



The bounds of votes of divisor electoral methods

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Abstract

In proportional electoral systems, party vote counts must be converted to seat allocations within a parliament of fixed size. Divisor methods are the most common approach to this problem, but different divisor methods often give different seat allocations. To highlight these differences, the effects of various divisor methods on a party's seat allocation are expressed as intervals of the party's vote count within which the seat allocation is unchanged, assuming other parties' votes are fixed. These bounds are applied to data from four recent European parliamentary elections, as well as one hypothetical dataset.

Keywords Voting · Proportional representation · Allocation of seats · Divisor methods

1 Introduction

Proportional electoral systems are used in many countries to convert the party vote count in an election to an allocation of parliamentary seats. The problem is that the numbers of seats assigned to each party must be integers, which means that the proportion of seats allocated to a party is almost always different from the party's proportion of the votes. The same problem arises in federations when numbers of

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representatives of districts or states are apportioned according to their populations (Cortona et al. 1999; Still 1979).

Divisor methods are the most common approach to the allocation problem (Balinski and Young 2001; Carstairs 1980; Pukelsheim 2014; Mackie and Rose 1982). But different divisor methods can give different seat allocations from the same vector of party vote counts, so the choice of divisor method may determine whether a particular coalition of parties forms a parliamentary majority (Marošević and Soldo 2018).

We use the following notation:

- $n \geq 2$ - the total number of parties (or the total number of districts or member states in the country),
- $I = \{1, 2, \dots, n\}$ - the set of parties (i.e., the set of districts or member states),
- S - the total number of seats in the parliament,
- $\mathbf{v} = (v_1, v_2, \dots, v_n)$ - the vector of votes, where $v_k \in \mathbb{N}$ is the number of votes received by the party k (or the population of the district or the member state k), $k \in I$, and the total number of votes $P = \sum_{k=1}^n v_k$,
- $\mathbf{s} = (s_1, s_2, \dots, s_n)$ - the vector of seat allocation, where $s_k \geq 0$ is the number of seats assigned to the party k , and $\sum_{k=1}^n s_k = S$.

The divisor methods are based on the increasing sequence of their divisors (i.e., the functions $d : \mathbb{N} \rightarrow [0, \infty)$): $d(1) < d(2) < \dots < d(S)$. For each party and each divisor, the following ratios are observed (provided that $d(1) \neq 0$):

$$\frac{v_k}{d(1)} > \frac{v_k}{d(2)} > \dots > \frac{v_k}{d(S)}, \quad \forall k = 1, \dots, n.$$

Then the S seats are assigned to the parties that have got the S largest such ratios. In the case of ties, additional rules for seat allocation must be defined.

The most popular divisor methods are listed in Table 1 (Cortona et al. 1999; D'Hondt 1878; Hill 1911; Huntington 1921; Sainte-Laguë 1910).

Table 1 The divisor methods

Divisor method	And its divisors
Belgian (BE)	$d(k) = \frac{k+1}{2} (1, 1.5, 2, 2.5, \dots)$
d'Hondt (DH) (i.e., Jefferson)	$d(k) = k (1, 2, 3, \dots)$
Modified Sainte-Laguë (MS)	$d(1) = 1.4, d(k) = 2k - 1, k \geq 2 (1.4, 3, \dots)$
Sainte-Laguë (SL) (i.e., Webster)	$d(k) = 2k - 1 (1, 3, 5, \dots)$
Equal Proportions (EP) (i.e., Hill)	$d(k) = \sqrt{k(k-1)} (0, \sqrt{2} \approx 1.41, \dots)$
Harmonic Mean (HM)	$d(k) = \frac{2k(k-1)}{2k-1} (0, \frac{4}{3} \approx 1.33, 2.4, \dots)$
Danish (DA)	$d(k) = 3k - 2 (1, 4, 7, \dots)$
Smallest Divisors (SD) (i.e., Adams)	$d(k) = k - 1 (0, 1, 2, \dots)$

Remark 1 The smallest divisors method (SD), the harmonic mean method (HM) and the equal proportions method (EP) all have $d(1) = 0$. The interpretation of a method with first divisor 0 is that every party automatically receives one seat, and all ratios $\frac{v_k}{d(1)}$ are dropped from the calculation. The remaining $S - n$ seats are allocated to the parties with the $S - n$ largest ratios. Note that these methods cannot be applied unless the condition $S \geq n$ is satisfied.

Another group of proportional electoral methods are the quota methods. By these methods seats are allocated on the basis of the integer part $\lfloor q_k \rfloor$ of the given quota q_k and after that, the remaining seats are allocated on the basis of the largest remainders $r_k \in [0, 1)$, where $q_k = \lfloor q_k \rfloor + r_k$, $k = 1, \dots, n$. The most popular quota methods are defined by the following quotas:

- (a) the largest remainders method (LAR), by the natural quota $q_k = \frac{v_k}{P} \cdot S$,
- (b) the Droop quota method (DRO), by the Droop quota $\frac{v_k}{P} \cdot (S + 1)$,
- (c) the Imperiali quota method (IMP), by the Imperiali quota $\frac{v_k}{P} \cdot (S + 2)$.

With regard to the principle of 'fair distribution' of seats, one can look at the properties of underrepresentation and overrepresentation that a particular proportional method gives, and at the characterization of these properties by means of the Lorenz curve and the Gini index, which are considered in the field of welfare economics (Cortona et al. 1999).

One can also observe proportional methods as algorithms that give a solution to integer optimization problems of special objective functions that represent various measures of proportionality between the obtained votes and the assigned seats (Cortona et al. 1999).

In this paper, we consider the bounds of votes of divisor methods, i.e., we consider intervals of vote counts within which different divisor methods give identical seat allocations. The same question for quota methods as a class different from divisor methods is complex and not considered in this paper.

