text{ and } x_i^{(S,k)} = \frac{1}{k}. \end{cases}$$

This is just the characteristic vector of the stable set S_j on $S_j \cup \Gamma(S_j)$ and equals to $x^{(S',k-1)}$ with $S' = S \setminus S_j$ on $G - S_j - \Gamma(S_j)$, thus it belongs to $N^m(G)$ by the induction hypothesis ($k - 1 \geq m + 2$) and Lemma 2.

It is easy to check that when we take the difference of the first and the jth columns, we get exactly the same vectors (in the opposite order), which completes the proof. □

Note that the edge deletion operation can increase the N_0-rank or the N-rank of a graph, an example is shown on Figure 4. If we delete the edge e from that graph, we get an odd-star subdivision of K_4, hence its N-rank is 2. However, that inequality is just the sum of two odd cycle inequalities in the original graph, hence its rank is just 1, and it can be checked that the graph indeed has rank 1, so the deletion of e has increased the rank.

Odd-star subdivisions of K_4 are the only minimal graphs we know which have N_0-rank 2. However, for $n \geq 5$ not only odd-star subdivisions of K_n have rank $n - 2$. In fact, we can construct graphs that have arbitrarily large rank while no vertex has degree higher than 3. To show this, we need the following generalizations of the odd subdivision of an edge and the subdivision of a star operations:

Let v be a vertex with neighbourhood $\Gamma(v)$. Partition $\Gamma(v)$ into two nonempty, disjoint sets A_1 and A_2 (so $A_1 \cup A_2 = \Gamma(v)$, and $A_1 \cap A_2 = \emptyset$). A *stretching* of the vertex v is obtained as follows:

Remove v, introduce two vertices instead, called v_1 and v_2, add an edge between v_i and every vertex in A_i for $i \in \{1, 2\}$, add the edge $\{v_1, v_2\}$, and then do one of the following:

(i) subdivide the edge $\{v_1, v_2\}$ with one vertex w; or
(ii) subdivide every edge between v_2 and A_2 with one vertex.

These operations are illustrated on Figure 5.

Notice that when A_1 contains a single vertex, the stretching of a vertex operation reduces to the odd subdivision of an edge (in case of (i)) and the subdivision of a star (in case of (ii)) operations, so it is really their common generalization. Now we can identify another class of graphs that have N-rank at least $n - 2$:

Theorem 23. *If G is a graph obtained from K_n using the stretching of a vertex operation finitely many times, then $r_0(G) \geq r(G) \geq n - 2$.*

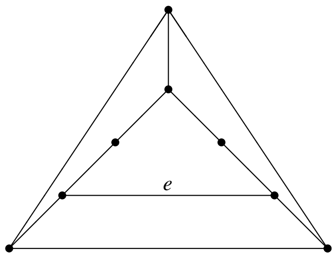

Fig. 4. Deleting e increases the rank

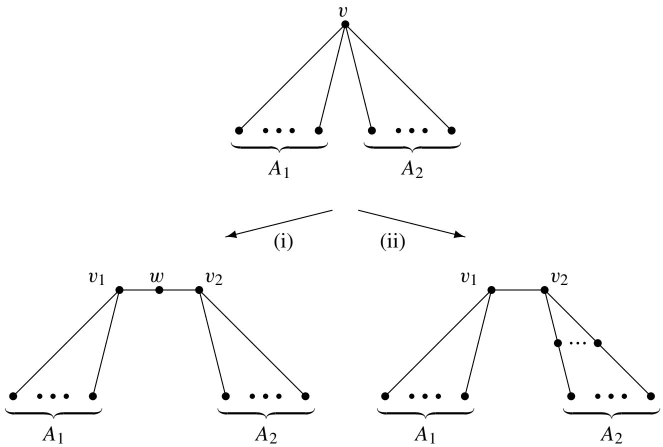

Fig. 5. Two types of stretching of v

Proof. We follow the idea of the proof of Theorem 21 by defining an inequality inductively that has N-rank at least $n-2$, which is proven by a point $x^{(S,n-1)} \in N^{(n-3)}(G) \setminus STAB(G)$ for some stable set S in G.

Clearly K_n has such an inequality, namely $\sum_{i=1}^{n} x_i \leq 1$ with $x^{(\emptyset,n-1)}$ (i.e. $S = \emptyset$). Now we assume inductively that after applying the stretching of a vertex operation a finite number of times we have an inequality $a^T x \leq b$ which has N-rank at least $n-2$ in G, and this is shown by the point $x^{(S,n-1)}$ for some stable set S. Now apply a stretching of $v \in V(G)$ to get the new graph $\tilde{G}$. Define the corresponding inequality $\tilde{a}^T x \leq \tilde{b}$ and the set $\tilde{S}$ as follows:

In case (i), let

$$\tilde{a}(u) := \begin{cases} a(v) \text{ if } u = v_1, v_2, \text{ or } w, \\ a(u) \text{ otherwise;} \end{cases}$$

$$\tilde{b} := b + a(v);$$

$$\tilde{S} := \begin{cases} (S \setminus \{v\}) \cup \{v_1, v_2\} & \text{if } v \in S, \\ S \cup \{w\} & \text{otherwise.} \end{cases}$$

In case (ii), let $\tilde{w}$ denote the vertex which was used to subdivide the edge $\{v_2, w\}$ for $w \in A_2$, recall that $a(A) := \sum_{u \in A} a(u)$, and let

$$\tilde{a}(u) := \begin{cases} a(v) & \text{if } u = v_1, \\ a(A_2) & \text{if } u = v_2, \\ a(w) & \text{if } u = \tilde{w} \text{ and } w \in A_2, \\ a(u) & \text{otherwise;} \end{cases}$$

$$\tilde{b} := b + a(A_2);$$

$$\tilde{S} := \begin{cases} (S \setminus \{v\}) \cup \{\tilde{w} : w \in A_2\} \cup \{v_1\} & \text{if } v \in S, \\ S \cup \{v_2\} & \text{otherwise.} \end{cases}$$

The new weights are illustrated on Figure 6.

