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Tight formulations for some simple mixed integer programs and convex objective integer programs

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Abstract. We study the polyhedral structure of simple mixed integer sets that generalize the two variable set $\{(s, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : s \geq z - b\}$. These sets form basic building blocks that can be used to derive tight formulations for more complicated mixed integer programs. For four such sets we give a complete description by valid inequalities and/or an integral extended formulation, and we also indicate what constraints can be added without destroying integrality.

We apply these results to provide tight formulations for certain piecewise–linear convex objective integer programs, and in a companion paper we exploit them to provide polyhedral descriptions and computationally effective mixed integer programming formulations for discrete lot-sizing problems.

Key words. mixed integer programming – valid inequalities – extended formulations

1. Introduction

Recently several authors have shown that two variable mixed integer rounding (MIR) inequalities can be used to generate interesting inequalities for a variety of models, and to generate the convex hull in certain special cases, see Magnanti et al. (1993), Pochet and Wolsey (1995), Marchand and Wolsey (2001). All the sets studied can be seen as simple generalizations of the basic two variable set $X = \{(s, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : s \geq z - b\}$, or the equivalent set $\{(x, z) \in \mathbb{R}^1 \times \mathbb{Z}^1 : x \geq b, x \geq z\}$ obtained by substituting $s = x - b$. In this case the addition of the MIR inequality $s \geq (1 - f)(z - \lfloor b \rfloor)$, where $f = b - \lfloor b \rfloor$, suffices to give a description of the convex hull of X. Here we look at some natural generalizations of the set X, attempt to describe the convex hulls, and give tight extended formulations where appropriate.

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We also observe that $(s, z) \in X$ if and only if $s \geq g(z)$ where $g(z) = \max\{0, z - b\}$ is a piecewise–linear convex function in the integer variable z. Thus a study of X and its generalizations is also a study of the convex objective integer program ($COIP$): $\min\{g(z) : z \in \mathbb{Z}^1\}$. Pursuing this direction therefore leads naturally to the study of tight formulations for integer programs with separable piecewise–linear convex objective functions.

Given these tight descriptions, the next question is whether these convex hull formulations remain tight, or whether these formulations of $COIPs$ still have integer solutions, when additional constraints are added to these integer programs. Specifically, if X^i with variables (s_i, z_i) for $i = 1, \dots, n$ are sets of the form X given above, and $Bz \leq d$ are constraints linking the integer variables $(z_1, \dots, z_n)$, when is $\bigcap_i \operatorname{conv}(X^i) \cap \{z : Bz \leq d\}$ integral?

Below we study four generalizations of the set X. The first two involve a single integer variable:

$$V = \{(\sigma, z) \in \mathbb{R}^K_+ \times \mathbb{Z}^1 : \sigma_k \geq z - b_k \text{ for } k = 1, \dots, K\}$$

and

$$W = \{(\phi, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : \phi \geq a_k z - c_k \text{ for } k = 1, \dots, K\}$$

to which we can associate the function $g(z) = \max\{0, \max_k(a_k z - c_k)\}$. Here we note that any $COIP$ with a separable piecewise–linear convex objective function can be written either as $\min\{\sum_j g_j(z_j) : Az \leq b, z \in \mathbb{Z}^n\}$ where each function g_j is of the above form, or as $\min\{\sum_j \phi_j : (\phi_j, z_j) \in W_j \text{ for all } j, Az \leq b\}$ where each set W_j has the structure of W.

In contrast, the last two generalizations give rise to nonseparable objective functions. Specifically with

$$Y = \{(\phi, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^n : \phi \geq b_i - y_i \text{ for } i = 1, \dots, n\},$$

we associate the function $g(y) = \max\{0, \max_i(b_i - y_i)\}$, and with

$$Z = \{(\phi, r, y) \in \mathbb{R}^1_+ \times \mathbb{R}^n_+ \times \mathbb{Z}^n : \phi \geq b_i - r_i - y_i \text{ for } i = 1, \dots, n\}$$

the function $g(r, y) = \max\{0, \max_i(b_i - r_i - y_i)\}$ with $r \geq 0$.

In Sections 2–5 we study the sets V, W, Y, Z in turn. In Section 2 we show that the MIR inequalities suffice to give a description of the convex hull of the set V. In Section 3 we use this result to give a tight formulation for W, and thus for all single variable piecewise–linear convex functions appearing in the objective function of an integer program. We also show that, after the addition of constraints on the integer variables with a totally unimodular constraint matrix, the reformulation still has integer solutions.

The latter results relate to those in Chapter 13 of Ahuja et al. (1993), where it is shown that it is possible to solve $COIPs$ of the form $\min\{g(z) : Bz \leq d, z \in \mathbb{Z}^n_+\}$, when $g(z) = \sum_j g_j(z_j)$ is separable and B is a network flow matrix, by breaking up the flow z_j on arc j into separate flows z_{kj} for each cost segment k of the integer closure $\bar{g}_j$, and then solving by a standard network flow algorithm. This approach extends to the case where B is totally unimodular (TU), since in this case it suffices to duplicate columns in the matrix. Also in Hochbaum and Shantikumar (1990) a polynomial algorithm is given for $COIPs$ where the objective g is separable, each function of a single variable g_j is an arbitrary convex function, and A is TU. The algorithm is based on solving successive linear programs and applying proximity results. In contrast, our approach is to find effective LP or MIP reformulations rather than specialized algorithms. For cases of multi–variate convex integer functions, we are not aware of research that describes polynomial algorithms for $COIPs$.

The set Y examined in Section 4 has already been studied by Pochet and Wolsey (1994) and Günlük and Pochet (2001) (see also Atamtürk et al. (2000)). Here we show that it is possible to retain integrality when adding dual network flow constraints. We also give a simple proof of a compact extended formulation for Y. For the set Z examined in Section 5 we have not succeeded in finding a description of the convex hull in the original space of variables, but we provide an extended formulation for it.

The approach used to prove integrality for formulations for each of the unbounded sets V, W, and Y consists of showing that all the bounded faces of the associated polyhedra are integral. This technique was first used in Pereira and Wolsey (2001), and it seems particularly suitable for analyzing $COIPs$.

In addition to enabling us to reformulate integer programs with piecewise affine convex objective functions, the results also have immediate applications to discrete lot-sizing, see e.g. Fleischmann (1990), van Hoesel et al. (1994) and van Eiji and van Hoesel (1997). These are explored and tested computationally in a companion paper of Miller and Wolsey (2002).