Table 2 Bounds of votes $l_k < v_k < u_k$ for the divisor methods where $I_0 = \emptyset$

The method of smallest divisors (SD)

$$\max_{\ell \in \Lambda(k)} \frac{s_k - 1}{s_\ell} v_\ell < v_k < \min_{\ell \in \Lambda(k)} \frac{s_k}{s_\ell - 1} v_\ell, \quad k = 1, \dots, n$$

The method of harmonic mean (HM)

$$\max_{\ell \in \Lambda(k)} \frac{s_k(s_k - 1)(2s_\ell + 1)}{(2s_k - 1)(s_\ell + 1)s_\ell} v_\ell < v_k < \min_{\ell \in \Lambda(k)} \frac{(s_k + 1)s_k(2s_\ell - 1)}{(2s_k + 1)s_\ell(s_\ell - 1)} v_\ell, \quad k = 1, \dots, n$$

The method of equal proportions (EP)

$$\max_{\ell \in \Lambda(k)} \frac{\sqrt{s_k(s_k - 1)}}{\sqrt{s_\ell(s_\ell + 1)}} v_\ell < v_k < \min_{\ell \in \Lambda(k)} \frac{\sqrt{s_k(s_k + 1)}}{\sqrt{s_\ell(s_\ell - 1)}} v_\ell, \quad k = 1, \dots, n$$

Table 3 Bounds of votes $l_k < v_k < u_k$ for the divisor methods in which $d(1) \neq 0$

The Danish method (DA)

$$\begin{aligned} \text{For any } k \in I \setminus I_0, & \quad \max_{\ell \in I \setminus \{k\}} \frac{3s_k - 2}{3s_\ell + 1} v_\ell < v_k < \min_{\ell \in I \setminus (I_0 \cup \{k\})} \frac{3s_k + 1}{3s_\ell - 2} v_\ell, \\ \text{For any } k \in I_0, & \quad v_k < \min_{\ell \in I \setminus I_0} \frac{1}{3s_\ell - 2} v_\ell \end{aligned}$$

The method of Sainte-Laguë (SL)

$$\begin{aligned} \text{For any } k \in I \setminus I_0, & \quad \max_{\ell \in I \setminus \{k\}} \frac{2s_k - 1}{2s_\ell + 1} v_\ell < v_k < \min_{\ell \in I \setminus (I_0 \cup \{k\})} \frac{2s_k + 1}{2s_\ell - 1} v_\ell \\ \text{for any } k \in I_0, & \quad v_k < \min_{\ell \in I \setminus I_0} \frac{1}{2s_\ell - 1} v_\ell, \end{aligned}$$

The d'Hondt method (DH)

$$\begin{aligned} \text{For any } k \in I \setminus I_0, & \quad \max_{\ell \in I \setminus \{k\}} \frac{s_k}{s_\ell + 1} v_\ell < v_k < \min_{\ell \in I \setminus (I_0 \cup \{k\})} \frac{s_k + 1}{s_\ell} v_\ell \\ \text{For any } k \in I_0, & \quad v_k < \min_{\ell \in I \setminus I_0} \frac{1}{s_\ell} v_\ell \end{aligned}$$

The Belgian method (BE)

$$\begin{aligned} \text{for any } k \in I \setminus I_0, & \quad \max_{\ell \in I \setminus \{k\}} \frac{s_k + 1}{s_\ell + 2} v_\ell < v_k < \min_{\ell \in I \setminus (I_0 \cup \{k\})} \frac{s_k + 2}{s_\ell + 1} v_\ell \\ \text{For any } k \in I_0, & \quad v_k < \min_{\ell \in I \setminus I_0} \frac{2}{s_\ell + 1} v_\ell \end{aligned}$$

In Sect. 2, the conditions of specified seat allocation for different divisor methods are considered by means of the bounds of votes. In Sect. 3, a few cases of elections and examples are given that illustrate the bounds of votes of different divisor methods. Finally, a few concluding remarks are made at the end of the paper.

2 The bounds of votes of divisor methods

Given the input data on the obtained votes of parties (i.e., on the number of population of each member state), various proportional methods can allocate seats differently.

In this section, we look at the conditions under which a divisor method will give specified seat allocation by means of the bounds of votes, i.e., we consider the following question: Assume a vote count vector $v = (v_1, v_2, \dots, v_n)$ is given and the application of a divisor method based on $d(1), d(2), \dots, d(S)$ results in the seat distribution $s = (s_1, s_2, \dots, s_n)$. What are the maximum and minimum values of v_k such that if v_i does not change for all $i \neq k$, then s (and, in particular, s_k) does not change after the same divisor method is applied?

The following theorem is the well-known min-max inequality, which describes the condition that a divisor method gives specified seat allocation (Balinski and Young 1975, 2001; Cortona et al. 1999).

Theorem 1 *Let there be given the vector of obtained votes (v_1, v_2, \dots, v_n) , the vector of seat allocation (s_1, s_2, \dots, s_n) , where $\sum_{k=1}^n s_k = S$, and the M divisor method determined by its divisors $d(m)$, $m = 1, \dots, S$.*

Let us denote the set $I_0 = \{j \in I : s_j = 0\}$. Then the M divisor method gives seat allocation (s_1, s_2, \dots, s_n) if and only if

$$\min_{k \in I \setminus I_0} \frac{v_k}{d(s_k)} > \max_{\ell \in I} \frac{v_\ell}{d(s_\ell + 1)}. \tag{1}$$

□

The next proposition follows from Theorem 1. It gives the bounds of votes, i.e., the vote intervals, within which one can change the number of votes v_k of the party k such that the M divisor method gives the same seat allocation (provided that the other votes v_i , $i \neq k$, are unchanged). Note that when v_k changes, but not v_i for $i \neq k$, then the total number of votes in the election changes.

