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Efficiency analysis in two-stage structures using fuzzy data envelopment analysis

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Abstract Two-stage data envelopment analysis (TsDEA) models evaluate the performance of a set of production systems in which each system includes two operational stages. Taking into account the internal structures is commonly found in many situations such as seller-buyer supply chain, health care provision and environmental management. Contrary to conventional DEA models as a black-box structure, TsDEA provides further insight into sources of inefficiencies and a more informative basis for performance evaluation. In addition, ignoring the qualitative and imprecise data leads to distorted evaluations, both for the subunits and the system efficiency. We present the fuzzy input and output-oriented TsDEA models to calculate the global and pure technical efficiencies of a system and sub-processes when some data are fuzzy. To this end, we propose a possibilistic programming problem and then convert it into a deterministic interval programming problem using the α-level based method. The proposed method preserves the link between two stages in the sense that the total efficiency of the system is equal to the product of the efficiencies derived from two stages. In addition to the study of technical efficiency, this research includes two further contributions

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to the ancillary literature; firstly, we minutely discuss the efficiency decompositions to indicate the sources of inefficiency and secondly, we present a method for ranking the efficient units in a fuzzy environment. An empirical illustration is also utilised to show the applicability of the proposed technique.

Keywords Data envelopment analysis · Efficiency · Two-level systems · Fuzzy data · Ranking

Mathematics Subject Classification 90C05 · 94D05 · 90C70 · 90C90

1 Introduction

What it is extremely vital for organisations in today's competitive market is to be fully aware how efficiently and effectively they are operating compared to rivals. For instance, a particular research and development (R&D) centre may desire to compare its performance with the similar R&D centres. In the ancillary literature, two major approaches are developed for measuring efficiency, namely, parametric and non-parametric frontier approaches. Contrary to the parametric techniques, non-parametric techniques do not require a *priori* assumptions for making up the production function. Data envelopment analysis (DEA) is a powerful and easy-to-use non-parametric technique to measure the relative efficiency of a set of decision-making units (DMUs) where each DMU consumes multiple inputs to produce multiple outputs. The original DEA under the constant returns to scale (CRS) was developed in Charnes et al. (1978) by the use of linear programming (LP) with respect to production economics concepts.

The emphasis of conventional DEA models is on a black-box function in a sense that internal or linking activities are neglected in these models. However, each DMU is in practice may be composed of a series of sequential activities (sub-DMUs) occurring in various sectors such as hospitals, universities, R&D and etc. One of the long-standing challenge in DEA is whether a "black box" treatment of efficient production behaviour without considering the internal structure is acceptable. Färe and Primont (1984)'s study is the starting point in the DEA literature for breaking down the unknown structure of electricity generation plants into several sub-processes. However, Färe and Grosskopf (2000)'s study received a great deal of attention of researchers such as Yu and Chen (2011) and Amirteimoori (2013) dealing with network DEA models.

A two-stage structure, namely two-stage data envelopment analysis (TsDEA), is one of the simplest and best-known techniques within network DEA in which the first stage uses inputs to produce outputs that then become the inputs to the second stage (Agrell and Hatami-Marbini 2013). The second stage thus consume these first stage outputs to produce its own outputs. One refers to the first stage outputs as *intermediate measures*. Agrell and Hatami-Marbini (2013) provided an overview of TsDEA models that are categorised into three groups; (1) two-stage process DEA models, (2) game theory DEA models, and (3) bi-level programming. The two-stage models are the special case of multi-stage framework where each DMU is composed of two divisions. The game theory DEA models exploit the concept of non-cooperative and cooperative games in game theory to treat the network structure of operations. The final group includes those methods which have been developed based on bi-level programming aiming to evaluate the performance of a two-stage process in decentralised organisations.