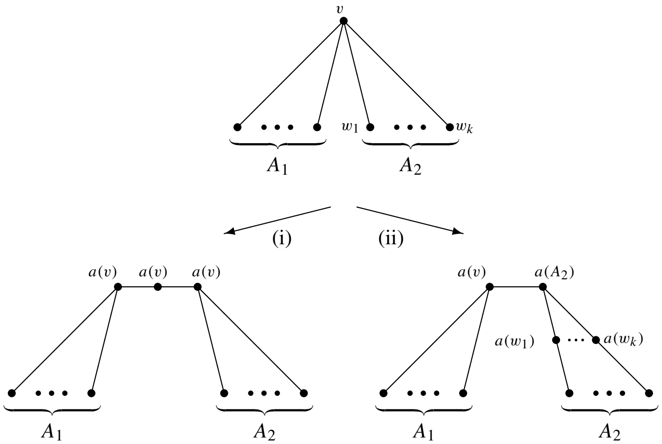

Fig. 6. New weights after the stretchings of v

First check that the new inequality $\tilde{a}^T x \leq \tilde{b}$ is valid for $STAB(\tilde{G})$ (though it might not be a facet). Consider type (i) stretching, and let M be a stable set in $\tilde{G}$. If M contains at most one of the vertices v_1, w, and v_2, then $M' := M \setminus \{v_1, v_2, w\}$ is stable in G, hence

$$\tilde{a}(M) \leq \tilde{a}(M') + a(v) = a(M') + a(v) \leq b + a(v).$$

The remaining possibility is that M contains both v_1 and v_2, but not w, and then $M' := (M \cup \{v\}) \setminus \{v_1, v_2\}$ is stable in G, hence

$$\tilde{a}(M) = a(M') + a(v) \leq b + a(v).$$

Now lets examine type (ii) stretching, and let M be a stable set in $\tilde{G}$ as before. If $v_2 \in M$, then $M' := M \setminus \{v_2\}$ is stable in G, hence

$$\tilde{a}(M) = \tilde{a}(M') + \tilde{a}(v_2) = a(M') + a(A_2) \leq b + a(A_2).$$

If $v_1, v_2 \notin M$, then $(M \cup \{v_2\}) \setminus \{\tilde{w}_1, \ldots, \tilde{w}_k\}$ is also stable in $\tilde{G}$ with at least as much weight (since $\tilde{a}(v_2) = a(A_2)$), so we are done by the first case. In the remaining case, when $v_1 \in M$ and $v_2 \notin M$, we have $M' := ((M \cap V(G)) \cup \{v\}) \setminus A_2$ stable in G, thus

$$\tilde{a}(M) \leq (a(M') - a(v)) + \tilde{a}(v_1) + a(A_2) = b - a(v) + a(v) + a(A_2) \leq b + a(A_2),$$

where the first inequality follows from the fact that we removed v_1 and at most one of w_i and $\tilde{w}_i$ for any i from M, and then added v to get M'.

This proves that $\tilde{a}^T x \leq \tilde{b}$ remains valid for $STAB(\tilde{G})$, and it is easy to check that $x^{(\tilde{S},n-1)}$ still violates this inequality if $x^{(S,n-1)}$ violated $a^T x \leq b$.

Since $x^{(\tilde{S},3)}$ will satisfy all odd cycle inequalities (every odd cycle going through some of the new vertices in $\tilde{G}$ corresponds to a shorter odd cycle in G), by Lemma 22 we proved that the N-rank of $\tilde{G}$ is also at least $n-2$. □

Even though we can only prove this lower bound for the rank of these graphs, we think that it is actually sharp:

Conjecture 24. *If G is obtained from K_n using the stretching of a vertex operation finitely many times, then $r_0(G) = r(G) = n - 2$.*

Theorem 23 also follows from the following generalization of Theorem 16:

Theorem 25. *If the graph G' is obtained from G using the stretching of a vertex operation, then $r_0(G') \geq r_0(G)$ (and similarly for the N- and the N_+-rank).*

Proof. The proof is analogous to that of Theorem 16. It is enough to prove the claim when G' is obtained from G by a single application of the stretching of a vertex operation. If $r_0(G) = k$, then there is a point $x \in \mathbb{R}^{V(G)}$ such that $x \in N_0^{k-1}(G)$ but $x \notin STAB(G)$. For type (i) stretching define $x' \in \mathbb{R}^{V(G')}$ as follows:

$$x'_u := \begin{cases} 1 - x_v & \text{if } u = w, \\ x_v & \text{if } u = v_1 \text{ or } u = v_2, \\ x_u & \text{otherwise;} \end{cases}$$

while for type (ii) stretching define

$$x'_u := \begin{cases} 1 - x_v & \text{if } u = v_2, \\ x_v & \text{if } u \in \Gamma_{G'}(v_2), \\ x_u & \text{otherwise.} \end{cases}$$

Now the claim will follow from the generalization of Lemma 17:

Lemma 26. *If $x \notin STAB(G)$, then $x' \notin STAB(G')$, and if $x \in N_0^k(G)$, then $x' \in N_0^k(G')$ (similarly for N and N_+).*

Proof. The first claim follows easily since if $a^T x \leq b$ is a valid nontrivial inequality for $STAB(G)$ violated by x, then we can define $\tilde{a}$ as in Theorem 23 to get a new valid inequality for $STAB(G)$ violated by x'.