Proposition 1 *Let there be given the vector of obtained votes (v_1, \dots, v_n) , the vector of seat allocation (s_1, \dots, s_n) , where $\sum_{k=1}^n s_k = S$, and the M divisor method determined by its divisors $d(m)$, $m = 1, \dots, S$.*

Let us denote the set $I_0 = \{j \in I : s_j = 0\}$. Then the M divisor method gives seat allocation (s_1, \dots, s_n) if and only if

$$\max_{\ell \in I \setminus \{k\}} \frac{d(s_k)}{d(s_\ell + 1)} v_\ell < v_k < \min_{\ell \in I \setminus (I_0 \cup \{k\})} \frac{d(s_k + 1)}{d(s_\ell)} v_\ell, \quad \forall k \in I \setminus I_0, \tag{2}$$

$$v_k < \min_{\ell \in I \setminus I_0} \frac{d(1)}{d(s_\ell)} v_\ell, \quad \forall k \in I_0. \tag{3}$$

Proof Proposition 1 has the same assumptions as Theorem 1. Inequality (1) holds if and only if the following inequalities hold:

$$d(s_k) \cdot \max_{\ell \in I} \frac{v_\ell}{d(s_\ell + 1)} < v_k, \quad \text{for any } k \in I \setminus I_0, \tag{4}$$

$$v_\ell < d(s_\ell + 1) \cdot \min_{k \in I \setminus I_0} \frac{v_k}{d(s_k)}, \quad \text{for any } \ell \in I. \tag{5}$$

It can be easily seen that inequalities (4) and (5) are equivalent to inequalities (2) and (3) provided that v_k does not have to be compared with itself. □

Remark 2 If $I_0 = \emptyset$, then expression (2) in Proposition 1 takes the form:

$$\max_{\ell \in I \setminus \{k\}} \frac{d(s_k)}{d(s_\ell + 1)} v_\ell < v_k < \min_{\ell \in I \setminus \{k\}} \frac{d(s_k + 1)}{d(s_\ell)} v_\ell, \quad \forall k \in I. \tag{6}$$

Table 4 Elections in Austria for the EuP in 2019: allocation of seats to 5 parties

Votes	
$v_1 = 1,305,956$ $v_2 = 903,151$ $v_3 = 650,114$ $v_4 = 532,193$ $v_5 = 319,024$	
Divisor method	Seats: $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5)$
BE	(8, 5, 3, 2, 1)
DH	(7, 5, 3, 3, 1)
SL, MS	(7, 4, 3, 3, 2)
SD, DA, HM, EP	(6, 5, 3, 3, 2)

From Proposition 1 we can determine the corresponding bounds of votes related to particular divisor methods.

The bounds of votes for those divisor methods where there holds $I_0 = \emptyset$ are listed in Table 2. (If $d(1) = 0$, then $I_0 = \emptyset$, i.e., then there holds $s_k \geq 1, \forall k = 1, \dots, n$). In these methods, the bounds of votes are obtained by putting the divisor formula of a particular divisor method in expression (6).

The bounds of votes for those divisor methods where there holds $d(1) \neq 0$ are listed in Table 3. In these methods, the bounds are obtained by putting the corresponding divisor formula for a particular divisor method in expressions (2) and (3).

Knowing the vote intervals within which different divisor methods give identical seat allocations could help in (the posteriori) analysis of elections, in future planning of an electoral strategy, when considering the properties of different divisor methods, and the like. For example, if an electoral system consists of several electoral districts, one can see in which districts some party would gain a seat if its vote count increased by a small percentage. So, the party could choose on which district it would focus its future election campaign more strongly, and vice versa.

Remark 3 Divisor methods can be compared according to the extent that they ‘favour large states’ (Balinski and Young 1975, 2001). In this approach, divisor method M'' , with divisors $d_{M''}$, favours large states relative to divisor method M' , with divisors $d_{M'}$, if the following inequalities hold:

$$\frac{d_{M''}(k)}{d_{M''}(\ell)} < \frac{d_{M'}(k)}{d_{M'}(\ell)}, \quad \forall k > \ell \geq 1. \quad (7)$$

Note that if $d_{M'}(1) = 0$, then one makes the convention that $\frac{d_{M'}(k)}{d_{M'}(1)} = +\infty, \forall k > 1$, where the ‘plus infinity’ symbol $+\infty$ is such that $\min\{+\infty, a\} = a$ and $\max\{+\infty, a\} = +\infty$ for every given real number a (Cortona et al. 1999).

For instance, with respect to the BE and SD methods, the following inequality holds:

Table 5 The lower bounds $l_k - v_k$ and upper bounds $u_k - v_k$ of votes for the corresponding divisor methods (Austria, EuP, 2019)

Method	$v_1 = 1, 305, 956 \quad s_1 = 6$ $\langle l_1 - v_1, u_1 - v_1 \rangle$	$v_2 = 903, 151 \quad s_2 = 5$ $\langle l_2 - v_2, u_2 - v_2 \rangle$	$v_3 = 650, 114 \quad s_3 = 3$ $\langle l_3 - v_3, u_3 - v_3 \rangle$
EP	$\langle -278036, 2835 \rangle$	$\langle -1956, 286869 \rangle$	$\langle -156510, 49464 \rangle$
HM	$\langle -271684, 7087 \rangle$	$\langle -4875, 306379 \rangle$	$\langle -165045, 46603 \rangle$
DA	$\langle -265774, 14034 \rangle$	$\langle -9603, 313291 \rangle$	$\langle -168973, 44618 \rangle$
SD	$\langle -222433, 48771 \rangle$	$\langle -32514, 402805 \rangle$	$\langle -214796, 27250 \rangle$
Method	$v_4 = 532, 193 \quad s_4 = 3$ $\langle l_4 - v_4, u_4 - v_4 \rangle$	$v_5 = 319, 024 \quad s_5 = 2$ $\langle l_5 - v_5, u_5 - v_5 \rangle$	
EP	$\langle -38589, 167385 \rangle$	$\langle -34042, 175653 \rangle$	
HM	$\langle -47124, 164524 \rangle$	$\langle -49542, 168678 \rangle$	
DA	$\langle -51052, 162539 \rangle$	$\langle -44086, 167289 \rangle$	
SD	$\langle -96875, 145171 \rangle$	$\langle -101365, 132552 \rangle$	