Setting aside the black-box and network DEA models, the negligence of uncertainty in measuring and collecting the data might have adverse effect on the results. There are four main approaches in the relevant literature to deal with uncertainty in DEA models. First, *interval DEA* initiated by Cooper et al. (1999) aims to evaluate the relative efficiency of units in the situations where the values of factors lie within the bounded interval. Second, *chance constrained DEA* originally proposed by Land et al. (1993) makes use of chance constrained programming to develop efficient frontiers where the outputs are often assumed to be stochastic by a joint distribution and the inputs are deterministic. Third, *stochastic frontier analysis* originated by Aigner et al. (1977) and Meeusen and van den Broeck (1977), puts forward the existence of technical inefficiencies of production of DMUs for producing a particular output, which can be carried out for cross-sectional and panel data. Lately, Olesen and Petersen (2016) reviewed stochastic DEA in three directions; deviations from the deterministic frontier, random noise, and the stochastic frontier based on the production possibility set (PPS). The last uncertainty approach in DEA is known as *fuzzy DEA* introduced by Sengupta (1992), which is of interest to us in this study.

Hatami-Marbini et al. (2011) and Emrouznejad et al. (2014) classified the fuzzy DEA models into six groups: the tolerance approach, the α-level based approach, the fuzzy ranking approach, the possibility and credibility approach, the fuzzy arithmetic, and the fuzzy random/type-2 fuzzy sets (see e.g., Saati et al. 2002; Lertworasirikul et al. 2003; Wen et al. 2011; Ruiz and Sirvent 2017).

Although the neglect of the internal linking activities can be observed in the above fuzzy DEA models, there are only few studies addressed the network DEA problems with fuzzy data. Kao and Liu (2011) and Liu (2014a, b) developed the fuzzy version of the relational two-stage model of Kao and Hwang (2008) to yield the fuzzy efficiency by making use of a pair of two-level mathematical programs introduced by Kao and Liu (2000). On the basis of the parallel production systems argued in Kao (2009, 2012), Kao and Lin (2012) utilised the idea of Kao and Liu (2000) to compute the fuzzy system and process efficiencies for parallel systems when some input and output data are characterised by fuzzy numbers. Taking into account Kao and Liu (2011, 2012), Lozano (2014a, b) proposed the alternative methods for computing the fuzzy efficiencies of the distinctive stages. The difficulty has been observed in all the above-mentioned fuzzy network DEA studies due to the nonlinear programming models.

In this paper, in line with the literature we put emphasis on the relational two-stage model of Kao and Hwang (2008) to develop several new fuzzy TsDEA models using the α-level based approach. Referring to Hatami-Marbini et al. (2011), the α-level based approach is known as the most prevalent fuzzy DEA approaches in terms of the number of existing studies. In addition, the α-level based approach takes various values of α into account to keep track of the efficiency change when the possibility level α varies. Importantly, the decision maker would benefit from the efficiency scores with different α values to comprehend the sensitivity of the results to small variations. Beyond technical efficiencies, we present the efficiency decompositions to determine the sources of inefficiency in the fuzzy environment as well as presenting a method for ranking the efficient DMUs.

The structure of the paper is organised as follows: In Sect. 2, we briefly review the two-stage DEA models and show how to compute the technical and scale efficiencies with precise data. In Sect. 3, we generalise the two-stage models to deal with the fuzzy inputs and/or outputs. In addition to global and pure technical efficiencies, we introduce the efficiency decomposition in a fuzzy environment. The case study of 24 non-life insurance companies in Taiwan is presented to explain the efficacy of the proposed method in Sect. 4. Finally, in Sect. 5, some conclusions are drawn.

2 Two-stage DEA model

Consider a set of n DMUs ($j=1,\ldots,n$) consisting of two stages where the first stage utilises m inputs $x_j = (x_{1j}, \ldots, x_{mj}) \in \mathbb{R}_+^m$ to produce q intermediate measures $z_j = (z_{1j}, \ldots, z_{qj}) \in \mathbb{R}_+^q$, and the second stage generates s outputs $y_j = (y_{1j}, \ldots, y_{sj}) \in \mathbb{R}_+^s$ using the intermediate measures. The structure is depicted in Fig. 1.