The proof of the second statement is very similar to the previous proofs seen in Lemmas 17 and 18 so we again only give a sketch of the induction. For $k = 0$ the statement is trivial (x' will satisfy the trivial and the edge inequalities if x did), so assume that the claim holds for $k \geq 0$. If $x \in N_0^{k+1}(G)$, then there is matrix $X = (X_{ij}) \in \mathbb{R}^{V(G)\times V(G)}$ such that

$$Y := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in M_0^{k+1}(G),$$

and we define the corresponding matrix Y' showing $x' \in N_0^{k+1}(G')$ by

$$Y' := \begin{pmatrix} 1 & (x')^T \\ x' & X' \end{pmatrix}$$

where the matrix $X' = (X'_{ij}) \in \mathbb{R}^{V(G')\times V(G')}$ is defined as follows: For type (i) stretching and $i = 1, 2$ let

$$X'_{v_i u} := \begin{cases} x_v & \text{if } u = v_1 \text{ or } u = v_2, \\ 0 & \text{if } u = w, \\ x_u & \text{otherwise;} \end{cases}$$

$$X'_{u v_i} := \begin{cases} x_v & \text{if } u = v_1 \text{ or } u = v_2, \\ 0 & \text{if } u = w, \\ x_u & \text{otherwise;} \end{cases}$$

while the remaining new row and column is

$$X'_{wu} := \begin{cases} 0 & \text{if } u = v_1 \text{ or } u = v_2, \\ 1 - x_v & \text{if } u = w, \\ x_u - X_{vu} & \text{otherwise;} \end{cases} \qquad X'_{uw} := \begin{cases} 0 & \text{if } u = v_1 \text{ or } u = v_2, \\ 1 - x_v & \text{if } u = w, \\ x_u - X_{uv} & \text{otherwise;} \end{cases}$$

and finally $X'_{uu'} := X_{uu'}$ if $u, u' \in V(G) \setminus \{v\}$.

To get Y' for type (ii) stretching first apply type (i) stretching to vertex v, then the subdivision of a star to vertex v_2, then delete w and $\tilde{w}$ and the corresponding rows and columns.

The proof showing $Y' \in M_0^{k+1}(G')$ is the same as in Lemmas 17 and 18. The proofs for the N- and N_+-operators are the same, too, finishing the proof. □

This finishes the proof of Theorem 25. □

5. Further facts about the N_0 and N operators

The polytope $N_0(P)$ has a nice geometric description as follows: for every coordinate x_i take the convex hull of those points of P that have $x_i = 0$ or 1, then the intersection of these convex hulls is $N_0(P)$ (see Balas et al. [3]). We know of no similar nice way to describe the polytope $N(P)$, except for $P \subset \mathbb{R}^2$:

Theorem 27. *When $P \subset [0, 1]^2$, the polytope $N(P)$ is defined by the following inequalities:*

(i) *The inequalities obtained by the N_0 operator.*

(ii) *Pick any vertex v of the unit square and a direction (clockwise or counterclockwise). Find the first points of P in the chosen direction on the two sides of the unit square not containing v. The two nontrivial coordinates of these two points give another point w (e.g. if the two points were $(a, 1)$, $(0, b)$, then $w = (a, b)$). The inequality defined by the line vw that contains the vertex* before *v in the chosen direction is valid for $N(P)$.*

Proof. We use the alternative definition of the N_0 and N operators using the derivation of the valid inequalities. An inequality is valid for $N_0(P)$ if we can obtain it from inequalities valid from P by multiplying them by x_i or $1 - x_i$, then replacing x_i^2 by x_i, and $x_i x_j$ by y_{ij}, and taking their nonnegative linear combination eliminating all y-variables. If we assume $y_{ij} = y_{ji}$ for all coordinates i, j, we get the N operator. For more details, see [11].

Without loss of generality we can assume that v is the origin, and the first two points of P in clockwise direction on its boundary are $(a, 1)$ and $(1, b)$ with $0 < a, b < 1$. Then with some $0 \leq c, d \leq 1$ the inequalities

$$0 \leq ca + (1 - c)x - ay \tag{5}$$

(going through the points $(0, c)$ and $(a, 1)$) and

$$0 \leq 1 - db - (1 - b)x - (1 - d)y \tag{6}$$

(going through the points $(d, 1)$ and $(1, b)$) are valid for P. To get the inequalities that may not be valid for $N_0(P)$, we need to use different variables when multiplying (5) and (6). Since we have only two variables, it is easy to see that to get a meaningful new inequality we need to multiply (5) by y and (6) by x. After replacing x^2 and y^2 by x and y, respectively, and replacing xy and yx by y_{xy} we get

$$0 \leq (c - 1)ay + (1 - c)y_{xy}$$

and

$$0 \leq b(1 - d)x - (1 - d)y_{xy}.$$

Divide these inequalities by $1 - c$ and $1 - d$, then add them to eliminate y_{xy} to get

$$0 \leq bx - ay,$$

which is exactly the line going through the origin and (a, b), and $(1, 0)$ clearly satisfies it. This proves the claim. □

It is unclear whether $N(P)$ admits an analogous description when $d \geq 3$.

