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Multistars, partial multistars and the capacitated vehicle routing problem

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Abstract. In an unpublished paper, Araque, Hall and Magnanti considered polyhedra associated with the Capacitated Vehicle Routing Problem (CVRP) in the special case of *unit demands*. Among the valid and facet-inducing inequalities presented in that paper were the so-called *multistar* and *partial multistar* inequalities, each of which came in several versions. Some related inequalities for the case of *general* demands have appeared subsequently and the result is a rather bewildering array of apparently different classes of inequalities.

The main goal of the present paper is to present two relatively simple procedures that can be used to show the validity of all known (and some new) multistar and partial multistar inequalities, in both the unit and general demand cases. The procedures provide a unifying explanation of the inequalities and, perhaps more importantly, ideas that can be exploited in a cutting plane algorithm for the CVRP.

Computational results show that the new inequalities can be useful as cutting planes for certain CVRP instances.

Key words. vehicle routing – valid inequalities – cutting planes

1. Introduction

This paper is concerned with the well-known (and $\mathcal{NP}$-hard) *Capacitated Vehicle Routing Problem* (CVRP), which can be formally defined as follows [5, 8, 14, 18, 23]. A complete undirected graph $G = (V, E)$ is given, with $V = \{0, \ldots, n\}$. Vertex $\{0\}$ represents the depot, the other vertices represent customers. The cost of travel between vertices i and j is denoted by c_{ij}, and we assume that costs are symmetric, i.e., that $c_{ij} = c_{ji}$. An unlimited fleet of identical vehicles, each of capacity $Q > 0$, is available. Each customer i has an integer demand q_i, with $0 < q_i \leq Q$. Each customer must be served by a single vehicle and no vehicle can serve a set of customers whose demand exceeds its capacity. The task is to find a set of vehicle routes of minimum cost, where each vehicle used leaves from and returns to the depot.

The most successful algorithms to date for solving the CVRP (or minor variations of it) are based on the *two-index* integer programming formulation [3, 5, 13, 15, 23]. In this formulation, x_{ij} represents the number of times a vehicle travels between vertices i and j. (Because the problem is undirected, x_{ij} and x_{ji} represent the same variable.)

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In order to present this formulation, we will need some further definitions and notation. Given a set of customers $S \subseteq V \setminus \{0\}$, $q(S)$ will denote $\sum_{i \in S} q_i$, $\delta(S)$ will denote the set of edges in G with exactly one end-vertex in S, $E(S)$ will denote the set of edges in G with both end-vertices in S and $k(S)$ will denote the minimum number of vehicles required to serve the customers in S. Note that calculating $k(S)$ exactly is an ($\mathcal{NP}$-Hard) Bin Packing Problem. Finally, given an arbitrary $F \subseteq E$, $x(F)$ will denote $\sum_{e \in F} x_e$. The integer programming formulation is then:

Minimize $\sum_{e \in E} c_e x_e$

Subject to:

$$x(\delta(\{i\})) = 2 \qquad (i = 1, \ldots, n) \tag{1}$$

$$x(\delta(S)) \geq 2k(S) \qquad (S \subseteq \{1, \ldots, n\}, |S| \geq 2) \tag{2}$$

$$x_{ij} \in \{0, 1\} \qquad (1 \leq i < j \leq n) \tag{3}$$

$$x_{ij} \in \{0, 1, 2\} \qquad (i = 0, j = 1, \ldots, n). \tag{4}$$

The *degree equations* (1) ensure that customers are visited exactly once. The *capacity inequalities* (2) impose the vehicle capacity restrictions and also ensure that the routes are connected. Note that the formulation remains valid if one replaces $k(S)$ on the right-hand side with the obvious lower bound $\lceil q(S)/Q \rceil$. Note also that the capacity inequalities can be re-written using the degree equations to take the form

$$x(E(S)) \leq |S| - k(S). \tag{5}$$

Finally, constraints (3) and (4) are the integrality conditions. Note that the x_{ij} are permitted to take the value 2 when $i = 0$, to allow routes in which a vehicle serves a single customer.

Some other variants of the CVRP can be easily incorporated into this framework. If there is an upper bound K on the number of vehicles to be used, then the inequality $x(\delta(\{0\})) \leq 2K$ can be added. If hiring a vehicle incurs a cost C, then $C/2$ can be added to the objective function coefficient of each edge incident on the depot. If *exactly* K vehicles must be used, then the equation $x(\delta(\{0\})) = 2K$ can be added. Finally, if routes containing only one customer are forbidden, then all variables can be made binary. The results in this paper remain valid for these other variants.

We are interested in the integer polytope associated with the above formulation, viz., the convex hull in $\mathbb{R}^{|E|}$ of incidence vectors x satisfying (1)–(4) (see [25]). Several papers in the literature present valid inequalities for this polytope or related polyhedra [1, 5, 11, 13, 15, 19]. Successful optimization algorithms based upon valid inequalities can be found in [5, 13, 18, 23]. In [2–4, 10], attention is given to the *unit demand* case, i.e., the special case where $q_i = 1$ for $i = 1, \ldots, n$. We will denote this special case by CVRPUD.

This paper is concerned with valid inequalities known as *multistar* and *partial multistar* inequalities, which we define formally in Section 2. These inequalities first appeared in [3], in the context of the CVRPUD. In that paper, three distinct classes of multistar inequalities and four distinct classes of partial multistar inequalities were defined, and conditions were given for the resulting inequalities to induce facets.

In more recent papers [1, 13, 15], some more classes of inequalities of multistar type have appeared for the *general* demand case. The result is a rather bewildering array of apparently different classes of inequalities.

The main goal of the present paper is to present two relatively simple procedures that can be used to show the validity of all known (and some new) multistar and partial multistar inequalities, in both the unit and general demand cases. The procedures provide a unifying explanation of the inequalities and, perhaps more importantly, ideas that can be exploited in a cutting plane algorithm for the CVRP.

The outline of the remainder of the paper is as follows. In Section 2, the terms *multistar* and *partial multistar* are defined and previous work on multistar and partial multistar inequalities is reviewed. An important distinction will be made between what we call *homogeneous* and *inhomogeneous* inequalities. In Section 3, the homogeneous case is examined in detail. A simple procedure is given that can be used to generate all known homogeneous inequalities in the literature, along with some new ones. In Section 4, the inhomogeneous case is examined. A connection between the CVRP and the *knapsack problem* is used to define new classes of inhomogeneous multistar inequalities that generalize all previously known ones. In Section 5, we describe a cutting plane algorithm based on capacity and (partial) multistar inequalities and give some computational results. Concluding comments are made in Section 6.

2. Previous work

The seminal paper on multistar and partial multistar inequalities is that of Araque, Hall & Magnanti [3]. In our opinion, this is an excellent paper, yet to date it remains unpublished. For this reason, we begin this section by reviewing [3] in some detail.

