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Probabilistic programming with discrete distributions and precedence constrained knapsack polyhedra

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Abstract. We consider stochastic programming problems with probabilistic constraints involving random variables with discrete distributions. They can be reformulated as large scale mixed integer programming problems with knapsack constraints. Using specific properties of stochastic programming problems and bounds on the probability of the union of events we develop new valid inequalities for these mixed integer programming problems. We also develop methods for lifting these inequalities. These procedures are used in a general iterative algorithm for solving probabilistically constrained problems. The results are illustrated with a numerical example.

Key words. stochastic programming – integer programming – valid inequalities

1. Introduction

Reliability and risk are key issues in models arising in insurance, finance, telecommunication and many other areas. One way to incorporate them into optimization problems are *probabilistic constraints*.

Stochastic programming problems with probabilistic constraints can be introduced as follows. We have a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and the space $\mathcal{X}$ of measurable mappings $x : \Omega \to \mathbb{R}^n$. Next, we are given a functional $f : \mathcal{X} \to \mathbb{R}$, a measurable constraint function $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m$, a random vector $\xi : \Omega \to \mathbb{R}^s$, and a set $X \subset \mathcal{X}$. The problem is to find

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & \mathbb{P}\{g(x(\omega), \xi(\omega)) \geq 0\} \geq 1 - \alpha, \\ & x \in X, \end{aligned} \tag{1}$$

where the symbol $\mathbb{P}$ denotes probability and $\alpha \in (0, 1)$ is some prescribed level.

The simplest case is the *here–and–now* problem in which the decision x is not allowed to depend on the random vector ξ, that is, $X \subseteq \mathbb{R}^n$.

A more involved situation occurs in the *two-stage* case, in which x has two subvectors, $x = (x_1, x_2)$, the first of which has to be determined without the knowledge of the random outcome, while the second one, x_2, can be decided upon *after* $\xi(\omega)$ is known. Then X can contain only decision rules of the form $x(\omega) = (x_1, x_2(\omega))$. In a more involved *multistage model* we have $x = (x_1, \ldots, x_T)$, where T is the number of stages,

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and each part x_t of the decision vector may use some partial information available at stage t. The reader is referred to the book of Birge and Louveaux[4] for an extensive treatment of different information structures in stochastic programming models.

Programming under probabilistic constraints has a long history. Charnes, Cooper and Symonds in [7] formulated probabilistic constraints individually for each stochastic constraint. Joint probabilistic constraints for independent random variables were used first by Miller and Wagner in [15]. The general case was first studied by Prékopa in [18].

Much is known about problem (1) in the case when the decisions x are deterministic vectors in $\mathbb{R}^n$, f is linear in x, and

$$g(x, \xi) = Tx - \xi, \tag{2}$$

with some random vector ξ and a deterministic matrix T. In particular, if ξ has a continuous distribution, [20] is an excellent reference. Much less is known in the case of a discrete distribution of ξ (see [8, 10, 21]). When the dependence of g on ξ is more involved, for example the matrix T in (2) is random, too, significant difficulties arise. We should mention here the works [13] and [12] on stochastic routing problems, where inequalities eliminating infeasible routes have been developed.

We shall focus our efforts on the case when there are only finitely many realizations $\xi^1, \dots, \xi^N$ of the random vector ξ, occurring with probabilities $p_1, \dots, p_N$. We shall call them *scenarios*. As a result, only finitely many solution realizations $x^i = x(\xi^i)$ may occur, $i = 1, \dots, N$. To facilitate formulation of probabilistic constraints in this case, let us introduce the indicator function $\chi : \mathbb{R}^m \to \{0, 1\}$:

$$\chi(u) = \begin{cases} 1 & \text{if } u \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Problem (1) can be then written in a more explicit form:

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i \chi(g(x^i, \xi^i)) \geq 1 - \alpha, \\ & x \in X. \end{aligned} \tag{3}$$

Let us keep in mind that the set X in the above formulation takes care of the information restrictions on x. For example, in the here-and-now problem, the set X contains only decisions x such that $x^1 = \cdots = x^N$. In this case, of course, there is no need to distinguish the scenario solutions x^i in (1) and the entire problem can be written in terms of just one vector x, common for all scenarios. In the two-stage case, where the decision vector has two parts $x = (x_1, x_2)$, the set X contains only decisions x such that $x_1^1 = \cdots = x_1^N$. The second part of the decision vector, x_2, may still depend on the scenario.

Discrete distributions arise frequently in applications, either directly, or as empirical approximations of the underlying distribution $\mathbb{P}$. In the latter case ξ^i are independent observations of ξ, and $p_i = 1/N$ for $i = 1, \dots, N$. If more than one observation have identical outcomes we may still formally treat them as different scenarios.

Throughout, we assume that the functions $f(\cdot)$ and $g(\cdot, \xi^i)$, $i = 1, \dots, N$, are continuous and the set X is compact. Thus, if (3) has a nonempty feasible set, an optimal solution exists.

The main observation around which we plan to focus our research is that in many cases one can define a partial order $\preceq$ on the set of scenarios: for some pairs of scenarios i and j we shall be able to say that i is 'not harder' than j. In the case when

$$g(x, \xi) = t(x) - \xi$$

for some function $t : \mathbb{R}^n \to \mathbb{R}^m$ (and with $s = m$) the order $\preceq$ is defined as the component-wise inequality between the right hand side realizations:

$$i \preceq j \;\Leftrightarrow\; \xi^i \leq \xi^j.$$

This has been extensively exploited in our recent works with D. Dentcheva and A. Prékopa [8–10] where we show that only a limited number of scenarios play a role in the problem. These are $(1 - \alpha)$-*efficient points* defined as the minimal points (in the sense of the partial order $\leq$) of the set of realizations ξ^i for which

$$\mathbb{P}\{\xi \leq \xi^i\} \geq 1 - \alpha.$$

In [8] we developed an algorithm that iteratively updates the set of relevant $(1 - \alpha)$-efficient points to generate tight lower and upper bounds for probabilistically constrained problems. The algorithm has been extended to general convex programming with probabilistic constraints in [9]. In [2, 3] we discuss methods based on partial or complete enumeration of efficient points for stochastic integer programming problems.

In section 2 we introduce a more general definition of a *consistent* order and we show that it can be defined for many classes of probabilistically constrained problems. This will be exploited in section 3 to formulate deterministic equivalents of probabilistically constrained problems with the use of *precedence constrained knapsack polyhedra*: a particular structure of combinatorial optimization problems, which will be defined and analyzed there. We shall discuss valid inequalities for probabilistic constraints and we shall formulate auxiliary separation problems to find valid inequalities of interest. Section 4 is devoted to specialized lifting procedures for these inequalities. In section 5 we shall construct a method for solving probabilistically constrained problems that uses valid inequalities developed in the preceding sections. Finally, in section 6 we shall have a numerical illustration.

We shall use the symbol $\preceq$ to denote a partial order relation in a set I; the strict relation $i \prec j$ will be understood in a usual way ($i \preceq j$ and $i \neq j$). The sets of minimal and maximal elements of I under the order $\preceq$ will be denoted $\mathcal{M}(I)$ and $\mathcal{S}(I)$, respectively.

