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Alper Atamtürk

On the facets of the mixed–integer knapsack polyhedron

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Abstract. We study the mixed–integer knapsack polyhedron, that is, the convex hull of the mixed–integer set defined by an arbitrary linear inequality and the bounds on the variables. We describe facet–defining inequalities of this polyhedron that can be obtained through sequential lifting of inequalities containing a single integer variable. These inequalities strengthen and/or generalize known inequalities for several special cases. We report computational results on using the inequalities as cutting planes for mixed–integer programming.

Key words. Mixed-integer programming – knapsack sets – polyhedral theory – lifting

1. Introduction

We investigate the facial structure of the convex hull of the *mixed–integer knapsack set*

$$K = \left\{ (x, w) \in \mathbb{Z}_+^I \times \mathbb{R}_+^C : ax + gw \leq b, \;\; x \leq u, \;\; w \leq v \right\},$$

where I is the index set of integer variables, C is the index set of continuous variables. The mixed–integer knapsack set K is the set of points in $\mathbb{Z}_+^I \times \mathbb{R}_+^C$ that satisfy an arbitrary linear inequality and the upper bounds on the variables. We assume that the data is rational, with the exception that u and v may have entries equal to infinite, so that the variables are not necessarily bounded. We impose no sign restriction on a, g, or b.

Since each constraint of a mixed–integer programming (MIP) formulation defines a mixed–integer knapsack set, strong valid inequalities for K can be used as cutting planes for MIP. There are many important polyhedral studies on special cases of the mixed–integer knapsack set K. The most studied is probably the 0–1 knapsack set ($u = 1$ and $C = \emptyset$) for which seminal works [5, 7, 19, 33] date back to 70's; see also [16, 28, 32, 37]. Crowder, Johnson, and Padberg [13] demonstrate the effectiveness of cutting planes from individual 0–1 knapsack constraints in solving 0–1 programming problems.

Carrying this line of research to mixed 0–1 programming, Marchand and Wolsey [22] give strong inequalities for the 0–1 knapsack set with a continuous variable. Recently, Richard et al. [31] study the mixed 0–1 knapsack set with bounded continuous variables.

Most of the research on polyhedral analysis of structured sets is done on (mixed) 0–1 problems. Polyhedral studies on problems with integer variables, even for the pure

A. Atamtürk*: Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, CA 94720–1777, e-mail: atamturk@ieor.berkeley.edu

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integer case, are rare; see [2, 8, 10, 20, 21, 25] for certain network design problems. Pochet and Weismantel [29] and Pochet and Wolsey [30] study the convex hull of the pure integer knapsack set with divisible coefficients. Ceria et al. [11] give an extension of the 0–1 knapsack cover inequalities for integer knapsacks.

Gomory mixed–integer cuts [14] or, the equivalent, mixed–integer rounding (MIR) cuts [27] are well–known valid inequalities for K, and, consequently, for mixed–integer programming. They are incorporated in leading optimization software systems after their computational effectiveness has been evidenced in a branch–and–cut framework [6, 23]. These general algebraic inequalities depend on the representation of the constraints rather than the geometry of the feasible set since multiplying the coefficients of a constraint by a constant may lead to a different MIR inequality [12]. Furthermore, Gomory mixed–integer cuts or mixed–integer rounding cuts are not only valid for an MIP problem, but also for its group relaxation [15], obtained by dropping the bounds of the basic variables. This suggests that stronger inequalities for K may be identified by studying directly K, rather than its group relaxation, as illustrated in Example 1 in Section 2. Our goal here is to derive strong inequalities based on the geometric structure of the convex hull of K $(conv(K))$.

One difficulty with studying (mixed) integer polyhedra is that simple extensions of combinatorial, disjunctive and/or rounding arguments, that give strong inequalities for (mixed) 0–1 programming, generally do not lead to inequalities that define high–dimensional faces for integer programming. For instance, even though for 0–1 knapsacks a minimal cover inequality is facet–defining on the space of the variables defining the cover, its extension to integer knapsacks may not define a high dimensional face. An intuitive reason for this is that integer points lie “deep” in the linear programming (LP) relaxation as opposed to on the “surface” as in the case for (mixed) 0–1 problems. Recall that all integer points of a 0–1 programming problem are among the extreme points of its LP relaxation.

In this study we make use of superadditive functions for defining strong inequalities for the mixed–integer knapsack set. It is well–known that the convex hull of the feasible region of any MIP problem can be described with inequalities defined by superadditive functions and convex functions [4]. However, from a practical perspective, the challenge is to identify the shape of specific functions that can be used effectively as cutting planes in branch–and–cut computations. See Gu et. al [17] and Marchand and Wolsey [22] for two successful works in this direction for mixed 0–1 programming.

In Section 2 we review recent developments that motivated this study and compute the lifting function of a simple MIR inequality with two variables. Section 3 contains the main results of the paper. Here we describe facet–defining inequalities for $conv(K)$ by building on the results in Section 2. In Section 4 we highlight the connections between the new inequalities and others defined earlier in the literature for certain special cases. In particular, if the coefficients of all integer variables (a_i $i \in I$) have the same sign, either positive or negative, then the new inequalities dominate mixed–integer rounding inequalities [14, 27]. For the special case of 0–1 knapsack with a single continuous variable, the inequalities reduce to the ones given in [22]. For the knapsack set with bounded integer variables, they generalize and strengthen integer cover inequalities [11] and weight inequalities [24]. In Section 5 we present a summary of computational experiments for testing the effectiveness of the new inequalities as cutting planes. The results indicate that the inequalities may be useful in branch–and–cut algorithms for MIP.

In order to simplify the notation, we assume (wlog) that $g_i \in \{-1, 1\}$ for all $i \in C$ since continuous variables can be rescaled. We define $C^+ = \{i \in C : g_i = 1\}$, $C^- = C \setminus C^+$, $I_B = \{i \in I : u_i < \infty\}$, and $C_B = \{i \in C : v_i < \infty\}$. We assume (wlog) that $g_i = 1$ for all $i \in C_B$, after complementing variables if necessary; thus $C_B \subseteq C^+$. Throughout we assume that $conv(K)$ is full–dimensional. We let a^+ denote $\max\{a, 0\}$ for $a \in \mathbb{R}$.

2. Preliminaries

We start with an example to illustrate that inequalities stronger than Gomory mixed–integer or mixed–rounding inequalities can be identified by studying $conv(K)$ directly, rather through its group relaxation.

Example 1. Suppose the mixed–integer knapsack set is given as

$$K' = \left\{(x, w) \in \mathbb{Z}^2 \times \mathbb{R} : x_1 + ax_2 - w \leq 1 + \varepsilon, \;\; x_1 \geq 0, \; x_2 \geq 0, \; w \geq 0\right\},$$

where $0 < \varepsilon < 1$ and $a > 2$. Although it is not necessary, in order to keep the example simple, we assume that a is integer. The Gomory mixed–integer inequality or mixed–integer rounding (MIR) inequality [6, 23]

$$x_1 + ax_2 - \frac{w}{1 - \varepsilon} \leq 1 \tag{1}$$

cuts off the fractional vertex $(x_1, x_2, w) = (1 + \varepsilon, 0, 0)$ of the LP relaxation of K'. Inequality (1) defines a facet of the convex hull of the group relaxation [15] of K'

$$conv(K'_G) = conv\{x \in \mathbb{Z}^2, w \in \mathbb{R} : x_1 + ax_2 - w \leq 1 + \varepsilon, \;\; x_2 \geq 0, \; w \geq 0\},$$

obtained by dropping the nonnegativity constraint $x_1 \geq 0$; however, it is not facet–defining for $conv(K')$. On the other hand, inequality

$$x_1 + \frac{a - 2\varepsilon}{1 - \varepsilon} x_2 - \frac{w}{1 - \varepsilon} \leq 1 \tag{2}$$

is stronger than (1) since $a < (a - 2\varepsilon)/(1 - \varepsilon)$ for $a > 2$ and $0 < \varepsilon < 1$. Indeed, inequality (2) defines a facet of $conv(K')$. Notice that the difference between the coefficients of x_2 in inequalities (1) and (2) becomes arbitrarily large as ε approaches to one. Inequality (2) is the special form of (15) for K'. □

This study is motivated by the following recent developments: the knowledge of a complete linear description of the convex hull of the restriction of K with a single integer variable, the existence of a polynomial–time separation algorithm for this restriction, and the possibility of sequence independent lifting for general mixed–integer programming. In the rest of Section 2 we review these developments and compute the lifting function of a simple MIR inequality with two variables as building blocks for studying $conv(K)$.

2.1. The convex hull of the restriction with a single integer variable

We start by describing a property of the facets of the convex hull of the mixed–integer knapsack set

$$K = \left\{ (x, w) \in \mathbb{Z}_+^I \times \mathbb{R}_+^C : \sum_{i \in I} a_i x_i + \sum_{i \in C^+} w_i - \sum_{i \in C^-} w_i \leq b, \quad x \leq u, \quad w \leq v \right\}$$

regarding the unbounded continuous variables. We call the nonnegativity constraints on the variables, the knapsack constraint, and the upper bound constraints as the *trivial inequalities* of $conv(K)$. The following property is useful.

Proposition 1. *Any non–trivial facet–defining inequality* $\pi x + \mu w \leq \pi_o$ *of* $conv(K)$ *satisfies*

1. $\mu_i = 0$ *for all* $i \in C^+ \setminus C_B$,
2. $\mu_i = \alpha$ *for all* $i \in C^-$, *where* α *is a negative scalar.*

Proof. Let $\pi x + \mu w \leq \pi_o$ be a non–trivial inequality defining facet F of $conv(K)$. For $i \in C^+ \setminus C_B$, since $\pi x + \mu w \leq \pi_o$ differs from $w_i \geq 0$, there exists a point (x', w') on F with $w'_i > 0$. Since reducing w'_i by a small $\epsilon > 0$ gives a feasible point, validity of the inequality implies $\mu_i \geq 0$. Also since the inequality differs from $ax + gw \leq b$, there exists $(\bar{x}, \bar{w})$ on F with $a\bar{x} + g\bar{w} < b$. Since by increasing $\bar{w}_i$ by small $\epsilon > 0$, we maintain feasibility, validity of $\pi x + \mu w \leq \pi_o$ implies $\mu_i \leq 0$.

On the other hand for $i \in C^-$, since w_i can be increased without violating feasibility, validity of $\pi x + \mu w \leq \pi_o$ implies that $\mu_i \leq 0$. However if $\mu_i = 0$, then the feasible point $(\hat{x}, \hat{w})$ with $\pi \hat{x} > \pi_0$, $\hat{w}_i$ large enough (as $C_B \subseteq C^+$), and $\hat{w}_j = 0$ for $j \in C \setminus \{i\}$ is violated by $\pi x + \mu w \leq \pi_o$. Notice that the point $(\hat{x}, \hat{w})$ exists, since otherwise $\pi x + \mu w \leq \pi_o$ is dominated by bound constraints, hence cannot define a facet. Thus we have $\mu_i < 0$. Since $\pi x + \mu y \leq \pi_o$ is different from $w_i \geq 0$, there exists a point $(\tilde{x}, \tilde{w})$ on F with $\tilde{w}_i > 0$. The point obtained from $(\tilde{x}, \tilde{w})$ by decreasing $\tilde{w}_i$ by small $\epsilon > 0$ and increasing $\tilde{w}_j$ for $j \in C^- \setminus \{k\}$ by ϵ is feasible. Since $(\tilde{x}, \tilde{y})$ is on F, we have $\mu_i \geq \mu_j$. Finally $\mu_i \leq \mu_j$ follows from symmetry. □

The first part of Proposition 1 is stated in [22] without proof. Now for some $\ell \in I$, consider the restriction of K obtained by fixing all integer variables but x_ℓ to zero:

$$T = \left\{ (x_\ell, w) \in \mathbb{Z}_+ \times \mathbb{R}_+^C : a_\ell x_\ell + \sum_{i \in C^+} w_i - \sum_{i \in C^-} w_i \leq b, \ x_\ell \leq u_\ell, \ w_i \leq v_i \ i \in C_B \right\}.$$

It follows from Proposition 1 and [21] that $conv(T)$ is given by the inequalities in the description of T and

$$(a_\ell - r)x + \sum_{i \in S} w_i - \sum_{i \in C^-} w_i \leq b - \eta r \quad \forall S \subseteq C_B \quad \text{if } a_\ell > 0, \tag{3}$$

$$rx + \sum_{i \in S} w_i - \sum_{i \in C^-} w_i \leq v_S + \eta r \quad \forall S \subseteq C_B \quad \text{if } a_\ell < 0, \tag{4}$$

where $\eta = \lceil (b - v_S)/a_\ell \rceil$, $r = b - v_S - \lfloor (b - v_S)/a_\ell \rfloor a_\ell$, and $v_S = \sum_{i \in S} v_i$. Moreover, an exact linear–time algorithm is given for separating inequalities (3)–(4) in [3]. This suggests that inequalities for K lifted from (3)–(4) may be potentially useful as cutting planes for K.

2.2. *Sequence independent lifting*

In this section we review a lifting technique for (general) integer variables. Consider a mixed–integer set $P = \{x \in \mathbb{Z}^I, y \in \mathbb{R}^C : Ax + Gy \le b\}$, where A, G, and b are rational matrices with m rows. Let (L, U, R) be a partition of I and $P_{L,U,R}(d) = \{x_R \in \mathbb{Z}^R, y \in \mathbb{R}^C : A_R x_R + Gy \le d\}$ be a nonempty restriction of P, obtained by fixing $x_i = l_i$ for $i \in L$ and $x_i = u_i$ for $i \in U$, where $l_i > -\infty$ and $u_i < +\infty$ are the minimum and maximum values x_i attains in P, respectively. Let

$$\pi_R x_R + \mu y \le \pi_o \tag{5}$$

be a valid inequality for $P_{L,U,R}(d)$ and the lifting function $\Phi : \mathbb{R}^m \mapsto \mathbb{R} \cup \{\infty\}$ of $\pi_R x_R + \mu y \le \pi_o$ be defined as

$$\Phi(a) = \pi_o - \max\left\{\pi_R x_R + \mu y : (x_R, y) \in P_{L,U,R}(d - a)\right\}.$$

We let $\Phi(a) = \infty$ if $P_{L,U,R}(d - a) = \emptyset$. Since (5) is valid for $P_{L,U,R}(d - a)$, the maximization problem above is bounded and consequently $\Phi(a) > -\infty$.