2. A simple extension

We first consider a very simple extension of the two variable mixed integer set X, namely

$$V = \{(\sigma, z) \in \mathbb{R}^K_+ \times \mathbb{Z}^1 : \sigma_k \geq z - b_k \text{ for } k = 1, \dots, K\}. \tag{1}$$

Note that we can also express V as

$$\{(\sigma, y) \in \mathbb{R}^K_+ \times \mathbb{Z}^1 : \sigma_k \geq d_k - y \text{ for } k = 1, \dots, K\}$$

by letting $y = -z$ and $d_k = -b_k$ for $k = 1, \dots, K$, but we have chosen to work with the formulation (1). We let $f_k = b_k - \lfloor b_k \rfloor$, the fractional part of b_k, for each $k = 1, \dots, K$.

To describe conv(V), we first establish a simple result.

Lemma 1. *For $z \in \mathbb{R}^1$, let $\sigma_k = \max\{0, z - b_k, (1 - f_k)(z - \lfloor b_k \rfloor)\}$.*

i) $\sigma_k = z - b_k$ if and only if $z \geq \lceil b_k \rceil$,
ii) $\sigma_k = 0$ if and only if $z \leq \lfloor b_k \rfloor$,
iii) $\sigma_k = (1 - f_k)(z - \lfloor b_k \rfloor)$ if and only if either $b_k \in \mathbb{Z}^1$ and $z \geq \lfloor b_k \rfloor$, or $b_k \notin \mathbb{Z}^1$ and $\lfloor b_k \rfloor \leq z \leq \lceil b_k \rceil$.

Proof. Suppose $b_k \notin \mathbb{Z}^1$. $\sigma_k = z - b_k$ holds if and only if $z - b_k \geq (1 - f_k)(z - \lfloor b_k \rfloor)$ and $z - b_k \geq 0$. But

$$z - b_k \geq (1 - f_k)(z - \lfloor b_k \rfloor) \iff f_k z \geq b_k - (1 - f_k)\lfloor b_k \rfloor \iff f_k z \geq f_k(1 + \lfloor b_k \rfloor) \iff z \geq \lceil b_k \rceil.$$

Similarly, $\sigma_k = 0$ holds if and only if $0 \geq (1-f_k)(z-\lfloor b_k \rfloor)$ and $0 \geq z - b_k$. The first inequality holds if and only if $z \leq \lfloor b_k \rfloor$. Finally the last possibility, $\sigma_k = (1-f_k)(z-\lfloor b_k \rfloor)$, holds if and only if $\lfloor b_k \rfloor \leq z \leq \lceil b_k \rceil$. The case with $b_k \in \mathbb{Z}^1$ is similar. □

It turns out that MIR inequalities suffice to give the convex hull of V.

Theorem 2. *i). conv(V) is described by the inequalities*

$$\sigma_k \geq z - b_k \quad \textit{for } k = 1, \ldots, K \tag{2}$$
$$\sigma_k \geq (1 - f_k)(z - \lfloor b_k \rfloor) \quad \textit{for } k = 1, \ldots, K \tag{3}$$
$$\sigma_k \geq 0 \quad \textit{for } k = 1, \ldots, K. \tag{4}$$

ii). Let $V^j \subseteq \mathbb{R}_+^{K_j} \times \mathbb{Z}^1$ be sets of the form (1) for $j = 1, \ldots, n$; then

$$conv(\bigcap_{j=1}^{n} V^j \cap \{z \in \mathbb{Z}^n : Bz \leq d\}) = \bigcap_{j=1}^{n} conv(V^j) \cap \{z \in \mathbb{R}^n : Bz \leq d\}$$

when the matrix B is totally unimodular, and d is an integer vector.

Proof. i). Let T be the polyhedron described by the inequalities (2)–(4). Clearly conv(V) $\subseteq T$. Consider now a bounded face T' of T of maximum dimension. As each σ_k must be minimal on such a face, $\sigma_k = \max\{0, z - b_k, (1 - f_k)(z - \lfloor b_k \rfloor)\}$. Let $K_1 = \{k : \sigma_k = 0\}$, $K_2 = \{k : \sigma_k = z = b_k\}$ and $K_3 = \{k : \sigma_k = (1 - f_k)(z - \lfloor b_k \rfloor)$. From the previous Lemma, it follows that $K_1 \cup K_2 \cup K_3 = \{1, \ldots, K\}$. Therefore the face T' can be written as

$$T' = \{(\sigma, z) : z \leq \lfloor b_k \rfloor, \sigma_k = 0, \quad k \in K_1, z \geq \lceil b_k \rceil, \sigma_k = z - b_k, \quad k \in K_2,$$
$$\lfloor b_k \rfloor \leq z \leq \lceil b_k \rceil, \sigma_k = (1 - f_k)(z - \lfloor b_k \rfloor), \quad k \in K_3\}.$$

Therefore $T' = \{(\sigma, z) \in \mathbb{R}^n \times \mathbb{R}^1 : \sigma = c + dz, l \leq z \leq u\}$ where $c, d \in \mathbb{R}^n$ and $l, u \in \mathbb{Z}^1 \cup \{\pm\infty\}$. Thus each bounded face of T has extreme points with z integral, and the claim follows.
ii). The same argument shows that each maximal bounded face is of the form $Bz \leq d$, $l_j \leq z_j \leq u_j$ for $j = 1, \ldots, n$. As the matrix $(B^T, I - I)$ is totally unimodular, the face again has extreme points with z integral, and the claim follows. □

Note that nonnegativity constraints $z \geq 0$, or simple bound constraints $p \leq z \leq q$ with $p, q \in \mathbb{Z}^1$ correspond to a trivial TU matrix, so the addition of such constraints to the set V does not alter the results.

Example 1.
An example of the set V is

$$V = \{(\sigma, y) \in \mathbb{R}^2_+ \times \mathbb{Z}^1 : \sigma_1 \geq z - \frac{1}{3}, \sigma_2 \geq z - \frac{5}{4}\}$$

In this case $b_1 = \frac{1}{3}$ and $b_2 = \frac{5}{4}$, so $f_1 = \frac{1}{3}$ and $f_2 = \frac{1}{4}$, and the additional inequalities needed to describe conv(V) are

$$\sigma_1 \geq \tfrac{2}{3}(z - 0) \tag{5}$$
$$\sigma_2 \geq \tfrac{3}{4}(z - 1). \tag{6}$$

□

3. Model W: Separable convex objectives

Here we consider sets of the form

$$W = \{(\phi, z) \in \mathbb{R}^1 \times \mathbb{Z}^1 : \phi \geq a_k z - c_k \text{ for } k = 0, \ldots, K\}, \tag{7}$$

where $0 = a_0 = c_0$ and $a_k \geq 0$ for $k = 1, \ldots, K$, which can also be written as $W = P_W \cap (\mathbb{R}^1 \times \mathbb{Z}^1)$, where

$$P_W = \{(\phi, z) \in \mathbb{R}^1 \times \mathbb{R}^1 : \phi \geq a_k z - c_k \text{ for } k = 0, \ldots, K\}.$$

Note that P_W is the epigraph of the function

$$g(z) = \max_k (a_k z - c_k),$$

which is a nonnegative, nondecreasing piecewise–linear convex function of the single variable z. Adding the restriction that z is integer, the set conv(W) that we are interested in is the epigraph of a function $\bar{g}$, the *integer closure* of g. (See Figure 1.)