$$\frac{d_{BE}(k)}{d_{BE}(\ell)} < \frac{d_{SD}(k)}{d_{SD}(\ell)}, \quad \forall k > \ell \geq 1.$$

So, compared to the SD method, the BE method tends to 'favour large states'. Note that inequality (7) does not hold for the DA and HM methods.

Divisor methods, ranging between the BE method and the SD method, are listed in Table 1 (Balinski and Young 1975, 2001; Marošević and Scitovski 2007). They can be compared by the following lemma.

Lemma 1 *If the BE method and the SD method give equal seat allocation $s = (s_1, \dots, s_n)$, then each divisor method listed in Table 1, i.e., the DH, MS, SL, EP, HM and DA methods, gives the equal seat allocation.* □

This lemma can also be proven by means of the bounds of votes for the corresponding divisor methods that are given in Tables 2 and 3. Lemma 1 is illustrated in Example 5.

3 Examples

In order to empirically study and illustrate the bounds of votes of different divisor methods, we look at a few cases of elections and an example with hypothetical data.

Example 1 We use data related to elections for the European Parliament in Austria in 2019, where $n = 5$ parties (or lists) passed the electoral threshold of 5% in the whole country as a single electoral unit (euparl 2021; wikiped 2021). The obtained votes

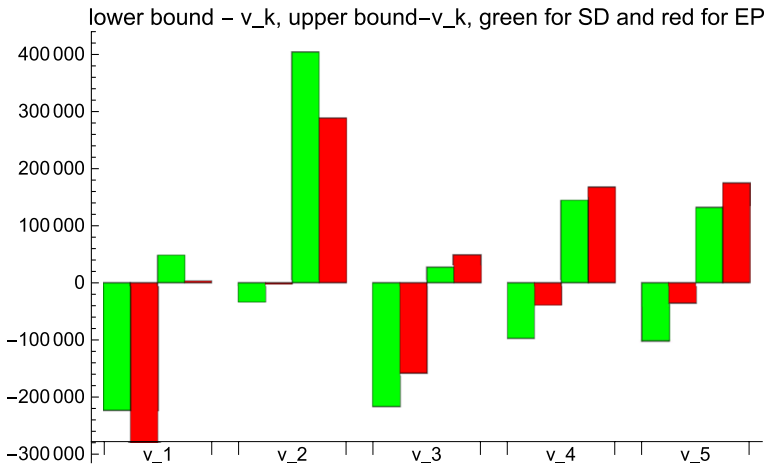


Fig. 1 Intervals $\langle l_k - v_k, 0 \rangle$, $\langle 0, u_k - v_k \rangle$, green (the first bar) for the SD method and red (the second bar) for the EP method, from Table 5 (Austria, EuP, 2019) (colour figure online)

Table 6 The bounds of votes for the d’Hondt method ($s_{DH} = (13, 4, 2, 1, 1)$) (Hungary, EuP, 2019)

Votes	$\langle l_k, u_k \rangle$	$\langle l_k - v_k, u_k - v_k \rangle$
$v_1 = 1,824,220$	$\langle 1492885, 1949784 \rangle$	$\langle -331335, 125564 \rangle$
$v_2 = 557,081$	$\langle 521205, 701624 \rangle$	$\langle -35876, 144543 \rangle$
$v_3 = 344,512$	$\langle 260602, 417811 \rangle$	$\langle -83910, 73299 \rangle$
$v_4 = 229,551$	$\langle 130301, 278541 \rangle$	$\langle -99250, 48990 \rangle$
$v_5 = 220,184$	$\langle 130301, 278541 \rangle$	$\langle -89883, 58357 \rangle$

$v_k, k = 1, \dots, 5$, are shown in descending order in Table 4. The allocation of seats $S = 19$ is done by the d’Hondt method: $s_{DH} = (7, 5, 3, 3, 1)$.

For the purpose of comparison and empirical research, we have applied other different divisor methods, and the corresponding seat allocations are shown in Table 4.

In this case, the SD, DA, HM and EP divisor methods give the same seat allocation $s = (6, 5, 3, 3, 2)$. According to expressions for these methods given in Tables 2 and 3, the corresponding lower bounds $l_k - v_k$ and upper bounds $u_k - v_k$ of votes are listed in Table 5.