Definition 1. *$\varphi : \mathbb{R}^m \mapsto \mathbb{R}$ is superadditive on $D \subseteq \mathbb{R}^m$ if $\varphi(a) + \varphi(b) \le \varphi(a + b)$ for all $a, b \in D$ such that $a + b \in D$.*

A valid inequality for P can be obtained from (5) by sequential lifting, i.e., introducing the fixed variables x_i $i \in L \cup U$ to the inequality one at a time in some sequence [34]. One difficulty with this approach is that it requires the solution of a *nonlinear* (fractional) mixed–integer problem for each fixed variable, as opposed to a *linear* mixed–integer problem as in 0–1 programming.

For monotone ($A \ge 0$) 0–1 programming and monotone mixed 0–1 programming, Wolsey [35] and Gu et al. [18] show that superadditive lifting functions lead to sequence independent lifting of valid inequalities, which reduces the computational burden of lifting significantly. The theorem below states that this property holds for general mixed–integer programming as well *if* lower dimensional restrictions are obtained by fixing integer variables to a bound rather than to some intermediate value.

Theorem 1. *[1] Let Φ be defined as before and let $\phi : \mathbb{R}^m \mapsto \mathbb{R}$ be a superadditive function such that $\phi \le \Phi$. Then inequality*

$$\pi_R x_R + \sum_{i \in L} \phi(A_i)(x_i - l_i) + \sum_{i \in U} \phi(-A_i)(u_i - x_i) + \mu y \le \pi_o \tag{6}$$

is valid for P. In addition, if $\phi(A_i) = \Phi(A_i)$ for all $i \in L$, $\phi(-A_i) = \Phi(-A_i)$ for all $i \in U$, and inequality (5) *defines a k–dimensional face of $conv(P_{L,U,R}(d))$, then inequality* (6) *defines an at least $k + |L| + |U|$–dimensional face of $conv(P)$.*

We note that if the lifting function is superadditive, nonlinearity of the lifting problems is resolved easily and lifting (5) in any sequence leads to a unique inequality for P. We use Theorem 1 for deriving strong inequalities for K in Section 3.

2.3. Lifting function of a simple MIR inequality

Here we compute the lifting function of a simple MIR inequality for a two–variable mixed–integer restriction of K as a building block for studying $conv(K)$. Let

$$S = \{x \in \mathbb{Z}, y \in \mathbb{R}_+ : cx - y \le d,\ l \le x \le u\}$$

with $c, d \in \mathbb{Q}$ and $l, u \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $l < u$. Let $\eta = \lceil d/c \rceil$ and $r = d - \lfloor d/c \rfloor c$. Observe that the LP relaxation of S has a fractional vertex $(d/c, 0)$ if and only if $d/c \notin \mathbb{Z}$ (or equivalently $r \neq 0$) and $l < d/c < u$. If $c > 0$, the fractional vertex $(d/c, 0)$ is cut off by the simple mixed–integer rounding (SMIR) inequality [27, 36]

$$(c - r)x - y \le d - \eta r. \tag{7}$$

On the other hand, if $c < 0$, it is cut off by inequality

$$rx - y \le \eta r. \tag{8}$$

As illustrated in Figure 1 inequalities (7) and (8) are sufficient to describe $conv(S)$ when added to the original inequalities of S in either case. Lifting these inequalities amounts to maximizing a linear function over S as a function of d.

Maximizing an arbitrary linear function over S is easy. Without loss of generality, we assume that the objective coefficient of y is negative and by scaling is -1, since otherwise the problem is unbounded, and write the optimization problem as

$$\zeta(\bar{d}) = \max\{ex - y : cx - y \le \bar{d},\ l \le x \le u,\ x \in \mathbb{Z},\ y \in \mathbb{R}_+\}. \tag{9}$$

If $e \le 0$ or $e > c$, problem (9) has a trivial optimal solution with $x = l$ or with $x = u$. Otherwise, an optimal solution, which is an extreme point of $conv(S)$, can be found graphically in Figure 1 as stated in the following lemma.

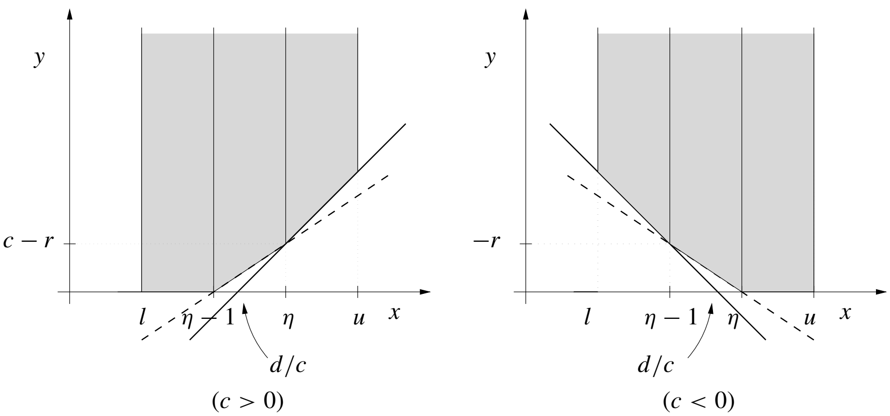

Fig. 1. SMIR inequality and the convex hull of S

Lemma 1. *If* $0 < e \le c$, *then problem* (9) *has an optimal solution* (x, y) *with objective value* $\zeta(\bar{d})$ *that can be expressed as*

$$(x, y, \zeta(\bar{d})) = \begin{cases} (u, 0, eu) & \text{if } \ \bar{d}/c \ge u, \\ (\bar{\eta}, c - \bar{r}, (e-c)\bar{\eta} + \bar{d}) & \text{if } \ c - \bar{r} < e \le c \ \& \ l < \bar{d}/c < u, \\ (\bar{\eta} - 1, 0, e(\bar{\eta} - 1)) & \text{if } \ 0 < e \le c - \bar{r} \ \& \ l < \bar{d}/c < u, \\ (l, lc - \bar{d}, (e-c)l + \bar{d}) & \text{if } \ \bar{d}/c \le l, \end{cases} \tag{10}$$

where $\bar{\eta} = \lceil \bar{d}/c \rceil$ *and* $\bar{r} = \bar{d} - \lfloor \bar{d}/c \rfloor c$.

Now we can compute the lifting function Φ of the SMIR inequality (7) over S. Lifting function of (8) can be computed similarly. Let

$$\Phi(a) = d - \eta r - \max\{(c-r)x - y : cx - y \le d - a, \ l \le x \le u, \ x \in \mathbb{Z}, \ y \in \mathbb{R}_+\}.$$

Theorem 2. *The lifting function* Φ *of inequality* (7) *can be expressed as*

$$\Phi(a) = \begin{cases} (\eta - u - 1)(c - r) & \text{if } \ a < d - uc, \\ k(c - r) & \text{if } \ kc \le a < kc + r, \\ a - (k+1)r & \text{if } \ kc + r \le a < (k+1)c, \\ a - (\eta - l)r & \text{if } \ a \ge d - lc. \end{cases} \qquad k \in \mathbb{Z} \tag{11}$$

Proof. The result follows from setting $\bar{d}$ in (10) equal to $d - a$ and evaluating the objective function.

1. If $a \le d - uc$, or equivalently $(d-a)/c \ge u$, then $\Phi(a) = d - \eta r - (c - r)u = (\eta - u - 1)(c - r)$.
2. Let $\bar{r} = d - a - \lfloor (d-a)/c \rfloor c$ and $\bar{\eta} = \lceil (d-a)/c \rceil$. If $kc \le a < kc + r$, or equivalently $\bar{r} \le r$, then $\Phi(a) = d - \eta r - (c - r)(\bar{\eta} - 1)$. Using $\bar{\eta} = \eta - k$ in this case, we get $\Phi(a) = k(c - r)$.
3. If $kc + r \le a < (k+1)c$, or equivalently $r < \bar{r}$, then $\Phi(a) = d - \eta r - (-r\bar{\eta} + d - a)$. Using $\bar{\eta} = \eta - (k+1)$ in this case, we obtain $\Phi(a) = a - (k+1)r$.
4. If $a \ge d - lc$, or equivalently $(d-a)/c \le l$, then $\Phi(a) = d - \eta r - (-lr + d - a) = a - (\eta - l)r$. □

A particular realization of Φ is depicted in Figure 2. Φ is superadditive on $\mathbb{R}_+$ and on $\mathbb{R}_-$ separately. However, it is superadditive on $\mathbb{R}$ if and only if $l = -\infty$ and $u = +\infty$. The function Φ depicted in Figure 2 is not superadditive on $\mathbb{R}$, as $\Phi(-c) + \Phi(3c) > \Phi(2c)$. Observe from (11) that the function values on intervals $a < (\eta - u - 1)c$ and $a > (\eta - l)c$ are due to the finite upper bound and lower bound on x. If we let $l = -\infty$ and $u = +\infty$, Φ equals its superadditive lower bound

$$\phi(a) = \begin{cases} k(c - r) & \text{if } \ kc \le a < kc + r, \\ a - (k+1)r & \text{if } \ kc + r \le a < (k+1)c. \end{cases} \qquad k \in \mathbb{Z} \tag{12}$$

It is shown in [1] that lifting (7) with the superadditive approximation ϕ gives exactly the MIR inequality [26]

$$\sum_{i \in I} \left(\lfloor a_i/c \rfloor + \frac{(f_i - f)^+}{1 - f} \right) x_i - \frac{y}{c(1-f)} \le \lfloor d/c \rfloor,$$

where $f_i = a_i/c - \lfloor a_i/c \rfloor$ and $f = d/c - \lfloor d/c \rfloor$.

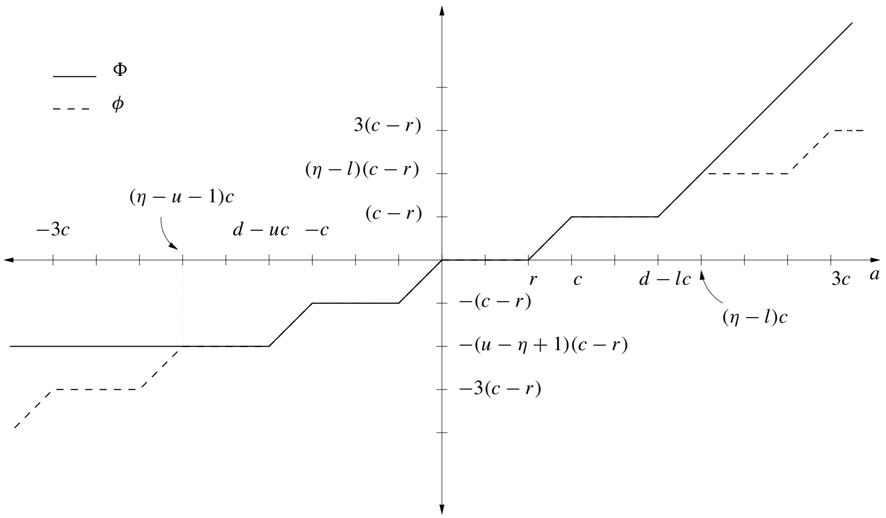

Fig. 2. Lifting function Φ and its superadditive approximation ϕ ($\eta = u - 1 = l + 2$)

3. Facets of *conv*(*K*)

In this section we describe valid inequalities for K defined by the exact lifting function Φ, rather than its superadditive approximation ϕ. Let $\ell \in I$ and $U \subseteq I_B \setminus \{\ell\}$. Defining $z_i = u_i - x_i$ for $i \in U$ and $z_i = x_i$ for $i \in I \setminus U$, we rewrite K as

$$K = \left\{ (z, w) \in \mathbb{Z}_+^I \times \mathbb{R}_+^C : \sum_{i \in I} \bar{a}_i z_i + \sum_{i \in C^+} w_i - \sum_{i \in C^-} w_i \le \bar{b}, \quad z \le u, \quad w \le v \right\},$$

where $\bar{a}_i = -a_i$ for $i \in U$, $\bar{a}_i = a_i$ for $i \in I \setminus U$, and $\bar{b} = b - \sum_{i \in U} a_i u_i$. Now by fixing all integer variables except z_ℓ to zero, we obtain the following restriction of K

$$K_{U,\ell} = \left\{ (z_\ell, w) \in \mathbb{Z}_+ \times \mathbb{R}_+^C : a_\ell z_\ell + \sum_{i \in C^+} w_i - \sum_{i \in C^-} w_i \le \bar{b}, \quad z_\ell \le u_\ell, \quad w \le v \right\}.$$

If $C^- = \emptyset$, we update u_ℓ as $\min\{u_\ell, \lfloor \bar{b}/a_\ell \rfloor\}$ and require that $u_\ell \ge 1$, so that $conv(K_{U,\ell})$ is full–dimensional.

The facets of this restriction have been presented in Section 2.1. Here we extend them to facets of $conv(K)$ using Theorem 1. Since the families of inequalities for $K_{U,\ell}$ depend on the sign of a_ℓ, we consider these cases separately in Sections 3.1 and 3.2.

3.1. *Case 1: $a_\ell > 0$*

In order to obtain facets of $conv(K)$ we lift the facet–defining inequalities of $conv(K_{U,\ell})$

$$(a_\ell - r)z_\ell + \sum_{i\in S} w_i - \sum_{i\in C^-} w_i \le \bar{b} - \eta r, \tag{13}$$

where $\eta = \lceil(\bar{b} - v_S)/a_\ell\rceil$ and $r = \bar{b} - v_S - \lfloor(\bar{b} - v_S)/a_\ell\rfloor a_\ell$ for $S \subseteq C_B$. Letting $y = \sum_{i\in C^-} w_i + \sum_{i\in S}(v_i - w_i)$, we write (13) as

$$(a_\ell - r)z_\ell - y \le d - \eta r, \tag{14}$$

where $d = \bar{b} - v_S$. Since this aggregation of variables has no impact on the objective value of the lifting problems, the lifting function of (13) over $K_{U,\ell}$ is equivalent to the lifting function of (14) over

$$K'_{U,\ell} = \{z_\ell \in \mathbb{Z}_+, y \in \mathbb{R}_+ : a_\ell z_\ell - y \le d,\ z_\ell \le u_\ell\}.$$

Hence, lifting inequality (13) reduces to lifting SMIR inequality (14).