2. Consistent orders of scenarios

We start from the definition of an 'easier' scenario.

Definition 1. *A partial order* $\preceq$ *on* $\{1, \dots, N\}$ *is* consistent *with problem (3) if for every* $x \in X$ *there exists* $\bar{x} \in X$ *such that*

(i) $f(\bar{x}) \leq f(x)$;

(ii) $\sum_{i=1}^{N} p_i \chi(g(\bar{x}^i, \xi^i)) \geq \sum_{i=1}^{N} p_i \chi(g(x^i, \xi^i))$; *and*

(iii) *for all* $i, j \in \{1, \dots, N\}$ *one has*

$$(i \prec j) \wedge (g(\bar{x}^j, \xi^j) \geq 0) \Rightarrow (g(\bar{x}^i, \xi^i) \geq 0).$$

The order $\preceq$ *is* strongly consistent *if condition (iii) holds for* $\bar{x} = x$.

Let us consider two practically important cases of probabilistically constrained stochastic programming problems when a consistent order can easily be defined.

We start from the linear problem with joint probabilistic constraints:

$$\begin{aligned} \min \quad & c^T x \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i \chi(T^i x - h^i) \geq 1 - \alpha, \\ & x \in X, \end{aligned} \tag{4}$$

with scenarios $i = 1, \dots, N$ characterized by realizations (T^i, h^i) of an $m \times n$ random matrix T and a random vector $h \in \mathbb{R}^m$. The convex closed polyhedron $X \subseteq \mathbb{R}^n$, the cost vector $c \in \mathbb{R}^n$ and the probability level $\alpha \in (0, 1)$ are given. From Definition 1 we obtain the following result.

Lemma 1. *The partial order* $\preceq$ *defined on* $\{1, \dots, N\}$ *as follows*

$$i \preceq j \;\Leftrightarrow\; h^i - T^i x \leq h^j - T^j x \text{ for all } x \in X$$

is strongly consistent with problem (4).

In a special case, if $X = \mathbb{R}^n_+$ we have

$$i \preceq j \;\Leftrightarrow\; T^i \geq T^j \text{ and } h^i \leq h^j.$$

When only the right hand side h is random, the order $\preceq$ is identical to the componentwise inequality $\leq$ in the space of realizations of h, whose implications for our problem are thoroughly analyzed in [8].

Our second example is the linear two-stage problem with probabilistic constraints. It has two groups of decision variables: first stage decisions $x \in \mathbb{R}^n$ and second stage decisions $y^i \in \mathbb{R}^l$ associated with each scenario $i = 1, \dots, N$. The problem is formulated as follows:

$$\begin{aligned} \min \quad & c^T x + \sum_{i=1}^{N} p_i \langle q, y^i \rangle \\ \text{subject to} \quad & \sum_{i=1}^{N} p_i \chi(T^i x + W y^i - h^i) \geq 1 - \alpha, \\ & x \in X, \\ & y^i \in Y, \quad i = 1, \dots, N. \end{aligned} \tag{5}$$

In addition to the notation explained at (4), $Y \subseteq \mathbb{R}^l$ is a convex closed polyhedron, and $q^i \in \mathbb{R}^l$ is a given second stage cost vector associated with scenario $i = 1, \dots, N$. The probabilities of scenarios are denoted $p_1, \dots, p_N$.

Lemma 2. *The partial order $\preceq$ defined on $\{1, \dots, N\}$ as follows*

$$(i \preceq j) \Leftrightarrow (p_i = p_j) \wedge (h^i - T^i x \leq h^j - T^j x) \quad \forall x \in X$$

is consistent with problem (5).

Proof. Let $x \in X$ and $y^i \in Y, i = 1, \dots, N$. Suppose that condition (iii) of Definition 1 is violated, that is, there exist two scenarios, k and l, such that $k \prec l$, $T^k x + W y^k \geq h^k$ but $T^l x + W y^l \not\geq h^l$. Let us consider the set of scenarios at which the probabilistic constraint is violated,

$$I = \{i : T^i x + W y^i \not\geq h^i\}, \tag{6}$$

the set of its minimal elements, $\mathcal{M}(I)$, and the set of scenarios dominated by the minimal elements,

$$A = \bigcup_{m \in \mathcal{M}(I)} \{j : m \preceq j\}. \tag{7}$$

Since $k \in I$, there must exist $m \in \mathcal{M}(I)$ such that $m \prec k \prec l$. Define a new second stage solution $\widetilde{y}$ by switching in $\hat{y}$ the values of y^m and y^l.

By the definition of $\preceq$ we have

$$T^m x + W \widetilde{y}^m \geq h^m. \tag{8}$$

The point $(x, \widetilde{y})$ is equally good as (x, y) and

$$\sum_{i=1}^{N} p_i \chi(T^i x + W \widetilde{y}^i - h^i) \geq \sum_{i=1}^{N} p_i \chi(T^i x + W y^i - h^i).$$

Let I' and A' be the sets (6) and (7) calculated at the new point $(x, \widetilde{y})$. Since m was a minimal element of I, relation (8) implies that $m \notin A'$. On the other hand $l \notin \mathcal{M}(I)$, so $A' \subset A$.

Consequently, we were able to modify the solution, without increasing the objective or the probability of violating the probabilistic constraint, in such a way that the cardinality of the set A decreased. Therefore, by carrying out the above transformation finitely many times we can construct a solution $(x, \bar{y})$ at which the order $\preceq$ satisfies Definition 1. □

In section 6 we shall use an example of a two-stage problem with probabilistic constraints to illustrate our results.

3. Mixed integer formulation and induced covers

Let us reformulate problem (3) as a mixed integer program. To this end we find for each $i = 1, \dots, N$ a vector $d^i \in \mathbb{R}^m$ such that

$$g(x^i, \xi^i) + d^i \geq 0, \quad \text{for all} \quad x \in X.$$

Such a vector exists, because $g(\cdot, \xi^i)$ is continuous and X compact.

This allows us to transform (3) into a mixed integer program:

$$\min \quad f(x) \tag{9}$$

$$\text{subject to} \quad g(x^i, \xi^i) + d^i z_i \geq 0, \quad i = 1, \dots, N, \tag{10}$$

$$\sum_{i=1}^{N} p_i z_i \leq \alpha, \tag{11}$$

$$x \in X, \tag{12}$$

$$z_i \in \{0, 1\}, \quad i = 1, \dots, N. \tag{13}$$

If f is convex and $g(\cdot, \xi^i)$ concave for all ξ^i, the above problem is a mixed integer convex program; its relaxation (with the integrality restriction (13) ignored) can be solved by convex programming methods. However, the full mixed integer programming problem appears to be very difficult, since the number of scenarios N may be very large. To reduce its complexity we shall use the partial order $\preceq$ associated with (3). From Definition 1 we obtain the following observation.