We now make some observations about such functions. We assume for simplicity that each of the $K + 1$ segments in the description of g is necessary for its description, in which case we can assume wlog that $0 = a_0 < a_1 < \ldots < a_K$.

Observation 1. The breakpoints of the function g, the points $b_k = \frac{c_k - c_{k-1}}{a_k - a_{k-1}}$ for $k = 1, \ldots, K$, satisfy $b_1 < \ldots < b_K$ as each pair (a_k, c_k) is necessary in the description of $g(z)$.

Observation 2. $a_k z - c_k = (a_{k-1} z - c_{k-1}) + (a_k - a_{k-1})(z - b_k) = \sum_{i=1}^{k} (a_i - a_{i-1})(z - b_i)$ for $k = 1, \ldots, K$. Let $b_0 = -\infty$ and $b_{K+1} = \infty$. Then $a_k z - c_k = \max_{i=0,\ldots,K}\{a_i z - c_i\}$ if and only if $b_k \leq z \leq b_{k+1}$ for $k = 0, \ldots, K$.

Observation 3. For $z \in \mathbb{R}^1$,

$$\begin{aligned} g(z) &= \sum\nolimits_{k=1}^{K} (a_k - a_{k-1})(z - b_k)^+ \\ &= \min\{\sum\nolimits_{k=1}^{K} (a_k - a_{k-1})\sigma_k : \sigma_k \geq z - b_k, \sigma_k \geq 0, \text{ for } k = 1, \ldots, K\} \\ &= \min\{\sum\nolimits_{k=1}^{K} a_k s_k : \sum_{k=1}^{K} s_k \geq z - b_1, 0 \leq s_k \leq b_{k+1} - b_k \text{ for } k = 1, \ldots, K\}, \end{aligned}$$

where $(x)^+$ denotes $\max\{x, 0\}$.

The last equality holds because $\sigma_k = \sum_{t=k}^{K} s_t$ for σ and s that minimize the second and third expressions, respectively, due to the fact that $\{a_k\}$ are strictly increasing.

Observation 4. In studying sets such as W, there is no loss of generality in assuming that the associated function g is nonnegative and nondecreasing. Assume that h is an arbitrary piecewise linear function, i.e., with a_k, c_k that are arbitrary, but with each segment necessary. Let $W' = \{(\psi, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : \psi \geq a_k z - c_k \text{ for } k = 1, \ldots, K\}$. To obtain a function that is nonnegative and nondecreasing, it suffices to find $a_0 = \min_{k=0,1,\ldots,K} a_k$ and define $g(z) = h(z) - (a_0 z - c_0)$; this corresponds to defining W via the change of variable $\phi = \psi - (a_0 z - c_0)$.

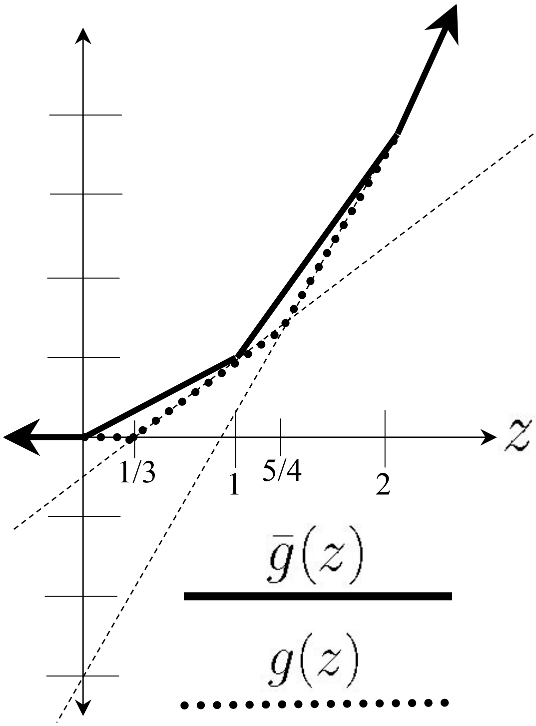

Fig. 1. Graph of $\bar{g}(z)$, the integer closure of $g(z)$

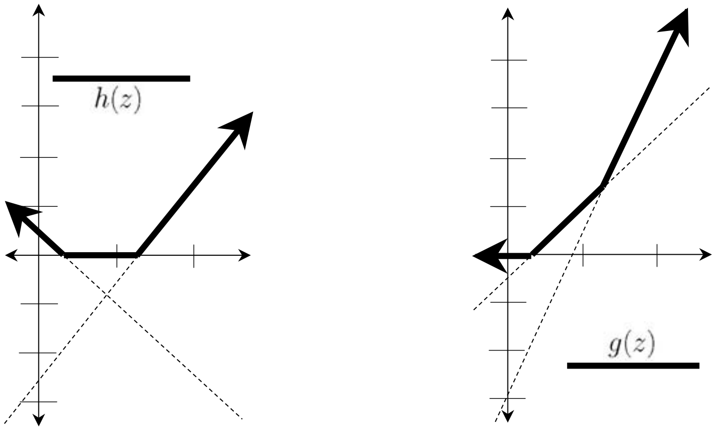

Fig. 2. Graph of the functions $h(z)$ and $g(z)$

Example 2.
Consider a set W' defined by

$$W' = \{(\psi, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : \psi \geq -\frac{3}{2}z - (-\frac{1}{2}), \psi \geq 2z - \frac{5}{2}, \psi \geq 0\}.$$

$W' = P_{W'} \cap (R_1 \times \mathbb{Z}_1)$, where $P_{W'}$ is the epigraph of the function $h(z) = \max\{-\frac{3}{2}z - (-\frac{1}{2}), 0z - 0, 2z - \frac{5}{2}\}$, which is convex but not nondecreasing. By setting $g(z) = h(z) - (-\frac{3}{2}z - (-\frac{1}{2}))$, we obtain

$$g(z) = \max\{0, \frac{3}{2}z - \frac{1}{2}, \frac{7}{2}z - 3\},$$

a nonnegative and nondecreasing function whose epigraph is

$$P_W = \{(\phi, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : \phi \geq \frac{3}{2}z - \frac{1}{2}, \phi \geq \frac{7}{2}z - 3\}.$$