3.1.1. Inequality class I Recall that the lifting function Φ of the SMIR inequality is not superadditive on $\mathbb{R}$. However, it is superadditive on $\mathbb{R}_+$ and on $\mathbb{R}_-$ separately, which allows us to use Theorem 1 in two phases. Let $I^+ = \{i \in I \setminus \{\ell\} : \bar{a}_i > 0\}$ and $I^- = \{i \in I \setminus \{\ell\} : \bar{a}_i < 0\}$. In order to simplify the notation, we define $\ell I^+ = \{\ell\} \cup I^+$ and $\ell I^- = \{\ell\} \cup I^-$. Since Φ is superadditive on $\mathbb{R}_+$, by Theorem 1, inequality (14) can be lifted to

$$\sum_{i\in \ell I^+} \Phi(\bar{a}_i)z_i - y \le d - \eta r \tag{15}$$

since $\Phi(\bar{a}_\ell) = a_\ell - r$. In order to lift (15) with x_i $i \in I^-$, we compute

$$\Omega(a) = d - \eta r - \max\left\{\sum_{i\in \ell I^+} \Phi(\bar{a}_i)z_i - y : (z, y) \in K_{I^-,\ell I^+}(a)\right\}, \tag{16}$$

where

$$K_{I^-,\ell I^+}(a) = \left\{(z, y) \in \mathbb{Z}_+^{\ell I^+} \times \mathbb{R}_+ : \sum_{i\in \ell I^+} \bar{a}_i z_i - y \le d - a,\ z_i \le u_i\ i \in \ell I^+\right\}$$

for $a \in \mathbb{R}_-$.

Lemmas 2 and 3 below are the central results on the structure of optimal solutions to problem (16) that lead to an explicit description of Ω in a special case, a superadditive lower bound on Ω, and the valid inequalities described in Theorem 4.

Let $I^{++} = \{i \in I^+ : \bar{a}_i \ge d\} = \{1, 2, \ldots, n'\}$ be indexed in nonincreasing order of $\bar{a}_i$, ties broken arbitrarily, and let $n = \min\{i \in I^{++} : u_i = \infty\}$ (if $I^{++} = \emptyset$, then let $n = 0$; if $u_i < \infty$ for all $i \in I^{++}$, then let $n = n'$). Also if $I^{++} \ne \emptyset$, let $J^{++} = \{i \in I^{++} : \bar{a}_i \ge \eta a_\ell\} = \{1, 2, \ldots, s'\}$, $s = \min\{i \in J^{++} : u_i = \infty\}$ (if $u_i < \infty$ for all $i \in J^{++}$, then let $s = s'$). Since $J^{++} \subseteq I^{++}$, we have $s' \le n'$ and if $u_i = \infty$ for some $i \in J^{++}$, then $n = s$.

Lemma 2. *For $a \leq 0$ the maximization problem* (16) *has an optimal solution* (z, y) *such that*

(i) $z_i = u_i$ *for all* $i \in \{1, 2, \dots, j-1\}$ *and* $z_i = 0$ *for all* $i \in \{j+1, j+2, \dots, n'\}$ *for some* $j \in \{1, 2, \dots, n'\}$,
(ii) $z_i = 0$ *for all* $i \in I^+$ *with* $0 \leq \bar{a}_i \leq r$, *and*
(iii) $z_i = 0$ *for all* $i \in I^+$ *with* $r < \bar{a}_i < \eta a_\ell$ *if the constraint* $x_\ell \leq u_\ell$ *is removed from* (16).

Proof. (i) For $i \in I^{++}$ we have $\Phi(\bar{a}_i) = \bar{a}_i - \eta r$. Let $\bar{a}_h > \bar{a}_i$ for $h, i \in I^{++}$ and suppose $z_h < u_h$ and $z_i > 0$. Decreasing z_i by one, increasing z_h by one and y by $\bar{a}_h - \bar{a}_i$, we obtain a feasible solution with the same objective value. (ii) Follows from $\Phi(\bar{a}_i) = 0$ for $i \in I^+$ if $0 \leq \bar{a}_i \leq r$. (iii) Suppose $z_i = q > 0$ for some $i \in I^+$ such that $r < \bar{a}_i < \eta a_\ell$. If $k a_\ell \leq \bar{a}_i < k a_\ell + r$, then $\Phi(\bar{a}_i) = (a_\ell - r)k$. Since $x_\ell \leq u_\ell$ is removed, decreasing z_i to 0 and increasing z_ℓ by kq, we obtain a feasible solution with the same objective value. Otherwise if $k a_\ell + r \leq \bar{a}_i < (k+1)a_\ell$, then $\Phi(\bar{a}_i) = \bar{a}_i - (k+1)r$. Similarly, decreasing z_i to 0 and increasing z_ℓ by $(k+1)q$ and y by $((k+1)a_\ell - \bar{a}_i)q$ we obtain a feasible solution with the same objective value. □

Lemma 3. *If $I^+ = I^{++}$, then for $a_2 < a_1 \leq 0$ the maximization problem* (16) *has optimal solutions* $(z(a_1), y(a_1))$ *and* $(z(a_2), y(a_2))$ *that satisfy* $z_i(a_1) \leq z_i(a_2)$ *for all* $i \in I^{++}$.

Proof. Let $(z(a_1), y(a_1))$ and $(z(a_2), y(a_2))$ be two solutions satisfying Lemma 2 (i) for $a_1 > a_2$. Suppose $z_k(a_1) > z_k(a_2)$ for some $k \in \{1, 2, \dots, n'\}$. Then, since $z_k(a_1) > 0$ and $z_k(a_2) < u_k$, Lemma 2 (i) implies that $z_i(a_1) = u_i$ for all $i \in \{1, 2, \dots, k-1\}$ and $z_i(a_2) = 0$ for all $i \in \{k+1, k+2, \dots, n'\}$. Thus $z_i(a_1) \geq z_i(a_2)$ holds for all $i \in I^{++}$.

Next we show that the knapsack constraint $\sum_{i \in I^{++}} \bar{a}_i z_i(a_2) + a_\ell z_\ell(a_2) - y(a_2) \leq d - a_2$ has a slack of at most r. Suppose the slack is $r + \epsilon$ with $\epsilon > 0$. If $z_\ell(a_2) < u_\ell$, then by increasing $z_\ell(a_2)$ by one and $y(a_2)$ by $(a_\ell - r - \epsilon)^+$, we obtain a solution with objective value $\min\{a_\ell - r, \epsilon\}$ larger than for $(z(a_2), y(a_2))$. On the other hand, if $z_\ell(a_2) = u_\ell$, then by decreasing $z_\ell(a_2)$ by $\eta - 1$, and increasing $z_k(a_2)$ by one and $y(a_2)$ by $(a_k - d - \epsilon)^+$, the objective value is increased by $\min\{a_k - d, \epsilon\}$. Both cases either contradict the optimality of $(z(a_2), y(a_2))$ or give an alternative optimal solution in which $z_k(a_2)$ is increased. Therefore, we may assume that the slack of the knapsack constraint for $(z(a_2), y(a_2))$ is at most r. Then, since $a_1 > a_2$ as well, feasibility of $(z(a_1), y(a_1))$ requires that

$$y(a_1) > y(a_2) + \sum_{i \in I^{++}} \bar{a}_i (z_i(a_1) - z_i(a_2)) + a_\ell z_\ell(a_1) - a_\ell z_\ell(a_2) - r \tag{17}$$

$$\geq \bar{a}_k + a_\ell z_\ell(a_1) - a_\ell z_\ell(a_2) - r. \tag{18}$$

The second inequality follows from $y(a_2) \geq 0$, $z_i(a_1) \geq z_i(a_2)$ for $i \in I^{++}$, and $z_k(a_1) > z_k(a_2)$. Now let $\delta_k = \bar{a}_k - \eta a_\ell$. We consider two cases depending on the sign of δ_k. In each case we either obtain a contradiction or change the value of $z_k(a_1)$ or $z_k(a_2)$ toward satisfying $z_k(a_1) \leq z_k(a_2)$.

First suppose that $\delta_k \geq 0$. If $z_\ell(a_2) \geq \eta$, then the solution obtained from $(z(a_2), y(a_2))$ by increasing $z_k(a_2)$ by one and $y(a_2)$ by δ_k, and decreasing $z_\ell(a_2)$ by η is another optimal solution in which $z_k(a_2)$ is increased. Therefore, we may assume that $z_\ell(a_2) \leq \eta - 1$. Then from (18) we have

$$y(a_1) > \bar{a}_k + a_\ell z_\ell(a_1) - a_\ell(\eta - 1) - r = \bar{a}_k - d + a_\ell z_\ell(a_1).$$

Thus $y(a_1) = \bar{a}_k - d + a_\ell z_\ell(a_1) + \epsilon$, where $\epsilon > 0$. If $z_\ell(a_1) \geq 1$, the solution obtained from $(z(a_1), y(a_1))$ by decreasing $z_\ell(a_1)$ by one and $y(a_1)$ by a_ℓ has an objective value r larger than for $(z(a_1), y(a_1))$. Otherwise $z_\ell(a_1) = 0$, and the solution obtained from $(z(a_1), y(a_1))$ by increasing $z_\ell(a_1)$ by $\eta - 1$, decreasing $z_k(a_1)$ by one and $y(a_1)$ by $\bar{a}_k - d + \min\{r, \epsilon\}$ has an objective value $\min\{r, \epsilon\}$ larger than for $(z(a_1), y(a_1))$. Both cases contradict the optimality of $(z(a_1), y(a_1))$.

Now suppose that $\delta_k < 0$. If $z_\ell(a_1) \leq u_\ell - \eta$, then the solution obtained from $(z(a_1), y(a_1))$ by decreasing $z_k(a_1)$ by one and increasing $z_\ell(a_1)$ by η and $y(a_1)$ by $-\delta_k$ is an alternative optimal solution in which $z_k(a_1)$ is decreased. Therefore, we may assume that $z_\ell(a_1) \geq u_\ell - \eta + 1$. Let $\kappa = z_\ell(a_1) - (u_\ell - \eta + 1)$. From (18) we have

$$\begin{aligned} y(a_1) &> \bar{a}_k + a_\ell(u_\ell - \eta + 1) + a_\ell \kappa - a_\ell z_\ell(a_2) - r \\ &= \bar{a}_k - a_\ell(\eta - 1) - r + a_\ell(u_\ell - z_\ell(a_2)) + a_\ell \kappa \geq \bar{a}_k - d + a_\ell \kappa. \end{aligned}$$

So $y(a_1) = \bar{a}_k - d + a_\ell \kappa + \epsilon$, where $\epsilon > 0$. If $\kappa \geq 1$, then the solution obtained from $(z(a_1), y(a_1))$ by decreasing $z_\ell(a_1)$ by one and $y(a_1)$ by a_ℓ has an objective value r larger than for $(z(a_1), y(a_1))$. Otherwise, $\kappa = 0$ or $z_\ell(a_1) = u_\ell - (\eta - 1)$, and the solution obtained from $(z(a_1), y(a_1))$ by increasing $z_\ell(a_1)$ by $\eta - 1$, decreasing $z_k(a_1)$ by one and $y(a_1)$ by $\bar{a}_k - d + \min\{r, \epsilon\}$ has an objective value $\min\{r, \epsilon\}$ larger than for $(z(a_1), y(a_1))$. Both cases contradict the optimality of $(z(a_1), y(a_1))$. □

Theorem 3. *If $\bar{a}_i \leq r$ or $\bar{a}_i \geq d$ for all $i \in I^+$, then*

$$\Omega(a) = \begin{cases} u_{ih}\eta r + a & \text{if } m_{ih} - \delta_i \leq a \leq m_{ih}, \\ (u_{ih}\eta + k)r + a & \text{if } m_{ih} - \delta_i - (k+1)a_\ell + r \leq a \leq m_{ih} - \delta_i - k a_\ell, \\ u_{ih}\eta r + m_{ih} - \delta_i - (k+1)(a_\ell - r) & \text{if } m_{ih} - \delta_i - (k+1)a_\ell \leq a \leq m_{ih} - \delta_i - (k+1)a_\ell + r, \\ u_{su_s}\eta r + pr + a & \text{if } m_{J^{++}} - (p+1)a_\ell + r \leq a \leq m_{J^{++}} - p a_\ell, \\ m_{J^{++}} + u_{su_s}\eta r - (p+1)(a_\ell - r) & \text{if } m_{J^{++}} - (p+1)a_\ell \leq a \leq m_{J^{++}} - (p+1)a_\ell + r, \\ u_{iu_i}\eta r + (\eta - u_\ell - 1)r + a & \text{if } \bar{m}_{ih} - \delta_i - (a_\ell - r) \leq a \leq \bar{m}_{ih}, \\ u_{iu_i}\eta r + (\eta - u_\ell)r + \bar{m}_{ih} - \delta_i - (k+1)a_\ell & \text{if } \bar{m}_{ih} - \delta_i - (k+1)a_\ell \leq a \leq \bar{m}_{ih} - \delta_i - (k+1)a_\ell + r, \\ u_{iu_i}\eta r + (\eta - u_\ell - 1)r + (a_\ell - r) + a & \text{if } \bar{m}_{ih} - \delta_i - (k+1)a_\ell + r \leq a \leq \bar{m}_{ih} - \delta_i - k a_\ell, \\ m_{I^{++}} + u_{nu_n}\eta r + (u_\ell - \eta + 1)r & \text{if } a \leq m_{I^{++}}, \end{cases}$$

where $\delta_i = \bar{a}_i - \eta a_\ell$ for $i \in I^{++}$, $u_{ih} = \sum_{k=1}^{i-1} u_k + h$, $m_{ih} = m_{(i-1)u_{i-1}} - h\bar{a}_i$ for $h \in \{0, 1, \ldots, u_i\}$, $i \in \{1, 2, \ldots, n\}$ with $m_{0u_0} = 0$, and $\bar{m}_{ih} = m_{ih} - (u_\ell - \eta + 1)a_\ell$ for $h \in \{0, 1, \ldots, u_i\}$, $i \in \{s, s+1, \ldots, n\}$, and $m_{J^{++}} = m_{su_s}$, $m_{I^{++}} = \bar{m}_{nu_n}$, $k \in \{0, 1, \ldots, \eta - 1\}$, and $p \in \{0, 1, \ldots, u_\ell - \eta\}$.