2.1. The work of Araque, Hall and Magnanti

Given a non-empty set $S \subseteq V \setminus \{0\}$ of customer vertices, let $\bar{S}$ denote the 'complementary' set of customer vertices $V \setminus (S \cup \{0\})$. Also, given two disjoint vertex sets S_1, S_2, let $E(S_1 : S_2)$ denote the set of edges 'crossing' from S_1 to S_2. That is, $E(S_1 : S_2) = \delta(S_1) \cap \delta(S_2)$.

A *multistar* is a subgraph of G with two sets of vertices, a *nucleus* $N \subset V \setminus \{0\}$ and a set of *satellites* $S \subseteq \bar{N}$. The multistar contains the edge-set $E(N) \cup E(N : S)$. Figure 1 shows a multistar with $|N| = |S| = 3$.

Araque, Hall & Magnanti [3] call any valid inequality of the form:

$$\lambda x(E(N)) + x(E(N : S)) \leq \mu, \ (N \subset V \setminus \{0\}, S \subseteq \bar{N}) \tag{6}$$

a *multistar inequality*. Here, λ and μ are constants that depend on N and S. Moreover, there may be more than one such inequality even for a given N and S. Note that the edges whose variables have a non-zero coefficient in the inequality induce a multistar in G.

It is sometimes more useful to write multistar inequalities in a slightly different form:

$$x(\delta(N)) \geq \rho + \sigma x(E(N : S)), \tag{7}$$

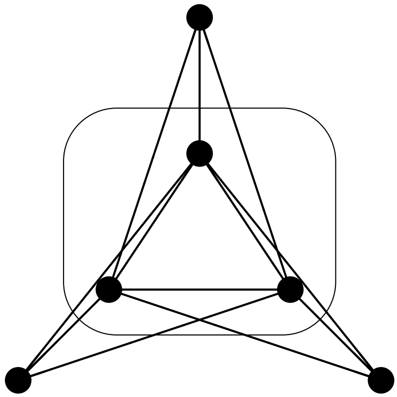

Fig. 1. A Multistar

where, again, ρ and σ are constants that depend on N and S. These two forms are easily shown to be equivalent using the degree equations.

In [3], three classes of multistar inequalities are given for the CVRPUD. The *large multistar* (LM) inequalities, valid for all N, are:

$$Qx(E(N)) + x(E(N:\bar{N})) \leq (Q-1)|N|. \tag{8}$$

The *intermediate multistar* (IM) inequalities are valid only for sets N satisfying certain conditions. For a given N, define $b = 2 + |N| \bmod (Q-2)$. Then, provided that $3 \leq b < 2\lceil |N|/(Q-2) \rceil$, the IM inequality is valid and takes the form

$$bx(E(N)) + x(E(N:\bar{N})) \leq b|N| - (b-2)\lceil |N|/(Q-2) \rceil. \tag{9}$$

Finally, the *small multistar* (SM) inequalities are valid for sets N and S satisfying certain conditions. For a given N and a given $S \subseteq \bar{N}$, define $d = |N \cup S| \bmod Q$. Then, provided that $|N \cup S| > Q$ and $2 \leq d < |S|$, the SM inequality is valid and takes the form:

$$dx(E(N)) + x(E(N:S)) \leq d(|N| - k(N \cup S)) + |S|. \tag{10}$$

The proofs of validity for the LM, IM and SM inequalities, and the conditions under which they induce facets of the associated integer polytope, are subtle and rather complex [3]. We will not go into details here. The important point is that a fractional vector x^* that satisfies all degree equations and capacity inequalities may violate a multistar inequality. Figure 2 gives an example of such a fractional point for a CVRPUD instance with $Q = 4$ and $n = 6$. The solid lines indicate edges whose variables have value 1, the dotted lines indicate edges whose variables have value 1/2. The depot and the edges incident on the depot have been omitted for clarity. Letting N be the three central vertices and S be the three surrounding vertices, we obtain the SM inequality $2x(E(N)) + x(E(N:S)) \leq 5$. This is violated because the left hand side is currently $(2 \times 2) + \frac{3}{2} = 5.5$. Similar examples can be constructed for the LM and IM inequalities.

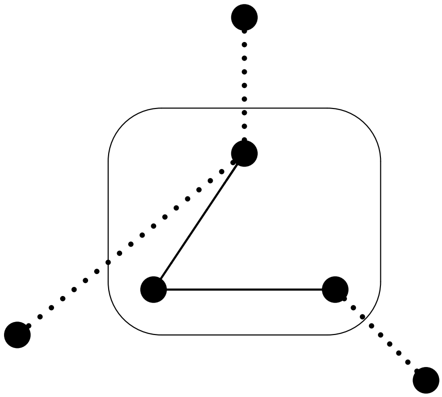

Fig. 2. Fractional point violating an SM inequality

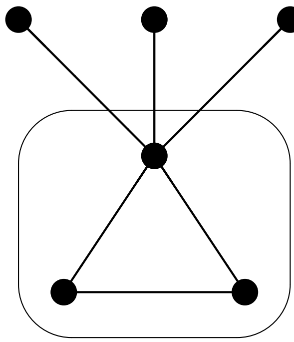

Fig. 3. A Partial Multistar

Araque, Hall & Magnanti also introduced the *partial* multistar inequalities. A *partial* multistar is like a multistar except that only some of the nucleus vertices (called *connector* vertices) are connected to the satellites. Figure 3 shows a partial multistar with $|N| = |S| = 3$ and one connector vertex. A general partial multistar inequality will take the form:

$$\lambda x(E(N)) + x(E(C : S)) \leq \mu, \tag{11}$$

where $C \subset N$ is the set of connector vertices and, again, λ and μ are constants that depend upon N, S, C and the type of partial multistar.

The partial multistar inequalities come in four 'flavours' as follows. Inequalities of the first kind are defined only when $Q \geq 3$, $|N|$ is a multiple of Q and $|C| = 1$. They take the form:

$$2x(E(N)) + x(E(C : \bar{N})) \leq 2(|N| - k(N)). \tag{12}$$

Inequalities of the second kind are defined only when $Q \geq 4$, $|N| \bmod Q = 1$ and $|C| = 2$. They take the form:

$$2x(E(N)) + x(E(C : \bar{N})) \leq 2(|N| - k(N) + 1). \tag{13}$$

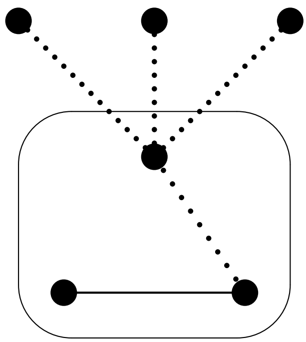

Fig. 4. Fractional point violating a partial multistar inequality

Inequalities of the third and fourth kinds are defined only when $Q \geq 4$, $|N|$ is a multiple of Q and $|C| = 3$. They take the form:

$$3x(E(N)) + x(E(C : \bar{N})) \leq 3(|N| - k(N)) \tag{14}$$

and

$$2x(E(N)) + x(E(C : \bar{N})) \leq 2(|N| - k(N)) + 1. \tag{15}$$

respectively.