According to the CCR model, the efficiency of the whole process and the two distinct stages for a specific DMU$_o$ under the constant returns to scale (CRS) can be computed as (Agrell and Hatami-Marbini 2013):

$$\begin{aligned} E_o &= \max\left\{ \frac{u^T y_o}{v^T x_o} \;\middle|\; \begin{array}{c} \dfrac{u^T y_j}{v^T x_j} \leq 1, \forall j, \\ u \geq 0, v \geq 0, \end{array} \right\} \\ \text{Stage (1) } E_o^1 &= \max\left\{ \frac{w^T z_o}{v^T x_o} \;\middle|\; \begin{array}{c} \dfrac{w^T z_j}{v^T x_j} \leq 1, \forall j, \\ w \geq 0, v \geq 0, \end{array} \right\} \\ \text{Stage (2) } E_o^2 &= \max\left\{ \frac{u^T y_o}{\bar{w}^T z_o} \;\middle|\; \begin{array}{c} \dfrac{u^T y_j}{\bar{w}^T z_j} \leq 1, \forall j, \\ u \geq 0, \bar{w} \geq 0, \end{array} \right\} \end{aligned} \tag{1}$$

where v and w are the weight vectors associated to the input and output (intermediate measure) for Stage 1, respectively, and $\bar{w}$ and u are the weight vectors associated to the input (intermediate measure) and output for Stage 2, respectively. Besides, the superscript T in the above models represents the transpose operator.

Kao and Hwang (2008) modified the conventional DEA model to define the link between the two stages within the whole production system by setting $\bar{w}^T = w^T$ which has not taken into account in the approach of Seiford and Zhu (1999). To make the two-stage model relational, Kao and Hwang (2008) proposed the overall efficiency of the system as the product of the efficiencies of the two stages with the identical weights

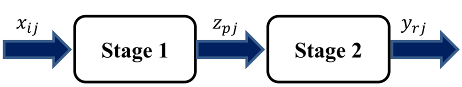

Fig. 1 A two-stage structure

for intermediate measures (i.e., $E_o = E_o^1 \times E_o^2$). As a consequence, the input-oriented model for measuring the overall efficiency of DMU$_o$ is expressed as follows:

$$E_o = \max \left\{ \frac{u^T y_o}{v^T x_o} \middle| \begin{array}{l} \frac{u^T y_j}{v^T x_j} \leq 1, \quad \forall j, \\ \frac{w^T z_j}{v^T x_j} \leq 1, \quad \forall j, \\ \frac{u^T y_j}{w^T z_j} \leq 1, \quad \forall j, \\ u \geq 0, v \geq 0, w \geq 0, \end{array} \right\}. \tag{2}$$

The constraint $u^T y_j \big/ v^T x_j \leq 1$ is redundant by the virtue of $w^T z_j / v^T x_j \leq 1$ and $u^T y_j / w^T z_j \leq 1$. Model (2) is therefore expressed as follows:

$$E_o = \max \left\{ u^T y_o \middle| \begin{array}{l} w^T z_j - v^T x_j \leq 0, \quad \forall j, \\ u^T y_j - w^T z_j \leq 0, \quad \forall j, \\ v^T x_o = 1, \\ u \geq 0, v \geq 0, w \geq 0, \end{array} \right\} \tag{3}$$

where E_o represents the overall efficiency of the production system. Let v^*, u^* and w^* be the optimal values of model (3). The efficiencies of Stages 1 and 2, respectively, can be calculated as $E_o^1 = w^{*T} z_o / v^{*T} x_o$ and $E_o^2 = u^{*T} y_o / w^{*T} z_o$. However, these efficiencies (E_o^1 and E_o^2) might not be unique because of non-uniqueness of the optimal weights derived from model (3). To deal with the problem, the following model is developed to seek a set of weights that produces the largest efficiency score for Stage 1 while preserving the overall efficiency at E_o^*:

$$E_o^1 = \max \left\{ w^T z_o \middle| \begin{array}{l} u^T y_o = E_o^*, \\ w^T z_j - v^T x_j \leq 0, \quad \forall j, \\ u^T y_j - w^T z_j \leq 0, \quad \forall j, \\ v^T x_o = 1, \\ u \geq 0, v \geq 0, w \geq 0, \end{array} \right\} \tag{4}$$

where E_o^* is the optimal value obtained from model (3). Given that the efficiency measures of the overall system and Stage 1 are obtained from models (3) and (4), one attains the efficiency measure of Stage 2 as $E_o^2 = E_o^* \big/ E_o^{1*}$. Note that model (4) can be reformulated for Stage 2 if the decision maker gives a higher priority to Stage 2.