The functions h and g for this example appear in Figure 2. □

Two extended formulations of conv(W) are easily obtained using Theorem 2 and Observation 3. Consider the polyhedron P:

$$\phi \geq \textstyle\sum_{k=1}^{K} a_k s_k \tag{8}$$
$$0 \leq s_k \leq b_{k+1} - b_k \quad \text{for } k = 1, \ldots, K \tag{9}$$
$$\textstyle\sum_{k=1}^{K} s_k \geq z - b_1 \tag{10}$$
$$\textstyle\sum_{i=k}^{K} s_i \geq (1 - f_k)(z - \lfloor b_k \rfloor) \text{ for } k = 1, \ldots, K, \tag{11}$$

and the polyhedron Q:

$$\phi \geq \textstyle\sum_{k=1}^{K}(a_k - a_{k-1})\sigma_k \tag{12}$$

$$\sigma_k \geq z - b_k \quad \text{for } k = 1, \ldots, K \tag{13}$$

$$\sigma_k \geq (1 - f_k)(z - \lfloor b_k \rfloor) \quad \text{for } k = 1, \ldots, K \tag{14}$$

$$\sigma_k \geq 0 \quad \text{for } k = 1, \ldots, K. \tag{15}$$

Theorem 3. *conv(W) =proj$_{\phi,z}(P)$ =proj$_{\phi,z}(Q)$.*

Proof. From Observation 3,

$$P_W = \text{proj}_{\phi,z}\{(\phi, z, \sigma) : \phi \geq \sum_{k=1}^{K}(a_k - a_{k-1})\sigma_k : \sigma_k \geq z - b_k, \sigma_k \geq 0,$$
$$\text{for } k = 1, \ldots, K\}.$$

Now by Theorem 2, any linear program over Q with an objective function in the ϕ, z variables will have an optimal solution with z integer. It follows directly that $\text{conv}(W) = \text{proj}_{\phi,z}(Q)$.

We now show that $\text{proj}_{\phi,z}(P)$ =$\text{proj}_{\phi,z}(Q)$. If $(\phi, z) \in \text{proj}_{\phi,z}(P)$, there exists an s such that $(\phi, z, s) \in P$. Setting $\sigma_k = \sum_{i=k}^{K} s_i$, it is readily checked that $(\phi, z, \sigma) \in Q$. So $\text{proj}_{\phi,z}(P) \subseteq \text{proj}_{\phi,z}(Q)$. Conversely we see from Observation 3 that

$$P_W = \text{proj}_{\phi,z}\{(\phi, z, s) : \phi \geq \sum_{k=1}^{K} a_k s_k : \sum_{k=1}^{K} s_k \geq z - b_1, 0 \leq s_k \leq b_{k+1} - b_k$$
$$\text{for } k = 1, \ldots, K\}.$$

Clearly the inequality $\sum_{i=k}^{K} s_i \geq z - b_k$ is valid for P_W and thus the MIR inequality $\sum_{i=k}^{K} s_i \geq (1 - f_k)(z - \lfloor b_k \rfloor)$ is valid for $\text{conv}(W)$. Thus $\text{conv}(W) \subseteq \text{proj}_{\phi,z}(P)$. □

The model obtained by taking (9)–(10) and requiring z to be integer has been studied before. Magnanti et al. (1993) showed that an exponential family of MIR inequalities describe the convex hull of this model, and Atamtürk and Rajan (2002) gave a linear time separation algorithm for these inequalities. Theorem 3 implies that optimizing a given linear objective function over this MIP model can be accomplished by linear programming by adding only the K constraints (11). These form a subset of the MIR inequalities described in the two papers just cited.

Example 2 (continued)

For $g(z)$, $a_0 = 0, a_1 = \frac{3}{2}, a_2 = \frac{7}{2}$, and $c_0 = 0, c_1 = \frac{1}{2}, c_2 = 3$. Thus

$$b_1 = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}, b_2 = \frac{3 - \frac{1}{2}}{\frac{7}{2} - \frac{3}{2}} = \frac{5}{4}, \text{ and } b_2 - b_1 = \frac{11}{12}.$$

Recall that $b_0 = -\infty$ and $b_3 = \infty$ by definition. So P takes the form

$$\phi \geq \tfrac{3}{2}s_1 + \tfrac{7}{2}s_2 \tag{16}$$
$$s_1 + s_2 \geq z - \tfrac{1}{3} \tag{17}$$
$$0 \leq s_1 \leq \tfrac{11}{12}, 0 \leq s_2 \tag{18}$$
$$s_1 + s_2 \geq \tfrac{2}{3}(z - 0) \tag{19}$$
$$s_2 \geq \tfrac{3}{4}(z - 1), \tag{20}$$

and Q becomes

$$\phi \geq \tfrac{3}{2}\sigma_1 + 2\sigma_2 \tag{21}$$
$$\sigma_1 \geq z - \tfrac{1}{3} \tag{22}$$
$$\sigma_2 \geq z - \tfrac{5}{4} \tag{23}$$
$$\sigma_1 \geq \tfrac{2}{3}(z - 0) \tag{24}$$
$$\sigma_2 \geq \tfrac{3}{4}(z - 1) \tag{25}$$
$$\sigma \geq 0. \tag{26}$$

Note the relationship between the MIR inequalities (5) and (6), (19) and (20), and (24) and (25). □

Finally we would like a description of conv(W) in the original (ϕ, z) space. Note that some of the linear segments defining g may not be needed in the description of $\bar{g}$. In particular only the segments k in which the interval $[b_k, b_{k+1})$ contains an integer point are supports of $\bar{g}$. Below we will initially assume that the segments that are not supports have been removed, and thus $\lfloor b_k \rfloor < \lfloor b_{k+1} \rfloor$ for all k.

We consider the following polyhedron R:

$$\phi \geq a_k z - c_k \quad \text{for } k = 1, \ldots, K \tag{27}$$
$$\phi \geq a_{k-1}z - c_{k-1} + (a_k - a_{k-1})(1 - f_k)(z - \lfloor b_k \rfloor) \quad \text{for } k = 1, \ldots, K \tag{28}$$
$$\phi \geq 0. \tag{29}$$

We derive a lemma for R resembling Lemma 1 that is easily checked.

Lemma 4. *For $z \in \mathbb{R}^1$, let*

$$\phi = \max\{0, \max_k(a_k z - c_k), \max_k(a_{k-1}z - c_{k-1} + (1 - f_k)(a_k - a_{k-1})(z - \lfloor b_k \rfloor))\}.$$

i) For each k, $\phi = a_k z - c_k$ if and only if $\lceil b_k \rceil \leq z \leq \lfloor b_{k+1} \rfloor$.
ii) For each k, $\phi = a_{k-1}z - c_{k-1} + (a_k - a_{k-1})(1 - f_k)(z - \lfloor b_k \rfloor)$ if and only if either $b_k \in \mathbb{Z}^1$ and $\lfloor b_k \rfloor \leq z \leq \lfloor b_{k+1} \rfloor$, or $b_k \notin \mathbb{Z}^1$ and $\lfloor b_k \rfloor \leq z \leq \lceil b_k \rceil$.
iii) $\phi = 0$ if and only if $z \leq \lfloor b_1 \rfloor$.