Now let us turn back to the graph ranks. We say that $e \in E$ is *r_0-critical* (*r-critical*, resp.) if $r_0(G - e) < r_0(G)$ ($r(G - e) < r(G)$, resp.). Lemmas 3 and 5 imply that if $e \in E$ is r_0-critical then $r_0(G - e) = r_0(G) - 1$, and similarly for the r-critical edges. The proof of the next fact is elementary and hence is omitted. Below, we say *odd-contraction* to mean the inverse of the odd subdivision of an edge.

Proposition 28. *Let G be a graph and $e := \{u, v\} \in E$ be such that $d(u) = d(v) = 2$. Name the other neighbors of u and v as w, z respectively (so that the odd-contraction of the edge e replaces the path $wuvz$ by the edge $\{w, z\}$). If e is r-critical then*

$$r(G) - 1 \leq r(G/e) \leq r(G),$$

where G/e denotes the graph obtained from G by the odd-contraction of the edge e. Moreover, if $G/e \in \mathcal{C}$ then $r(G) = r(G/e)$. Finally, $r(G/e) = r(G) - 1$ if and only if the edge $\{w, z\}$ is not r-critical in G/e.

Analogous statement holds for r_0, $\mathcal{C}_0$, etc. The last statement of Proposition 28 is not empty. Let G denote the graph obtained from the graph in Figure 2 (p. 331) by the subdivision of a star operation applied to node 7. Let e denote the edge in G defined by node 4 and a new node. Then $r(G) = r_0(G) = 2$, but $r(G/e) = r_0(G/e) = 1$. (Of course, the edge $\{1, 7\}$ is neither r-critical nor r_0-critical in G/e.)

Let $\alpha(G)$ denote the size of a maximum stable set in G. An edge $e \in E$ is called *α-critical* if $\alpha(G-e) > \alpha(G)$, while the graph G is called *α-critical* if all of its edges are α-critical. Lovász and Schrijver [16] proved that $\alpha(G)$ can be used to bound the N_0-rank:

$$r_0(G) \leq |V| - \alpha(G) - 1, \tag{7}$$

since we can repeatedly delete vertices outside a maximum stable set. What graphs have the property that there is equality in (7)? Clearly every complete graph. Odd subdivision of an edge destroys this property, since it increases $\alpha(G)$ only by 1, while the number of vertices increases by 2, unless the N_0-rank is also increased. The subdivision of a star operation also destroys it (apply the subdivision of a star operation to any vertex of K_4).

We conclude this section with two very elementary theorems.

Theorem 29. *If $r_0(G) = 1 = |V| - \alpha(G) - 1$ in the connected graph G, then G has the following property: there is an edge $\{u, v\}$ such that the vertices of $V(G) \setminus \{u, v\}$ can be partitioned into three disjoint sets A, B, and C such that vertices of A are joined only to u, the vertices in B are joined only to v, while the vertices of C (which must be nonempty) are joined to both u and v.*

Proof. Suppose $G = (V, E)$ satisfies the assumptions. Then $|V| = \alpha(G) + 2$. Let S be a maximum stable set in G, so $|V \setminus S| = 2$. Define $\{u, v\} := V \setminus S$. There must be at least one path from u to v, since G is connected. Since S is a stable set, such a path cannot have more than one intermediate node. If all such paths have one intermediate node, then $r_0(G) = r(G) = 0$ (since G is bipartite). If there is only an edge connecting u and v, then G is again bipartite and has N_0-rank zero. Therefore $\{u, v\} \in E$, and there is at least one path of length two between u and v. These intermediate nodes make up the set C. Since G is connected, $V \setminus \{u, v\}$ is partitioned as in the statement of the theorem. □

Theorem 30. *If $r_0(G) = |V| - \alpha(G) - 1$, then the deletion of an edge e cannot increase the N_0-rank. Moreover, if the edge e is α-critical, then $r_0(G-e) = |V| - \alpha(G-e) - 1 = r_0(G) - 1$.*

Proof. The first claim follows from (7), since the deletion of an edge cannot decrease $\alpha(G)$. If e is α-critical, then $\alpha(G-e) = \alpha(G)+1$, hence $r_0(G-e) \leq |V| - \alpha(G-e) - 1 = r_0(G) - 1$. But by Lemma 3 the deletion of an edge can decrease the rank by at most 1, hence we have equality. □

6. On the ranks of some sparse graphs

Even though the stable set problem on line graphs and related graphs is rather trivial to solve by matching techniques, such graphs seem quite important in our current understanding of the inefficiencies of N_0, N, and N_+ operators. Therefore, a better understanding of these operators' behaviour on such graphs seems relevant.

The *line graph* $L(G)$ of a graph $G = (V(G), E(G))$ is defined by $V(L(G)) := E(G)$ and $E(L(G)) := \{\{e, f\} : e, f \in E(G)$ are adjacent in $G\}$. Consider $L(K_{2k+1})$, the line graph of the complete graph K_{2k+1}. Then $r_+(L(K_{2k+1})) = \alpha(L(K_{2k+1})) = k$

(see [18]—also see Aguilera et al. [1], which includes a study of Balas–Ceria–Cornuéjols ranks in addition to N- and N_+-ranks for matching and very closely related problems), and this graph is *maximal* in the sense that the addition of an edge to it decreases the stability number and hence r_+. In an effort to understand graphs with high r_+, we would like to understand very sparse graphs with as high an r_+ as possible.