Writing the dual of model (3) for DMU$_o$, we arrive at the following formulation which is of interest particularly to economists as it pertains to Farrell's measure and a number of axioms:

$$E_o = \min \left\{ \theta \middle| \begin{array}{l} \theta x_o - \sum_j \lambda_j x_j \geq 0, \\ \sum_j (\lambda_j - \mu_j) z_j \geq 0, \\ \sum_j \mu_j y_j \geq y_o, \\ \lambda_j, \mu_j \geq 0, \quad \forall j. \end{array} \right\} \tag{5}$$

As a special case, the conventional DEA model as a black box is made when $\mu_j = \lambda_j$ for all DMUs in the above model. However, the inequality between μ_j and λ_j majorly occurs for some DMUs to study the internal efficiencies. In addition, the efficiency of Stage 1 can be measured using the dual of model (4) when E_o^* is the overall efficiency of model (5):

$$E_o^1 = \min\left\{\theta + hE_0^* \left| \begin{array}{l} \theta x_o - \sum_j \lambda_j x_j \geq 0, \\ \sum_j (\lambda_j - \mu_j) z_j \geq z_0, \\ \sum_j \mu_j y_j + hy_o \geq 0, \\ \lambda_j, \mu_j \geq 0, \quad \forall j. \end{array} \right.\right\} \tag{6}$$

Obviously, the efficiency of Stage 2 can be computed as $E_o^2 = E_o^* / E_o^{1*}$.

For the sake of computing the scale efficiency of each stage of system, we need to formulate two different models for two processes of the production system under the variable returns to scale (VRS) assumption. Kao and Hwang (2011) developed the following models to preserve the relation between two processes in a way that models (7) and (8) are the input- and output-oriented models, respectively:

$$B_o^1 = \max\left\{w'^T z_o - g_o \left| \begin{array}{ll} u^T y_o = E_o^*, & \\ w^T z_j - v^T x_j \leq 0, & \forall j, \\ u^T y_j - w^T z_j \leq 0, & \forall j, \\ (w'^T z_j - g_o) - v^T x_j \leq 0, & \forall j, \\ v^T x_o = 1, & \\ u \geq 0, v \geq 0, w \geq 0, w' \geq 0, & \end{array} \right.\right\} \tag{7}$$

$$B_o^2 = \max\left\{u^T y_o \left| \begin{array}{ll} u^T y_o = E_o^* v^T x_o, & \\ w^T z_o = E_o^{1*} v^T x_o, & \\ w^T z_j - v^T x_j \leq 0, & \forall j, \\ u^T y_j - w^T z_j \leq 0, & \forall j, \\ u^T y_j - (w'^T z_j - g_o) \leq 0, & \forall j, \\ w'^T z_o - g_o = 1, & \\ u \geq 0, v \geq 0, w \geq 0, w' \geq 0, & \end{array} \right.\right\} \tag{8}$$

where E_o^* and E_o^{1*} are the optimal objective value of (3) and (4), respectively. Conventionally, the ratio of the CRS technical efficiency to the output VRS technical efficiency is called *scale efficiency*. Adapted for the scale efficiency from the conventional viewpoint, the ratio of the CRS efficiency to the VRS efficiency is the *(input) scale efficiency* for Stage 1 as $S_o^1 = E_o^1 / B_o^1$ and the *(output) scale efficiency* for Stage 2 as $S_o^2 = E_o^2 / B_o^2$ where E_o^1 and E_o^2 are called the *global technical efficiencies* for Stages 1 and 2, and B_o^1 and B_o^2 are called the *pure (input and output) technical efficiencies* for Stages 1 and 2. Therefore, the scale efficiency of the system is the product of the scale efficiencies of the two stages, i.e., $S_o = S_o^1 \times S_o^2$.