Lemma 3. *If $\preceq$ is a consistent order for (9)–(13), then there exists an optimal solution $(\hat{x}, \hat{z})$ of (9)–(13) such that for all $i, j \in \{1, \dots, N\}$*

$$(i \preceq j) \Rightarrow (z_i \leq z_j).$$

Therefore, adding to (9)–(13) the constraints

$$z_i \leq z_j \quad \text{for all } i, j \in \{1, \dots, N\} \text{ such that } i \preceq j \tag{14}$$

does not cut off *all* optimal solutions.

Inequalities (11) and (14), together with the integrality restriction (13), define a *precedence constrained knapsack polyhedron* (PCKP), extensively studied in combinatorial optimization [6, 17, 14]. We shall adapt and develop some of the ideas introduced for PCKPs in order to gain more insight into problem (9)–(14) and to create specialized methods for its solution.

Let us define the sets

$$A_i = \{j \in \{1, \dots, N\} : i \preceq j\}, \quad i = 1, \dots, N.$$

Germane to our research is the concept of the *induced cover*, which generalizes the classical notion of a cover for knapsack constraints (see [16, 22] and the references therein).

Definition 2. *A set $C \subseteq \{1, \ldots, N\}$ is called an* induced cover *if*

$$\mathbb{P}\Big\{\bigcup_{i\in C} A_i\Big\} > \alpha. \tag{15}$$

An induced cover C is proper*, if for every $j \in C$*

$$\mathbb{P}\Big\{\bigcup_{i\in C\setminus\{j\}} A_i\Big\} \le \alpha \tag{16}$$

and minimal *if for every $j \in C$*

$$\mathbb{P}\Big\{\bigcup_{i\in C} A_i \setminus \{j\}\Big\} \le \alpha. \tag{17}$$

For any induced cover C we have a valid inequality:

$$\sum_{i\in C} z_i \le |C| - 1. \tag{18}$$

Indeed, if $z_i = 1$ for all $i \in C$ then (14) and the definition of A_i imply that $z_k = 1$ for $k \in \bigcup_{i\in C} A_i$. Consequently, (15) contradicts (11).

To illustrate the notion of an induced cover let us consider the realizations $\xi^1, \ldots, \xi^{20}$ of a random variable $\xi = (\xi_1, \xi_2)$, displayed in Figure 1 (for simplicity we number only 8 of them). Let us assume that the consistent order $\preceq$ is defined as follows:

$$(i \preceq j) \;\Leftrightarrow\; (\xi^i \le \xi^j),$$

where the inequality between the realizations is understood componentwise. Suppose that all the realizations are equally probable, $p_i = 0.05$, for all i, and that $\alpha = 0.25$. We have

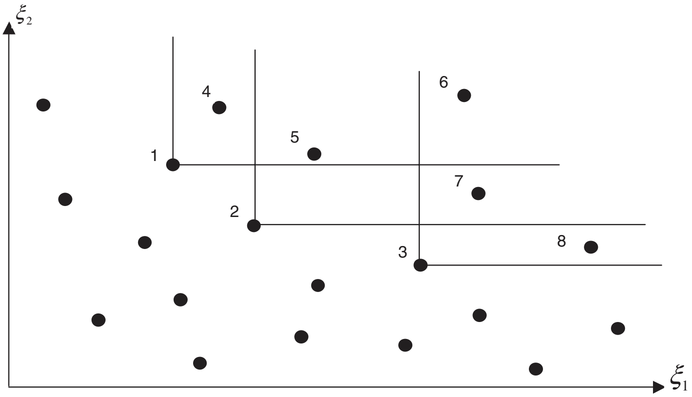

Fig. 1. The sets A_1, A_2 and A_3

$$A_1 = \{1, 4, 5, 6\},$$
$$A_2 = \{2, 5, 6, 7\},$$
$$A_3 = \{3, 6, 7, 8\}, \quad \text{etc..}$$

The set $C = \{1, 2\}$ is an induced cover, because $\mathbb{P}\{A_1 \cup A_2\} = 0.3$. It is proper and minimal. There are many other induced covers, for example, $\{1, 2, 3\}$, $\{1, 3\}$, $\{2, 3\}$, $\{4, 2, 8\}$ etc.

The notion of a minimal induced cover for PCKP has been introduced in [6] and analyzed in [14, 17]. Van de Leensel, van Hoesel and van de Klundert prove in [14] that inequalities (18) generated by minimal induced covers are facet defining for subsets of PCKP and they use the general lifting algorithm of Balas [1] to obtain facet defining inequalities for the entire PCKP.

In the context of probabilistic programming, though, the application of these results encounters difficulties due to the large number N of possible scenarios. The enumeration of all proper induced covers is practically impossible. Lifting of the covers, as shown in [14], requires the solution of very many knapsack subproblems, each of them NP-hard. We shall therefore concentrate on two issues: finding cover inequalities that cut-off a given fractional solution (the separation problem), and determining their lifting. Our main objective is to incorporate these techniques into a specialized method for solving probabilistically constrained problems of form (3).

The first question we are interested in is the following: given a set $I \subseteq \{1, \dots, N\}$ and a fractional point $\tilde{z} \in [0, 1]^N$ find an induced cover $C \subseteq I$ such that the inequality (18) cuts-off $\tilde{z}$, that is,

$$\sum_{i \in C} \tilde{z}_i > |C| - 1. \tag{19}$$

Of course, the only interesting case is I being an induced cover itself. To find a cover C for which the difference between the two sides of (19) is the largest, we introduce binary variables v_i, $i \in I$, to decide whether scenario i will be included in C or not, and we formulate the optimization problem:

$$\min \quad \sum_{i \in I} (1 - \tilde{z}_i) v_i \tag{20}$$

$$\text{subject to} \quad \mathbb{P}\Big\{ \bigcup_{i \,:\, v_i = 1} A_i \Big\} > \alpha, \tag{21}$$

$$v_i \in \{0, 1\}, \quad i \in I. \tag{22}$$

From Definition 2 we deduce the following result.

Lemma 4. *Assume that I is an induced cover. If the optimal value of (20)–(22) is smaller than 1, the set $C = \{i \in I : v_i = 1\}$ defines an induced cover for which inequality (19) is satisfied. If the optimal value is greater or equal than 1, there is no induced cover $C \subseteq I$ such that inequality (19) holds.*

Problem (20)–(22) is still a difficult combinatorial optimization problem, especially due to the implicit constraint (21). We shall derive two restrictions of this problem in form of linear programs. They both use the classical Boole–Bonferroni inequality (see, e.g., [5, 19] and the references therein)

$$\mathbb{P}\Big\{\bigcup_{i\in C} A_i\Big\} \geq \sum_{i\in C}\mathbb{P}\{A_i\} - \sum_{\substack{i,j\in C\\ i<j}} \mathbb{P}\{A_i \cap A_j\}. \tag{23}$$

Our first approximation applies this inequality directly to constraint (21). For the example illustrated in Figure 1 the Boole–Bonferroni inequality yields $\mathbb{P}\{A_1 \cup A_2 \cup A_3\} \geq 0.35$, so its application will allow $\{1, 2, 3\}$ as a cover, if $\alpha = 0, 25$. However, if $\alpha = 0.35$, we shall miss it, although $\mathbb{P}\{A_1 \cup A_2 \cup A_3\} = 0.4$.