Let $FRAC_M(G)$ denotes the usual LP relaxation of the matching polytope of G, i.e. in addition to the restriction $x \in [0, 1]^{V(G)}$, $FRAC_M(G)$ is defined by the following constraints:

$$\sum_{e \in E(G):\, e \text{ is incident to } v \text{ in } G} x_e \le 1 \qquad \text{for all } v \in V(G). \tag{8}$$

Note that $\{e, f\} \in E(L(G))$ implies that the edges e and f are incident to some node $v \in V(G)$, thus (8) for v implies the inequality $x_e + x_f \le 1$.

Proposition 31. *We have $FRAC(L(G)) \supseteq FRAC_M(G)$, where equality holds if and only if G is a cycle or a path or a disjoint union of these.*

Proof. The inclusion is clear from the definitions above. It is also clear that if G is a cycle or a path, then equality holds. For the converse, suppose that the equality holds. Then every node in G has degree less than or equal to 2 (otherwise $\frac{1}{2}\bar{e} \in FRAC(L(G)) \setminus FRAC_M(G)$, a contradiction). Thus G is a cycle or a path or a disjoint union of these. □

From Proposition 31 we can conclude that

$$N_\sharp^k(FRAC(L(G))) \supseteq N_\sharp^k(FRAC_M(G))$$

for all $k \ge 0$ and for all $N_\sharp \in \{N_0, N, N_+\}$. We know from [18] that for $G = K_{2k+1}$ the N_+-rank of the matching polytope relative to $FRAC_M(G)$ is k. This proves that there exist graphs G with

$$r_+(G) \ge \left\lfloor \frac{1}{4}\left(\sqrt{8|V(G)| + 1} - 1\right) \right\rfloor.$$

This is the best lower bound known to date for $r_+(G)$.

Motivated by the results of the previous section, we are interested in determining r_0 and r values of blossom inequalities on very sparse graphs (before we consider r_+ in the next section). In this section, an *odd subdivision* of a graph is obtained by replacing an edge by a path of length 3. Let the graph shown on Figure 7 be denoted by G_k. If we replace the path $v_{3k-2}v_{3k-1}v_{3k}v_{3k-3}$ with the edge $\{v_{3k-2}, v_{3k-3}\}$ in the graph G_k, denote the resulting graph by $\tilde{G}_k$ (e.g. $\tilde{G}_2 = K_4$). Then G_k is an odd subdivision of $\tilde{G}_k$, and $\alpha(\tilde{G}_k) = \alpha(G_k) - 1$. To motivate these graphs, consider the *house* on five nodes for the matching problem (see Figure 8). Then its line graph has six nodes and the removal of an appropriate edge makes it α-critical (actually, the resulting α-critical graph is exactly G_2). In general, G_k can also be obtained from an *apartment* with $k - 1$ floors (or a *blossom ladder* with k steps—see Figure 8) after taking the line graph and removing $k - 1$ special edges. The motivation for studying these graphs came from the desire

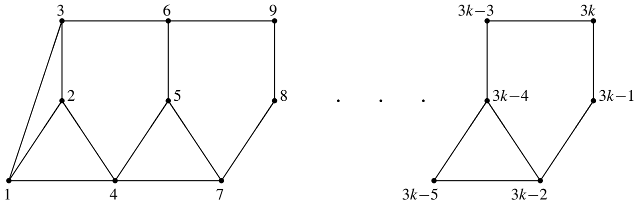

Fig. 7. The graph G_k.

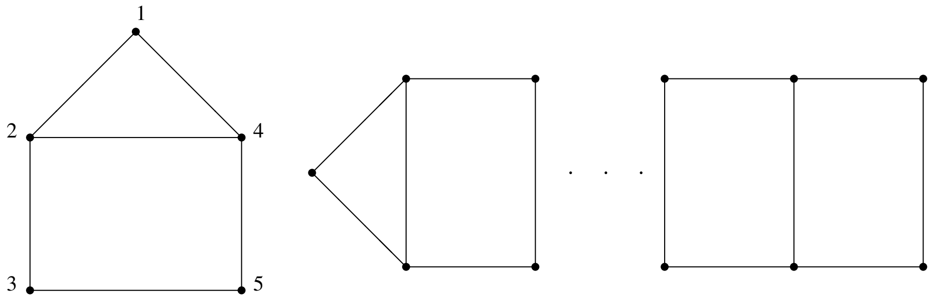

Fig. 8. A house and an apartment

to understand how the lift-and-project procedures behave under the nested blossom inequality structures. We see below in Theorem 35 that the N_0-rank grows logarithmically (as a function of k). As a result, we also see that for this class of graphs, the information given by the destruction lemma (Lemma 4) is extremely weak while the deletion lemma (Lemma 3) can be used to obtain the sharp, logarithmic upper bound.