Intervals $\langle l_k - v_k, 0 \rangle$ and $\langle 0, u_k - v_k \rangle, k = 1, \dots, 5$, green (the first bar) for the SD method and red (the second bar) for the EP method, are illustrated in Fig. 1. For the first party with the largest number of votes $v_1 = 1,305,956$, the SD method has got greater bounds than the EP method. A reverse relation holds for the fifth party with the smallest number of votes $v_5 = 319,024$. For instance, it can be seen that if the

Table 7 Elections in Hungary for the EuP in 2019:allocation of seats to 5 parties

Votes	
$v_1 = 1,824,220 v_2 = 557,081 v_3 = 344,512 v_4 = 229,551 v_5 = 220,184$	
Divisor method	Seats: $(s_1, s_2, s_3, s_4, s_5)$
BE	(15, 3, 2, 1, 0)
DH	(13, 4, 2, 1, 1)
SL, EP	(12, 4, 2, 2, 1)
SD, HM, DA	(11, 4, 2, 2, 2)

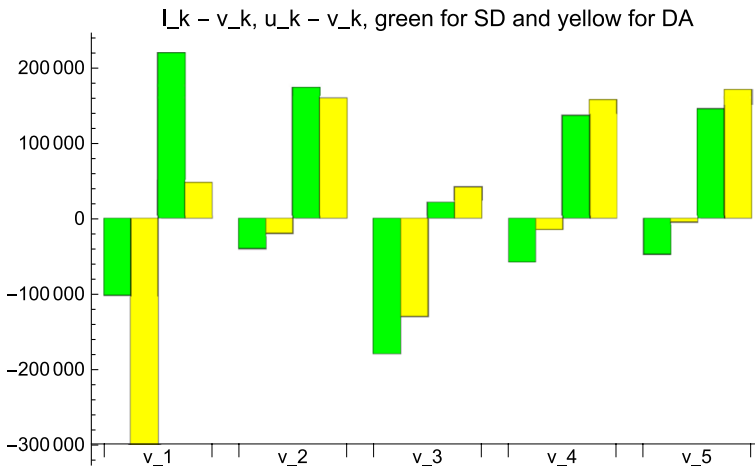


Fig. 2 Intervals $\langle l_k - v_k, 0 \rangle, \langle 0, u_k - v_k \rangle$, green (the first bar) for the SD method and yellow (the second bar) for the DA method (Hungary, EuP, 2019) (colour figure online)

first party with the largest number of votes $v_1 = 1,305,956$ had slightly larger number of votes $v'_1 = u_1 = 1,308,791$, then the EP method would give one seat more to the first party (i.e., $s'_1 = 7$), while the SD method would give the unchanged number of seats $s_1 = 6$ to the first party (provided that the other number of votes $v_i, i \neq 1$, are unchanged). This is in accordance with the property that the EP method 'tends to favour large states' over the SD method.

Example 2 We use data related to elections for the European Parliament in Hungary in 2019, where $n = 5$ parties (or lists) passed the electoral threshold of 5% in the whole country as a single electoral unit (euparl (2021); wikiped (2021)). The obtained votes $v_k, k = 1, \dots, 5$, are shown in descending order in Table 6.

The allocation of seats $S = 21$ is done by the d'Hondt method: $s_{DH} = (13, 4, 2, 1, 1)$. According to expressions in Table 3 for the d'Hondt method, the corresponding lower bounds l_k and upper bounds u_k of votes are shown in Table 6. Furthermore, the corresponding intervals $\langle l_k - v_k, u_k - v_k \rangle$ are also shown in Table 6. For instance, one can see that if the second party obtains $v_2 - l_2 = 35,876$ votes less

Table 8 Elections in Croatia for the EuP in 2019: allocation of seats to 6 parties

Votes	
$v_1 = 244,076, v_2 = 200,976, v_3 = 91,546, v_4 = 84,765, v_5 = 60,847, v_6 = 55,806$	
Divisor method	$s = (s_1, s_2, s_3, s_4, s_5, s_6)$
BE	(6, 4, 1, 1, 0, 0)
DH	(4, 4, 1, 1, 1, 1)
SL, MS, EP, HM, DA	(4, 3, 2, 1, 1, 1)
SD	(3, 3, 2, 2, 1, 1)

Table 9 The lower and upper bounds of votes for the corresponding divisor methods (Croatia, EuP, 2019)

Method	$v_1 = 244,076, s_1 = 4$ $\langle l_1 - v_1, u_1 - v_1 \rangle$	$v_2 = 200,976, s_2 = 3$ $\langle l_2 - v_2, u_2 - v_2 \rangle$	$v_3 = 91,546, s_3 = 2$ $\langle l_3 - v_3, u_3 - v_3 \rangle$
SL	$\langle -43100, 30562 \rangle$	$\langle -59701, 12632 \rangle$	$\langle -5414, 82794 \rangle$
EP	$\langle -36446, 45418 \rangle$	$\langle -54159, 23265 \rangle$	$\langle -6781, 81042 \rangle$
HM	$\langle -26109, 61078 \rangle$	$\langle -48399, 34428 \rangle$	$\langle -6781, 79308 \rangle$
DA	$\langle -32164, 53449 \rangle$	$\langle -52638, 27889 \rangle$	$\langle -6781, 79308 \rangle$
Method	$v_4 = 84,765, s_4 = 1$ $\langle l_4 - v_4, u_4 - v_4 \rangle$	$v_5 = 60,847, s_5 = 1$ $\langle l_5 - v_5, u_5 - v_5 \rangle$	$v_6 = 55,806, s_6 = 1$ $\langle l_6 - v_6, u_6 - v_6 \rangle$
SL	$\langle -56055, 6781 \rangle$	$\langle -32137, 30699 \rangle$	$\langle -27096, 35740 \rangle$
EP	$\langle -84765, 6781 \rangle$	$\langle -60847, 30699 \rangle$	$\langle -55806, 35740 \rangle$
HM	$\langle -84765, 6781 \rangle$	$\langle -60847, 30699 \rangle$	$\langle -55806, 35740 \rangle$
DA	$\langle -64668, 6781 \rangle$	$\langle -39656, 30699 \rangle$	$\langle -34615, 35740 \rangle$

than $v_2 = 557,081$ votes, then the second party will get one seat less (provided that the other votes $v_i, i \neq 2$, are unchanged). Note that the second party has the smallest lower bound ($v_2 - l_2 = 35,876$). So, the party which is most sensitive to the lowering of its vote count is the second party, in this case for the d'Hondt method.