To approximate the separation problem (20)–(22) by a mixed integer linear programming problem using the Boole–Bonferroni inequality, we introduce additional decision variables y_{ij}, $i, j \in I$, $i < j$, representing the decision that both scenarios, i and j, are in the cover. We obtain the formulation:

$$\min \quad \sum_{i\in I}(1 - \tilde{z}_i)v_i \tag{24}$$

$$\text{subject to} \quad \sum_{i\in I} v_i\mathbb{P}\{A_i\} - \sum_{\substack{i,j\in I\\ i<j}} y_{ij}\mathbb{P}\{A_i \cap A_j\} \geq \alpha + \epsilon, \tag{25}$$

$$y_{ij} \geq v_i + v_j - 1, \quad y_{ij} \geq 0, \quad i, j \in I, \quad i < j, \tag{26}$$

$$v_i \in \{0, 1\}, \quad i \in I, \tag{27}$$

with $0 < \epsilon < \min_{1\leq i\leq N} p_i$. The role of ϵ is to allow replacing the strong inequality (21) by the weak inequality (25).

Proposition 1. *If problem (24)–(27) has a solution, the set $C = \{i \in I : v_i = 1\}$ is an induced cover. Moreover, if the optimal value is smaller than 1, then inequality (19) is satisfied.*

Proof. Let $(\hat{v}, \hat{y})$ be the optimal solution of (24)–(27). With no loss of feasibility we may assume that $\hat{y}_{ij} = \hat{v}_i \wedge \hat{v}_j$. Then (25) takes on the form

$$\sum_{i\in C}\mathbb{P}\{A_i\} - \sum_{\substack{i,j\in C\\ i<j}} \mathbb{P}\{A_i \cap A_j\} \geq \alpha + \epsilon.$$

Recalling the Boole–Bonferroni inequality (23) we conclude that (15) holds, that is, C is an induced cover. By assumption, the value of (24) is smaller than 1, so $\sum_{i\in C}(1-\tilde{z}_i) < 1$ which is identical to (19). □

The Boole–Bonferroni inequality is not sharp, but problem (24)–(27) can be refined by clustering the sets A_i.

Definition 3. *A collection $J_k \subseteq I$, $k \in K$ is called a* proper partition *of I, if*

(i) $\bigcup_{k \in K} J_k = I$;

(ii) $B_k = \bigcap_{i \in J_k} A_i \neq \emptyset$, $k \in K$; *and*

(iii) $B_k \cap A_i = \emptyset$ *for all* $k \in K$, $i \in J_l$, $l \neq k$.

For a proper partition we must have $J_k \cap J_l = \emptyset$ if $k \neq l$. Indeed, suppose that $i \in J_k \cap J_l$. By (ii), $B_k \subseteq A_i$ is nonempty. Since $i \in J_l$, condition (iii) implies $B_k \cap A_i = \emptyset$, a contradiction.

A proper partition can be found by the following greedy algorithm. We find J_1 as the largest subset of I for which $B_1 = \bigcap_{i \in J_1} A_i \neq \emptyset$. Then $B_1 \cap A_j = \emptyset$ for all $j \in I \setminus J_1$. After replacing I by $I \setminus J_1$ we repeat this operation, etc. For the example illustrated in Figure 1, a proper partition of $I = \{1, 2, 3\}$ has only one cluster, $J_1 = I$.

We shall use proper partitions to sharpen the bound provided by the Boole–Bonferroni inequality. Let $k(i)$ be such that $i \in J_{k(i)}$ for all $i \in I$.

Lemma 5. *If J_k, $k \in K$, is a proper partition of I, then*

$$\mathbb{P}\Big\{\bigcup_{i \in I} A_i\Big\} \geq \sum_{k \in K} \mathbb{P}\{B_k\} + \sum_{i \in I} (\mathbb{P}\{A_i\} - \mathbb{P}\{B_{k(i)}\}) \\ - \sum_{k \in K} \sum_{\substack{i,j \in J_k \\ i<j}} \Big(\mathbb{P}\{A_i \cap A_j\} - \mathbb{P}\{B_k\}\Big) - \sum_{\substack{i,j \in I, i<j \\ k(i) \neq k(j)}} \mathbb{P}\{A_i \cap A_j\}. \quad (28)$$

Proof. We have

$$\bigcup_{i \in I} A_i = \bigcup_{k \in K} B_k \cup \bigcup_{i \in I} (A_i \setminus B_{k(i)}).$$

Applying the Boole–Bonferroni inequality to the union on the right hand side and noting that Definition 3(iii) implies

$$\mathbb{P}\{(A_i \setminus B_{k(i)}) \cap (A_j \setminus B_{k(j)})\} = \begin{cases} \mathbb{P}\{A_i \cap A_j\} - \mathbb{P}\{B_{k(i)}\} & \text{if } k(i) = k(j), \\ \mathbb{P}\{A_i \cap A_j\} & \text{if } k(i) \neq k(j), \end{cases}$$

we obtain the required result. □

Inequality (28) is stronger than (23) by the quantity

$$\sum_{k \in K} |J_k|(|J_k| - 2)\mathbb{P}\{B_k\}.$$

For the set $I = \{1, 2, 3\}$ in Figure 1, it provides the perfect estimate $\mathbb{P}\{A_1 \cup A_2 \cup A_3\} \geq 0.35$.

We shall use Lemma 5 to refine problem (24)–(27). Let us denote for brevity, $\mu_i = \mathbb{P}\{A_i\}$, $\mu_{ij} = \mathbb{P}\{A_i \cap A_j\}$, $\rho_k = \mathbb{P}\{B_k\}$ and consider the linear programming problem

$$\min \quad \sum_{i\in I}(1-\tilde{z}_i)v_i \tag{29}$$

$$\text{subject to} \quad \sum_{k\in K}\rho_k\lambda_k + \sum_{i\in I} v_i(\mu_i - \rho_{k(i)})$$

$$-\sum_{k\in K}\sum_{\substack{i,j\in J_k\\ i<j}} y_{ij}\Big(\mu_{ij}-\rho_k\Big) - \sum_{\substack{i,j\in I, i<j\\ k(i)\neq k(j)}} y_{ij}\mu_{ij} \;\geq\; \alpha+\epsilon, \tag{30}$$

$$y_{ij} \geq v_i + v_j - 1, \quad y_{ij} \geq 0, \quad i,j\in I, \quad i<j, \tag{31}$$

$$\lambda_k \leq \sum_{i\in J_k} v_i, \quad \lambda_k \leq 1, \quad k\in K, \tag{32}$$

$$v_i \in \{0,1\}, \quad i\in I. \tag{33}$$

Proposition 2. *If problem (29)–(32) has a solution, then the set $C = \{i \in I : v_i = 1\}$ is an induced cover. Moreover, if the optimal value is smaller than one, then inequality (19) is satisfied.*

Proof. Let us observe that with no loss of feasibility we may set $y_{ij} = v_i \wedge v_j$ and $\lambda_k = \bigvee_{i\in J_k} v_i$. Define

$$\tilde{J}_k = J_k \cap C, \qquad \tilde{K} = \{k\in K : \tilde{J}_k \neq \emptyset\}.$$

The sets $\tilde{J}_k$, $k\in\tilde{K}$, define a proper partition of C. Using Lemma 5 and the inclusion

$$\tilde{B}_k = \bigcap_{i\in\tilde{J}_k} A_i \supseteq B_k, \quad k\in\tilde{K},$$

we conclude that (30) implies (15). The remaining part of the proof is identical with the proof of Proposition 1. □

4. Lifting

Let us now consider the issue of lifting a cover inequality (see [1, 16]). We are not necessarily interested in the optimal lifting, which is known to be a very difficult problem, but rather in a lifting that can be accomplished relatively easy, by linear programming.