Figure 4 shows a fractional point for a CVRPUD instance with $Q = 3$ and $n = 6$. Again, the bold line indicates a variable with value 1 and the dotted lines indicate variables with value 1/2, and edges incident on the depot have been omitted. Letting N be the three vertices at the bottom, $S = \bar{N}$ be the three vertices at the top, and C be the single vertex in the nucleus that is x^*-adjacent to the satellites, we obtain a partial multistar inequality of the first kind, $2x(E(N)) + x(E(C : \bar{N})) \leq 4$. This is violated because the left hand side is currently $(2 \times \frac{3}{2}) + \frac{3}{2} = 4.5$. Similar examples can be constructed for the other kinds of partial multistar inequalities.

2.2. *Subsequent work on multistar inequalities*

The multistar and partial multistar inequalities were originally defined in the context of the CVRPUD. A natural question is whether they have a simple generalization to the general CVRP. To our knowledge, no one has attempted to generalize the partial multistar inequalities in this way. (We do this in Section 3.) However, several classes of multistar inequalities have appeared in the literature for the general CVRP.

The LM inequalities have a natural generalization for the general demand case, namely, the inequalities

$$Qx(E(S)) + \sum_{j \in \bar{S}} q_j x(E(S : \{j\})) \leq |S| - q(S)/Q \qquad (\forall S \subset V \setminus \{0\} : |S| \geq 2), \tag{16}$$

or, equivalently,

$$x(\delta(S)) \geq \frac{2}{Q}\left(q(S) + \sum_{j \in \bar{S}} q_j x(E(S : \{j\}))\right) \qquad (\forall S \subset V \setminus \{0\} : |S| \geq 2). \quad (17)$$

We will call these *generalized large multistar* (GLM) inequalities. Note that the GLM inequalities are rather different from the other inequalities mentioned so far, in that the coefficients of the edges in $E(N : S)$ vary depending on the satellite vertex involved. We will say that such inequalities are *inhomogeneous*.

The GLM inequalities have been discovered by several authors independently [1, 13, 15], and the proof of validity is fairly straightforward. The authors of [1] attempted to generalize the IM and SM inequalities in a similar way. However, this was not entirely successful and, for the sake of brevity, we do not go into details here.

Some *homogeneous* multistar inequalities for the general demand case are also presented in [1]. For a given N and $S \subseteq \bar{N}$ such that $k(N \cup S) > k(N)$, let $1 \leq p \leq |S|$ be the smallest integer such that $k(N \cup W) > k(N)$ for all subsets $W \subseteq S$ with $|W| = p$. Then the inequality

$$(|S| - p + 1)x(E(N)) + x(E(N : S)) \leq (|S| - p + 1)(|N| - k(N)) + (p - 1) \quad (18)$$

is valid. Moreover, if $p - 1 \leq 2k(N)$, then the inequality

$$(2k(N) - p + 3)x(E(N)) - x(E(N : S)) \leq (2k(N) - p + 3)(|N| - k(N)) + (p - 1) \quad (19)$$

is also valid and dominates (18) when $|S| \geq 2k(N) + 3$.

The inequalities (18) and (19) collectively dominate and generalize some inequalities introduced in [13]. However, we wish to point out that, in the unit demand case, (18) and (19) are themselves dominated by the LM, IM and SM inequalities. This is summarized in the following two propositions, which we state without proof for the sake of brevity.

Proposition 1. *In the case of unit demands, the inequality (18) is:*

- *dominated by the capacity inequality on N and the LM inequality on N if $p \leq |S| - Q + 1$,*
- *dominated by the SM inequality on N and S if $|S| - Q + 1 < p \leq |S| - 1$,*
- *dominated by the capacity inequality on $N \cup S$ if $p = |S|$.*

Proposition 2. *In the case of unit demands, the inequality (19) is:*

- *dominated by the capacity inequality on N and the LM inequality on N if $p \leq 2k(N) - Q + 3$,*
- *dominated by the IM inequality on N if $2k(N) - Q + 3 < p \leq 2k(N) + 1$.*

3. Generating homogeneous inequalities

In this section, we present a procedure for deriving *homogeneous* multistar and partial multistar inequalities for the CVRP. The procedure is fairly simple and intuitive, yet, as we will show, it yields all of the homogeneous inequalities mentioned in Section 2, and some new ones besides. It also provides ideas that can be used in a cutting plane algorithm, as shown in Section 5.

For brevity, from now on we will view a multistar inequality as a special kind of partial multistar inequality; namely, as one in which $C = N$.

For a fixed N, S and C, the set of all valid homogeneous partial multistar inequalities could in theory be found by projecting the CVRP polyhedron onto a planar subspace having $x(E(N))$ and $x(E(C:S))$ as axes. Of course, computing this projection is $\mathcal{NP}$-hard. Fortunately, however, it is not difficult to compute a good *approximation* to this projection. This is the main task of this section.

We begin by computing an upper bound on the value of $x(E(C:S))$.

Lemma 1. *All feasible CVRP solutions satisfy:*

$$x(E(C:S)) \leq \min\{2|C|, 2|S|, |C| + |S| - k(C \cup S)\}. \tag{20}$$

Proof. The degree equations imply $x(E(C:S)) \leq \min\{2|C|, 2|S|\}$ and the capacity inequality for $C \cup S$ (in the form (5)) implies that $x(E(C:S)) \leq |C| + |S| - k(C \cup S)$.

For CVRPUD instances in which Q is even, it is possible to produce a slightly stronger upper bound:

Lemma 2. *All feasible CVRPUD solutions with Q even satisfy:*

$$x(E(C:S)) \leq \min\left\{2|C|, 2|S|, |C| + |S| - \left\lceil \frac{2\max\{|C|, |S|\}}{Q} \right\rceil\right\}. \tag{21}$$

Proof. In any feasible CVRPUD solution x^*, consider the graph with vertex set $C \cup S$ and an edge for each $e \in E(C:S)$ with $x^*_e = 1$. This graph is a simple kind of forest in which each connected component is either a path or a single vertex. Each connected component contains at most $Q/2$ vertices in C and at most $Q/2$ vertices in S. Therefore the number of connected components is at least $\left\lceil \frac{2\max\{|C|,|S|\}}{Q} \right\rceil$. The result follows from the well-known fact that a forest with p nodes and q connected components contains $p - q$ edges.