Lemma 32. *The graphs $\tilde{G}_k$ and G_k are α-critical for $k \geq 1$, and $\alpha(G_k) = k$.*

To prove the above lemma we use the following *union* operation used to obtain new α-critical graphs (see Wessel [19]): Suppose that H_1 and H_2 are disjoint graphs. Let $\{x_1, x_2\} \in E(H_1)$, $\{y_1, y_2\} \in E(H_2)$. Take the disjoint union of the two graphs, delete the edges $\{x_1, x_2\}$ and $\{y_1, y_2\}$, add the edge $\{x_1, y_1\}$, and identify x_2 with y_2. Denote the resulting graph by $H := H_1 \oplus H_2$. The operation is demonstrated on Figure 9 with $H_1 = H_2 = K_4$. Note that odd subdivision is the same as the union with K_3. We also utilize the following theorem of Wessel [19]:

Theorem 33. *If H_1 and H_2 are α-critical, then so is $H = H_1 \oplus H_2$, and $\alpha(H) = \alpha(H_1) + \alpha(H_2)$.*

Proof. (of Lemma 32) The proof goes by induction. The cases $k = 1, 2$ are trivial, since $G_1 = K_3$ and $\tilde{G}_2 = K_4$, hence its subdivision is also α-critical. Now assume that we have shown that G_{k-1} is α-critical for $k \geq 2$. We will show that so is G_{k+1}. Notice that $\tilde{G}_{k+1} = G_{k-1} \oplus K_4$. Hence Theorem 33 implies that $\tilde{G}_{k+1}$ is α-critical, thus so is G_{k+1}, and it is easy to see that $\alpha(G_k) = k$. □

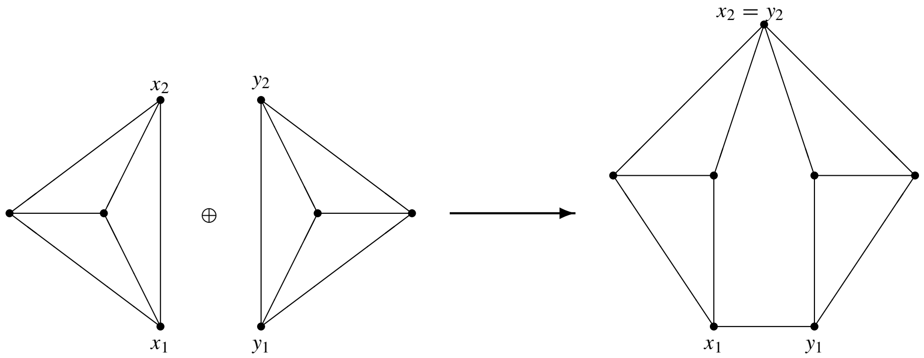

Fig. 9. The union of two K_4

Notice that we can have the edge $v_{3j-4}v_{3j}$ instead of the edge $v_{3j-4}v_{3j-2}$ for any $2 \le j \le k$, and the conclusion of Lemma 32 would still apply, so the resulting graph is also α-critical. Chvátal [4] proved the following nice property of α-critical graphs:

Theorem 34. *If the graph G is α-critical, then $\bar{e}^T x \le \alpha(G)$ defines a facet of $STAB(G)$.*

We can now establish the N_0-rank of the graphs G_k:

Theorem 35. $r_0(G_k) = \lfloor \log_2 \frac{k+1}{3} \rfloor + 2$ *for any* $k \ge 1$.

Proof. Since by Lemma 32 the graph G_k is α-critical for any $k \ge 1$, Theorem 34 implies that $\bar{e}^T x \le \alpha(G_k)$ defines a facet of $STAB(G_k)$. We will show by induction on k that this inequality achieves the N_0-rank of G_k, and this will enable us to find a recursion for $r_0(G_k)$.

The case $k = 1$ is easy to check, so let $k > 1$, and assume for any $1 \le m < k$ that the N_0-rank of $\bar{e}^T x \le \alpha(G_m)$ is equal to $r_0(G_m)$. We use the following property of the N_0-rank of an inequality, proved in [11]: If $a^T x \le b$ is a facet of $STAB(G)$ with N_0-rank k, then there is a vertex $v \in V(G)$ such that the deletion and the destruction of v give rise to inequalities with N_0-rank strictly less than k.

Turning this around and by using Lemma 3 we get that if $k - 1$ is the minimum over all vertices of G of the maximum of the ranks of the inequalities obtained by the deletion and the destruction of v, then the rank of the original inequality is *exactly* k.

Clearly there are basically three different choices for the vertex v to be deleted and destroyed:

(i) $v = v_{3m+1}$ for some $0 \le m \le k - 1$;
(ii) $v = v_{3m+2}$ for some $0 \le m \le k - 1$;
(iii) $v = v_{3m+3}$ for some $0 \le m \le k - 1$.

Consider case (i). If we delete v_{3m+1}, then in the remaining graph v_{3m} is a cut vertex, so by Lemma 5 we can also delete the edge $v_{3m}v_{3m+3}$, then the remaining two disjoint subgraphs will be α-critical, since one of them is G_m, the other one is an odd subdivision of $G_{k-(m+1)}$ as shown on Figure 10 (the cases $m = 0$ or $k - 1$ are simpler). Similarly, if we destroy v_{3m+1}, then the remaining graph will have the following

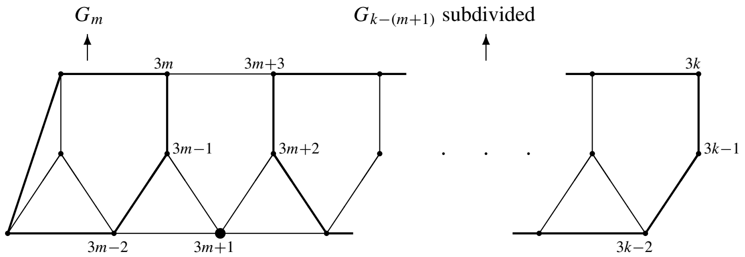

Fig. 10. The deletion of v_{3m+1}.