Suppose that we have an induced β-cover: a set C such that

$$\sum_{i\in C} z_i \leq \beta \tag{34}$$

is a valid inequality, where $\beta \leq |C|-1$. For a scenario $s\notin C$ we want to find (γ_s, β_s) such that the inequality

$$\sum_{i\in C} z_i + \gamma_s z_s \leq \beta_s \tag{35}$$

is valid for the PCKP. Of course, we want to construct a stronger inequality than (34) so such solutions like $\gamma_s = 1$, $\beta_s = \beta + 1$ are of no interest.

For the example illustrated in Figure 1, where all realizations are equally likely and $\alpha = 0.25$, the set $C = \{1, 2\}$ is an induced cover, because $\mathbb{P}\{A_1 \cup A_2\} = 0.3$. We thus have the valid inequality $z_1 + z_2 \leq 1$. But also $\{1, 3\}$ and $\{2, 3\}$ are induced covers, so we may lift this inequality to $z_1 + z_2 + z_3 \leq 1$. Our aim is to find a lifting algorithm using this idea.

Let us first consider the case when

$$s \notin \bigcup_{i \in C} A_i.$$

We shall search for a lifting in a form of a β-cover inequality, assuming $\beta_s = \beta$ and checking whether we can set $\gamma_s = 1$ in (35). This can be decided by solving the following combinatorial problem

$$\max \quad \sum_{i \in C} v_i \tag{36}$$

$$\text{subject to} \quad \mathbb{P}\Big\{A_s \cup \bigcup_{i\,:\,v_i = 1} A_i\Big\} \leq \alpha, \tag{37}$$

$$v_i \in \{0, 1\}, \quad i \in C. \tag{38}$$

If the optimal value of this problem is smaller than β we can set $\gamma_s = 1$; otherwise $\gamma_s = 0$ (lifting is unsuccessful). After that, we can process the next candidate variable, etc.

Problem (36)–(38) is a difficult combinatorial optimization problem. It was considered in [14] (with a different notation) and proved to be NP-hard. In our setting, in view of a very large number of scenarios, solving it in its pure form appears to be very difficult, especially because it has to be carried out for every candidate variable to be included in the valid inequality.

We shall develop relaxations of problem (36)–(38) which will be easier to solve and which will generate valid liftings, although (possibly) missing some lifting opportunities. To this end, we shall replace the left hand side of (37) by a lower bound which is easier to compute. One way to do it is to use the Boole–Bonferroni inequality (23), as in the previous section. To illustrate another class of bounds that can be used here, and to improve the quality of the bounds, we shall adapt and modify the probability bounding approach based on binomial moments (see [5, 19]).

For random events A_i, $i \in I$, we define p_m to be the probability that exactly m out of $n = |I|$ events happen. The probabilities p_m, $m = 1, \ldots, n$, satisfy the binomial moment equations

$$\sum_{m=r}^{n} \binom{m}{r} p_m = \sum_{i_1 < i_2 < \cdots < i_r} \mathbb{P}\{A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}\}, \quad r = 1, \ldots, n. \tag{39}$$

The probability that at least one of these events happens equals

$$\mathbb{P}\Big\{\bigcup_{i \in I} A_i\Big\} = \sum_{m=1}^{n} p_m. \tag{40}$$

Using these relations, we construct the following linear programming problem

$$\max \quad \sum_{i\in I} v_i \tag{41}$$

$$\text{subject to} \quad \sum_{m=1}^{n} p_m \le \alpha, \tag{42}$$

$$\sum_{m=1}^{n} m p_m = \sum_{i\in I} v_i \mathbb{P}\{A_i\}, \tag{43}$$

$$\sum_{m=2}^{n} \binom{m}{2} p_m = \sum_{i<j} y_{ij} \mathbb{P}\{A_i \cap A_j\}, \tag{44}$$

$$y_{ij} \ge v_i + v_j - 1, \quad 0 \le y_{ij} \le \min(v_i, v_j), \quad i, j \in I, \quad i < j, \tag{45}$$

$$v_i \in \{0, 1\}, \quad i \in I. \tag{46}$$

$$p_m \ge 0, \quad m = 1, \dots, n. \tag{47}$$

Proposition 3. *Let $\bar{\beta}$ be the optimal value of problem (41)–(47). Then $\sum_{i\in I} z_i \le \bar{\beta}$ is a valid inequality.*

Proof. Suppose that the assertion is not true. Then there exists a set $J \subseteq I$ of cardinality $|J| > \bar{\beta}$ such that

$$\mathbb{P}\Big\{\bigcup_{i\in J} A_i\Big\} \le \alpha.$$

Define $v_i = 1$ if $i \in J$, and $y_{ij} = v_i \wedge v_j$. Also, let p_m be the probability that exactly m events out of the collection $A_i, i \in J$, happen. Then (39)–(40) imply that the constraints (42)–(44) are satisfied. The other constraints (45)–(47) are satisfied by construction. Thus $|J| = \sum_{i\in I} v_i \le \bar{\beta}$, a contradiction. □

To lift the cover C in (34) we apply the above result with $I = C \cup \{s\}$ and we enforce $v_s = 1$ (we already have a valid inequality without s). If the optimal value $\bar{\beta}$ does not exceed β, we can add z_s to the inequality; that is, replace C with $C \cup \{s\}$ in (34). For the example of Figure 1 with $C = \{1, 2\}$ and $s = 3$ problem (41)–(47) has the optimal value of 1, so $\{1, 2, 3\}$ is a 1-cover, too.

In (43)–(44) we use only two first binomial moment constraints, rather than all of them, and therefore constraint (42) is a relaxation of (37). We could have included higher order binomial moment constraints to improve the quality of this relaxation, but in the context of stochastic programming it would be highly unrealistic, due to the large number of combinations of events A_i to be considered. Instead of that, we shall try to refine problem (41)–(47) by using the information that is readily available.

First, it is easy to calculate for each A_i the probability

$$\delta_i = \mathbb{P}\Big\{A_i \setminus \bigcup_{j\in I\setminus\{i\}} A_j\Big\}.$$

Then we must have $p_1 \geq \sum_{i\in I} \delta_i v_i$; inequality is needed here because C is a subset of I.