We have applied other different divisor methods, and the corresponding seat allocations are shown in Table 7.

In this case, the SD, HM and DA divisor methods give the same seat allocation $s = (11, 4, 2, 2, 2)$. Intervals $\langle l_k - v_k, 0 \rangle$ and $\langle 0, u_k - v_k \rangle, k = 1, \dots, 5$, green (the first bar) for the SD method and yellow (the second bar) for the DA method, are illustrated in Fig. 2. One can see that for the first party with the largest number of votes $v_1 = 1,824,220$, the SD method has got greater bounds than the DA method. A reverse relation holds for the fifth party with the smallest number of

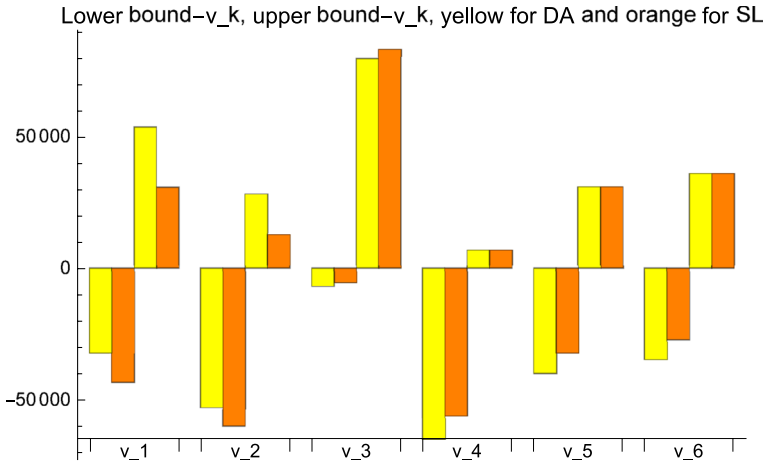


Fig. 3 Intervals $\langle l_k - v_k, 0 \rangle$, $\langle 0, u_k - v_k \rangle$, yellow (the first bar) for the DA method and orange (the second bar) for the SL method, from Table 9 (Croatia, EuP, 2019) (colour figure online)

Table 10 Elections in Spain for the EuP in 2019: allocation of seats to 8 parties

Divisor method	$\mathbf{s} = (s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8)$
BE	(23, 13, 8, 6, 3, 3, 2, 1)
DH	(21, 13, 8, 6, 4, 3, 3, 1)
SL, MS, EP	(20, 13, 8, 6, 4, 3, 3, 2)
SD, HM, DA	(20, 12, 8, 6, 4, 4, 3, 2)

votes $v_5 = 220, 184$. This is in accordance with the property that the DA method tends to favour large states over the SD method.

Example 3 We use data related to elections for the European Parliament in Croatia in 2019, where $n = 6$ parties (or lists) passed the electoral threshold of 5% in the whole country as a single electoral unit (euparl 2021; wikiped 2021). The obtained votes $v_k, k = 1, \dots, 6$, are shown in descending order in Table 8. The allocation of seats $S = 12$ is done by the d’Hondt method: $\mathbf{s}_{DH} = (4, 4, 1, 1, 1, 1)$.

For the purpose of comparison, we have applied other different divisor methods, and the corresponding seat allocations are shown in Table 8.

In this case, the SL, MS, EP, HM and DA divisor methods give the same seat allocation $s = (4, 3, 2, 1, 1, 1)$. According to expressions for these methods given in

Tables 2 and 3, the corresponding lower bounds $l_k - v_k$ and upper bounds $u_k - v_k$ of votes are listed in Table 9.

Intervals $\langle l_k - v_k, 0 \rangle$ and $\langle 0, u_k - v_k \rangle$, $k = 1, \dots, 5$, yellow (the first bar) for the DA method and orange (the second bar) for the SL method are illustrated in Fig. 3. Let us note that for the first party with the largest number of votes $v_1 = 244,076$, the DA method has got greater bounds than the SL method. A reverse relation holds for the sixth party with the smallest number of votes $v_6 = 55,806$. This is in accordance with the property that the SL method 'tends to favour large states' over the DA method.

Note that the fourth party has the smallest upper bound ($u_4 - l_4 = 6,781$), in this case. So, the party which is most sensitive to the increment of its vote count is the fourth party (i.e., the fourth party would gain a seat if its vote count increased by the smallest amount, provided that the other votes v_i , $i \neq 4$, are unchanged).

Example 4 We use data related to elections for the European Parliament in Spain in 2019, that have got no electoral threshold. The allocation of seats $S = 59$ was done by the d'Hondt method: $s_{DH} = (21, 13, 8, 6, 4, 3, 3, 1)$ (euparl 2021; wikiped 2021). Therefore, we take into account $n = 8$ parties (or lists) that have got at least one seat.

For the purpose of comparison, we have also applied other different divisor methods, and the corresponding seat allocations are shown in Table 10.

In this case, the SL and EP divisor methods give the same seat allocation $s = (20, 13, 8, 6, 4, 3, 3, 2)$. According to expressions for these methods given in Tables 2 and 3, the corresponding lower bounds $l_k - v_k$ and upper bounds $u_k - v_k$ of votes are listed in Table 11.

Intervals $\langle l_k - v_k, 0 \rangle$ and $\langle 0, u_k - v_k \rangle$, $k = 1, \dots, 8$, red (the first bar) for the EP method and orange (the second bar) for the SL method are illustrated in Fig. 4. Let us note that for the first party with the largest number of votes $v_1 = 7,369,789$, the EP method has got greater bounds than the SL method. A reverse relation holds for the eighth party with the smallest number of votes $v_8 = 633,090$. This is in accordance with the property that the SL method 'tends to favour large states' over the EP method.