Note that if $b_k \in \mathbb{Z}$ for a given k, because of Observation 2, i) and ii) reduce to the same statement for that k. Using this Lemma, we can show that R is the formulation required.

Theorem 5. *$conv(W) = R$.*

Proof. $\text{proj}_{\phi,z}(Q) \subseteq R$ as every inequality describing R is valid for Q. We now show that $R \subseteq \text{proj}_{\phi,z}(Q)$. Given $(\phi, z) \in R$, let

$$\phi^* = \max\{0, \max_k(a_k z - c_k), \max_k(a_{k-1}z - c_{k-1} + (a_k - a_{k-1})(1 - f_k)(z - \lfloor b_k \rfloor))\}.$$

We just verify the case where $b_k \notin \mathbb{Z}^1$.
i). If $\phi^* = a_k z - b_k$ for some k, $\lceil b_k \rceil \leq z \leq \lfloor b_{k+1} \rfloor$ from Lemma 4. Set $\sigma_i = (z - b_i)^+$ for $i = 1, \ldots, K$, so (13) and (15) are satisfied. Also for $i \leq k$, $z \geq \lfloor b_k \rfloor + 1 \geq \lfloor b_i \rfloor + 1$ which implies $\sigma_i \geq (1 - f_i)(z - \lfloor b_i \rfloor)$, and for $i > k$, $z \leq \lfloor b_{k+1} \rfloor \leq \lfloor b_{i+1} \rfloor$, which implies that $\sigma_i = 0 \geq (1 - f_i)(z - \lfloor b_i \rfloor)$, so (14) hold. Finally

$$\sum_{i=1}^{K}(a_i - a_{i-1})\sigma_i = \sum_{i=1}^{k}(a_i - a_{i-1})(z - b_i) = a_k z - c_k = \phi^* \leq \phi$$

and (12) holds.
ii). If $\phi^* = a_{k-1}z - c_{k-1} + (a_k - a_{k-1})(1 - f_k)(z - \lfloor b_k \rfloor)$ for some k, $\lfloor b_k \rfloor \leq z \leq \lceil b_k \rceil$ by Lemma 4. Set $\sigma_i = z - b_i$ for $i \leq k-1$, $\sigma_k = (1 - f_k)(z - \lfloor b_k \rfloor)$ and $\sigma_i = 0$ for $i > k$. Again for $i \leq k - 1$, $z \geq \lfloor b_k \rfloor \geq \lfloor b_i \rfloor + 1$ implies $\sigma_i = z - b_i \geq (1 - f_i)(z - \lfloor b_i \rfloor)$. For $i = k$, $\lfloor b_k \rfloor \leq z_k \leq \lceil b_k \rceil$ implies $\sigma_i = (1 - f_i)(z - \lfloor b_i \rfloor) \geq z - b_i$, and for $i > k$, $\sigma_i = 0 \geq z - b_i$ and $\sigma_i = 0 \geq (1 - f_i)(z - \lfloor b_i \rfloor)$. Finally $\sum_{i=1}^{K}(a_i - a_{i-1})\sigma_i = \phi^* \leq \phi$.
iii). When $\phi^* = 0$, it suffices to take $\sigma_k = 0$ for all k. □

Example 2 (continued)

By Theorem 5, the additional inequalities needed to describe conv(W) are

$$\phi \geq z - 0 \tag{30}$$

$$\phi \geq \tfrac{3}{2}(z - \tfrac{1}{3}) + \tfrac{3}{2}(z - 1) \tag{31}$$

These inequalities also allow us to form $\bar{g}$, the integer closure of g:

$$\bar{g}(z) = \max\{0, z - 0, \frac{3}{2}z - \frac{1}{2}, \frac{3}{2}z - \frac{1}{2} + \frac{3}{2}(z - 1), \frac{7}{2}z - 3\}.$$

This numerical example is illustrated in Figure 1. □

Now we drop the assumption $\lfloor b_k \rfloor < \lfloor b_{k+1} \rfloor$ for all k. Let $U = \{k : k > 1, \lfloor b_{k-1} \rfloor < \lfloor b_k \rfloor\}$. For $k \in U$, let $p(k)$ denote the predecessor of k in U. The constraints needed to describe conv(W) now take the form

$$\phi \geq a_{p(k)}z - c_{p(k)} + \sum_{j=p(k)+1}^{k}(a_j - a_{j-1})(1 - f_j)(z - \lfloor b_k \rfloor) \quad \text{for } k \in U. \tag{32}$$

Finally, we consider what additional constraints on the integer variables can be added without destroying integrality.

Theorem 6. *Let $W^j \subseteq \mathbb{R}^1_+ \times \mathbb{Z}^1$ be sets of the form (7) with variables (ϕ_j, z_j). The polyhedron*

$$\bigcap_{j=1}^{n} conv(W^j) \cap \{z : Bz \leq d\}$$

is integral when B is totally unimodular and d is integral.

Proof. Using the same argument as in the proof of Theorem 2, it follows from Lemma 4 that all maximal bounded faces of R are of the form : $l_j \leq z_j \leq v_j$ for $j = 1, \ldots, K$, $Bz \leq d$ where l_j, v_j are integers or infinite. As B is TU, the face is integral, and the claim follows. □

Thus, given an IP with a separable, piecewise–linear objective function and a TU constraint matrix, we can reformulate this IP as an LP with integral extreme points by adding the MIR constraints (28) (or if necessary (32)) for each set W^j. Again it follows from Theorem 6 that nonnegativity of z, or integral bounds on z do not affect the results.

4. Model Y: $g(y_1, \ldots, y_n) = \max\{0, \max_{j=1,\ldots,n}(b_j - y_j)\}$.

Here we consider the set

$$Y = \{(\phi, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^n : \phi \geq b_j - y_j \text{ for } j = 1, \ldots, n\}. \tag{33}$$

We can also express Y as

$$\{(\phi, z) \in \mathbb{R}^1_+ \times \mathbb{Z}^n : \phi \geq z_j - d_j \text{ for } j = 1, \ldots, n\}$$

by substituting $y_j = -z_j$ and $b_j = -d_j$ for $j = 1, ..., n$, but we work with (33) because this form was used in earlier research.

We associate with Y the polyhedron

$$P_Y = \{(\phi, y) \in \mathbb{R}^1_+ \times \mathbb{R}^n : \phi \geq b_j - y_j \text{ for } j = 1, \ldots, n\}.$$

P_Y can be viewed as the epigraph of the convex function $g(y) = \max\{0, \max_j(b_j - y_j)\}$, in which case conv$(Y)$ is the epigraph of the integer closure of g_Y.

First we present the convex hull of Y in the original space, and then we address the question of what additional constraints can be added without losing integrality. We let $f_j = b_j - \lfloor b_j \rfloor$ for $j = 1, \ldots, n$.