Second, a substantial refinement can be gained by employing clustering. Let, again J_k, $k \in K$, be a proper partition of I. As before, we denote $\mu_i = \mathbb{P}\{A_i\}$, $\mu_{ij} = \mathbb{P}\{A_i \cap A_j\}$, $\rho_k = \mathbb{P}\{B_k\}$. Consider the problem

$$\max \quad \sum_{i\in I} v_i \tag{48}$$

$$\text{subject to} \quad \sum_{m=1}^{n} p_m + \sum_{k\in K} \rho_k \lambda_k \leq \alpha \tag{49}$$

$$\sum_{m=1}^{n} m p_m = \sum_{i\in I} (\mu_i - \rho_{k(i)}) v_i, \tag{50}$$

$$\sum_{m=2}^{n} \binom{m}{2} p_m = \sum_{k\in K} \sum_{\substack{i,j\in J_k \\ i<j}} y_{ij} \Big(\mu_{ij} - \rho_k\Big) + \sum_{\substack{i,j\in I, i<j \\ k(i)\neq k(j)}} y_{ij}\mu_{ij}, \tag{51}$$

$$y_{ij} \geq v_i + v_j - 1, \quad 0 \leq y_{ij} \leq \min(v_i, v_j), \quad i, j \in I, \quad i < j, \tag{52}$$

$$\sum_{i\in J_k} v_i \leq |J_k| \lambda_k, \quad k \in K, \tag{53}$$

$$v_i \in \{0, 1\}, \quad i \in I, \tag{54}$$

$$\lambda_k \in \{0, 1\}, \quad k \in K, \tag{55}$$

$$p_1 \geq \sum_{i\in I} \delta_i v_i, \tag{56}$$

$$p_m \geq 0, \quad m = 2, \ldots, n. \tag{57}$$

The role of inequality (53) is to enforce $\lambda_k = 1$ whenever some of the sets in the cluster J_k are selected, so ρ_k has to be added to the left hand side of (49).

Similarly to Proposition 3, using the observations from the proof of Proposition 2 we obtain the following result.

Proposition 4. *Let $\bar{\beta}$ be the optimal value of problem (48)–(57). Then $\sum_{i\in I} z_i \leq \bar{\beta}$ is a valid inequality.*

Although (48)–(57) appears rather complicated, it is a mixed integer linear programming problem. It is much easier to solve than the ‘compact’ formulation (36)–(38), which involves a nonlinear probabilistic constraint. Furthermore, the number of integer variables in (48)–(57) is equal to the size of the cover I plus the number of clusters in I, and these are typically small numbers, as compared to the total number of scenarios.

Let us now consider lifting with respect to scenarios

$$s \in \bigcup_{i\in C} A_i. \tag{58}$$

The case when C is a minimal induced cover is well studied in [14] and the ideas employed there are readily applicable to our problem. To illustrate them in our context, we can formulate the following result.

Lemma 6. *Let C be an induced cover, J_k, $k \in K$, be a proper partition of C, and let $j_k \in \bigcap_{i \in J_k} A_i$. Then the inequality*

$$\sum_{i \in C} z_i + \sum_{k \in K} (|J_k| - 1)(1 - z_{j_k}) \leq |C| - 1 \tag{59}$$

is a valid inequality for the PCKP.

Proof. The assertion follows from the observation that $z_{j_k} = 0$ implies $z_i = 0$ for all $i \in J_k$. □

Unfortunately, the practical relevance of the cover inequalities lifted with respect to the scenarios s satisfying (58) is rather limited. Indeed, consider the continuous relaxation of problem (9)–(14) (obtained by ignoring (13)) and suppose that $(\tilde{x}, \tilde{z})$ is its optimal solution. Define $V = \{i : \tilde{z}_i > 0\}$. Clearly, we need valid inequalities only if $\sum_{i \in V} p_i > \alpha$; otherwise the current solution is optimal for (9)–(14).

Let $C \subset V$ be an induced cover satisfying the assumptions of lemma 6. If the lifted inequality (59) can be satisfied by setting $z_{j_k} = 1$ for all clusters k, we shall obtain a new optimal solution of the relaxed problem. At this solution, the values of decision variables x, the set V and the objective value are exactly the same as before. On the other hand, if making $z_{j_k} = 1$ does not restore feasibility, the same effect can be obtained from the basic cover inequality (18), to which (59) reduces in this case.

For these reasons we shall not explore the lifting with respect to scenarios satisfying (58).

5. Cut and branch method for probabilistic constraints

Let us now turn to ways of solving the mixed integer formulation (9)–(14) with the application of valid inequalities developed in sections 3 and 4. Define the sets

$$S_0 = \{z \in \mathbb{R}^N : \sum_{i=1}^{N} p_i z_i \leq \alpha,\ z_i \leq z_j \text{ for all } i, j \in \{1, \ldots, N\} \text{ such that } i \preceq j\},$$

$$B_0 = \{z \in \mathbb{R}^N : 0 \leq z_i \leq 1,\ i = 1, \ldots, N\},$$

$$L_0 \subseteq \{1, \ldots, N\}.$$

We shall construct sequences of sets S_k, B_k and L_k, $k = 1, 2, \ldots$, by adding valid inequalities to the definition of S_k, restricting to $\{0, 1\}$ some variables in B_k, and selecting subsets of relevant scenarios to be included into L_k.

Step 0 Set $k = 0$.
Step 1 Solve the master problem

$$\min \quad f(x) \tag{60}$$

$$\text{subject to} \quad g(x^i, \xi^i) + d^i z_i \geq 0, \quad i \in L_k, \tag{61}$$

$$x \in X, \tag{62}$$

$$z \in S_k \cap B_k. \tag{63}$$

Let $(\hat{x}^k, \hat{z}^k)$ denote the solutions found, with scenario solutios $(\hat{x}^{ki}, \hat{z}_i^k), i = 1, \ldots, N$.

Step 2 Define the sets

$$H_k = \{i \in \{1, \dots, N\} : g(\hat{x}^{ki}, \xi^i) \geq 0\},$$
$$I_k = \{1, \dots, N\} \setminus H_k.$$

If $\sum_{i \in I_k} p_i \leq \alpha$ then stop; otherwise continue.

Step 3 Find an induced cover $C_k \subseteq \mathcal{M}(I_k)$ (recall that $\mathcal{M}(I_k)$ is the set of minimal elements in I_k).

Step 4 For each $s \in \mathcal{M}(I_k) \setminus C_k$ lift the cover C_k to obtain a $|C_k|$-cover $\hat{C}_k \subseteq \mathcal{M}(I_k)$.

Step 5 Set

$$S_{k+1} = S_k \cap \Big\{ z \in \mathbb{R}^N : \sum_{i \in \hat{C}_k} z_i \leq |C_k| - 1 \Big\}.$$

Step 6 If $\mathcal{M}(I_k) \subseteq L_k$ and $\hat{z}^k \in S_{k+1}$, choose $b_k \in \mathcal{M}(I_k)$ such that $z^k_{b_k} \in (0, 1)$ and set $B_{k+1} = \{z \in B_k : z_{b_k} \in \{0, 1\}\}$; otherwise set $B_{k+1} = B_k$.

Step 7 Choose $L_{k+1} \supseteq L_k \cup \mathcal{M}(I_k)$ increase k by one and go to Step 1.