Looking only at the EP method in this case, one can see that the second party has the smallest lower bound ($v_2 - l_2 = 4,554$). So, the party which is most sensitive to the lowering of its vote count is the second party (i.e., the second party would lose a seat if its vote count decreased by the smallest amount).

Example 5 We use data from Marošević et al. (2013), where $n = 5$ hypothetical electoral units in Croatia are suggested on the basis of cluster analysis. The corresponding populations v_k , $k = 1, \dots, 5$ of these 5 hypothetical electoral units are given in descending order in Table 12. (The Croatian Parliament has a total of 151 seats: $S = 140$ seats for 10 actual electoral units, 8 seats for national minorities and 3 seats for Croatian citizens living abroad.) In order to apportion $S = 140$ seats to the 5 hypothetical electoral units proportionally, based on their populations, we applied

Table 11 The lower and upper bounds of votes for the corresponding divisor methods (Spain, EuP, 2019)

Method	$v_1 = 7,369,789$ $s_1 = 20$ $\langle l_1 - v_1, u_1 - v_1 \rangle$	$v_2 = 4,519,205$ $s_2 = 13$ $\langle l_2 - v_2, u_2 - v_2 \rangle$	$v_3 = 2,731,825$ $s_3 = 8$ $\langle l_3 - v_3, u_3 - v_3 \rangle$
SL	$\langle -393586, 41708 \rangle$	$\langle -25432, 398080 \rangle$	$\langle -35561, 341235 \rangle$
EP	$\langle -323610, 45437 \rangle$	$\langle -4554, 405663 \rangle$	$\langle -26896, 338371 \rangle$
Method	$v_4 = 2,258,857$ $s_4 = 6$ $\langle l_4 - v_4, u_4 - v_4 \rangle$	$v_5 = 1,393,684$ $s_5 = 4$ $\langle l_5 - v_5, u_5 - v_5 \rangle$	$v_6 = 1,252,139$ $s_6 = 3$ $\langle l_6 - v_6, u_6 - v_6 \rangle$
SL	$\langle -281597, 91130 \rangle$	$\langle -135428, 233230 \rangle$	$\langle -353385, 13239 \rangle$
EP	$\langle -279052, 86044 \rangle$	$\langle -141545, 224451 \rangle$	$\langle -371281, 1263 \rangle$
Method	$v_7 = 1,018,435$ $s_7 = 3$ $\langle l_7 - v_7, u_7 - v_7 \rangle$	$v_8 = 633,090$ $s_8 = 2$ $\langle l_8 - v_8, u_8 - v_8 \rangle$	
SL	$\langle -119681, 246943 \rangle$	$\langle -93838, 270751 \rangle$	
EP	$\langle -133040, 234967 \rangle$	$\langle -121907, 253200 \rangle$	

different divisor methods. In this case, the condition of Lemma 1 is satisfied. Therefore, the BE and SD methods give equal seat allocation $s = (35, 29, 27, 26, 23)$, and all other mentioned divisor methods give the same seat apportionment. According to expressions in Tables 2 and 3, the corresponding lower and upper bounds of populations are also shown in Table 12.

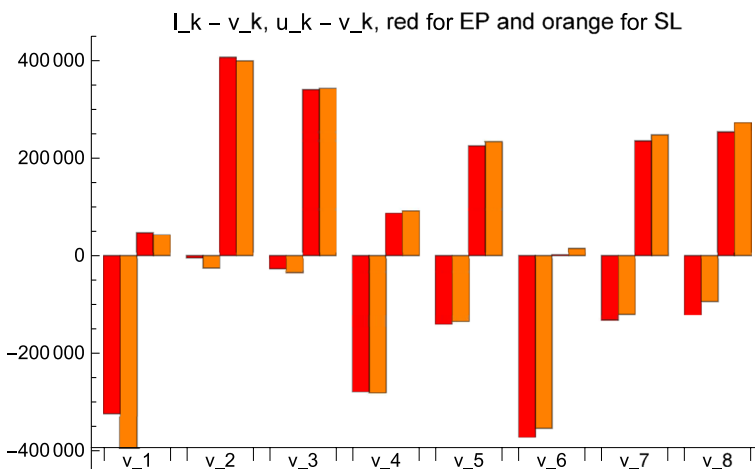


Fig. 4 Intervals $\langle l_k - v_k, 0 \rangle, \langle 0, u_k - v_k \rangle$, red (the first bar) for the EP method and orange (the second bar) for the SL method, from Table 11 (Spain, EuP, 2019)