Theorem 7. *i). (Günlük and Pochet (2001)). conv(Y) is described by the polyhedron*

$$\phi \geq b_j - y_j \quad \textit{for } j = 1, \ldots, n \tag{34}$$

$$\phi \geq \textstyle\sum_{p=1}^{P}(f_{j_p} - f_{j_{p-1}})(\lceil b_{j_p} \rceil - y_{j_p})$$
$$\text{for all } \{j_1, \ldots, j_P\} \subseteq \{1, \ldots, n\}, 0 = f_{j_0} < f_{j_1} < \ldots < f_{j_P} \tag{35}$$

$$\phi \geq \textstyle\sum_{p=1}^{P}(f_{j_p} - f_{j_{p-1}})(\lceil b_{j_p} \rceil - y_{j_p}) + (1 - f_{j_P})(\lceil b_{j_1} \rceil - y_{j_1} - 1)$$
$$\text{for all } \{j_1, \ldots, j_P\} \subseteq \{1, \ldots, n\}, 0 = f_{j_0} < f_{j_1} < \ldots < f_{j_P} \tag{36}$$

$$\phi \in \mathbb{R}^1_+, \; y \in \mathbb{R}^n. \tag{37}$$

ii). Let $Y^k \subseteq \mathbb{R}^1_+ \times \mathbb{Z}^{n_k}$ for $k = 1, \ldots, K$ be sets of the form (33) in the variables (ϕ_k, y^k). Then

$$\bigcap_{k=1}^{K} conv(Y^k) \cap \{y : By \leq d\} \tag{38}$$

is integral if B is the transpose of a network flow matrix and d is integer, where $y = (y^1, \ldots, y^K)$.

Proof. i). We consider the different faces of the polyhedron (34)–(37) in which ϕ is bounded. Note that in such a face, at least one of the inequalities (35), (36) and $\phi \geq 0$ holds at equality. The following claims essentially follow from the separation algorithm for the mixing inequalities (Pochet and Wolsey (1994), Günlük and Pochet (2001)).

a) $\phi = 0$ if and only if $y_j \geq \lceil b_j \rceil$ for all j.
b) $\phi = \sum_{p=1}^{P} (f_{i_p} - f_{i_{p-1}})(\lceil b_{i_p} \rceil - y_{i_p})$ if and only if

$$1 \geq \lceil b_{i_1} \rceil - y_{i_1} \geq \ldots \geq \lceil b_{i_P} \rceil - y_{i_P} \geq 0$$

$$\lceil b_{i_p} \rceil - y_{i_p} \geq \lceil b_j \rceil - y_j \quad \text{for } i_{p-1} < j < i_p \text{ and } p = 1, \ldots, P$$

$$0 \geq \lceil b_j \rceil - y_j \quad \text{for } j > i_P.$$

c) $\phi = \sum_{p=1}^{P} (f_{i_p} - f_{i_{p-1}})(\lceil b_{i_p} \rceil - y_{i_p}) + (1 - f_{i_P})(\lceil b_{i_1} \rceil - 1 - y_{i_1})$ if and only if

$$\lceil b_{i_1} \rceil - y_{i_1} \geq \ldots \geq \lceil b_{i_P} \rceil - y_{i_P} \geq \lceil b_{i_1} \rceil - 1 - y_{i_1} \geq 0$$

$$\lceil b_{i_p} \rceil - y_{i_p} \geq \lceil b_j \rceil - y_j \text{ for } i_{p-1} < j < i_p \text{ and } p = 1, \ldots, P$$

$$\lceil b_{i_1} \rceil - 1 - y_{i_1} \geq \lceil b_j \rceil - y_j \text{ for } j > i_P.$$

Thus every face is of the form

$$l_{ij} \leq y_i - y_j \leq v_{ij} \text{ for } i, j \in \{1, \ldots, n\}$$
$$l_j \leq y_j \leq v_j \text{ for } j \in \{1, \ldots, n\}$$

with l_{ij}, l_j either integer or $-\infty$, and v_{ij}, v_j either integer or $+\infty$. So the constraint matrix has the form of the transpose of a network flow matrix, which is totally unimodular, and the claim follows.

ii). With the multiple sets and the additional constraints, each face is of the form

$$l_{ij} \leq y_i - y_j \leq v_{ij} \text{ for } i, j \in \{1, \ldots, n\} \tag{39}$$
$$l_j \leq y_j \leq v_j \text{ for } j \in \{1, \ldots, n\} \tag{40}$$
$$By \leq d, \tag{41}$$

and the constraint matrix is still the dual of a network flow matrix. □

Since adding integer bounds on y_j preserves the dual network flow structure of the constraint matrix, this result easily extends to the case when bounds are present.

Even if B is not the dual of a network flow matrix, ii) of Theorem 7 can be slightly generalized. In particular it suffices to show that each of the faces (39)–(41) is integral. An example of this is provided by the single-item constant capacity lot-sizing model with Wagner-Whitin costs – when modelled in the form (38), B is not a network dual matrix, but the constraint matrix of the face (39)–(41) can be shown to be totally unimodular, an observation of Constantino (2002).

As noted above, a polynomial separation algorithm for the mixing inequalities (35)–(36) is known. Now as an alternative to the exponential description of conv(Y) provided by the mixing inequalities, we give a simple derivation of a compact extended formulation. This formulation is similar to an extended formulation presented for the constant capacity lot–sizing problem in Pochet and Wolsey (1994), which was used there to define the separation algorithm for the mixing inequalities.

Proposition 8. *An extended formulation for conv(Y) is*

$$\phi = \textstyle\sum_{i=0}^{n} f_i \delta_i + \mu \tag{42}$$

$$y_j \geq \textstyle\sum_{i=0}^{n} \lceil b_j - f_i \rceil \delta_i - \mu \text{ for } j = 1, \ldots, n \tag{43}$$

$$\textstyle\sum_{i=0}^{n} \delta_i = 1 \tag{44}$$

$$\mu \geq 0, \delta_i \geq 0 \text{ for } i = 0, \ldots, n, \tag{45}$$

where $f_0 = 0$.

Proof. The $n+1$ extreme points $\{\phi^i, y^i\}_{i=0}^n$ of conv(Y) are given by $\phi^0 = 0$, $y_j^0 = \lceil b_j \rceil$ for $j = 1, \ldots, n$, and $\phi^i = f_i$, $y_j^i = \lceil b_j - f_i \rceil$ for $j = 1, \ldots, n$, as i varies over $\{1, \ldots, n\}$. The convex hull of the union of these points is given by the constraints (42)–(45) with (42) and (43) as equalities, and $\mu = 0$. Letting e_j be the j^{th} unit vector, and $\underline{1}$ be the vector of all 1's, the claim follows as the extreme rays of Y are $(\phi, y) = (0, e_j)$, for $j = 1, \ldots, n$, and $(\phi, y) = (1, -\underline{1})$ represented by the variable μ. □

Note that ii) of Theorem 7 does not hold when B is TU (counterexamples include those arising from discrete lot–sizing problems discussed in the companion paper of Miller and Wolsey (2002)). It is not clear whether there is a polynomial algorithm for optimization over such sets when B is TU.