Proof. From Lemma 2 (ii), we may assume that $z_i = 0$ for all $i \in I^+$ with $\bar{a}_i \leq r$. Consequently, the condition of Lemma 3 is satisfied. If $i \in I^{++} = \emptyset$, then (15) equals (14); hence $\Omega(a) = \Phi(a)$ for $a \leq 0$ as $m_{J^{++}} = 0$ and $m_{I^{++}} = (\eta - u_\ell - 1)a_\ell$. Otherwise, from Lemma 2 (i) and Lemma 3, there exist optimal solutions to (16), in which z_i $i \in I^{++}$ increase monotonically in nonincreasing order of $\bar{a}_i$ as a decreases.

That is, as a decreases from 0 there exists optimal solutions, where first z_1 is incremented from 0 to u_1 and then z_2 and so on. Thus by fixing z_i $i \in I^{++}$ in the order described in Lemma 2 (i) and Lemma 3, the lifting problem reduces to optimizing over the remaining variables z_ℓ and y. Suppose $z_i = u_i$ for $i \in \{1, 2, \dots, j-1\}$, $z_j = \rho$, and $z_i = 0$ for $i \in \{j+1, j+2, \dots, n'\}$. So the right hand side of the knapsack constraint for the reduced problem in two variables is $d - m_{j(\rho-1)} - a$. Then, if $\delta_j \geq 0$, similar to the discussion in Section 2.3, for $m_{j\rho} \leq a + m_{j(\rho-1)} \leq m_{j(\rho-1)}$ an optimal solution for the maximization problem (16) is given by

$$(z_j, z_\ell, y) = \begin{cases} (\rho, 0, a + \delta_j + a_\ell - r) & \text{if } -\delta_j \leq a \leq 0, \\ (\rho, k, a + \delta_j + (k+1)a_\ell - r) & \\ \quad\quad \text{if } -\delta_j - ka_\ell - (a_\ell - r) \leq a \leq -\delta_j - ka_\ell, & \\ (\rho, k, 0) \quad \text{if } -\delta_j - (k+1)a_\ell \leq a \leq -\delta_j - ka_\ell - (a_\ell - r), & \end{cases}$$

where $k \in \{0, 1, \dots, \eta - 1\}$. Since $(\rho, \eta - 1, 0)$ and $(\rho + 1, 0, \delta_j + a_\ell - r)$ both have the same objective value, as a decreases further, the structure of optimal values of z_ℓ and y repeats for $z_j \in \{\rho + 1, \rho + 2, \dots, u_j\}$. On the other hand if $\delta_j < 0$, then for $\bar{m}_{j\rho} \leq a + \bar{m}_{j(\rho-1)} \leq \bar{m}_{j(\rho-1)}$

$$(z_j, z_\ell, y) = \begin{cases} (\rho, u_\ell - \eta + 1, a + \delta_j + a_\ell - r) & \\ \quad\quad \text{if } -(a_\ell - r) - \delta_j \leq a \leq 0, & \\ (\rho, u_\ell - \eta + 1 + k, 0) & \\ \quad\quad \text{if } -\delta_j - (k+1)a_\ell \leq a \leq -\delta_j - (k+1)a_\ell + r, & \\ (\rho, u_\ell - \eta + 1 + k, a + (k+1)a_\ell - r + \delta_j) & \\ \quad\quad \text{if } -\delta_j - (k+1)a_\ell + r \leq a \leq -ka_\ell - \delta_j, & \end{cases}$$

where $k \in \{0, 1, \dots, \eta - 1\}$ is optimal for the lifting problem (16). Since $(\rho, u_\ell, 0)$ and $(\rho + 1, u_\ell - \eta + 1, \delta_j + a_\ell - r)$ have the same objective value, as a decreases further, z_j increases to $\rho + 1$ and the structure of optimal values of z_ℓ and y repeats. Evaluating Ω for these optimal solutions by incrementing $z_1, z_2, \dots, z_n$ one at a time in the order given in Lemma 2 (i) and Lemma 3 we obtain the expression in the statement of the theorem for Ω. □

An example lifting function Ω is depicted in Figure 4. Observe that the last case in the definition of Ω applies if $u_n < \infty$ and the three cases before that apply if $u_\ell < \infty$. Also note that if $I^{++} = \emptyset$, $\Omega(a) = \Phi(a)$ for $a \leq 0$ as $m_{J^{++}} = 0$ and $m_{I^{++}} = (\eta - u_\ell - 1)a_\ell$.

Giving an explicit description of Ω is difficult in general, because the properties described in Lemma 2 (i) and Lemma 3 do not hold for x_i $i \in I^+$ with $r < \bar{a}_i < d$. Therefore, instead, we give a lower bound on Ω, which equals Ω over a significant part of its domain. The lower bound is obtained by dropping the upper bound constraint $z_\ell \leq u_\ell$ from the lifting problem (16) so that there is an easy description of the optimal solutions to this relaxed problem as described in part (iii) of Lemma 2. Dropping $z_\ell \leq u_\ell$ from

(16), we obtain the following lower bound on Ω:

$$\omega(a) = d - \eta r - \max\Bigg\{ \sum_{i \in \ell I^+} \Phi(\bar{a}_i) z_i - y : \sum_{i \in \ell I^+} \bar{a}_i z_i - y \le d - a,\ z_i \le u_i\ i \in I^+,\ z \in \mathbb{Z}_+^{\ell I^+}, y \in \mathbb{R}_+ \Bigg\}$$

$$= d - \eta r - \max\Bigg\{ (a_\ell - r) z_\ell + \sum_{i \in J^{++}} (\bar{a}_i - \eta a_\ell) z_i - y : a_\ell z_\ell + \sum_{i \in J^{++}} \bar{a}_i z_i - y \le d - a,\ z_i \le u_i\ i \in J^{++},\ z \in \mathbb{Z}_+^{\ell J^{++}}, y \in \mathbb{R}_+ \Bigg\}.$$

The last equality follows from part (iii) of Lemma 2. Since $\bar{a}_i \ge d$ for all $i \in J^{++}$ and the upper bound on z_ℓ is dropped, it follows from Theorem 3 that ω can be expressed as

$$\omega(a) = \begin{cases} u_{ih}\eta r + a & \text{if } m_{ih} - \delta_i \le a \le m_{ih}, \\ (u_{ih}\eta + k) r + a & \text{if } m_{ih} - \delta_i - (k+1)a_\ell + r \le a \le m_{ih} - \delta_i - k a_\ell, \\ u_{ih}\eta r + m_{ih} - \delta_i - (k+1)(a_\ell - r) & \text{if } m_{ih} - \delta_i - (k+1)a_\ell \le a \le m_{ih} - \delta_i - (k+1)a_\ell + r, \\ u_{su_s}\eta r + pr + a & \text{if } m_{J^{++}} - (p+1)a_\ell + r \le a \le m_{J^{++}} - p a_\ell, \\ m_{J^{++}} + u_{su_s}\eta r - (p+1)(a_\ell - r) & \text{if } m_{J^{++}} - (p+1)a_\ell \le a \le m_{J^{++}} - (p+1)a_\ell + r, \end{cases}$$

where $k \in \{0, 1, \ldots, \eta - 1\}$ and $p \in \mathbb{Z}_+$.

Proposition 2. *Let Ω and ω be defined as above:*
1. *ω is a superadditive lower bound on Ω on $\mathbb{R}_-$.*
2. *$\Omega(a) = \omega(a)$ for $\bar{m}_{J^{++}} \le a \le 0$; hence Ω is superadditive on $[\bar{m}_{J^{++}}, 0]$, where $\bar{m}_{J^{++}} = m_{J^{++}} - (u_\ell - \eta + 1)a_\ell$.*
3. *Ω is superadditive on $\mathbb{R}_-$ under any of the following conditions:* *$(i)\, u_\ell = \infty$, $(ii)\, \eta = 1$, $(iii)\, \nexists i \in I^+$ s.t. $r < \bar{a}_i < \eta a_\ell$, $(iv)\, \exists i \in I^{++}$ s.t. $u_i = \infty$.*

Proof. 1. Since $\delta_i \ge 0$ for all $i \in J^{++}$, ω is a special case of the superadditive function ψ introduced in Section 3.3 with parameters $b_i = \delta_i$, $e = a_\ell$, $\rho = r$, and $\tau = \eta$. As ω is obtained by dropping the constraint $z_\ell \le u_\ell$ from the lifting problem (16), ω is a lower bound on Ω. 2. Follows from the descriptions of the functions Ω and ω. 3. In case (i) $\Omega = \omega$ on $\mathbb{R}_-$. In case (ii) Ω is a special case of the superadditive function $\bar{\psi}$ in Section 3.3 with parameters $b_i = \delta_i + a_\ell - r$, $e = r$, $\rho = r$, and $\tau = 1$. In cases (iii) and (iv) Ω is a special case $\bar{\psi}$ with the same parameters as in part 1. □

Remark 1. Observe that whenever $x_\ell \in \{0, 1\}$ and (13) is facet–defining for $conv(K_{U,\ell})$, we have $r = d$ and $\eta = 1$. Therefore the condition of Theorem 3 is satisfied and, from Proposition 2, Ω is superadditive on $\mathbb{R}_-$. It is possible to construct Ω that is not superadditive on $\mathbb{R}_-$ if none of the conditions of part 3 of Proposition 2 is satisfied.

From Theorem 1 and Proposition 2 we obtain the valid inequalities described in Theorem 4.

Theorem 4. *For $\ell \in I$ such that $a_\ell > 0$, $U \subseteq I_B \setminus \{\ell\}$, and $S \subseteq C_B$ let Φ, Ω, and ω be defined as before. Then inequality*

$$\sum_{i \in \ell I^+ \setminus U} \Phi(a_i)x_i + \sum_{i \in I^+ \cap U} \Phi(-a_i)(u_i - x_i) + \sum_{i \in I^- \setminus U} \omega(a_i)x_i + \sum_{i \in I^- \cap U} \omega(-a_i)(u_i - x_i) + \sum_{i \in S} w_i - \sum_{i \in C^-} w_i \le \bar{b} - \eta r \quad (19)$$

is valid for K. It is facet–defining for $conv(K)$ if inequality (13) *is facet–defining for $conv(K_{U,\ell})$ and $\bar{a}_i \ge \bar{m}_{J^{++}}$ for all $i \in I^-$. Moreover, when any one of the conditions of part 3 of Proposition 2 is satisfied, ω may be replaced with the exact lifting function Ω so that* (19) *is facet–defining for $conv(K)$.*

Example 2. Let $K = \{x \in \mathbb{Z}^3_+, w \in \mathbb{R}^2_+ : 3x_1 + 10x_2 - 4x_3 + w_1 - w_2 \le 8,\ x_1 \le 3,\ w_1 \le 1\}$. For $\ell = 1$ consider the restriction $K_{\emptyset,1} = \{x_1 \in \mathbb{Z}_+, w \in \mathbb{R}^2_+ : 3x_1 + w_1 - w_2 \le 8,\ x_1 \le 3,\ w_1 \le 1\}$. From Section 2.1 the two additional inequalities needed to describe $conv(K_{\emptyset,1})$ are

$$2x_1 + w_1 - w_2 \le 5, \quad (20)$$

$$x_1 - w_2 \le 2 \quad (21)$$

with $S = \{1\}$ and $S = \emptyset$, respectively. Lifting (20) first with x_2 using Φ, and then with x_3 using Ω gives us the facet–defining inequality (19)

$$2x_1 + 7x_2 - 3x_3 + w_1 - w_2 \le 5. \quad (22)$$

Note that here $\eta = \lceil (h - v_1)/a_\ell \rceil = 3$, $r = 1$, and consequently $\Phi(10) = 7$, $\delta_1 = a_2 - \eta a_\ell = 1$, and $\Omega(-4) = -3$. The lifting functions Φ and Ω for (20) are drawn in Figures 3 and 4. Observe that the MIR inequality

$$2x_1 + 6x_2 - 3x_3 + w_1 - w_2 \le 5, \quad (23)$$

obtained by lifting (20) using the lower bound ϕ is weaker than (22).

Similarly, lifting (21) first with x_2 using Φ, and then with x_3 using Ω gives us the facet–defining inequality (19)

$$x_1 + 4x_2 - 2x_3 - w_2 \le 2 \quad (24)$$

as $\eta = \lceil h/a_\ell \rceil = 3$, $r = 2$, and consequently $\Phi(10) = 4$ and since $\delta_1 = 1$ we have $\Omega(-4) = -2$. Lifting functions for this inequality are not drawn. Again the corresponding MIR inequality

$$x_1 + 3x_2 - 2x_3 - w_2 \le 2 \quad (25)$$

obtained by lifting (21) using ϕ is weaker than (24). □

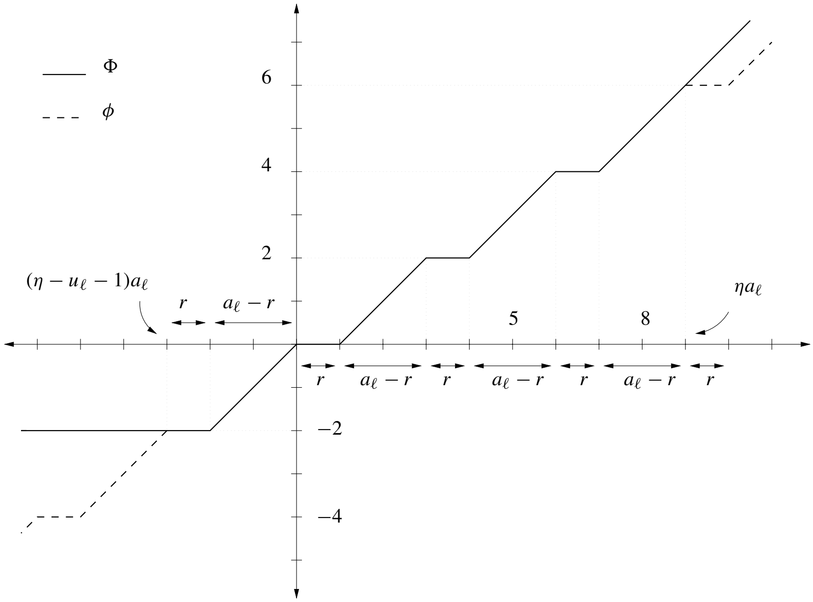

Fig. 3. Lifting function Φ in Example 2 ($\eta = u_\ell = 3$)

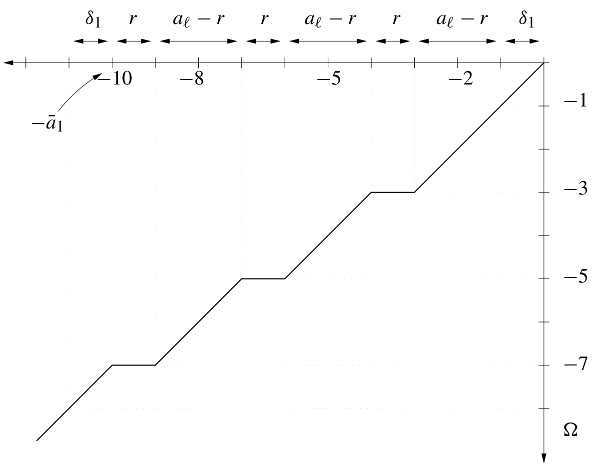

Fig. 4. Lifting functions Ω in Example 2 ($\eta = 3$)

3.1.2. Inequality class II Since Φ is superadditive also on $\mathbb{R}_-$, this time we lift the SMIR inequality (14) first with x_i $i \in I^-$ to obtain the intermediate inequality

$$\sum_{i \in \ell I^-} \Phi(\bar{a}_i) z_i - y \leq d - \eta r \tag{26}$$

since $\Phi(\bar{a}_\ell) = a_\ell - r$. Next we lift (26) with x_i $i \in I^+$. Let us define the lifting function of (26) as

$$\Gamma(a) = d - \eta r - \max \left\{ \sum_{i \in \ell I^-} \Phi(\bar{a}_i) z_i - y : (z, y) \in K_{I^+, \ell I^-}(a) \right\}, \qquad (27)$$

where

$$K_{I^+, \ell I^-}(a) = \left\{ (z, y) \in \mathbb{Z}_+^{\ell I^-} \times \mathbb{R}_+ : \sum_{i \in \ell I^-} \bar{a}_i z_i - y \le d - a,\ z_i \le u_i\ i \in \ell I^- \right\}$$

for $a \in \mathbb{R}_+$.