In the next section, we look into the fuzzy TsDEA models associated to the above-mentioned models to obtain the *global and pure technical efficiencies* in which the inputs, outputs and intermediate measures are fuzzy numbers.

3 Fuzzy efficiency measurement

We presume that the observations $(\tilde{x}_j, \tilde{z}_j, \tilde{y}_j)$ associated to a given DMU$_j$ ($j=1,\ldots,n$) can be represented by $\tilde{x}_j = (x_j^l, x_j^m, x_j^u)$, $\tilde{z}_j = (z_j^l, z_j^m, z_j^u)$, and $\tilde{y}_j = (y_j^l, y_j^m, y_j^u)$ to deal adequately with the uncertainty in the performance assessment.[1] The objective of this section is to measure the performance of a two-stage production system where the inputs and outputs are characterised by fuzzy numbers.

3.1 Global technical efficiency

Here, we develop the fuzzy DEA models to evaluate the overall efficiency of the system and two stages efficiency scores in the input- and output-orientation cases.

By applying fuzzy inputs, intermediate measures and outputs to model (3), we arrive at the following model for measuring the [input-oriented] overall efficiency of DMU$_o$:

$$\tilde{E}_o = \max \left\{ \sum_r u_r \tilde{y}_{ro} \left| \begin{array}{l} \sum_p w_p \tilde{z}_{pj} - \sum_i v_i \tilde{x}_{ij} \leq 0, \quad \forall j, \\ \sum_r u_r \tilde{y}_{rj} - \sum_p w_p \tilde{z}_{pj} \leq 0, \quad \forall j, \\ \sum_i v_i \tilde{x}_{io} = 1, \\ u_r, v_i, w_p \geq 0, \quad \forall r, i, p, \end{array} \right. \right\} \tag{9}$$

where "~" stands for the fuzziness in the above model. Due to the observations with fuzzy numbers, the resulting efficiency $\tilde{E}_o$ should also be a fuzzy number. Making use of Zadeh's extension principle (Zadeh 1978) is one of popular ways to determine the membership function of $\tilde{E}_o$, denoted by $\mu_{\tilde{E}_o}$, which interprets the link between $\mu_{\tilde{E}_o}$ and the membership function of $\tilde{x}_{ij}$, $\tilde{y}_{rj}$ and $\tilde{z}_{pj}$ as:

$$\mu_{\tilde{E}_o}(e) = \sup_{x,y,z} \min\{\mu_{\tilde{x}_{ij}}(x_{ij}), \mu_{\tilde{y}_{rj}}(y_{rj}), \mu_{\tilde{z}_{pj}}(z_{pj}), \forall i, r, p, j | e = E_o(x, y, z)\}$$

where $E_o(x, y, z)$ can be obtained from Model (5). The idea of extension principle can be employed with respect to the α-levels of $\tilde{E}_o$ (Nguyen 1978). Obviously, the minimum of $\mu_{\tilde{x}_{ij}}(x_{ij})$, $\mu_{\tilde{y}_{rj}}(y_{rj})$ and $\mu_{\tilde{z}_{pj}}(z_{pj})$, $\forall i, r, p, j$ results in $\mu_{\tilde{E}_o}(e)$. Given a certain α-level, it is requisite that the values of $\mu_{\tilde{x}_{ij}}(x_{ij})$, $\mu_{\tilde{y}_{rj}}(y_{rj})$ and $\mu_{\tilde{z}_{pj}}(z_{pj})$, $\forall i, r, p, j$ are equal to or greater than the value of α where one

[1] The overview of fuzzy sets theory and fuzzy numbers are briefly provided in "Appendix 1".