Let us denote by UB_{CS} the upper bound resulting from an application of these two lemmas. Our next step will be to compute, for $\alpha = 0, 1, \ldots, UB_{CS}$, an upper bound $UB(\alpha)$ on the value of $x(E(N))$ for any feasible CVRP solution with $x(E(C:S)) = \alpha$.

Lemma 3. *All feasible CVRP solutions with $x(E(C:S)) = \alpha$ satisfy*

$$x(E(N)) \leq |N| - \lceil \alpha/2 \rceil. \tag{22}$$

Proof. Obviously, $x(\delta(N)) \geq x(E(C:S))$. Since $x(\delta(N))$ must be even, we have $x(\delta(N)) \geq 2\lceil x(E(C:S))/2 \rceil$. The result then follows from the degree equations on N.

Lemma 4. *Sort the vertices in S in order of non-decreasing demand and let s_j for $j = 1, \ldots, |S|$ be the index of the jth vertex in the sorted list. Then the following are valid upper bounds on $x(E(N))$ for various values of $\alpha = x(E(C:S))$:*

- *When $\alpha = 0$, $x(E(N)) \leq |N| - k(N)$,*
- *When $1 \leq \alpha \leq |S|$, $x(E(N)) \leq |N| - k(N \cup \{s_1, \ldots, s_\alpha\})$,*
- *When $\alpha > |S|$, $x(E(N)) \leq |N \cup S| - k(N \cup S) - \alpha$.*

Proof. The first inequality is the capacity inequality on N (in the form (5)) and the third inequality follows from the capacity inequality on $N \cup S$. To show the validity of the second inequality, consider a feasible CVRP solution with $1 \leq x(E(C : S)) \leq |S|$. For $i = 1, 2$, let $S^i := \{v \in S : x(E(C : \{v\})) = i$ in the feasible solution$\}$. Then, we have that

$$\begin{aligned} x(E(N)) &\leq x(E(N \cup S^1 \cup S^2)) - |S^1| - 2|S^2| \\ &\leq |N \cup S^1 \cup S^2| - k(N \cup S^1 \cup S^2) - |S^1| - 2|S^2|, \\ &= |N| - k(N \cup S^1 \cup S^2) - |S^2|, \\ &\leq |N| - k(N \cup s_1 \cup \cdots \cup s_{|S^1|+2|S^2|}), \\ &\leq |N| - k(N \cup \{s_1, \ldots, s_\alpha\}), \end{aligned}$$

where the second inequality follows from the capacity inequality on $N \cup S^1 \cup S^2$ (in the form (5)).

When $|S| > |C|$, the following upper bounds also may be useful.

Lemma 5. *Sort the vertices in S in order of non-decreasing demand and let s_j for $j = 1, \ldots, |S|$ be the index of the jth vertex in the sorted list. Similarly, let c_j for $j = 1, \ldots, |C|$ be the index of the jth smallest vertex in C. Then the following is also a valid upper bound on $x(E(N))$ for a fixed value of $\alpha = x(E(C : S))$ when $|C| \leq \alpha \leq 2|C|$:*

$$x(E(N)) \leq |N| - \alpha + |C| - k((N \setminus C) \cup \{c_1, \ldots, c_{2|C|-\alpha}\} \cup \{s_1, \ldots, s_{2|C|-\alpha}\}). \quad (23)$$

Proof. By contradiction. Suppose that (23) is violated by a feasible CVRP solution vector x^* for a given N, C, S and $\alpha = x^*(C : S)$. If $\alpha > |C|$, then there must be at least one vertex $j \in C$ such that $x^*(E(\{j\} : S)) = 2$. In this case, we can obtain another inequality of the form (23), which is violated by at least as much, by:

- Setting $N := N \setminus \{j\}$,
- Setting $C := C \setminus \{j\}$ and
- Reducing α by two.

To see this, note that $x^*(E(N))$, $|N| - \alpha + |C|$ and $N \setminus C$ all remain unchanged. Moreover, $2|C| - \alpha$, and therefore $\{s_1, \ldots, s_{2|C|-\alpha}\}$, remain unchanged also. The only possible change is that one of the demands in $\{c_1, \ldots, c_{2|C|-\alpha}\}$ has *increased.*

Each time this argument is applied, $|C|$ drops by one and α drops by two. Eventually we cannot repeat the procedure because $\alpha = |C|$. But the inequality (23) cannot be violated when $\alpha = |C|$, because it reduces to

$$x(E(N)) \leq |N| - k(N \cup \{s_1, \ldots, s_\alpha\}), \quad (24)$$

which is valid by Lemma 4.

These lemmas provide a way of generating homogeneous partial multistar inequalities for a fixed N, C and S. For $\alpha = 0, 1, \ldots, UB_{CS}$, compute an upper bound $UB(\alpha)$ on the value of $x(E(N))$ given that $x(E(C : S)) = \alpha$. Once these have been calculated, the desired inequalities can easily be found in $\mathcal{O}(n \log n)$ time by computing the convex hull of a polygon in $(\alpha, UB(\alpha))$-space.

Figure 5 illustrates the procedure for the first example given in Section 2 (shown in Figures 1 and 2). Because $|N| = |S| = 3$ and $k(N \cup S) = 2$, we have $UB(N : S) = 4$. Apart from the trivial inequalities $x(E(N : S)) \geq 0$ and $x(E(N)) \geq 0$, we obtain three other inequalities: $x(E(N)) \leq 2$, which is the capacity inequality on N; $2x(E(N) + x(E(N : S)) \leq 5$, which is the SM inequality mentioned in Section 2; and $x(E(N) + x(E(N : S)) \leq 4$, which is redundant, being dominated by the capacity inequality on $N \cup S$.

Figure 6 illustrates the procedure for the second example given in Section 2 (shown in Figures 3 and 4). Because $|C| = 1$ and $|S| = 3$ we have $UB(C : S) = 2$. Apart from the trivial inequalities, we obtain one inequality, $2x(E(N)) + x(E(C : S)) \leq 4$, which is the partial multistar inequality of the first kind mentioned in Section 2.

Let us call the entire procedure for generating homogeneous multistar and partial multistar inequalities the *polygon* procedure. Providing necessary and sufficient conditions for the polygon procedure to yield facets of the CVRP polytope seems to be an extremely difficult task, even for the CVRPUD. However, it turns out that the procedure generates all known homogeneous multistar and partial multistar inequalities, in both the general demand and unit demand cases. This is summarized in the following five theorems. The proof of each theorem involves a consideration of the possible values of $x(E(C : S))$ in a feasible CVRPUD solution.