Lemmas 4 and 5 below are the central results on the structure of optimal solutions to problem (27). They lead to an explicit description of Γ in a special case, a superadditive lower bound on Γ, and the valid inequalities described in Theorem 6.

Let $I^{--} = \{i \in I^- : \bar{a}_i \le d - u_\ell a_\ell\} = \{1, 2, \ldots, n'\}$ be indexed in nondecreasing order of $\bar{a}_i$, ties broken arbitrarily. Let $n = \min\{i \in I^{--} : u_i = \infty\}$ (if $I^{--} = \emptyset$, then let $n = 0$; if $u_i < \infty$ for all $i \in I^{--}$, then let $n = n'$). If $I^{--} \neq \emptyset$, let $J^{--} = \{i \in I^{--} : \bar{a}_i \le (\eta - 1 - u_\ell) a_\ell\} = \{1, 2, \ldots, s'\}, s = \min\{i \in J^{--} : u_i = \infty\}$ (if $u_i < \infty$ for all $i \in J^{--}$, then let $s = s'$). Since $J^{--} \subseteq I^{--}$, we have $s' \le n'$ and if $u_i = \infty$ for some $i \in J^{--}$, then $n = s$.

Lemma 4. *For $a \ge 0$ the maximization problem* (27) *has an optimal solution* (z, y) *such that*

(i) $z_i = u_i$ *for all* $i \in \{1, 2, \ldots, j - 1\}$ *and* $z_i = 0$ *for all* $i \in \{j + 1, j + 2, \ldots, n'\}$ *some* $j \in \{1, 2, \ldots, n'\}$,
(ii) $z_i = 0$ *for all* $i \in I^-$ *with* $r - a_\ell \le \bar{a}_i \le 0$, *and*
(iii) $z_i = 0$ *for all* $i \in I^-$ *with* $(\eta - 1 - u_\ell) a_\ell < \bar{a}_i < r - a_\ell$ *if* $z_\ell \ge 0$ *is removed from* (27).

Proof. (i) For $i \in I^{--}$, we have $\Phi(\bar{a}_i) = (\eta - 1 - u_\ell)(a_\ell - r)$. Let $\bar{a}_h < \bar{a}_i$ for $h, i \in I^{--}$ and suppose $z_h < u_h$ and $z_i > 0$. Decreasing z_i by one, increasing z_h by one, we obtain a feasible solution with the same objective value. (ii) Since $\Phi(\bar{a}_i) = \bar{a}_i$ for $i \in I^-$ with $r - a_\ell \le \bar{a}_i \le 0$, if $z_i = p > 0$, we obtain a feasible solution, with the same objective value by increasing y by $-\bar{a}_i p$ and decreasing z_i to zero. (iii) Suppose $z_i = p > 0$. If $k a_\ell \le \bar{a}_i < k a_\ell + r$, then $\Phi(\bar{a}_i) = (a_\ell - r)k$. Notice that since $\bar{a}_i < 0$ and $\bar{a}_\ell > 0$, k is a negative integer. Since $z_\ell \ge 0$ is removed, by decreasing z_i to 0 and decreasing z_ℓ by $-kp$, we obtain a feasible solution with the same objective value. Else if $k a_\ell + r \le \bar{a}_i < (k + 1) a_\ell$, then $\Phi(\bar{a}_i) = \bar{a}_i - (k + 1) r$. Similarly, decreasing z_i to 0 and decreasing z_ℓ by $-(k + 1)p$ and y by $((k + 1) a_\ell - \bar{a}_i)p$, we obtain a feasible solution with the same objective value. □

Lemma 5. *If $I^- = I^{--}$, then for $a_1 > a_2 \ge 0$ the maximization problem* (27) *has optimal solutions* $(z(a_1), y(a_1))$ *and* $(z(a_2), y(a_2))$ *that satisfy* $z_i(a_1) \ge z_i(a_2)$ *for all* $i \in I^{--}$.

Proof. Let $a_1 > a_2$ and $(z(a_1), y(a_1))$ and $(z(a_2), y(a_2))$ be two optimal solutions satisfying Lemma 4 (i). Suppose $z_k(a_1) < z_k(a_2)$ for some $k \in \{1, 2, \dots, n'\}$. Then $z_k(a_1) < u_k$, $z_k(a_2) > 0$, and from Lemma 4 (i), we have $z_i(a_1) = 0$ for all $i \in \{k+1, k+2, \dots, n'\}$ and $z_i(a_2) = u_i$ for all $i \in \{1, 2, \dots, k-1\}$, implying $z_i(a_1) \leq z_i(a_2)$ for all $i \in I^{--}$.

Let φ be the slack of the knapsack constraint $\sum_{i \in I^{--}} \bar{a}_i z_i(a_2) + a_\ell z_\ell(a_2) - y(a_2) \leq d - a_2$. Since $a_1 > a_2$, feasibility of $(z(a_1), y(a_1))$ requires that

$$y(a_1) > y(a_2) + \sum_{i \in I^{--}} \bar{a}_i (z_i(a_1) - z_i(a_2)) + a_\ell z_\ell(a_1) - a_\ell z_\ell(a_2) - \varphi \tag{28}$$

$$\geq -\bar{a}_k + a_\ell z_\ell(a_1) - a_\ell z_\ell(a_2) - \varphi. \tag{29}$$

The last inequality follows from $y(a_2) \geq 0$, $z_k(a_1) < z_k(a_2)$, $\bar{a}_i < 0$, and $z_i(a_1) \leq z_i(a_2)$ for all $i \in I^{--}$. Let $\delta_k = (\eta - u_\ell - 1)a_\ell - \bar{a}_k$. The upper bound we give on the slack φ is a function of the sign of δ_k and the value of $z_\ell(a_2)$. If $z_\ell(a_2) < u_\ell$, then $\varphi \leq r$. Since otherwise $\varphi = r + \epsilon$ with $\epsilon > 0$ and increasing $z_\ell(a_2)$ by one and increasing $y(a_2)$ by $(a_\ell - r - \epsilon)^+$ gives a solution with objective value $\min\{a_\ell - r, \epsilon\}$ larger than for $(z(a_2), y(a_2))$. On the other hand, if $z_\ell(a_2) \geq u_\ell - \eta$, then $\varphi \leq r + \delta_k$. Since otherwise $\varphi = \delta_k + r + \epsilon$, with $\epsilon > 0$ and the solution obtained from $(z(a_2), y(a_2))$ by decreasing $z_\ell(a_2)$ by $u_\ell - \eta$ and $y(a_2)$ by $(a_\ell - r - \epsilon)^+$ and increasing $z_k(a_2)$ by one has an objective value $\min\{a_\ell - r, \epsilon\}$ larger than for $(z(a_2), y(a_2))$. Hence, we conclude that when $\delta_k \geq 0$, we have $\varphi \leq r + \delta_k$ if $z_\ell(a_2) = u_\ell$ and $\varphi \leq r$ otherwise; and when $\delta_k < 0$, we have $\varphi \leq r + \delta_k$ if $z_\ell(a_2) \geq u_\ell - \eta$ and $\varphi \leq r$ otherwise.

Next we use the bounds on φ either to obtain a contradiction or to change the value of $z_k(a_1)$ or $z_k(a_2)$ toward satisfying $z_k(a_1) \geq z_k(a_2)$. First consider the case $\delta_k \geq 0$. If $z_\ell(a_1) \leq \eta - 1$, then the solution obtained from $(z(a_1), y(a_1))$ by increasing $z_k(a_1)$ by one and increasing $z_\ell(a_1)$ by $u_\ell - \eta + 1$ is feasible since $\delta_k \geq 0$ and has the same objective value. So we may assume that $z_\ell(a_1) \geq \eta$. Then, using $\varphi \leq r + \delta_k$, from (29) we get

$$y(a_1) > -\bar{a}_k - a_\ell(u_\ell - \eta) - \delta_k - r \geq a_\ell - r.$$

That is, $y(a_1) = a_\ell - r + \epsilon$, where $\epsilon > 0$. But then the solution obtained from $(z(a_1), y(a_1))$ by decreasing $z_\ell(a_1)$ by one and decreasing $y(a_1)$ by $a_\ell - r + \min\{r, \epsilon\}$ is feasible since $z_\ell(a_1) \geq 1$ and has an objective value $\min\{r, \epsilon\}$ larger than for $(z(a_1), y(a_1))$, which contradicts the optimality of $(z(a_1), y(a_1))$.

Now consider the case $\delta_k < 0$. If $z_\ell(a_2) \geq u_\ell - \eta + 1$, then the solution obtained from $(z(a_2), y(a_2))$ by decreasing $z_k(a_2)$ by one and decreasing $z_\ell(a_2)$ by $u_\ell - \eta + 1$ is feasible has the same objective value. Therefore, we may assume that $z_\ell(a_2) \leq u_\ell - \eta$. We need to consider three subcases depending on the values of $z_\ell(a_1)$ and $z_\ell(a_2)$.

(i) $z_\ell(a_1) > 0$: Using $z_\ell(a_2) \leq u_\ell - \eta$ and $\varphi \leq r$, from (29) we get

$$y(a_1) > -\bar{a}_k + a_\ell z_\ell(a_1) - a_\ell(u_\ell - \eta) - r \geq a_\ell z_\ell(a_1) - r \geq a_\ell - r.$$

So $y(a_1) = a_\ell - r + \epsilon$ with $\epsilon > 0$. Since $z_\ell(a_1) \geq 1$, the solution obtained from $(z(a_1), y(a_1))$ by decreasing $z_\ell(a_1)$ by one and decreasing y_ℓ by $a_\ell - r +$

$\min\{r, \epsilon\}$ gives a feasible solution with objective value $\min\{r, \epsilon\}$ larger than for $(z(a_1), y(a_1))$, contradicting its optimality.

(*ii*) $z_\ell(a_1) = 0$ and $z_\ell(a_2) < u_\ell - \eta$: Let $\kappa = u_\ell - \eta - z_\ell(a_2)$. Using $\varphi \le r$, (29) gives

$$\begin{aligned} y(a_1) &> -\bar{a}_k - a_\ell(u_\ell - \eta - \kappa) - r \ge (a_\ell(\eta - u_\ell) - \bar{a}_k) + a_\ell\kappa - r \\ &\ge a_\ell\kappa - r \ge a_\ell - r. \end{aligned}$$

Thus again $y(a_1) = a_\ell - r + \epsilon$ with $\epsilon > 0$. Since $z_\ell(a_1) = 0$, in this case the solution obtained from $(z(a_1), y(a_1))$ by increasing $z_k(a_1)$ by one and increasing $z_\ell(a_1)$ by $u_\ell - \eta$ and $y(a_1)$ by $a_\ell - r + \min\{\epsilon, \delta_k + r\}$ is feasible and improves the objective value by $\min\{\epsilon, \delta_k + r\}$. If $\delta_k + r > 0$, this contradicts the optimality of $(z(a_1), y(a_1))$. If $\delta_k + r = 0$, we have an alternative solution in which $z_k(a_1)$ is one larger.