of them must be equal to α, resulting in the efficiency score which is equal to e. Note that all α-levels have a nested structure, i.e., for $0 < \alpha_2 < \alpha_1 \leq 1$, $\left[(x_{ij})^L_{\alpha_1},\ (x_{ij})^U_{\alpha_1}\right] \subseteq \left[(x_{ij})^L_{\alpha_2},\ (x_{ij})^U_{\alpha_2}\right]$, $\left[(y_{rj})^L_{\alpha_1},\ (y_{rj})^U_{\alpha_1}\right] \subseteq \left[(y_{rj})^L_{\alpha_2},\ (y_{rj})^U_{\alpha_2}\right]$ and $\left[(z_{pj})^L_{\alpha_1},\ (z_{pj})^U_{\alpha_1}\right] \subseteq \left[(z_{pj})^L_{\alpha_2},\ (z_{pj})^U_{\alpha_2}\right]$. We therefore exert the α-level based approach on the objective function and constraints of model (9) to get the following interval programming model:

$$E_o = \max\left\{\sum_r u_r \widehat{y}_{ro} \left| \begin{array}{l} \sum_p w_p \widehat{z}_{pj} - \sum_i v_i \widehat{x}_{ij} \leq 0, \quad \forall j, \\ \sum_r u_r \widehat{y}_{rj} - \sum_p w_p \widehat{z}_{pj} \leq 0, \quad \forall j, \\ \sum_i v_i \widehat{x}_{io} = 1, \\ u_r, v_i, w_p \geq 0, \quad \forall r, i, p. \end{array} \right.\right\} \tag{10}$$

where

$$\widehat{x}_{ij} \in [\alpha x^m_{ij} + (1-\alpha)x^l_{ij}, \alpha x^m_{ij} + (1-\alpha)x^u_{ij}], \quad \forall i, j,$$

$$\widehat{z}_{pj} \in [\alpha z^m_{pj} + (1-\alpha)z^l_{pj}, \alpha z^m_{pj} + (1-\alpha)z^u_{pj}], \quad \forall p, j,$$

$$\widehat{y}_{rj} \in [\alpha y^m_{rj} + (1-\alpha)y^l_{rj}, \alpha y^m_{rj} + (1-\alpha)y^u_{rj}], \quad \forall r, j.$$

It is essential to note that model (10) is a non-linear programming model. To create the linear programming model, we substitute the new variables in model (10) in the way that enables us to not only satisfy the constraint, but also maximise the objective function. The resulting model is formulated below:

$$E^\alpha_o = \max\left\{\sum_r \bar{y}_{ro} \left| \begin{array}{ll} \sum_p \bar{z}_{pj} - \sum_i \bar{x}_{ij} \leq 0, \quad \forall j, & \\ \sum_r \bar{y}_{rj} - \sum_p \bar{z}_{pj} \leq 0, \quad \forall j, & \\ \sum_i \bar{x}_{io} = 1, & \\ v_i(\alpha x^m_{ij} + (1-\alpha)x^l_{ij}) \leq \bar{x}_{ij} \leq v_i(\alpha x^m_{ij} + (1-\alpha)x^u_{ij}), & \forall i, j, \\ u_r(\alpha y^m_{rj} + (1-\alpha)y^l_{rj}) \leq \bar{y}_{rj} \leq u_r(\alpha y^m_{rj} + (1-\alpha)y^u_{rj}), & \forall r, j, \\ w_p(\alpha z^m_{pj} + (1-\alpha)z^l_{pj}) \leq \bar{z}_{pj} \leq w_p(\alpha z^m_{pj} + (1-\alpha)z^u_{pj}), & \forall p, j, \\ u_r, v_i, w_p, \bar{x}_{ij}, \bar{y}_{rj}, \bar{z}_{pj} \geq 0, \quad \forall r, i, p, j, & \end{array} \right.\right\} \tag{11}$$