Theorem 1. *After finitely many iterations the algorithm stops at a point $(\hat{x}^k, \hat{z}^k)$ such that $\hat{x}^k$ is optimal for (3).*

Proof. Let us show that if the algorithm does not stop at iteration k, Steps 3–6 can be executed. Since $\sum_{i \in I_k} p_i > \alpha$, the set $\mathcal{M}(I_k)$ is an induced cover, so Step 3 can be carried out. The induced cover C_k is a legitimate outcome of Step 4, too. Step 5 defines a nonempty set S_{k+1}, because it always contains 0. It remains to analyze Step 6.

Suppose that $\mathcal{M}(I_k) \subseteq L_k$. By (61), $\hat{z}^k_i > 0$ for all $i \in \mathcal{M}(I_k)$. Then, by the definition of S_0, $\hat{z}^k_i > 0$ for all $i \in I_k$. If a fractional component $\hat{z}^k_{b_k}$ cannot be found, we must have $\hat{z}^k_i = 1$ for all $i \in I_k$. But then $\hat{z}^k$ violates the cover inequality $\sum_{i \in \hat{C}_k} z_i \leq |C_k| - 1$, so $\hat{z}^k \notin S_{k+1}$. Consequently, if $\hat{z}^k \in S_{k+1}$, a fractional coordinate $\hat{z}^k_{b_k}$ exists.

The above argument shows that the algorithm is well defined. If it does not stop, then $S_{k+1} \subseteq S_k$, $B_{k+1} \subseteq B_k$, and $L_{k+1} \supseteq L_k$, and at least one of these inclusions is strict.

There are finitely many covers possible, so finitely many different sets S_k may occur. The number of possible sets B_k and L_k is finite, too. Therefore, the algorithm must stop at Step 2 at some iteration k^*.

Problem (60)–(63) is a relaxation of (9)–(14). By setting $z_i = 1$ if $\hat{z}^{k^*}_i > 0$, and $z_i = 0$ otherwise, we can satisfy all constraints of (9)–(14) without changing the objective value. Therefore the solution $\hat{x}^{k^*}$ is optimal for (3). □

6. Numerical illustration

Let us consider a stochastic multicommodity network flow problem with the node set $\mathcal{V}$ and arc set $\mathcal{A} \subset \mathcal{V} \times \mathcal{V}$. For each pair of nodes $(k, l) \in \mathcal{V} \times \mathcal{V}$ there is a random quantity d_{kl} to be shipped from k to l. Our objective is to find arc capacities $x(a)$, $a \in \mathcal{A}$, such that the network can carry the flows with a sufficiently large probability $1 - \alpha$ and the capacity expansion cost $\langle c, x \rangle$ is minimized.

Denote the demand scenarios by d^i_{kl}, $i = 1, \dots, N$, and their probabilities by p_i. Introducing the variables $y^i_{kl}(a)$ to denote the flow from k to l passing arc a in scenario i, we can formulate the problem as follows:

$$\min \quad \sum_{a\in\mathcal{A}} c(a)x(a) \tag{64}$$

$$\text{subject to} \quad \sum_{a\in\mathcal{A}^+(\nu)} y^i_{kl}(a) - \sum_{a\in\mathcal{A}^-(\nu)} y^i_{kl}(a) = \begin{cases} -d^i_{kl} & \text{if } \nu = k \\ d^i_{kl} & \text{if } \nu = l \\ 0 & \text{otherwise,} \end{cases} \tag{65}$$

$$\nu, k, l \in \mathcal{V}, \quad i = 1, \dots, N,$$

$$\sum_{i=1}^{N} p_i\, \chi\Big(x - \sum_{k,l\in\mathcal{V}} y^i_{kl}\Big) \geq 1 - \alpha, \tag{66}$$

$$x \geq 0, \quad y \geq 0. \tag{67}$$

In the flow balance equations (65) we use $\mathcal{A}^-(\nu)$ and $\mathcal{A}^+(\nu)$ to denote the sets of arcs going out of node ν and coming into node ν, respectively.

As an illustration, consider the network shown in Figure 2. We assume that the demand is symmetric, that is, $d_{kl} = d_{lk}$ for all pairs (k, l). For $k < l$ we set:

$$d_{kl} = 0.1D + \xi_{kl},$$

where D (the total traffic) has a normal distribution with the expected value 30 and standard deviation 5, and ξ_{kl} are independent normal variables with zero expectation and standard deviation 0.25.

The expansion costs are symmetric, too. Table 1 gives their values for $k < l$.

Our original formulation, with normal distributions, is extremely difficult to solve. Therefore, we used sample-based optimization. To find an approximate solution of the problem we randomly generated a sample of size N from the demand distribution. These were our scenarios. Two versions of the problem have been solved: with 100 and with 200 scenarios. In both cases we set $\alpha = 0.1$. These problems are not easy from the point of view of mixed integer programming; for example, the 200-scenario version has 28000

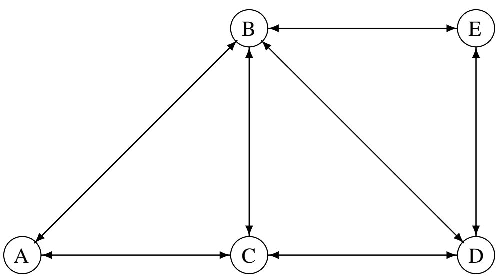

Fig. 2. The graph of the stochastic multicommodity network flow example

Table 1. Expansion costs

From	To	Cost
A	B	310
A	C	230
B	C	250
B	D	180
B	E	350
C	D	400
D	E	270

continuous variables, 200 binary variables, and 20001 constraints. They are already too difficult for the standard MIP solver CPLEX. We have to admit here that the choice of the number of scenarios incorporated into the model was fairly arbitrary here. The statistical analysis of the approximation error involved is far beyond the scope of this paper.

The analysis of the sets A_i for the purpose of cover generation, clustering and lifting has been implemented in a rather straightforward way as follows. By carrying out pairwise comparisons of scenarios we defined a binary matrix Q with