(*iii*) $z_\ell(a_1) = 0$ and $z_\ell(a_2) = u_\ell - \eta$: Using $\varphi \le \delta_k + r$ from (29) we have

$$\begin{aligned} y(a_1) &> -\bar{a}_k - a_\ell u_\ell + a_\ell\eta - \delta_k - r \\ &= (-\bar{a}_k + (\eta - u_\ell - 1)a_\ell - \delta_k) + a_\ell - r = a_\ell - r. \end{aligned}$$

Since and $z_\ell(a_1) = 0$ and $y(a_1) = a_\ell - r + \epsilon$ with $\epsilon > 0$ the case reduces to case 2 above. □

Theorem 5. *If $\bar{a}_i \le d - u_\ell a_\ell$ or $\bar{a}_i \ge r - a_\ell$ for all $i \in I^-$, then*

$$\Gamma(a) = \begin{cases} u_{ih}(u_\ell - \eta + 1)(a_\ell - r) & \text{if } m_{ih} \le a \le m_{ih} + \delta_i, \\ (u_{ih}(u_\ell - \eta + 1) + k)(a_\ell - r) & \text{if } m_{ih} + \delta_i + k a_\ell \le a \le m_{ih} + \delta_i + k a_\ell + r, \\ u_{ih}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{ih} - \delta_i - (k+1)r & \text{if } m_{ih} + \delta_i + k a_\ell + r \le a \le m_{ih} + \delta_i + (k+1)a_\ell, \\ (u_{su_s}(u_\ell - \eta + 1) + p)(a_\ell - r) & \text{if } m_{J^{--}} + p a_\ell \le a \le m_{J^{--}} + p a_\ell + r, \\ u_{su_s}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{J^{--}} - (p+1)r & \text{if } m_{J^{--}} + p a_\ell + r \le a \le m_{J^{--}} + (p+1)a_\ell, \\ (u_{iu_i}(u_\ell - \eta + 1) + \eta)(a_\ell - r) & \text{if } \bar{m}_{ih} \le a \le \bar{m}_{ih} + \delta_i + r \\ (u_{iu_i}(u_\ell - \eta + 1) + \eta)(a_\ell - r) + a - \bar{m}_{ih} - \delta_i - (k+1)r & \text{if } \bar{m}_{ih} + \delta_i + k a_\ell + r \le a \le \bar{m}_{ih} + \delta_i + (k+1)a_\ell \\ (u_{iu_i}(u_\ell - \eta + 1) + \eta + k)(a_\ell - r) & \text{if } \bar{m}_{ih} + \delta_i + k a_\ell \le a \le \bar{m}_{ih} + \delta_i + k a_\ell + r \\ u_{nu_n}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{I^{--}} - \eta r & \text{if } a \ge m_{I^{--}}, \end{cases}$$

where $\delta_i = (\eta - u_\ell - 1)a_\ell - \bar{a}_i$ for $i \in I^{--}$, $u_{ih} = \sum_{k=1}^{i-1} u_k + h$, $m_{ih} = m_{(i-1)u_{i-1}} - h\bar{a}_i$ for $h \in \{0, 1, \dots, u_i\}$ and $i \in \{1, 2, \dots, n\}$ with $m_{0u_0} = 0$, $\bar{m}_{ih} = m_{ih} + \eta a_\ell$ for $h \in \{0, 1, \dots, u_i\}$ and $i \in \{s, s+1, \dots, n\}$, $m_{J^{--}} = m_{su_s}$, $m_{I^{--}} = \bar{m}_{nu_n}$, $k \in \{0, 1, \dots, u_\ell - \eta\}$, and $p \in \{0, 1, \dots, \eta - 1\}$.

Proof. From Lemma 4 (*ii*), we may assume that $z_i = 0$ for all $i \in I^-$ such that $\bar{a}_i \ge r - a_\ell$. Hence the condition of Lemma 5 is satisfied. Observe that if $I^{--} = \emptyset$, inequality (26) equals (14). Consequently $\Gamma(a) = \Omega(a)$ for $a \in \mathbb{R}_+$ as $m_{J^{--}} = 0$ and $m_{I^{--}} = \eta a_\ell$. Otherwise, from Lemma 4 (*i*) and Lemma 5, as a increases, there exist optimal solutions in which z_i $i \in I^-$ is incremented monotonically in nondecreasing order of $\bar{a}_i$. Thus after fixing $z_1, z_2, \dots, z_{n'}$ in this order, the problem reduces to one with two variables z_ℓ and y as in Section 2.3. Suppose $z_i = u_i$ for $i \in \{1, 2, \dots, j-1\}$, $z_j = \rho$, and $z_i = 0$ for $i \in \{j+1, j+2, \dots, n'\}$. For the restricted problem the right hand side of the knapsack constraint becomes $d - m_{j(\rho-1)} - a$. Then, if $\delta_j \ge 0$, for

$m_{j(\rho-1)} \le m_{j(\rho-1)} + a \le m_{j\rho}$ an optimal solution for the restricted problem is given by

$$(z_j, z_\ell, y) = \begin{cases} (\rho, u_\ell, 0) & \text{if } 0 \le a \le \delta_j \\ (\rho, u_\ell - k, 0) & \text{if } \delta_j + ka_\ell \le a \le \delta_j + ka_\ell + r \\ (\rho, u_\ell - k, a - \delta_j - ka_\ell - r) & \text{if } \delta_j + ka_\ell + r \le a \le \delta_j + (k+1)a_\ell, \end{cases}$$

where $k \in \{0, 1, \ldots, u_\ell - \eta\}$. Since $(\rho, \eta, a_\ell - r)$ and $(\rho + 1, u_\ell, 0)$ are alternative optimal solutions when a increases further, the values of z_ℓ and y repeat the same pattern for $z_j \in \{\rho + 1, \rho + 2, \ldots, u_j\}$. On the other hand, if $\delta_j < 0$, then similarly for $\bar{m}_{j(\rho-1)} \le \bar{m}_{j(\rho-1)} + a \le \bar{m}_{j\rho}$

$$(z_j, z_\ell, y) = \begin{cases} (\rho, u_\ell - \eta, 0) & \text{if } 0 \le a \le \delta_j + r \\ (\rho, u_\ell - \eta - k, a - \delta_j - ka_\ell - r) & \text{if } \delta_j + ka_\ell + r \le a \le \delta_j + (k+1)a_\ell \\ (\rho, u_\ell - \eta - k, 0) & \text{if } \delta_j + ka_\ell \le a \le \delta_j + ka_\ell + r, \end{cases}$$

where $k \in \{0, 1, \ldots, u_\ell - \eta\}$ is an optimal solution to the restricted lifting problem. Since $(\rho, 0, a_\ell - r)$ and $(\rho+1, u_\ell - \eta, 0)$ have the same objective value, when a increases further, the structure of optimal solutions is repeated for $z_j \in \{\rho + 1, \rho + 2, \ldots, u_j\}$. Using the equality $d - \eta r = (\eta - 1)(a_\ell - r)$ in (27) and evaluating Γ for these optimal solutions by in incrementing $z_1, z_2, \ldots, z_n$ one at a time in the order described in Lemma 4 (i) and Lemma 5 gives the expression of the theorem for Γ. □

An example lifting function Γ is depicted in Figure 5. Observe that the last case in the definition of Γ applies if $u_n < \infty$. Also note that if $I^{--} = \emptyset$, $\Gamma(a) = \Phi(a)$ for $a \ge 0$ as $m_{J^{--}} = 0$ and $m_{I^{--}} = \eta a_\ell$.

The properties described in Lemma 4 (i) and Lemma 5 do not hold for x_i $i \in I^{--}$ with $d - u_\ell a_\ell < \bar{a}_i < r - a_\ell$, which makes it hard to characterize Γ in general. Therefore, we give a lower bound γ on Γ by dropping the nonnegativity constraint $z_\ell \ge 0$ from (27) so that an optimal solution can be easily described based on part (iii) of Lemma 4. So consider the relaxation of problem (27)

$$\gamma(a) = d - \eta r - \max \Bigg\{ \sum_{i \in I^-} \Phi(\bar{a}_i) z_i + (a_\ell - r) z_\ell - y :$$

$$\sum_{i \in I^-} \bar{a}_i z_i + a_\ell z_\ell - y \le d - a,\ z_i \le u_i\ i \in \ell I^-,\ z_\ell \in \mathbb{Z},\ z \in \mathbb{Z}_+^{I^-},\ y \in \mathbb{R}_+ \Bigg\}$$

$$= (\eta - 1)(a_\ell - r) - \max \Bigg\{ \sum_{i \in J^{--}} (\eta - 1)(a_\ell - r) z_i + (a_\ell - r) z_\ell - y :$$

$$z_i \le u_i\ i \in \ell J^{--},\ z_\ell \in \mathbb{Z},\ z \in \mathbb{Z}_+^{J^{--}},\ y \in \mathbb{R}_+$$

$$\sum_{i \in J^{--}} \bar{a}_i z_i + a_\ell z_\ell - y \le (\eta - 1) a_\ell + r - a \Bigg\}.$$

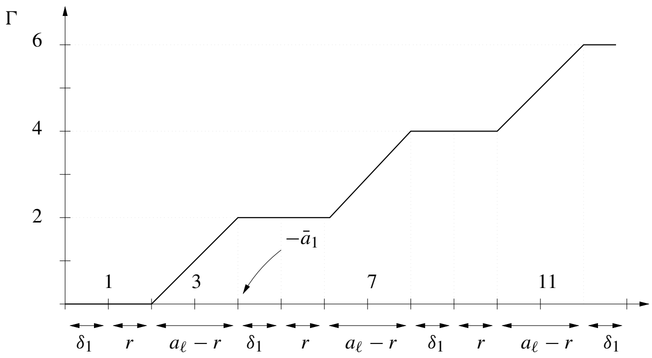

Fig. 5. Lifting function Γ in Example 2 (cont.) ($\eta = 3$)

The second equality follows from Lemma 4 (iii). Since $\bar{a}_i \leq d - u_\ell a_\ell$ for all $i \in J^{--}$ and the lower bound on z_ℓ is dropped, from Theorem 5 we get

$$\gamma(a) = \begin{cases} u_{ih}(u_\ell - \eta + 1)(a_\ell - r) & \text{if } m_{ih} \leq a \leq m_{ih} + \delta_i, \\ (u_{ih}(u_\ell - \eta + 1) + k)(a_\ell - r) & \text{if } m_{ih} + \delta_i + k a_\ell \leq a \leq m_{ih} + \delta_i + k a_\ell + r, \\ u_{ih}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{ih} - \delta_i - (k+1)r & \text{if } m_{ih} + \delta_i + k a_\ell + r \leq a \leq m_{ih} + \delta_i + (k+1)a_\ell, \\ (u_{su_s}(u_\ell - \eta + 1) + p)(a_\ell - r) & \text{if } m_{J^{--}} + p a_\ell \leq a \leq m_{J^{--}} + p a_\ell + r, \\ u_{su_s}(u_\ell - \eta + 1)(a_\ell - r) + a - m_{J^{--}} - (p+1)r & \text{if } m_{J^{--}} + p a_\ell + r \leq a \leq m_{J^{--}} + (p+1)a_\ell, \end{cases}$$

where $k \in \{0, 1, \ldots, u_\ell - \eta\}$ and $p \in \mathbb{Z}_+$.

Proposition 3. *Let Γ and γ be defined as above:*

1. *γ is a superadditive lower bound on Γ on $\mathbb{R}_+$.*
2. *$\Gamma(a) = \gamma(a)$ for $0 \leq a \leq \bar{m}_{J^{--}}$; hence Γ is superadditive on $[0, \bar{m}_{J^{--}}]$, where $\bar{m}_{J^{--}} = m_{J^{--}} + \eta a_\ell$.*
3. *Γ is superadditive on $\mathbb{R}_+$ under any of the following conditions: (i) $\eta = u_\ell$, (ii) $\nexists i \in I^-$ s.t. $(\eta - u_\ell - 1)a_\ell < \bar{a}_i < r - a_\ell$, (iii) $\exists i \in I^{--}$ s.t. $u_i = \infty$.*

Proof. 1. Since $\delta_i \geq 0$ for all $i \in J^{--}$, γ is a special case of the superadditive function χ in Section 3.3 with parameters $b_i = \delta_i$, $e = a_\ell$, $\rho = r$, and $\tau = u_\ell - \eta + 1$. Since γ is obtained by solving a relaxation of the lifting problem obtained by dropping the constraint $z_\ell \geq 0$, it is a lower bound on Γ. 2. Immediate from the descriptions of Γ and γ. 3. In case (i) Γ is a special case of the superadditive function $\bar{\chi}$ in Section 3.3 with $b_i = \delta_i + r$, $e = a_\ell - r$, $\rho = 0$, and $\tau = 1$. In cases (ii) and (iii) Γ is a special case of $\bar{\chi}$ with the same parameters as in part 1. □

Remark 2. Observe that if $x_\ell \in \{0,1\}$ and (13) is facet–defining for $conv(K_{U,\ell})$, then $r = d$ and $\eta = u_\ell$. Therefore the condition of Theorem 5 is satisfied and by Proposition 3 Γ is superadditive on $\mathbb{R}_+$. It is possible to construct Γ that is not superadditive on $\mathbb{R}_+$ if none of the conditions of part 3 of Proposition 3 is satisfied.

Finally Theorem 1 and Proposition 3 lead to the valid inequalities described in Theorem 6.

Theorem 6. *For $\ell \in I$ s.t. $a_\ell > 0$, $U \subseteq I_B \setminus \{\ell\}$, and $S \subseteq C_B$ let Φ, Γ, and γ be defined as before. Then inequality*

$$\sum_{i\in I^+\setminus U} \gamma(a_i)x_i + \sum_{i\in I^+\cap U} \gamma(-a_i)(u_i - x_i) + \sum_{i\in \ell I^-\setminus U} \Phi(a_i)x_i + \sum_{i\in I^-\cap U} \Phi(-a_i)(u_i - x_i) + \sum_{i\in S} w_i - \sum_{i\in C^-} w_i \le h - \eta r \quad (30)$$

is valid for K. It is facet–defining for $conv(K)$ if inequality (13) *is facet–defining for $conv(K_{U,\ell})$ and $\bar{a}_i \le \bar{m}_{J^{--}}$ for all $i \in I^+$. Moreover, when any one of the conditions of part 3 of Proposition 3 is satisfied, γ may be replaced with Γ so that* (30) *is facet–defining for $conv(K)$.*

Example 2 (cont.) When we lift (20) first with x_3 using Φ, and then with x_2 using Γ we obtain the facet–defining inequality (30)

$$2x_1 + 4x_2 - 2x_3 + w_1 - w_2 \le 5 \quad (31)$$

since $\Phi(-4) = -2$ and $\Gamma(10) = 4$ for $\eta = 3$, $r = 1$, and $\delta_1 = (\eta - u_\ell - 1)a_\ell - a_3 = 1$. The lifting functions Φ and Γ for (20) are depicted in Figures 3 and 5.

On the other hand lifting (21) first with x_3 using Φ, then with x_2 using Γ gives us the facet–defining inequality (30)

$$x_1 + 2x_2 - x_3 - w_2 \le 2 \quad (32)$$

as $\eta = 3$, $r = 2$ and consequently $\Phi(-4) = -1$ and since $\delta_1 = 1$, we have $\Gamma(10) = 2$. □

3.2. *Case 2: $a_\ell < 0$*

In this case the inequalities

$$-rz_\ell + \sum_{i\in S} w_i - \sum_{i\in C^-} w_i \le v_S - \eta r \quad \forall\, S \subseteq C_B \quad (33)$$

where $\eta = \lceil(\bar{b} - v_S)/a_\ell\rceil$ and $r = v_S - \bar{b} + \lfloor(\bar{b} - v_S)/a_\ell\rfloor a_\ell$ are sufficient to describe $conv(K_{U,\ell})$ when added to formulation with $a_\ell z_\ell + \sum_{i\in C^+} w_i - \sum_{i\in C^-} w_i \le \bar{b}$ and the bounds. Lifting them in a similar way as in Section 3.1, we obtain the inequalities described in Theorems 7 and 8.