Table 12 The lower l_k and upper bounds u_k of populations for divisor methods

Method	$v_1 = 1,013,125$ $s_1 = 35$ $\langle l_1, u_1 \rangle$	$v_2 = 851,876$ $s_2 = 29$ $\langle l_2, u_2 \rangle$	$v_3 = 788,584$ $s_3 = 27$ $\langle l_3, u_3 \rangle$
BE	$\langle 989275, 1042058 \rangle$	$\langle 821452, 872414 \rangle$	$\langle 769436, 816129 \rangle$
DH	$\langle 993855, 1051446 \rangle$	$\langle 818419, 868393 \rangle$	$\langle 766688, 810500 \rangle$
SL	$\langle 996261, 1056405 \rangle$	$\langle 821422, 866296 \rangle$	$\langle 765244, 807564 \rangle$
EP	$\langle 996300, 1056489 \rangle$	$\langle 821481, 866262 \rangle$	$\langle 765218, 807515 \rangle$
HM	$\langle 996338, 1056572 \rangle$	$\langle 821541, 866229 \rangle$	$\langle 765191, 807467 \rangle$
DA	$\langle 997082, 1058101 \rangle$	$\langle 822451, 865583 \rangle$	$\langle 764752, 806566 \rangle$
SD	$\langle 1001245, 1061556 \rangle$	$\langle 824554, 864137 \rangle$	$\langle 765658, 804541 \rangle$
Method	$v_4 = 761,239$ $s_4 = 26$ $\langle l_4, u_4 \rangle$	$v_5 = 677,313$ $s_5 = 23$ $\langle l_5, u_5 \rangle$	
BE	$\langle 741956, 787987 \rangle$	$\langle 659516, 703560 \rangle$	
DH	$\langle 738292, 781554 \rangle$	$\langle 653104, 694715 \rangle$	
SL	$\langle 736367, 778198 \rangle$	$\langle 649735, 690100 \rangle$	
EP	$\langle 736331, 778141 \rangle$	$\langle 649668, 690016 \rangle$	
HM	$\langle 736295, 778084 \rangle$	$\langle 649601, 689933 \rangle$	
DA	$\langle 735711, 777058 \rangle$	$\langle 648587, 688532 \rangle$	
SD	$\langle 736209, 774743 \rangle$	$\langle 646250, 685350 \rangle$	

In Table 12, one can see that intervals of populations by particular divisor methods are in accordance with the property of 'favouring large states'. For instance, for the first electoral unit with the largest population $v_1 = 1,013,125$, the BE method (that has the tendency of 'favouring large states' over other divisor

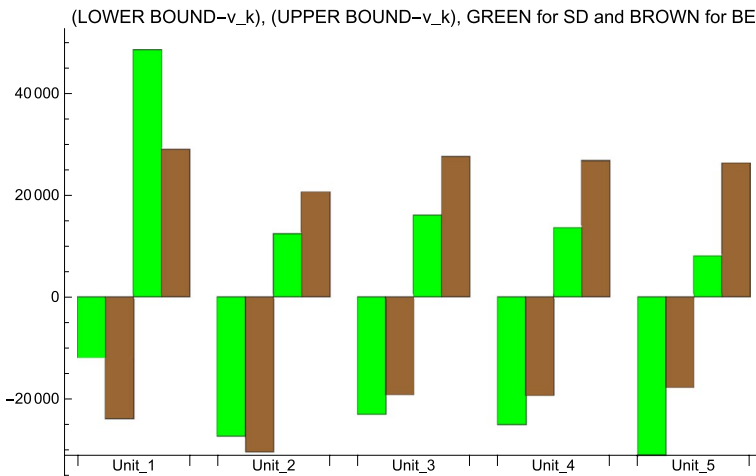


Fig. 5 Intervals $\langle l_k - v_k, 0 \rangle$, $\langle 0, u_k - v_k \rangle$, green (the first bar) for the SD method and brown (the second bar) for the BE method, from Table 12

methods) gets smaller bounds of population, while the SD method gets slightly greater bounds of population. On the other hand, a reverse relation holds for the fifth electoral district with the smallest population $v_5 = 677,313$.

Intervals $\langle l_k - v_k, 0 \rangle$ and $\langle 0, u_k - v_k \rangle$, $k = 1, \dots, 5$, green (the first bar) for the SD method and brown (the second bar) for the BE method are illustrated in Fig. 5. For instance, it can be seen that if the first electoral unit with the largest population $v_1 = 1,013,125$ had slightly smaller population $v'_1 = 1,001,245$, then the SD method would give one seat less to the first electoral unit, while the BE method would give the unchanged number of seats $s_1 = 35$ to the first unit (provided that the other populations v_i , $i \neq 1$, are unchanged). This is in accordance with the property that the BE method 'tends to favour large states' over the SD method.

Looking only at the SD method and five hypothetical electoral units in this case, one can see that the most sensitive upper bound of the population is in the fifth electoral unit (i.e., the smallest upper bound is $u_5 - v_5 = 8,037$). So, the fifth electoral unit would gain a seat if its population increased by a small percentage, provided that the other populations v_i , $i \neq 5$, are unchanged. Note that the least sensitive upper bound of the population is in the first electoral unit (i.e., the largest upper bound is $u_1 - v_1 = 48,431$).

4 Concluding remarks

Seat allocation is based on votes obtained by parties (i.e., seat apportionment is based on populations of electoral units). Various proportional methods can give different seat allocations. We consider the following proportional divisor electoral methods: the method of smallest divisors (SD), the Danish method (DA), the method of harmonic mean (HM), the method of equal proportions (EP), the Sainte-Laguë method (SL), the modified Sainte-Laguë method (MS), the d'Hondt method (DH), and the Belgian method (BE).

We look at divisor methods on the basis of reframing the well-known theorem on the min-max inequality. By using bounds of votes, we give conditions under which different divisor methods give specified seat allocation. These conditions are expressed by means of the lower and upper bounds of votes, i.e., by means of the intervals of vote counts, within which one can change the number of votes v_k of the party k such that the specified divisor method gives the same seat allocation (provided that the other votes v_i , $i \neq k$, are unchanged). Knowing these intervals of vote counts could help in the analysis of elections, planning of an electoral strategy and understanding of different divisor methods.

For the purpose of empirical research and illustration, we consider the bounds of votes of a specified seat assignment for divisor methods in a few cases of elections (in Austria, Hungary, Croatia and Spain). Comparing the cases when particular divisor methods give equal seat allocation, we can conclude that the bounds of votes for these divisor methods are in line with their tendency of 'favouring large states'.

By connecting the obtained proposition and assertions, one can analogously state similar assertions about bounds of votes for specified seat allocation of the divisor methods that would be taken into account. The related problem for quota methods may be an area for future research.

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Declarations

Conflict of interest The authors have no affiliation with any organization with a direct or indirect financial interest in the subject matter discussed in the manuscript.

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