Theorem 1. *LM inequalities are generated by the polygon procedure.*

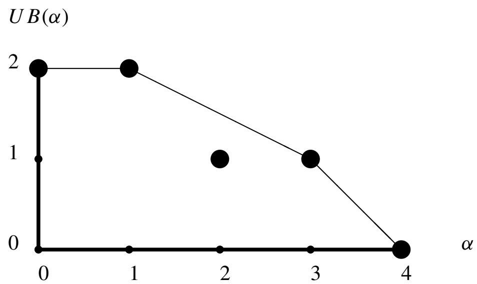

Fig. 5. Generating multistar inequalities

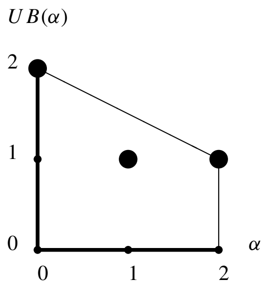

Fig. 6. Generating a partial multistar inequality

Proof. LM inequalities are defined only for the CVRPUD and only when $S = \bar{N}$ and $C = N$. We consider two cases:

- *Case 1:* $0 \leq x(E(N : \bar{N})) \leq |\bar{N}|$. In this case, Lemma 4 implies that $x(E(N)) \leq |N| - \lceil (N + x(E(N : \bar{N})))/Q \rceil$. If we weaken this by removing the integer rounding, and rearrange, we get the LM inequality.
- *Case 2:* $x(E(N : \bar{N})) > |\bar{N}|$. In this case, Lemma 4 implies that $x(E(N)) \leq |N| + |\bar{N}| - \lceil (|N| + |\bar{N}|)/Q \rceil - x(E(N : \bar{N}))$. If we weaken this by removing the integer rounding, and rearrange, we get the inequality $Qx(E(N)) + Qx(E(N : \bar{N})) \leq (Q-1)|N| + (Q-1)|\bar{N}|$, which dominates the LM inequality when $x(E(N : \bar{N})) > |\bar{N}|$.

Theorem 2. *IM inequalities are generated by the polygon procedure.*

Proof. Throughout this proof we let ϕ denote $\lceil |N|/(Q-2) \rceil$ for the sake of brevity.

By Theorem 1, all integer points in the polygon satisfy the LM inequality, and therefore satisfy the inequality:

$$x(E(N)) \leq |N| - \lceil (|N| + x(E(N : \bar{N})))/Q \rceil. \tag{25}$$

On the other hand, the IM inequality can be written as:

$$x(E(N)) \leq |N| - \phi + (2\phi - x(E(N : \bar{N})))/b. \tag{26}$$

Hence, we have that all integer points in the polygon such that

$$\lceil (|N| + x(E(N : \bar{N})))/Q \rceil \geq (x(E(N : \bar{N})) - 2\phi)/b + \phi \tag{27}$$

satisfy the IM inequality. Since, by definition, $b = 2 + |N| \bmod (Q-2) = Q + |N| - (Q-2)\phi$, we can rewrite (27) as:

$$\lceil (b - 2\phi + x(E(N : \bar{N})))/Q \rceil - 1 \geq (x(E(N : \bar{N})) - 2\phi)/b. \tag{28}$$

This holds provided $x(E(N : \bar{N})) \leq 2\phi$. (Indeed, if $0 \leq 2\phi - x(E(N : \bar{N})) < b$, the left hand side of (28) is zero and the right hand side is negative. On the other hand, if $2\phi - x(E(N : \bar{N})) \geq b$, then (28) holds even without integer rounding on the left.)

It remains to be shown that the integer points in the polygon such that $x(E(N : \bar{N})) > 2\phi$ also satisfy the IM inequality. But this is easy to show by comparing the IM inequality (26) with the inequality (22) and noting that $b > 2$.

Theorem 3. *SM inequalities are generated by the polygon procedure.*

Proof. As in the proof of Theorem 2, we have that all integer points in the polygon satisfy (25). On the other hand, the SM inequality (10) can be re-written as:

$$x(E(N)) \leq |N| - k(N \cup S) + (|S| - x(E(N : S)))/d. \tag{29}$$

Hence, we have that all integer points in the polygon such that

$$\lceil (|N| + x(S))/Q \rceil \geq k(N \cup S) + (x(E(N : S)) - |S|)/d \tag{30}$$

satisfy the SM inequality. From the definition of d and the assumption that $d > 0$, we have $d = |N \cup S| + Q - Qk(N \cup S)$, or, equivalently, $|N| = d - |S| - Q + Qk(N \cup S)$. Hence, the condition (30) can be re-written as:

$$-1 + \lceil (d - |S| + x(E(N : S)))/Q \rceil \geq (x(E(N : S)) - |S|)/d. \tag{31}$$

This holds provided $x(E(N : S)) \leq |S|$. (Indeed, if $0 \leq |S| - x(E(N : S)) < d$, the left hand side of (31) is zero and the right hand side is negative. On the other hand, if $|S| - x(E(N : S)) \geq d$, then (31) holds even without integer rounding on the left.)

It remains to be shown that the integer points in the polygon such that $x(E(N : S)) \geq |S|$ also satisfy the SM inequality. But this is easy to show by comparing the SM inequality (29) with the third inequality in Lemma 4 and noting that $d \geq 2$ in the definition of SM inequalities.

Theorem 4. *All four kinds of PM inequalities are generated by the polygon procedure.*

Proof (Sketch): For a PM inequality of the first kind, $x(E(C : \bar{N})) \in \{0, 1, 2\}$. For each of these three values, the PM inequality is implied by Lemma 4.

Similarly, for a PM inequality of the second kind, $x(E(C : \bar{N})) \in \{0, \ldots, 4\}$. For $x(E(C : \bar{N})) \in \{0, 1, 2\}$ the PM inequality is implied by the capacity inequality on N, whereas for $x(E(C : \bar{N})) \in \{3, 4\}$ it is implied by Lemma 5.

Finally, for PM inequalities of the third and fourth kinds, $x(E(C : \bar{N})) \in \{0, \ldots, 6\}$. Both of these inequalities are implied by Lemma 4 when $x(E(C : \bar{N})) \in \{0, \ldots, 3\}$ and by Lemma 5 when $x(E(C : \bar{N})) \in \{4, 5, 6\}$.

Theorem 5. *The inequalities (18) and (19) are dominated by the inequalities generated by the polygon procedure, and this dominance may be strict, even in the case of the CVRPUD.*

Proof (Sketch): The inequality (18) is implied by the first two inequalities in Lemma 4 when $x(E(N : S)) \leq |S|$, and by the third inequality in Lemma 4 when $x(E(N : S)) > |S|$.

The inequality (19) is implied by the first two inequalities in Lemma 4 when $x(E(N : S)) \leq 2k(N) + 1$, and by the inequality (22) when $x(E(N : S)) \geq 2k(N) + 2$.

The possibility of strict dominance is seen by comparing Propositions 1 and 2 with Theorems 1, 2 and 3.