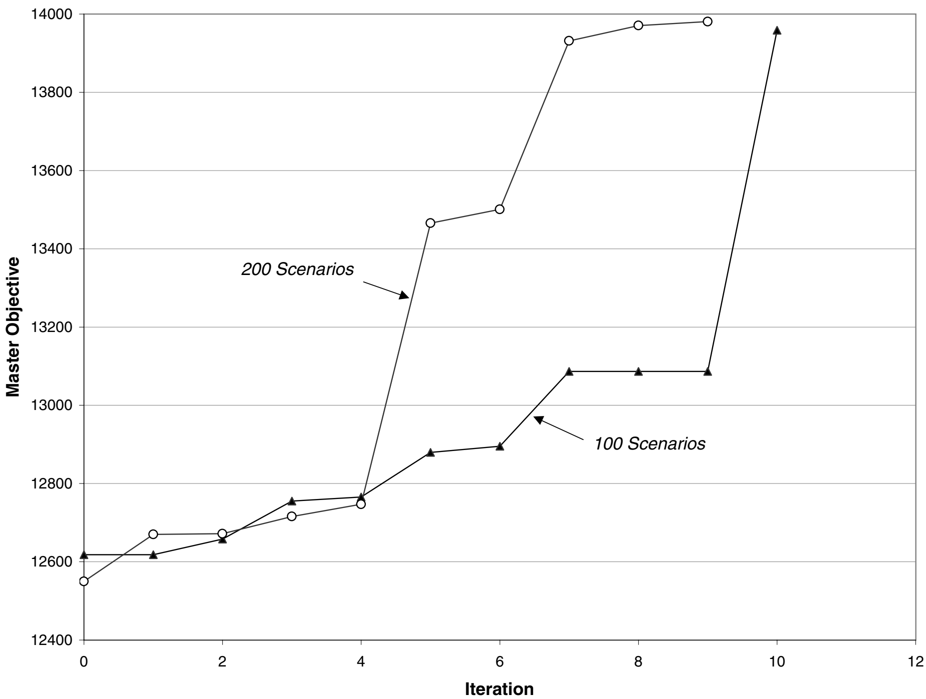

Fig. 3. The objective value of the master problem

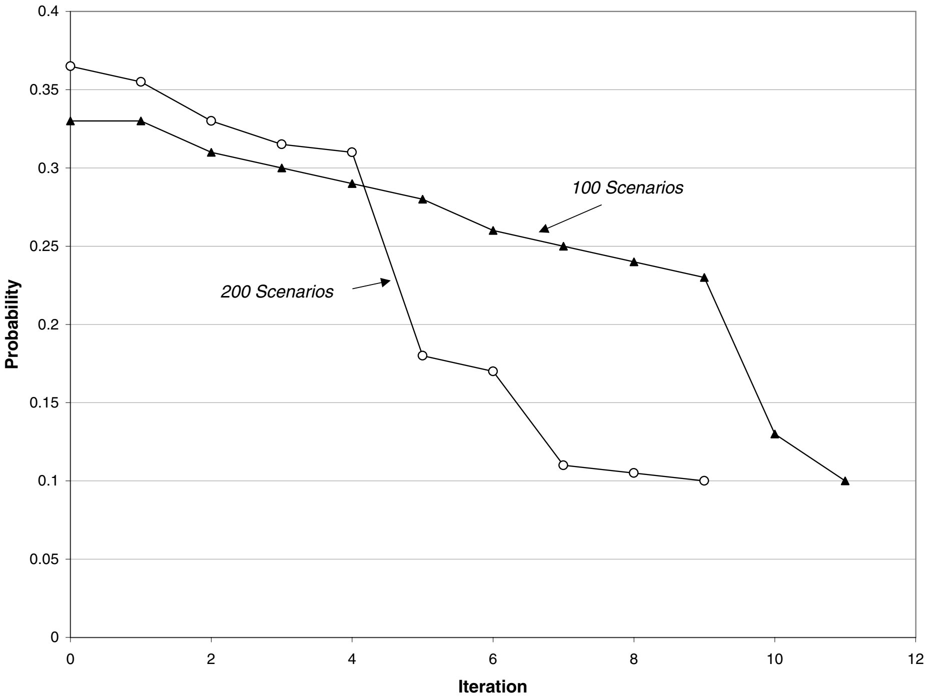

Fig. 4. The probability that no feasible flow exists

$$Q_{ij} = \begin{cases} 1 & \text{if } i \preceq j, \\ 0 & \text{otherwise.} \end{cases}$$

No more than $N(N-1)$ such comparisons are necessary. In fact, the required number is smaller due to the transitivity of the order: if $i \preceq j$ and A_j is already known, then $i \preceq k$ for all $k \in A_j$. Once the matrix Q has been calculated, all other operations are straightforward. For example, $\mu_{ij} = \mathbb{P}\{A_i \cap A_j\}$ is the scalar product of the i-th and the j-th row of Q. Clearly, for a larger problem a more advanced implementation, based on the representation of the partial order by an acyclic graph, would be advisable.

We have implemented the cut and branch method of Section 5 in the modeling language AMPL [11]. CPLEX was used as the MIP solver for the master problem at Step 1. It had much fewer binary variables than the full formulation, and could be solved rather effectively.

Figure 3 shows the master objective value in successive iterations for both cases. In Figure 4 we give the probability that the demand cannot be carried by the capacities equal to the current master's solution. Finally, Figure 5 shows the number of variables that are restricted to be binary at the current master's solution.

We see that the method converges rapidly in this example, and the number of binary variables remains moderate. This is due to the fact that the method tries to identify the

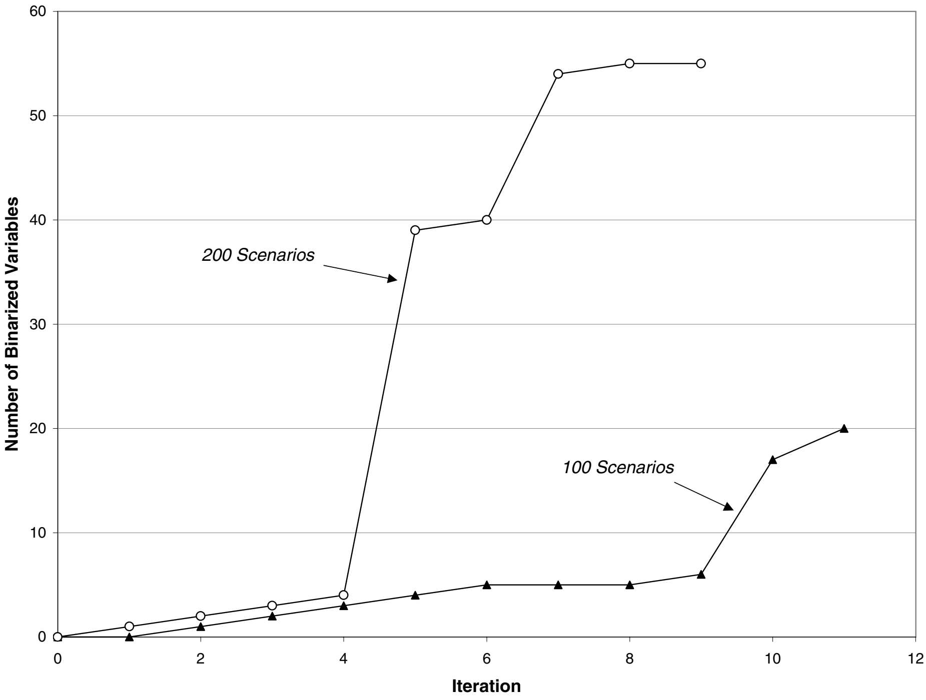

Fig. 5. The number of variables which are restricted to be binary

Table 2. Optimal arc capacities

From	To	Capacity 100 Scenarios	Capacity 200 Scenarios
1	2	11.25	10.86
1	3	3.63	3.89
2	3	7.37	7.11
2	4	7.54	7.22
2	5	11.08	10.53
3	4	3.76	3.97
4	5	3.84	4.35

key scenarios which are located on the boundary of the set of manageable demand realizations. It is worth mentioning that our lifting procedure generated 8 successful liftings in the 100 scenario example, and 10 successful liftings in the 200 scenario example.

The solutions obtained are similar, as can be seen from Table 2 (by symmetry, we give only the capacities $x(i, j)$ for $i < j$).

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