5. Model Z: $g(y, r) = \max_{j=1,\ldots,n}(b_j - r_j - y_j)^+$.

Here

$$Z = \{(\phi, r, y) \in \mathbb{R}_+^1 \times \mathbb{R}_+^n \times \mathbb{Z}^n : \phi \geq b_j - r_j - y_j \text{ for } j = 1, \ldots, n\}, \tag{46}$$

and its associated polyhedron P_Z represents the epigraph of the function $g(r, y) = \max\{0, \max_j(b_j - r_j - y_j)\}$. For such sets, we can define valid "mixing" inequalities for conv(Z) that are analogous to the inequalities (35) and (36):

$$\phi + \sum_{p=1}^{P} r_{j_p} \geq \sum_{p=1}^{P} (f_{j_p} - f_{j_{p-1}})(\lceil b_{j_p} \rceil - y_{j_p})$$
$$\text{for all } \{j_1, ..., j_P\} \subseteq \{1, ..., n\}, 0 = f_{j_0} < f_{j_1} < \ldots < f_{j_P} \tag{47}$$

$$\phi + \sum_{p=1}^{P} r_{j_p} \geq \sum_{p=1}^{P} (f_{j_p} - f_{j_{p-1}})(\lceil b_{j_p} \rceil - y_{j_p}) + (1 - f_{j_P})(\lceil b_{j_1} \rceil - y_{j_1} - 1)$$
$$\text{for all } \{j_1, ..., j_P\} \subseteq \{1, ..., n\}, 0 = f_{j_0} < f_{j_1} < \ldots < f_{j_P}. \tag{48}$$

However, these inequalities do not suffice to give a description of conv(Z).

Example 3. Let $n = 3$ and $b = (2.1, 0.4, 0.6)$, and consider an instance of the set Z

$$\begin{aligned} \phi + r_1 + y_1 &\geq 2.1 \\ \phi + r_2 + y_2 &\geq 0.4 \\ \phi + r_3 + y_3 &\geq 0.6 \\ \phi, r \geq 0, y &\in \mathbb{Z}^3. \end{aligned}$$

It is straightforward to check that the inequality

$$\phi + 0.5r_1 + 0.5r_2 + 0.5r_3 + 0.35y_1 + 0.2y_2 + 0.25y_3 \geq 1.2$$

is valid and facet-defining for conv(Z). Clearly it is not a mixing inequality of the form (47) or (48), since the coefficients of ϕ and the r_j variables are different. □

We do not know a characterization of the facets of conv(Z), but there is an extended formulation.

Theorem 9. *An extended formulation for conv(Z) is*

$$\phi = \sum_{i=0}^{n} f_i \delta_i + \mu \tag{49}$$
$$r_j = \sum_{i=0}^{n} f_i^j \beta_i^j + \nu_j \textit{ for } j = 1, \ldots, n \tag{50}$$
$$y_j \geq \lfloor b_j \rfloor + \sum_{i: f_i < f_j} (\delta_i - \beta_i^j) - \sum_{i: f_i > f_j} \beta_i^j - \mu - \nu_j \textit{ for } j = 1, \ldots, n \tag{51}$$
$$\beta_i^j \leq \delta_i \textit{ for } j = 1, \ldots, n, i = 0, \ldots, n \tag{52}$$
$$\sum_{i=0}^{n} \delta_i = 1 \tag{53}$$
$$\beta, \delta, \mu, \nu \geq 0, \tag{54}$$

where $f_0 = 0$, $f_i^j = f_j - f_i$ *if* $f_i \leq f_j$ *and* $f_i^j = 1 + f_j - f_i$ *if* $f_i > f_j$.

To prove this, we need the following lemma.

Lemma 10. *In each extreme point of conv(Z), either* $\phi = 0$, *or else there exists a* $j \in \{1, ..., n\}$ *such that both* $r_j = 0$ *and* $\phi + y_j = b_j$.

Proof. Let $(\bar{\phi}, \bar{r}, \bar{y}) \in$ conv(Z) be such that $\bar{\phi} > 0$, and either $\bar{r}_j > 0$ or $\bar{\phi} + \bar{y}_j > b_j$, $j = 1, \ldots, n$. If $\bar{\phi} + \bar{r}_j + \bar{y}_j > b_j$ for all j, then $(\bar{\phi}, \bar{r}, \bar{y})$ is clearly not an extreme point of conv(Z). So let $\mathcal{T} = \{j = 1, ..., n : \bar{\phi} + \bar{r}_j + \bar{y}_j = b_j\}$, and let $\bar{\mathcal{T}} = \{j = 1, ..., n : j \notin \mathcal{T}\}$. By hypothesis, for each $j \in \mathcal{T}$, $\bar{r}_j > 0$ must hold. Now define $\epsilon = \min\{\bar{\phi}, \min_{j \in \mathcal{T}}[\bar{r}_j], \min_{j \in \bar{\mathcal{T}}}[\bar{\phi} + \bar{r}_j + \bar{y}_j - b_j]\}$. It is clear that $\epsilon > 0$, and thus the following two points are in $Y(g)$:

$$\phi = \bar{\phi} - \epsilon; r_j = \bar{r}_j + \epsilon, j \in \mathcal{T}, r_j = \bar{r}_j, j \in \bar{\mathcal{T}}; y_j = \bar{y}_j, j = 1, \ldots, n$$
$$\phi = \bar{\phi} + \epsilon; r_j = \bar{r}_j - \epsilon, j \in \mathcal{T}, r_j = \bar{r}_j, j \in \bar{\mathcal{T}}; y_j = \bar{y}_j, j = 1, \ldots, n$$

Moreover, $(\bar{\phi}, \bar{y}, \bar{z})$ is a convex combination of these points and so cannot be an extreme point of $\operatorname{conv}(Z)$. The claim follows. □

Proof of Theorem 9. Given Lemma 10, it follows that $\operatorname{conv}(Z)$ has the following extreme points:

Case 1. $\phi = 0$. (Variable $\delta_0 = 1$).

For each $j = 1, \ldots, n$, either $r_j = 0$ and $y_j = \lceil b_j \rceil$ (variable $\beta_0^j = 0$), or $r_j = f_j = f_0^j$ and $y_j = \lfloor b_j \rfloor$ (variable $\beta_0^j = 1$). Thus, in this case, both (50)–(51) hold at equality for $j = 1, \ldots, n$, where δ and β take the values indicated.