We finish this section by showing that, even for the CVRPUD, the polygon procedure can generate facet-inducing inequalities that are not equivalent to any of those in the literature. Consider a CVRPUD instance with $Q = 4$ and $n = 8$, and a multistar with $|N| = 3$ and $|S| = 5$. Since Q is even, Lemma 2 applies and we have $x(E(N : S)) \leq 5$. The associated polygon is shown in Figure 7. The non-trivial inequalities defining the polygon are the capacity inequality $x(E(N)) \leq 2$, the IM inequality $3x(E(N)) + x(E(N : S)) \leq 7$, and the new inequality $x(E(N)) + x(E(N : S)) \leq 5$. This new inequality can be shown to be facet-inducing using standard techniques, such as enumerating affinely independent solutions to the CVRPUD instance.

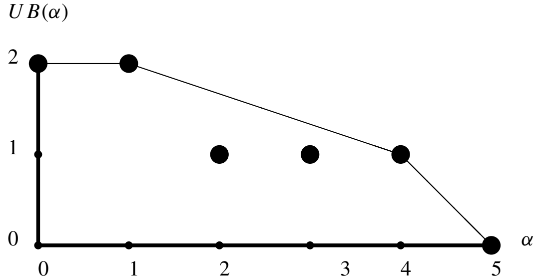

Fig. 7. Generating a new multistar inequality

4. New inhomogeneous inequalities

In the previous section we explored the homogeneous multistar and partial multistar inequalities in great detail. In this section we move on to consider multistar inequalities that are not necessarily homogeneous.

Consider the following polytope, which we denote by $K(q, Q)$:

Definition 1. $K(q, Q) := conv\left(y \in \{0, 1\}^n : \sum_{i=1}^n q_i y_i \le Q\right)$.

This is a 0-1 knapsack polytope (see, e.g., [7, 30]), the extreme points of which represent feasible allocations of customers to a single (arbitrary) vehicle. Now suppose also that we allow VRP instances in which one or more customers have $q_i = 0$. (This does not affect the formulation (1)–(4), except that it is necessary to define $k(S) = 1$ when $q(S) = 0$). Then we have the following lemma:

Lemma 6. *Let $\sum_{i=1}^n \gamma_i y_i \le \Gamma$ be a valid inequality for $K(q, Q)$, with $\Gamma > 0$ and integer, and $\gamma_i \ge 0$ and integer for all i. Then any feasible CVRP solution must also be feasible for a modified CVRP instance in which (q, Q) has been replaced by (γ, Γ).*

Proof. Call γ_i the *pseudo-demand* of customer i and call Γ the *pseudo-capacity*. By the definition of $K(q, Q)$, no vehicle can serve a set of customers whose collective pseudo-demand exceeds the pseudo-capacity. Thus, any single vehicle route obeying the capacity constraint for the original CVRP instance also obeys the capacity constraint for the modified CVRP instance.

The important implication of Lemma 6 for our purposes is that any class of inequalities whose coefficients are expressed in terms of the q_i and Q can be generalized. In particular, we obtain the following result.

Theorem 6. *If $\sum_{i=1}^n \gamma_i y_i \le \Gamma$, with $\Gamma > 0$ and $\gamma_i \ge 0$ for all i, is valid for $K(q, Q)$, then, for any S, the* knapsack large multistar *(KLM) inequality*

$$x(\delta(S)) \ge \frac{2}{\Gamma}\left(\sum_{i \in S} \gamma_i + \sum_{j \in \bar{S}} \gamma_j x(E(S : \{j\}))\right) (19) \qquad (32)$$

is valid for the CVRP.

Proof. Follows from Lemma 6, since (32) is the GLM inequality for the modified CVRP instance.

The inequalities (32) were first presented in Letchford & Eglese [20], though Hall [17] independently discovered some similar inequalities for the Capacitated Minimum Spanning Tree Problem. It is obvious that the KLM inequalities generalize GLM inequalities, since we can always set $\gamma_i = q_i$ and $\Gamma = Q$. It is also easy to show that the KLM inequalities for which $\sum_{i=1}^{n} \gamma_i y_i \leq \Gamma$ induces a facet of $K(q, Q)$ dominate all other KLM inequalities.

We could produce a number of other Theorems from Lemma 6, yielding, for example, knapsack versions of the Generalized IM and SM inequalities [1]. However, we do not examine this issue further for the sake of brevity.

Note that the KLM inequalities yield nothing new in the case of unit demands, as they reduce to either LM inequalities (when the knapsack inequality is $\sum_{i=1}^{n} y_i \leq Q$) or weakened capacity inequalities (when the knapsack inequality is $y_i \leq 1$ for some i). However, it is not difficult to find examples of facet-inducing KLM inequalities for certain instances of the CVRP with general-demands. Details are omitted for the sake of brevity.

5. Computational experiments

The authors have implemented a linear programming-based cutting-plane algorithm [4, 5, 25] that uses capacity, multistar and partial multistar inequalities to produce lower bounds for CVRP instances. The initial LP relaxation consists of the degree equations (1) and the bounds on the variables implied by (3), (4) and is solved by the primal simplex method. *Separation algorithms* [16, 25] are then invoked to generate violated inequalities. Any violated inequalities found are added to the LP as cutting planes and the LP is reoptimized using the dual simplex method. This process is repeated until no more violated inequalities can be found.

For capacity separation, we use heuristics that are very similar to ones found in [4–6]. They are based on the computation of connected components and maximum flows in suitable graphs, along with a greedy search procedure. The *trivial* bin packing lower bound (see Section 1) is used to estimate $k(S)$ when computing the right-hand sides. A full description is given in [21].

Our separation heuristic for Homogeneous Multistar (HM) inequalities is as follows. We use a greedy heuristic to find a number of 'candidates' for the nucleus, N. For each such candidate, we use a greedy heuristic to find a number of 'candidates' for the satellite set, S. For each pair N, S generated in this way we run the polygon procedure and check the resulting HM inequalities for violation.

To be more precise, we store the nucleus candidates in an array called **NUCLEI**, which is constructed as follows. (Here, x^* denotes the current LP vector to be cut off.)

- 1. Set **NUCLEI :=** $\emptyset$ and set $i = 1$.
- 2. Set $N := \{i\}$.
- 3. Let $W := \{j \in \bar{N} : x^*(E(N : \{j\})) > 0, N \cup \{j\} \notin$ **NUCLEI**$\}$. If W is empty, go to 6. Otherwise, among all vertices $j \in W$, choose the one that maximizes $x^*(E(N : \{j\})) - x^*_{0j}$ and set $N := N \cup \{j\}$.

- 4. If $N = V \setminus \{0\}$, go to 6.
- 5. Add N to **NUCLEI** and return to 3.
- 6. Set $i = i + 1$. If $i > n$, stop. Otherwise, return to 2.