α-critical components: G_{m+1}, the edge $v_{3m}v_{3m+3}$, and an odd subdivision of $G_{k-(m+2)}$, as demonstrated on Figure 11. Since clearly $r_0(G_m) \leq r_0(G_{m+1})$ for any $m \geq 1$, we obtain using the induction hypothesis that the N_0-rank of $\bar{e}^T x \leq \alpha(G_k)$ is at most $1 + \max\left\{r_0(G_m), r_0(G_{k-(m+1)})\right\}$, and by Lemma 3 it follows that

$$r_0(G_k) \leq 1 + \max\left\{r_0(G_m), r_0(G_{k-(m+1)})\right\}.$$

When we delete and destroy v_{3m+2} we obtain similarly that (notice that the resulting graph can be obtained as the union of G_m and G_{k-m}, hence it is α-critical)

$$r_0(G_k) \leq 1 + \max\left\{r_0(G_{k-1}), r_0(G_{m-1}), r_0(G_{k-(m+1)})\right\},$$

and in case of v_{3m+3} we get

$$r_0(G_k) \leq 1 + \max\left\{r_0(G_m), r_0(G_{k-(m+2)})\right\}.$$

Clearly the smallest N_0-rank (hence the N_0-rank of $\bar{e}^T x \leq \alpha(G_k)$) is obtained in this last case when $m = \lfloor \frac{k-1}{2} \rfloor$. Since this N_0-rank is also an upper bound for $r_0(G_k)$, we get that

$$r_0(G_{2k}) = 1 + r_0(G_{k-1})$$

and

$$r_0(G_{2k+1}) = 1 + r_0(G_k).$$

This recurrence is easy to solve by finding the indices when the N_0-rank is first n. It can be seen easily that for $n \geq 2$

$$r_0(G_k) \geq n \quad \text{if and only if} \quad k \geq 3 \cdot 2^{n-2} - 1,$$

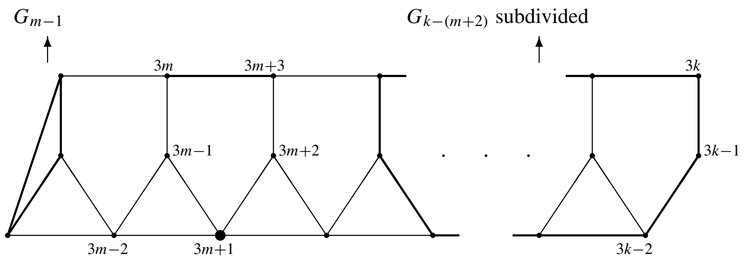

Fig. 11. The destruction of v_{3m+1}.

hence

$$r_0(G_k) = \left\lfloor \log_2 \frac{k+1}{3} \right\rfloor + 2$$

for $k \geq 2$, and this formula gives correct result for $k = 1$, too. □

This example shows that Lemma 4 itself can be rather weak. However, one can combine the N_+ operator with the weaker operators to achieve stronger bounds:

Theorem 36. *For every graph $G = (V, E)$, we have*

$$r_+(G) \leq \min \left\{ \max_{v \in V} \{r_+(G \ominus v)\}, \min_{v \in V} \{r_+(G - v)\} \right\} + 1.$$

Proof. Since we already have Lemma 4, we only need to prove that if there exists $v \in V$ such that $r_+(G - v) \leq k$, then $r_+(G) \leq k + 1$. To prove this, consider $N_+^k(G)$. Then $N(N_+^k(G)) = STAB(G)$ by Lemma 1.3 of [16] since both

$$N_+^k(G) \cap \{x \in \mathbb{R}^V : x_v = 0\} \text{ and } N_+^k(G) \cap \{x \in \mathbb{R}^V : x_v = 1\}$$

are integral polytopes by Lemma 1 and the assumption that $r_+(G - v) \leq k$. Since $N_+(N_+^k(G)) \subseteq N(N_+^k(G))$, we conclude that $r_+(G) \leq k + 1$. □

7. On the N_+-rank of graphs

Let us continue with the thread of investigation from the previous section. With a slightly different slant, we can ask "what is the smallest graph whose N_+-rank is 1?" The answer is "the triangle." The next question in the sequence is a bit harder and its answer exposes a significant amount of new insights into the behaviour of the N_+-rank under the fundamental graph operations.

Proposition 37. *For the graph G_2 on Figure 7 we have $r_+(G_2) = 2$. Moreover, every graph G with $V(G) \leq 5$ or $E(G) \leq 7$ satisfies $r_+(G) \leq 1$. Therefore, G_2 is the smallest graph with N_+-rank equal to 2.*

Proof. Let

$$Y(\xi) := \begin{bmatrix} 1 & \frac{17}{40} & \frac{17}{40} & 1/3 & 1/3 & 1/4 & 1/4 \\ \frac{17}{40} & \frac{17}{40} & 0 & \xi & 0 & 1/8 & 1/8 \\ \frac{17}{40} & 0 & \frac{17}{40} & 0 & \xi & 1/8 & 1/8 \\ 1/3 & \xi & 0 & 1/3 & 1/3-\xi & 0 & 0 \\ 1/3 & 0 & \xi & 1/3-\xi & 1/3 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/8 & 1/8 & 0 & 0 & 0 & 1/4 \end{bmatrix},$$

where the corresponding weights on G_2 are illustrated in Figure 12. It can be easily checked that

$$Y(\xi) \in M(G_2) \text{ if and only if } \frac{11}{60} \leq \xi \leq \frac{3}{10}.$$

Setting $\xi := \frac{497+\sqrt{609}}{2400} \in \left(\frac{11}{60}, \frac{3}{10}\right)$ makes Y positive semidefinite. Since the projection of Y onto the space of $FRAC(G)$ corresponds to the vector $\bar{x} := (\frac{17}{40}, \frac{17}{40}, 1/3, 1/3, 1/4, 1/4)^T$ with $\bar{e}^T\bar{x} = 2 + \frac{1}{60}$, and $\alpha(G_2) = 2$, we have $r_+(G_2) \geq 2$. Using Corollary 2.19 of [16], it is easy to see that $r_+(G_2) \leq 2$. Therefore, $r_+(G_2) = 2$.