Case 2. $r_i = 0$ and $\phi + y_i = b_i$ (variable $\delta_i = 1$) for some $i \in \{1, \ldots, n\}$.
If $f_j > f_i$, either $r_j = 0$, $y_j = \lceil b_j - f_i \rceil = \lceil b_j \rceil$ (variable $\beta_i^j = 0$), or $r_j = f_j - f_0^i$, $y_j = \lfloor b_j \rfloor$ (variable $\beta_i^j = 1$). If $f_j < f_i$, either $r_j = 0$, $y_j = \lceil b_j - f_i \rceil = \lceil b_j - 1 \rceil$ (variable $\beta_i^j = 0$), or $r_j = 1 + f_j - f_i$, $y_j = \lfloor b_j - 1 \rfloor$ (variable $\beta_i^j = 1$). If $f_j = f_i$, then $r_j = 0$, $y_j = b_j - f_i = \lfloor b_j \rfloor$ must hold, or else (ϕ, r, z) is not an extreme point of $\operatorname{conv}(Z)$. Thus, in this case as well, (49)–(51) hold at equality for $j = 1, \ldots, n$, where δ and β take the values indicated.

Therefore, similarly to the proof of Proposition 8, the convex hull of the extreme points of $\operatorname{conv}(Z)$ is exactly the projection of (49)–(54), with (49)–(51) at equality and $\mu = 0 = \nu_j$ for $j = 1, \ldots, n$, onto the (ϕ, r, y) space. The result follows as the extreme rays of $\operatorname{conv}(Z)$ are $(\phi, r, y) = (0, 0, e_j)$ for $j = 1, \ldots, n$, $(\phi, r, y) = (1, 0, -\underline{1})$ represented by the variable μ, and $(\phi, r, y) = (0, e_j, -e_j)$ represented by the variable ν_j for $j = 1, \ldots, n$. □

This extended formulation is related to that presented for $\operatorname{conv}(Y)$ in Section 4, but because of the extra continuous variable r_j, it requires a quadratic rather than linear number of extra variables. It is not known if dual network flow constraints can be added to link the y variables of multiple sets of the form Z without destroying the integrality provided by this extended formulation.

6. Closing remarks

There are other mixed integer sets that are related to convex integer objective functions. For instance, study of the continuous knapsack set $\{(y, s) \in \mathbb{Z}_+^n \times \mathbb{R}_+^1 : \sum_{j=1}^n a_j y_j \leq b + s\}$ can be interpreted as study of the integer hull of the function $g(y) = \max\{0, \sum_j a_j y_j - b\}$. Here results are known for the divisible knapsack case when $a_1 | \ldots | a_n$ (Pochet and Wolsey (1995)), and for the case $n = 2$ (Agra and Constantino (2001)). However some two variable functions seem to be inherently complicated, such as the function $g(z_1, z_2) = \max\{-z_1, -z_2, z_1 + z_2 - 2\}$ based on the IP $\max\{s : y_1 + y_2 + s \leq 2, s \leq y_1, s \leq y_2, s \in \mathbb{R}^1_+, y \in \mathbb{Z}^2\}$. This is a problem for which an infinite set of Gomory mixed or MIR inequalities is apparently needed to attain the optimal value (White (1961), see also Salkin (1975)).

In a companion paper Miller and Wolsey (2002), we discuss how these results can be used to derive tight formulations for various discrete lot–sizing problems. In particular, results for W provide tight formulations for problems with backlogging and/or safety stock requirements, results for Y provide tight formulations for problems in which initial inventory variables are present, and results for Z provide tight formulations for problems with both backlogging and initial inventory variables. These reformulations turn out to be effective on a set of industrial problems.

References

[1] Agra, A., Constantino, M.: Polyhedral description of basic knapsack problems with a continuous variable, University of Lisbon. 2001

[2] Ahuja, R.L., Magnanti, T.L., Orlin, J.B.: Network Flows: Theory, Algorithms, and Applications. Prentice-Hall, Englewood Cliffs, NJ. 1993

[3] Atamtürk, A., Nemhauser, G.L., Savelsbergh, M.W.P.: The Mixed Vertex Packing Problem. Math. Program. **89**, 35–53 (2000)

[4] Atamtürk, A., Rajan, D.: On splittable and unsplittable flow capacitated network design arc–set polyhedra. Mathematical Programming **92**, 315–333 (2002)

[5] Constantino, M.: Private communication, 2002

[6] Fleischmann, B.: The discrete lot–sizing and scheduling problem. Eur. J. Oper. Res. **44**, 337–348 (1990)

[7] Gomory, R.E.: An algorithm for the mixed integer problem. The RAND Corporation. American Mathematical Society, RM-2597. 1960

[8] Günlük, O., Pochet, Y.: Mixing MIR inequalities for mixed integer programs. Math. Program. **90**, 429–458 (2001)

[9] Hochbaum, D.S., Shantikumar, J.G.: Convex separable optimization is not much harder than linear optimization. Journal of ACM **37**, 843–862 (1990)

[10] Magnanti, T.L., Mirchandani, P., Vachani, R.: The convex hull of two core capacitated network design problems. Math. Program. **60**, 233–250 (1993)

[11] Marchand, H., Wolsey, L.A.: Aggregation and Mixed Integer Rounding to solve MIPs. Oper. Res. **49**, 363–371 (2001)

[12] Miller, A., Wolsey, L.A.: Tight MIP Formulations for Multi–Item Discrete Lot–Sizing Problems, CORE, Louvain-la-Neuve, Belgium. To appear in Operations Research. (2002)

[13] Pereira, O., Wolsey, L.A.: On the Wagner–Whitin lot–sizing polyhedron. Math. of Oper. Res. **26**, 591–600 (2001)

[14] Pochet, Y., Wolsey, L.A.: Polyhedra for lot–sizing with Wagner–Whitin costs. Math. Program. **67**, 297–323 (1994)

[15] Pochet, Y., Wolsey, L.A.: Integer Knapsacks and Flow Covers with Divisible Coefficients: Polyhedra, Optimization and Separation. Discrete Applied Mathematics **59**, 57–74 (1995)

[16] Salkin, H.M.: Integer Programming. Addison Wesley, Reading, Mass. 1975

[17] van Eijl, C.A., van Hoesel, C.P.M.: On the discrete lot–sizing and scheduling problem with Wagner–Whitin costs, Operations Research Letters **20**, 7–13 (1997)

[18] van Hoesel, C.P.M., Kuik, R., Salomon, M., van Wassenhove, L.N.: The single item discrete lot–sizing and scheduling problem: optimization by linear and dynamic programming. Discrete Appl. Math. **48**, 289–303 (1994)

[19] White, W.: On Gomory's mixed integer algorithm. Senior's thesis, Princeton University. 1961