For a given nucleus candidate N, we use the following simple procedure to generate candidate satellite sets S. Initially S is set to $\{i \in \bar{N} : x^*(E(\{i\} : N)) > 0\}$. Then we iteratively remove, one at a time, the satellite i with minimum value of $x^*(E(\{i\} : N))$.

In fact we also experimented with the following variant of the HM separation heuristic. In step 3 of the procedure for constructing **NUCLEI**, instead of adding the vertex that maximizes $x^*(E(N : \{j\})) - x^*_{0j}$, we added the vertex that reduced the slack of the GLM inequality associated with N by the most. It did not lead to better bounds by itself, but we found that it is worthwhile calling this second version when the original version fails. This is what we did in the experiments reported below.

Next, we describe our heuristic separation procedure for homogeneous partial multistar (HPM) inequalities. In some limited experiments, we found that the majority of HPM inequalities that actually led to improved bounds were either 'mere' HM inequalities (with $C = N$), or had $2 \le |C| \le 4$. So, given that we already had a satisfactory separation heuristic for HM inequalities, it seemed worthwhile to tailor the HPM separation algorithm to this latter case. So, if HM separation fails, we take the same set **NUCLEI** as before, and, for each set N in **NUCLEI**, we do the following:

- 1. Let $p = \min(4, |N| - 1)$.
- 2. If $p < 2$ stop.
- 3. Let C contain the p nodes in N with biggest values of $x^*(E(\{j\} : \bar{N}))$.
- 4. Let S contain those customers in $\bar{N}$ that are x^*-connected to C.
- 5. Use the polygon procedure to check for violated HPM inequalities for the given N, C, S.
- 6. If $|S| \ge 2$, drop one customer from S (the one with smallest $x^*(E(\{j\} : C))$) and return to 5.
- 7. Set $p := p - 1$ and return to 2.

Finally, we describe our separation heuristic for KLM inequalities. This is based on the following result:

Theorem 7. *Let x^* satisfy all degree equations and non-negativity inequalities and assume that $x^*_{ij} = 0$ if $q_i + q_j > Q$. For a fixed knapsack inequality $\sum_{i=1}^n \gamma_i y_i \le \Gamma$, the separation problem for KLM inequalities is solvable in polynomial time.*

Proof. First, note that the lhs of (32) is equivalent to $x(E(\{0\} : S)) + x(E(S : \bar{S}))$. Second, note that, by the degree equations, the first term on the rhs of (32) can be rewritten as $\sum_{i \in S} \gamma_i (x_{0i} + x(E(\{i\} : S)) + x(E(\{i\} : \bar{S})))/\Gamma$.
Therefore, (32) can be written as:

$$\sum_{i \in S} \left(x_{0i}(1 - \gamma_i/\Gamma) - \sum_{j \in V \setminus \{0,i\}} \gamma_j x_{ij}/\Gamma \right) + \sum_{i \in S, j \in \bar{S}} x_{ij}(1 - (\gamma_i + \gamma_j)/\Gamma) \ge 0. \quad (33)$$

Now, for each $i \in V \setminus \{0\}$ let us define a new 0-1 variable t_i that takes the value 1 if $i \in N$, 0 otherwise. Then (33) can be re-written as:

$$\sum_{i \in N} \left(x_{0i}(1 - \gamma_i/\Gamma) - \sum_{j \in V \setminus \{0,i\}} \gamma_j x_{ij}/\Gamma \right) t_i$$

$$+ \sum_{\{i,j\} \subset N} t_i(1 - t_j)\left(x_{ij}(1 - (\gamma_i + \gamma_j)/\Gamma\right) \geq 0. \quad (34)$$

The assumption that $x^*_{ij} = 0$ if $q_i + q_j > Q$ implies that $x^*_{ij} = 0$ if $\gamma_i + \gamma_j > \Gamma$. This means that the separation problem for (34) reduces to the minimization of a quadratic function of t in which all quadratic terms are negative. Using a standard construction [27] this can be reduced to a max-flow/min-cut problem and therefore solved in polynomial time.

Given Theorem 7, the question then arises as to which knapsack inequalities to use. Initially, we experimented with lifted cover (LC) inequalities (see, e.g., [7, 25, 30]. However, we were unable to find any violated KLM inequalities at all for any of our test instances when we used LC inequalities, even though we tried several ways of generating them. Instead, we found that much better results were obtained by the naïve option of merely setting $\gamma = q$ and $\Gamma = Q$. (That is, just separating GLM inequalities). We discuss this phenomenon in the next section.

We ran the cutting plane algorithm on eleven instances taken from the literature [6, 13, 28]. Statistics for these problems are shown in Table 1. The first three headings are self-explanatory. The column headed 'K' shows the number of vehicles (which is fixed in these problems at the minimum possible); the loading is the total demand divided by KQ; and the column 'UB' shows the best known upper bounds at the time of writing, taken from [5, 6, 24, 29], with a * if the upper bound is known to be optimal. The reason for choosing these instances is that they cannot be solved to optimality using capacity inequalities alone. Moreover, all have been solved to optimality except eilA76 and eilB76 [5, 6, 24, 29].

Table 2 displays various lower bounds for each instance. The first column labelled 'CAP' gives the best bound obtained using capacity separation heuristics only according to Augerat et al. [5, 6]. (The figure is missing for eilB76 because it does not appear in those papers.) Next, 'ALL' is the bound obtained by Augerat et al. [5, 24] when several sophisticated separation heuristics are invoked for various inequalities (generalized capacity, comb, hypotour, etc.). Then we have various columns that report lower bounds obtained by our cutting plane algorithm: 'CAP' is the bound obtained by our capacity separation heuristics; 'GLM' is the bound obtained when GLM separation is called after capacity separation fails; 'HM' gives the lower bound obtained by calling HM separation after capacity separation fails; 'HPM' gives the bound obtained by calling HPM separation after both capacity and HM separation fail. Finally, 'ALL' gives the lower bound obtained by calling GLM separation after capacity, HM and HPM separation fail. A * means that a lower bound is optimal.