Next we prove $r_+(G) \leq 1$ for $|V(G)| \leq 5$ or $|E(G)| \leq 7$. By Lemma 5 we can assume G is 2-connected (otherwise we have a cut vertex and a smaller graph with the same N_+-rank). Thus the degree of every node is at least 2, so for $|V(G)| \leq 5$ the destruction of any node leaves at most an edge, thus the remaining graph has N_+-rank 0, and we get $r_+(G) \leq 1$. Now we can assume $|V(G)| \geq 6$ and $|E(G)| \leq 7$. Since the degree of any node is at least 2, $|V(G)| \in \{6, 7\}$. If $|V(G)| = 7$, then G is an odd hole, which has N_+-rank 1, while if $|V(G)| = 6$, then G is a 6-cycle with a chord. The deletion of any of the degree 3 nodes leaves a path (having N_+-rank 0), therefore $r_+(G) \leq r_0(G) \leq 1 + \min_{v \in V}\{r_0(G - v)\} = 1$. □

Now we consider an arbitrary graph G. First, we present a summarizing theorem, then give a new upper bound on $r_+(G)$ in general. The operation of *cloning a vertex* $v \in G$ is replacing v by a clique of size at least 2 and connecting every new vertex to every original neighbour of v.

Theorem 38. *Each of the following operations can increase $r_+(G)$: odd subdivision of an edge, subdivision of an edge, cloning of a vertex, adding an edge, deleting an edge, contracting an edge.*

Proof. We proved in Proposition 37 that $r_+(G_2) = 2$. Since $r_+(K_4) = 1$ and G_2 can be obtained from K_4 by an odd subdivision of any edge, odd subdivision and consequently subdivision can increase r_+. Next consider the odd hole on 5 nodes (N_+-rank is 1), and clone any node. Since we again get G_2, cloning can increase r_+. Next, take a path of length 2 (N_+-rank is 0) and add the edge that joins the endpoints of the path, yielding a triangle which has N_+-rank 1. To see that deleting an edge can increase the N_+-rank, start with K_6 (which has N_+-rank 1) and notice that G_2 can be obtained from K_6 by deleting some edges. Since $r_+(G_2) = 2$, at some point throughout these deletions, the N_+-rank must increase. Finally, usual contraction can also increase the N_+-rank, since contracting an edge from a 4-cycle (N_+-rank is zero), results in a triangle. □

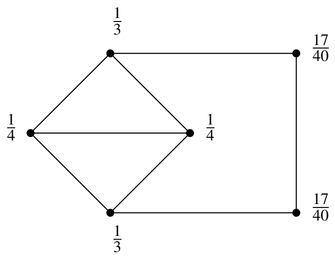

Fig. 12. G_2 with corresponding weights

Using Corollary 2.8 of [16], we get

$$r_+(G) \leq r(G) \leq r_0(G) \leq |V| - \alpha(G) - 1.$$

(The last two inequalities above are tight for $G = K_n$.) Using Corollary 2.19 of [16], we have

$$r_+(G) \leq \alpha(G).$$

(The inequality above is tight for $G = K_n$ and $G = L(K_n)$.) Therefore

$$r_+(G) \leq \left\lfloor \frac{|V| - 1}{2} \right\rfloor.$$

We can further improve the above upper bound:

Theorem 39. *Let $G = (V, E)$. Then $r_+(G) \leq \left\lfloor \frac{|V|}{3} \right\rfloor$.*

Proof. For each $k \geq 1$, let $n_+(k)$ denote the minimum number of nodes needed in a graph G to have $r_+(G) = k$. Clearly, $n_+(1) = 3$. Let G' be a graph with $n_+(k+1)$ nodes such that $r_+(G') = k + 1$. Then G' cannot contain a leaf node or an isolated node (since removing the isolated node or the leaf node does not decrease the N_+-rank, this would contradict the minimality of G'). Thus every node in G' has degree at least 2. Now, there must exist a node in G' whose destruction leaves a graph $\bar{G}$ with $r_+(\bar{G}) \geq k$ (otherwise by Lemma 4 we have $r_+(G') \leq k$, a contradiction). Since G' is a graph with $n_+(k+1)$ nodes such that $r_+(G') = k + 1$, the N_+-rank of $\bar{G}$ must be exactly k. So

$$n_+(k) \leq |V(\bar{G})| \leq |V(G')| - 3 = n_+(k+1) - 3.$$

Since $n_+(1) = 3$, we have the desired result. □

Note that $n_+(2) = 6$ was proved using the graph G_2 in Proposition 37. It is an interesting open question whether the relation $r_+(G) \leq \left\lfloor \frac{|V|}{3} \right\rfloor$ is tight for an infinite family of graphs, or whether $n_+(k) = 3k$ for all $k \geq 1$.

Conjecture 40. $n_+(k) = 3k$ *for all* $k \geq 1$. *Moreover, the equality is attained by a subdivision of the clique* K_{k+2}.

Acknowledgements. We thank the referees for their suggestions that helped greatly improve the presentation of this paper.

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