Theorem 7. *For $\ell \in I$ such that $a_\ell < 0$, $U \subseteq B_I \setminus \{\ell\}$, and $S \subseteq C_B$ let Φ and ω be defined as before. Then inequality*

$$\sum_{i \in \ell I^- \setminus U} (a_i + \Phi(-a_i))x_i + \sum_{i \in \ell I^- \cap U} (-a_i + \Phi(a_i))(u_i - x_i)$$
$$+ \sum_{i \in I^+ \cap U} (-a_i + \omega(a_i))(u_i - x_i) + \sum_{i \in I^+ \setminus U} (a_i + \omega(-a_i))x_i + \sum_{i \in S} w_i - \sum_{i \in C^-} w_i \leq v_S - \eta r$$

is valid for K. It is facet–defining for $conv(K)$ if inequality (33) *is facet–defining for $conv(K_{U,\ell})$ and $\bar{a}_i \leq \bar{m}_{J^{--}}$ for all $i \in I^+$.*

Theorem 8. *For $\ell \in I$ such that $a_\ell < 0$, $U \subseteq B_I \setminus \{\ell\}$, and $S \subseteq C_B$ let Φ and γ be defined as before. Then inequality*

$$\sum_{i \in \ell I^+ \setminus U} (a_i + \Phi(-a_i))x_i + \sum_{i \in \ell I^+ \cap U} (-a_i + \Phi(a_i))(u_i - x_i)$$
$$+ \sum_{i \in I^- \setminus U} (a_i + \gamma(-a_i))x_i + \sum_{i \in I^- \cap U} (-a_i + \gamma(a_i))(u_i - x_i) + \sum_{i \in S} w_i - \sum_{i \in C^-} w_i \leq v_S - \eta r$$

is valid for K. It is facet–defining for $conv(K)$ if inequality (33) *is facet–defining for $conv(K_{U,\ell})$ and $\bar{a}_i \geq \bar{m}_{J^{++}}$ for all $i \in I^-$.*

3.3. Four superadditive functions

Here we prove the superadditivity of four general piecewise–linear continuous functions of which the lifting functions Ω, Γ, ω, and γ introduced in Sections 3.1.1 and 3.1.2 are particular cases. Let $b_i \in \mathbb{R}_+$ $i = 1, 2, \ldots, m$ be such that $b_i \geq b_{i+1}$. Let $e \geq \rho \geq 0$ and $a_i = \tau e + b_i$ for some nonnegative integer τ. Define the partial sums $A_0 = 0$, $A_i = \sum_{k=1}^{i} a_k$, and $B_i = A_{i-1} + b_i$ for $1 \leq i \leq m$. Let $\chi : [0, A_m] \mapsto \mathbb{R}_+$ be defined as

$$\chi(a) = \begin{cases} i\tau(e-\rho) & \text{if } A_i \leq a \leq B_{i+1}, \\ (i\tau + k)(e-\rho) & \text{if } B_{i+1} + ke \leq a \leq B_{i+1} + ke + \rho, \\ i\tau(e-\rho) + a - B_{i+1} - (k+1)\rho & \text{if } B_{i+1} + ke + \rho \leq a \leq B_{i+1} + (k+1)e, \end{cases}$$

where $k \in \{0, 1, \ldots, \tau - 1\}$ and $\psi : [-A_m, 0] \mapsto \mathbb{R}_-$ be $\psi(a) = a + \chi(-a)$ for $a \in [-A_m, 0]$. Also let $\bar{\chi} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be defined as

$$\bar{\chi}(a) = \begin{cases} \chi(a) & \text{if } 0 \leq a \leq A_m, \\ \chi(A_m) + a - A_m & \text{if } A_m \leq a. \end{cases}$$

and $\bar{\psi} : \mathbb{R}_- \mapsto \mathbb{R}_-$ as $\bar{\psi}(a) = a + \bar{\chi}(-a)$ for $a \in \mathbb{R}_-$.

As stated in Proposition 2, ω is a special case of ψ and and Ω is a special case of $\bar{\psi}$ under the conditions of part 3 of the same proposition. Also γ is a special case of χ as described in Proposition 3 and Γ is a spacial case of $\bar{\chi}$ under the conditions of part 3 of the same proposition. By the following lemmas, χ, $\bar{\chi}$, ψ, and $\bar{\psi}$ are superadditive on their domain.

Lemma 6. *χ is superadditive on* $[0, A_m]$.

Proof. Throughout we will make use of the following observations:

1. χ is nondecreasing.
2. $A_{i+j} \le A_i + A_j$ for $i, j \in \{0, 1, \ldots, m\}$ such that $i + j \le m$ (since $0 \le b_{i+1} \le b_i$).

The proof consists of verifying that $\chi(a) + \chi(b) \le \chi(a+b)$ for all $a, b \ge 0$ such that $a + b \le A_m$, which reduces to the verification of the following six cases by symmetry.

(i) $A_i \le a \le B_{i+1}$ and $A_j \le b \le B_{j+1}$.

$$\chi(a) + \chi(b) = (i+j)\tau(e-\rho) = \chi(A_{i+j}) \le \chi(A_i + A_j) \le \chi(a+b).$$

(ii) $A_i \le a \le B_{i+1}$ and $B_{j+1} + ke \le b \le B_{j+1} + ke + \rho$.

$$\begin{aligned}\chi(a) + \chi(b) &= ((i+j)\tau + k)(e-\rho) = \chi(A_{i+j} + b_{i+j+1} + ke)\\ &\le \chi(A_i + A_j + b_{j+1} + ke) \le \chi(a+b).\end{aligned}$$

(iii) $A_i \le a \le B_{i+1}$ and $B_{j+1} + ke + \rho \le b \le B_{j+1} + (k+1)e$.

$$\begin{aligned}\chi(a) + \chi(b) &= (i+j)\tau(e-\rho) + b - B_{j+1} - (k+1)\rho\\ &= ((i+j)\tau + k)(e-\rho) + b - B_{j+1} - ke - \rho\\ &= \chi(A_{i+j} + b_{i+j+1} + ke + \rho + \Delta)\\ &\le \chi(A_i + A_j + b_{j+1} + ke + \rho + \Delta) \le \chi(a+b),\end{aligned}$$

where $\Delta = b - B_{j+1} - ke - \rho \le e - \rho$.

(iv) $B_{i+1} + ke \le a \le B_{i+1} + ke + \rho$ and $B_{j+1} + te \le b \le B_{j+1} + te + \rho$.

$$\begin{aligned}\chi(a) + \chi(b) &= ((i+j)\tau + k + t)(e-\rho) \le \chi(A_{i+j} + b_{i+j+1} + b_{i+j+2} + (k+t)e)\\ &\le \chi(A_i + A_j + b_{i+1} + b_{j+1} + (k+t)e) \le \chi(a+b).\end{aligned}$$

The first inequality above follows from $k + t \le 2\tau - 1$.

(v) $B_{i+1} + ke \le a \le B_{i+1} + ke + \rho$ and $B_{j+1} + te + \rho \le b \le B_{j+1} + (t+1)e$.

$$\begin{aligned}\chi(a) + \chi(b) &= ((i+j)\tau + k + t)(e-\rho) + b - B_{j+1} - te - \rho\\ &\le \chi(A_{i+j} + b_{i+j+1} + b_{i+j+2} + (k+t)e + \Delta)\\ &\le \chi(A_i + A_j + b_{i+1} + b_{j+1} + (k+t)e + \Delta) \le \chi(a+b),\end{aligned}$$

where $\Delta = b - B_{j+1} - te - \rho \le e - \rho$.

(vi) $B_{i+1} + ke + \rho \le a \le B_{i+1} + (k+1)e$ and $B_{j+1} + te + \rho \le b \le B_{j+1} + (t+1)e$. Let $\Delta_a = a - B_{i+1} - ke - \rho$ and $\Delta_b = b - B_{j+1} - te - \rho$. Note that $\Delta_a, \Delta_b \le e - \rho$.

$$\chi(a) + \chi(b) = ((i+j)\tau + k + t)(e-\rho) + \Delta_a + \Delta_b.$$

If $\Delta_a + \Delta_b \le e - \rho$, the result is obtained as in (v). Otherwise let $\Delta = \Delta_a + \Delta_b - (e - \rho)$. Then

$$\chi(a) + \chi(b) = ((i+j)\tau + k + t + 1)(e-\rho) + \Delta.$$

Since $k + t + 1 \le 2\tau - 1$ and $\Delta \le e - \rho$, the result is obtained as in (v). □

Lemma 7. *$\bar{\chi}$ is superadditive on $\mathbb{R}_+$.*

Proof. $\bar{\chi}$ is superadditive over $[0, A_m]$ since χ is. In order to prove that $\bar{\chi}$ is superadditive over $\mathbb{R}_+$, it suffices to show that $\bar{\chi}(a) + \bar{\chi}(b) \leq \bar{\chi}(a+b)$ for all $a, b \geq 0$ such that $a + b > A_m$, because $\bar{\chi}$ is superadditive over $[0, A_m]$. Observe that since b_i $i = 1, 2, \ldots, m$ are nonincreasing, $\bar{\chi}(a) \leq \alpha a$, where $\alpha = \frac{m\tau(e-\rho)}{A_m} < 1$.

(i) $0 \leq a, b \leq A_m$ and $A_m \leq a + b$.

$$\begin{aligned}\bar{\chi}(a) + \bar{\chi}(b) &\leq \alpha(a+b) = m\tau(e-\rho) + \alpha(a+b-A_m) \\ &\leq m\tau(e-\rho) + (a+b-A_m) = \bar{\chi}(a+b).\end{aligned}$$

(ii) $0 \leq a \leq A_m$ and $A_m \leq b$.

$$\begin{aligned}\bar{\chi}(a) + \bar{\chi}(b) &\leq \alpha a + m\tau(e-\rho) + (b - A_m) \\ &\leq m\tau(e-\rho) + (a+b-A_m) = \bar{\chi}(a+b).\end{aligned}$$

(iii) $A_m \leq a, b$.

$$\begin{aligned}\bar{\chi}(a) + \bar{\chi}(b) &= 2m\tau(e-\rho) + (a+b-2A_m) \\ &\leq m\tau(e-\rho) + (a+b-A_m) = \bar{\chi}(a+b).\end{aligned}$$

□

Lemma 8. *Let $\chi' : \mathbb{R}_+ \mapsto \mathbb{R}$ and $\psi' : \mathbb{R}_- \mapsto \mathbb{R}$ be such that $\psi'(a) = a + \chi'(-a)$ for $a \in \mathbb{R}_-$. Then $\bar{\chi}$ is superadditive on $\mathbb{R}_+$ if and only if ψ' is superadditive on $\mathbb{R}_-$.*

Proof. Suppose χ' is superadditive on $\mathbb{R}_+$. Then $\psi'(a) + \psi'(b) = a + \chi'(-a) + b + \chi'(-b) \leq a + b + \chi'(-a-b) = \psi'(a+b)$. The other direction is proven similarly, since $\chi'(a) = a + \psi'(-a)$ for $a \in \mathbb{R}_+$. □

4. Special cases

In this section we highlight some special cases and show the connection between the inequalities introduced in Section 3 and inequalities already known for these cases.

4.1. Mixed integer rounding inequalities

Consider the mixed–integer knapsack K' with either positive or negative coefficients for all integer variables. That is, $K' = \{(x, y) \in \mathbb{Z}_+^I \times \mathbb{R}_+^C : \sum_{i \in I} a_i x_i + \sum_{i \in C} g_i y_i \leq b\}$, where either $a_i > 0$ for all $i \in I$ or $a_i < 0$ for all $i \in I$. By introducing an artificial integer variable z to K with coefficient $c > 0$, writing inequality (15) if coefficients are positive, or inequality (26) if coefficients are negative, and then fixing z back to zero, we obtain

$$\sum_{i \in I} \Phi(a_i) x_i + \sum_{i \in C^-} g_i y_i \leq b - \eta r, \tag{34}$$

where $\eta = \lceil b/c \rceil$ and $r = b - \lfloor b/c \rfloor c$ as a valid inequality for K'. For any $c > 0$, (34) is stronger than the MIR inequality [26]

$$\sum_{i \in I} (\lfloor a_i/c \rfloor + \frac{(f_i - f)^+}{1 - f}) x_i + \sum_{i \in C^-} \frac{g_i}{c(1 - f)} y_i \le \lfloor b/c \rfloor, \tag{35}$$

where $f_i = a_i/c - \lfloor a_i/c \rfloor$, and $f = b/c - \lfloor b/c \rfloor$, or equivalently [1]

$$\sum_{i \in I} \phi(a_i) x_i + \sum_{i \in C^-} g_i y_i \le b - \eta r \tag{36}$$

because $\phi \le \Phi$. This is illustrated in Example 1 with $c = 1$.

Also observe that for the general knapsack set K if $I^{++} = \emptyset$, then $\Omega(a) = \Phi(a)$ and $\omega(a) = \phi(a)$ for $a \le 0$; consequently, inequality (19) reduces to the MIR inequality (36). Similarly, if $I^{--} = \emptyset$, then $\Gamma(a) = \Phi(a)$ and $\gamma(a) = \phi(a)$ for $a \ge 0$; consequently, inequality (30) reduces to the MIR inequality (36).