The results here are interesting. Very good bounds are obtained with the HM inequalities for the 76 node instances — which are widely recognized as the hardest instances

Table 1. Eleven CVRP instances

Name	n	Q	K	Loading	UB
eil30	29	4500	3	.94	534*
eil33	32	8000	4	.92	835*
eil51	50	160	5	.97	521*
eilA76	75	140	10	.97	832
eilB76	75	100	14	.97	1032
eilC76	75	180	8	.95	735*
eilD76	75	220	7	.89	682*
eilA101	100	200	8	.91	815*
cmt12	100	200	10	.91	820*
fisher72	71	30000	4	.96	237*
fisher135	134	2210	7	.95	1162*

Table 2. Lower bounds for CVRP instances

	Augerat et al.		Our cutting plane algorithm				
Name	CAP	ALL	CAP	GLM	HM	HPM	ALL
eil30	508.5	534*	508.5	508.5	508.5	508.5	508.5
eil33	833.5	835*	833.5	833.5	833.5	833.8	833.8
eil51	514.524	517.142	514.524	514.524	514.556	514.556	514.556
eilA76	789.416	793.384	789.441	793.008	795.58	796.386	796.674
eilB76	—	953.794	947.952	961.827	963.627	964.15	966.511
eilC76	711.17	713.746	711.201	712.441	714.279	714.327	714.327
eilD76	661.299	664.355	661.36	661.36	663.241	663.248	663.248
eilA101	796.314	799.656	796.405	796.405	798.728	798.771	798.771
cmt12	819.5	820*	819.5	819.5	819.5	820*	820*
fisher72	232.5	235	232.5	232.5	232.5	232.5	232.5
fisher135	1158.25	1159.06	1158.25	1158.304	1159.645	1159.65	1159.65

to solve. Also cmt12 can be solved to optimality using capacity, HM and HPM inequalities alone (without branching). On the other hand, the multistar inequalities make no difference at all for eil30 and fisher72 and little difference for eil33 and eil51. For these instances, cutting planes of a different kind appear to be necessary. For eil30 and fisher72, for example, the so-called *generalized capacity* inequalities work very well (see [5]).

Table 3 reports the time taken to compute the CAP, HM and HPM bounds. The hardware used was a 350 MHz PC Pentium II with 64MB of RAM running under Microsoft Windows 95. We used the ILOG CPLEX 6.0 callable library and the Watcom C/C++ compiler V. 11.0.

We also ran the cutting plane algorithm on some unit demand instances, taken from [4], all of which have been solved to optimality. For these instances, the HM and HPM separation heuristics could be improved somewhat. First, we could use the improved bound on $x(E(C : S))$ given by Lemma 2. Second, we wrote a dynamic programming code that enables us, for a given Q, n and N, to eliminate various possible values of S and C from consideration on the basis of dominance arguments.

Table 4 contains some information on these unit demand instances. Here, the number of vehicles is not fixed and therefore we just report $\lceil n/Q \rceil$ — the minimum number of vehicles necessary — for interest. (Note: the optimal solution to AKMP13 is 688, rather than 689 as reported in [4].)

Table 3. Running times (in seconds) for CVRP instances

Name	CAP	GLM	HM	HPM	ALL
eil30	3	3	3	3	3
eil33	7	8	8	11	11
eil51	7	7	8	8	8
eilA76	29	62	121	215	318
eilB76	39	177	262	313	862
eilC76	16	28	55	80	80
eilD76	13	13	29	36	36
eilA101	19	19	74	88	88
cmt12	44	44	44	49	49
fisher72	6	6	6	6	6
fisher135	76	77	256	266	266

Table 4. Ten unit demand instances from Araque et al.

Name	n	Q	$\lceil n/Q \rceil$	OPT
AKMP1	40	10	4	647
AKMP7	50	8	7	875
AKMP8	50	15	4	678
AKMP13	60	30	2	688
AKMP15	21	3	7	530
AKMP16	21	7	3	341
AKMP17	29	4	8	832
AKMP18	29	6	5	639
AKMP19	32	5	7	627
AKMP20	32	8	4	497

Table 5. Lower bounds for unit demand instances

Name	AKMP	CAP	GLM	HM	HPM
AKMP1	638	640.667	640.667	641	642.123
AKMP7	857.38	858.75	859.381	862.858	864.291
AKMP8	669.62	672.333	672.333	672.818	672.818
AKMP13	678.25	682.75	682.75	682.75	682.75
AKMP15	530*	530*	530*	530*	530*
AKMP16	341*	340.5	341*	341*	341*
AKMP17	830.8	832*	832*	832*	832*
AKMP18	639*	639*	639*	639*	639*
AKMP19	625.27	625.833	626.037	627*	627*
AKMP20	497*	497*	497*	497*	497*

Table 5 shows various lower bounds: ‘AKMP’ represents the bound obtained by Araque et al. [4] using separation heuristics for capacity, LM, IM and SM inequalities; the remaining four columns show bounds obtained with our algorithm, with the same headings as before. (There is no ‘ALL’ column because running GLM separation after HPM separation led to no increase in the bounds.)

It should be noted that our capacity separation heuristic on its own (column ‘CAP’) generally gives better bounds than those in column ‘AKMP’. Together with our HM and

Table 6. Running times (in seconds) for unit demand instances

Name	CAP	GLM	HM	HPM
AKMP1	6	6	7	11
AKMP7	9	12	15	22
AKMP8	5	5	6	6
AKMP13	4	4	4	4
AKMP15	5	5	5	5
AKMP16	5	6	5	5
AKMP17	7	7	7	7
AKMP18	7	7	7	7
AKMP19	7	9	9	9
AKMP20	5	5	5	5

HPM separation heuristics, we are able to solve many of the instances to optimality with no branching. (Table 6 gives the running times.)

6. Conclusion

The inequalities generated by the 'polygon' procedure, together with the KLM inequalities, generalize all other known multistar and partial multistar inequalities in the literature and are therefore interesting from a theoretical point of view. They also appear to be promising as cutting planes, especially the homogeneous ones. In our view, there are three areas of interest for further research:

- Finding better ways of choosing (γ, Γ) for KLM separation.
- Integrating the HM, HPM and KLM separation algorithms in a branch-and-cut solver for the CVRP.
- Deriving multistar (and perhaps partial multistar) inequalities for problems with more complex constraints, such as time windows (see, e.g., [12]).

The second and third of these issues are addressed in [21] and [22], respectively. We close this section with a discussion of the first issue, namely, the reason why the naïve method of choosing (γ, Γ) gave better results than using lifted cover (LC) inequalities. A partial explanation for this is as follows. With any given (γ, Γ) pair we can associate a measure of strength, $\sum_{i=1}^{n} \gamma_i / \Gamma$. We would expect a (γ, Γ) pair with large strength to yield better KLM inequalities, on average, than a pair with small strength. We found that using LC inequalities to generate (γ, Γ) invariably gave a measure of strength that was at least 30% smaller than the strength obtained under the naïve option.

To see why this might be the case, we computed a strongest possible (γ, Γ) for 8 out of the 10 problem instances, by optimizing over the 1-polar of $K(q, Q)$ using the algorithm of [9]. (For the two Fisher instances, this was not possible, because Q was too large). For 7 out of the 8 problems, it is optimal to set $(q, Q) = (\gamma, \Gamma)$. The only instance where there is some scope for increasing the strength is eil33: there is a demand of 40 that can be increased to 50 and a demand of 80 that can be increased to 100.

This suggests that, if one wishes to generate good facets of $K(q, Q)$ for KLM separation, one should not use LC inequalities.

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