4.2. Mixed 0–1 knapsack inequalities

For the special case $\bar{K} = \{(x, y) \in \{0, 1\}^I \times \mathbb{R}_+ : \sum_{i \in I} a_i x_i - y \le b\}$ with $a_i > 0$ inequalities (30) and (19) reduce to the continuous cover and continuous reverse cover inequalities introduced in [22]. Observe that for the 0–1 case, SMIR inequality (7) with $r > 0$ is facet defining for $conv(S)$ if and only if $0 < d/c < 1$, hence $\eta = 1$ and $r = d$. Consequently, (7) reduces to $(c - d)x - y \le 0$, which goes through the origin, and since $u_\ell = 1$ as well, Φ reduces to

$$\Phi(a) = \begin{cases} (a - d)^+ & \text{if } a \ge 0, \\ \max\{a, d - c\} & \text{if } a < 0. \end{cases} \tag{37}$$

Let $C \subseteq I$ and $\ell \in C$ be such that $\lambda = \sum_{i \in C} a_i - b > 0$ and $\mu = b - \sum_{i \in C \setminus \{\ell\}} a_i > 0$. Fixing all x_i $i \in I \setminus C$ to zero and all x_i $i \in C \setminus \{\ell\}$ to one, we obtain the restriction $a_\ell x_\ell - y \le \mu$. Lifting the corresponding SMIR inequality $\lambda x_\ell - y \le 0$ first with x_i $i \in C \setminus \{\ell\}$ using Φ and then with x_i $i \in I \setminus C$ using Γ, we get the continuous cover inequality

$$\lambda x_\ell - \sum_{i \in C \setminus \{\ell\}} \min\{a_i, \lambda\}(1 - x_i) + \sum_{i \in I \setminus C} \Gamma(a_i) x_i - y \le 0, \tag{38}$$

which is facet–defining by Theorem 6 since $\eta = u_\ell$. On the other hand, lifting $\lambda x_\ell \le y$ first with x_i $i \in I \setminus C$ using Φ and then with x_i $i \in C \setminus \{\ell\}$ using Ω, we get this time the reverse continuous cover inequality

$$\lambda x_\ell + \sum_{i \in C \setminus \{\ell\}} \Omega(-a_i)(1 - x_i) + \sum_{i \in I \setminus C} (a_i - \mu)^+ x_i - y \le 0, \tag{39}$$

which is facet–defining by Theorem 4 since $\eta = 1$.

4.3. Mixed bounded integer knapsack inequalities

Let us now consider the case $\hat{K} = \{(x, y) \in \mathbb{Z}_+^I \times \mathbb{R}_+ : \sum_{i \in I} a_i x_i - y \leq b, \quad x_i \leq u_i \quad i \in I\}$ with bounded integer variables. Since $u_i < \infty$, we have, if necessary after complementing variables, $a_i > 0$ for all $i \in I$. Let $C \subseteq I$ and $\ell \in C$ be such that $\lambda = \sum_{i \in C} a_i u_i - b > 0$ and $\mu = b - \sum_{i \in C \setminus \{\ell\}} a_i u_i > 0$. The set C is called a cover, whereas $C \setminus \{\ell\}$ is called a packing. Observe that $\lambda + \mu = a_\ell u_\ell$. Fixing all x_i $i \in C \setminus \{\ell\}$ to u_i and all x_i $i \in I \setminus C$ to zero, we obtain the restriction $a_\ell x_\ell - y \leq \mu$. Let $\eta = \lceil \mu / a_\ell \rceil$ and $r = \mu - \lfloor \mu / a_\ell \rfloor a_\ell$. Suppose the LP relaxation of the restriction has a fractional vertex, i.e., $r > 0$; thus we have the nontrivial SMIR inequality $(a_\ell - r) x_\ell - y \leq \mu - \eta r$. The lifting function Φ for this inequality is not as simple as (37) since $\eta = 1 = u_\ell$ may not hold. Two special cases lead to inequalities that generalize and strengthen the ones defined in the literature.

1. If $\eta = 1$, then $r = \mu$ and $\Phi(a) = (a - \mu)^+$ for $a > 0$. In this case, lifting the SMIR inequality first with x_i $i \in I \setminus C$ using Φ and then with x_i $i \in C \setminus \{\ell\}$ using Ω, gives inequality

$$(a_\ell - \mu) x_\ell + \sum_{i \in C \setminus \{\ell\}} \Omega(-a_i)(u_i - x_i) + \sum_{i \in I \setminus C} (a_i - \mu)^+ x_i - y \leq 0, \tag{40}$$

which is facet–defining by Theorem 4 since $\eta = 1$. If $\Omega(-a_i) = -a_i$ for all $i \in C \setminus \{\ell\}$, then (40) reduces to the weight inequality [24]

$$\sum_{i \in C \setminus \{\ell\}} a_i x_i + \sum_{i \in \{\ell\} \cup I \setminus C} (a_i - \mu)^+ x_i - y \leq b - \mu; \tag{41}$$

otherwise (40) is stronger than the weight inequality (41). Also by lifting SMIR first with x_i $i \in C \setminus \{\ell\}$, then with x_i $i \in I \setminus C$, we obtain inequality (30).

2. If $\eta = u_\ell$, then $a_\ell - r = \lambda$ and $\Phi(a) = \max\{a, \lambda\}$ for $a < 0$. Lifting the SMIR inequality first with x_i $i \in C \setminus \{\ell\}$ using Φ and then with x_i $i \in I \setminus C$ using Γ, we obtain inequality

$$\lambda x_\ell - \sum_{i \in C \setminus \{\ell\}} \min\{a_i, \lambda\}(u_i - x_i) + \sum_{i \in I \setminus C} \Gamma(a_i) x_i - y \leq \mu - \eta r, \tag{42}$$

which is facet–defining by Theorem 6 since $\eta = u_\ell$. If $\lambda \leq a_i$ for all $i \in C$ and $\Gamma(a_i) = 0$ for all $i \in I \setminus C$, then (42) reduces to the integer cover inequality [11],

$$\sum_{i \in C} (u_i - x_i) \geq \alpha - y/(a_\ell - r), \tag{43}$$

where $\alpha = \lceil \lambda / a_\ell \rceil = u_\ell - \eta + 1$; otherwise, (42) is stronger than the integer cover inequality (43). Also by lifting SMIR first with x_i $i \in I \setminus C$ and then with x_i $i \in C \setminus \{\ell\}$ we obtain inequality (19).

Recall that inequalities (19) and (30) are applicable more generally than the two cases listed above, that is when $\eta \in \{1, 2, \ldots, u_\ell\}$ as well. The examples in this section illustrate that the functions Φ, Ω, and Γ are fundamental in providing a common explanation for the inequalities given for special cases of the mixed–integer knapsack set K.

5. Computational experiments

In this section we report our computational results on using inequalities (19) and (30) as cutting planes. In the experiments we compare these inequalities with MIR cuts and default CPLEX[1] MIP solver cuts. Toward this end two data sets of instances of the form

$$\begin{aligned} \max \sum_{i \in I} c_i x_i &- \sum_{i \in C} d_i w_i \\ \text{s.t.} \sum_{i \in I} a_{ir} x_i &- \sum_{i \in C} w_{ir} \leq b_r \quad r \in R, \\ x \leq u, \ w &\leq v, \ x \in \mathbb{Z}_+^I, \ w \in \mathbb{R}_+^C \end{aligned}$$

are prepared. The coefficients of the instances are randomly generated integers from the following intervals: $b_r \in [951, 999]$, $a_{ir} \in [1, 2b_r]$, $c_i \in [\lfloor 0.1 a_{i1} \rfloor, \lfloor 0.9 a_{i1} \rfloor]$, $d_i \in [101, 200]$. In the first set variables are unbounded, whereas in the second set they are bounded with $u_i \in [1, 4]$ and $v_i \in [1, 20]$. Recall that if $I^{++} = \emptyset$, inequality (19) reduces to the MIR inequality. In order to ensure that there is at least one inequality different from the MIR inequality for each row r, we require that $a_{i_1 r} > a_{i_2 r} \lceil b_r / a_{i_2 r} \rceil$ for two variables. This allows us to make a comparison between the MIR inequalities and the new inequalities. The data set is available for download at `http://ieor.berkeley.edu/~atamturk/data`.

After solving the LP relaxation of an instance, we attempt to find violated inequalities from each constraint $r \in R$ in the following way. Given a fractional solution $(\bar{x}, \bar{w})$, let $U = \{i \in I : \bar{x}_i = u_i\}$. For all single integer variable restrictions $K_{U,\ell}$ with $\ell \in \{i \in I : 0 < \bar{x}_i < u_i\}$, the separation method described in [3] is used to find a violated inequality (13). Once such an inequality is found, it is lifted to (19) and (30) and the most violated one is added to the formulation. If no violated inequality is found this way, U is augmented with $i \in I \setminus (\{\ell\} \cup U)$ in nondecreasing order of $u_i - \bar{x}_i (> 0)$ one at a time and the procedure is repeated at most five times. If we fail to find cuts we resort to the branch–and–bound algorithm. In the current implementation, cuts are not added after branching. Implementation of the cut generation procedure is done using callback functions of CPLEX version 7.5. All experiments are performed on a 2 GHz Intel Pentium 4 / Linux workstation with 1 GB memory.

In Tables 1 and 2 we compare the performance of the algorithm when only the new cuts (19) and (30) are added versus when only the mixed–integer rounding (MIR) cuts (36) are added. MIR cuts are implemented also as described in the previous paragraph. CPLEX MIP solver also generates MIR cuts as well as other classes of cuts. The purpose of comparing the new cuts with our implementation of MIR cuts is to isolate the impact of merely using ϕ (12) versus the lifting functions Φ, ω, and γ, by eliminating other factors due to differences in the implementations. For instance, CPLEX generates MIR cuts not only from individual rows of the formulation, but also from mixed–integer knapsack set relaxations obtained by aggregation of constraints and

[1] CPLEX is a trademark of ILOG, Inc.

Table 1. Experiments with unbounded variables

$\|I\| : \|C\| : \|R\|$	CPLEX7.5				MIR cuts (36)				New cuts (19) & (30)			
	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time
250:1:50	111	67	68611	42	99	79	4416	4	101	83	1612	2
500:1:50	50	47	83061	63	99	77	6777	8	101	80	2929	4
250:1:75	177	56	196053	98	147	76	199883	183	153	79	40014	32
500:1:75	178	55	113745	92	147	75	176551	221	153	77	69242	79
250:1:100	220	71	251725	184	170	74	500125	417	204	75	226633	166
500:1:100	225	71	208047	283	196	73	256791	391	204	76	134211	187

substitution of variables similar to [23]. It also has several adaptive rules to decide when to, how often, how many cuts to add depending on the progress of the improvement of the bounds and size of the formulation, among others. Our implementation is simple and does not perform variable substitution or row aggregation. All CPLEX cuts are disabled when running these two cut generation procedures. In the tables we also present results with default CPLEX cuts. In all experiments CPLEX heuristic is turned off.

Under the columns for CPLEX7.5, MIR cuts, and new cuts, we report the averages (rounded to a nearest integer) for the number of cuts added (`cuts`), the percentage improvement in the integrality gap with the addition of the cuts (`gapimp`), the number of branch–and–bound nodes explored (`nodes`), and the CPU time elapsed in seconds (`time`) for five random instances. The `gapimp` is calculated as `100×(zlp-zroot)/(zlp-zopt)`, where `zlp`, `zroot`, and `zopt` refer to the objective values of the initial LP relaxation, of the LP relaxation after the cuts are added, and of an optimal MIP solution.

In Tables 1 and 2 we observe that the new cuts are more effective in closing the integrality gap than MIR cuts. They lead to a significant reduction in the number of branch–and–bound nodes and in the CPU time when compared with the MIR implementation. For the experiments with unbounded variables (Table 1) the algorithm with the new cuts is on the average 2.5 faster than the one with MIR cuts. For the experiments with bounded variables (Table 2) the average speed–up factor increases with the number of continuous variables, ranging from 3.8 for $|C| = 5$ to 16.6 for $|C| = 20$.

Interestingly for instances with large number of constraints default CPLEX7.5 is faster than our MIR implementation, even though it generally explores more nodes. Note that we generate cuts only at the root node, whereas CPLEX adds MIR cuts as well as others throughout the search tree. An important observation is that in both MIR and new cut implementations we add generally many more cuts than CPLEX does. It seems that adding many MIR cuts slows down the progress of computations, whereas the addition of a similar number of new cuts has the opposite effect.

Generating cuts (19) and (30) not only from individual constraints of the formulation, but also from mixed–integer knapsack relaxations obtained by variable substitution and row aggregation – as it is done for MIR cuts in [23] and in CPLEX – should improve the effectiveness of these cuts further. Substitution and aggregation appear to be essential especially for problems with network substructures as the ones in MIPLIB [9]. Another implementation strategy might be to generate them from the rows of the simplex tableau as in Gomory mixed–integer cuts.

Table 2. Experiments with bounded variables

	CPLEX7.5				MIR cuts (36)				New cuts (19) & (30)			
$\|I\| : \|C\| : \|R\|$	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time	cuts	gapimp	nodes	time
250:5:50	95	68	151252	63	213	79	7538	7	199	84	1070	2
500:5:50	50	42	128830	76	201	78	5045	7	206	83	1359	4
250:5:75	121	60	293392	144	408	77	86952	114	389	81	34412	38
500:5:75	122	59	207083	148	467	76	107139	213	440	80	17009	33
250:5:100	183	65	334504	203	691	80	179059	418	588	84	19257	29
500:5:100	130	68	305987	280	664	77	174394	446	613	81	33292	78
250:10:50	100	68	142438	67	245	77	15892	15	228	82	2443	3
500:10:50	50	38	166758	111	198	74	43397	46	235	80	2696	6
250:10:75	130	63	615537	370	474	75	213042	370	406	78	27591	38
500:10:75	105	61	228484	182	452	73	238559	519	477	78	33170	79
250:10:100	170	72	439032	361	777	75	301035	920	731	80	16689	55
500:10:100	136	72	302262	317	776	79	193303	489	678	81	18722	64
250:20:50	80	68	122693	73	228	76	186723	206	280	80	3247	6
500:20:50	48	45	131422	75	241	72	79649	101	252	79	3895	9
250:20:75	115	69	283218	176	416	73	424453	769	418	79	12852	22
500:20:75	118	69	390818	361	495	70	355781	779	479	78	20645	60
250:20:100	146	74	642986	531	849	70	383760	1594	656	79	26496	93
500:20:100	122	70	340536	434	702	71	305311	972	632	80	23930	88

6. Concluding remarks

We identified facet–defining inequalities of a very general mixed–integer knapsack polyhedron, which is the convex hull of the feasible set of an arbitrary linear inequality on integer and continuous variables. These facets are described through superadditive functions and are closely related to the MIR inequalities. Interestingly, the new inequalities strengthen and/or generalize known inequalities for special cases of the mixed–integer knapsack set studied earlier. Our computational results suggest that the inequalities can be useful in branch–and–cut algorithms for mixed–integer programming.

The polyhedral structure of the mixed–integer knapsack set deserves further investigation. The inequalities described in this paper are from restrictions of the mixed–integer knapsack set with a single integer variable. Restrictions with more than one integer variable have facets that are different from the ones identified here. Characterizing facets of the mixed–integer knapsack polyhedron with a small number of integer variables and lifting them may be an effective way of deriving new classes of strong valid inequalities. We are currently exploring this strategy to identify other facets of the mixed–integer knapsack polyhedron.

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