where $\bar{x}_{ij} = v_i \widehat{x}_{ij}$, $\bar{z}_{pj} = w_p \widehat{z}_{pj}$ and $\bar{y}_{rj} = u_r \widehat{y}_{rj}$. The objective function of model (11) proceeds with the identical idea of the conventional TsDEA model (3) to measure the overall efficiency of the system E^α_o for a given $\alpha \in [0, 1]$. That is, the system is efficient $E^\alpha_o = 1$ for a given α if both the first and second stages are efficient $E^{1\alpha}_o = E^{2\alpha}_o = 1$. Over and above, model (11) is capable of computing the optimal values of v^*, u^* and w^* of DMU_o for a particular α. These optimal weights and considering the decomposition $E_o^{\alpha} = E_o^{1\alpha} \times E_o^{2\alpha}$ make use of obtaining two stages efficiency scores as $E_o^{1\alpha} = w^{*T} z_o / v^{*T} x_o$ and $E_o^{2\alpha} = u^{*T} y_o / w^{*T} z_o$ for all α-levels. However, analogous to model (3), model (11) may have alternative optimal solutions which lead to the identical optimal objective value and satisfying all constraints. Assuming high priority in Stage 1 contrary to Stage 2, we propose the following program for a given α level to compute the largest efficiency measure of Stage 1 while maintaining the overall efficiency at $E_o^{\alpha *}$:

$$E_o^{1\alpha} = \max \left\{ \sum_p \bar{z}_{po} \left| \begin{array}{ll} \sum_r \bar{y}_{ro} = E_o^{\alpha *}, & \\ \sum_p \bar{z}_{pj} - \sum_i \bar{x}_{ij} \le 0, \quad \forall j, & \\ \sum_r \bar{y}_{rj} - \sum_p \bar{z}_{pj} \le 0, \quad \forall j & \\ \sum_i \bar{x}_{io} = 1, & \\ v_i(\alpha x_{ij}^m + (1-\alpha) x_{ij}^l) \le \bar{x}_{ij} \le v_i(\alpha x_{ij}^m + (1-\alpha) x_{ij}^u), & \forall i, j, \\ u_r(\alpha y_{rj}^m + (1-\alpha) y_{rj}^l) \le \bar{y}_{rj} \le u_r(\alpha y_{rj}^m + (1-\alpha) y_{rj}^u), & \forall r, j, \\ w_p(\alpha z_{pj}^m + (1-\alpha) z_{pj}^l) \le \bar{z}_{pj} \le w_p(\alpha z_{pj}^m + (1-\alpha) z_{pj}^u), & \forall p, j \\ u_r, v_i, w_p, \bar{x}_{ij}, \bar{y}_{rj}, \bar{z}_{pj} \ge 0, \quad \forall r, i, p, j. & \end{array} \right. \right\} \tag{12}$$

After we calculate the efficiency measure of Stage 1 ($E_o^{1\alpha *}$) using the above model alongside preserving the overall efficiency ($E_o^{\alpha *}$), the efficiency score of Stage 2 is worked out as $E_o^{2\alpha} = E_o^{\alpha *} \big/ E_o^{1\alpha *}$. It should be emphasised that the same idea can be accommodated to the circumstance where one gives priority to the second stage. A DMU as whole and its Stage 1 (Stage 2) are called *efficient* if $E_j^{\alpha *} = 1$ and $E_j^{1\alpha *} = 1$ ($E_j^{2\alpha *} = 1$). Since the overall efficiency of DMU is the product of the efficiencies of the first stage and the second stage, in the case of the *efficient* DMU leads to *efficient* Stage 1 and *efficient* Stage 2.

It is interesting to investigate the output-oriented fuzzy TsDEA model that attempts to maximise the outputs of the production system while consuming no more than the observed value of any input is formulated as follows:

$$F_o^{\alpha} = \min \left\{ \sum_i \bar{x}_{io} \left| \begin{array}{ll} \sum_p \bar{z}_{pj} - \sum_i \bar{x}_{ij} \le 0, \quad \forall j, & \\ \sum_r \bar{y}_{rj} - \sum_p \bar{z}_{pj} \le 0, \quad \forall j, & \\ \sum_r \bar{y}_{ro} = 1, & \\ v_i(\alpha x_{ij}^m + (1-\alpha) x_{ij}^l) \le \bar{x}_{ij} \le v_i(\alpha x_{ij}^m + (1-\alpha) x_{ij}^u), & \forall i, j, \\ u_r(\alpha y_{rj}^m + (1-\alpha) y_{rj}^l) \le \bar{y}_{rj} \le u_r(\alpha y_{rj}^m + (1-\alpha) y_{rj}^u), & \forall r, j, \\ w_p(\alpha z_{pj}^m + (1-\alpha) z_{pj}^l) \le \bar{z}_{pj} \le w_p(\alpha z_{pj}^m + (1-\alpha) z_{pj}^u), & \forall p, j, \\ u_r, v_i, w_p, \bar{x}_{ij}, \bar{y}_{rj}, \bar{z}_{pj} \ge 0, \quad \forall r, i, p, j. & \end{array} \right. \right\} \tag{13}$$

Likewise, a DMU and its Stage 1 (Stage 2) are called *efficient* if $F_j^{\alpha *} = 1$ and $F_j^{1\alpha *} = 1$ ($F_j^{2\alpha *} = 1$).

Proposition *The inverse of the optimal objective value of the input-oriented fuzzy TsDEA model (11) equals to the optimal objective value of the output-oriented fuzzy TsDEA model (13), i.e.,* $F_o^{\alpha *} = \frac{1}{E_o^{\alpha *}}$.

3.2 Pure technical efficiency and efficiency decomposition

Let us proceed with the evaluation of the system and two processes under the VRS assumption when the input, output and intermediate data are represented by fuzzy numbers. Models (7) and (8) can be expressed by the following fuzzy LP models to obtain the pure technical efficiency measures for the first and second stages, respectively:

$$\tilde{B}_o^1 = \max \left\{ w'^T \tilde{z}_o - g_o \left| \begin{array}{ll} u^T \tilde{y}_o = E_o^*, & \\ w^T \tilde{z}_j - v^T \tilde{x}_j \leq 0, & \forall j, \\ u^T \tilde{y}_j - w^T \tilde{z}_j \leq 0, & \forall j, \\ (w'^T \tilde{z}_j - g_o) - v^T \tilde{x}_j \leq 0, \forall j, & \\ v^T \tilde{x}_o = 1, & \\ u \geq 0, v \geq 0, w \geq 0, w' \geq 0. & \end{array} \right. \right\} \tag{14}$$

$$\tilde{B}_o^2 = \max \left\{ u^T \tilde{y}_o \left| \begin{array}{ll} u^T \tilde{y}_o = v^T \tilde{x}_o E_o^*, & \\ w^T \tilde{z}_o = v^T \tilde{x}_o E_o^{1*}, & \\ w^T \tilde{z}_j - v^T \tilde{x}_j \leq 0, & \forall j, \\ u^T \tilde{y}_j - w^T \tilde{z}_j \leq 0, & \forall j, \\ u^T \tilde{y}_j - (w'^T \tilde{z}_j - g_o) \leq 0, \forall j, & \\ w'^T \tilde{z}_o - g_o = 1, & \\ u \geq 0, v \geq 0, w \geq 0, w' \geq 0. & \end{array} \right. \right\} \tag{15}$$

where $\tilde{x}_j = (x_j^l, x_j^m, x_j^u)$, $\tilde{z}_j = (z_j^l, z_j^m, z_j^u)$, and $\tilde{y}_j = (y_j^l, y_j^m, y_j^u)$ are the fuzzy input, intermediate and output values of the jth DMU, respectively. The objective value of (14), $\tilde{B}_o^1$, is the pure [input] technical efficiency of Stage 1 and the objective value of (15), $\tilde{B}_o^2$, is the pure [output] technical efficiency of Stage 2. Likewise, the following models are developed using the α-level based approach to solve the above fuzzy LP models:

$$B_o^{1\alpha} = \max\left\{ \sum_p \bar{z}'_{po} - \bar{g}_o \;\middle|\; \begin{array}{ll} \sum_r \bar{y}_{ro} = E_o^{\alpha*}, & \\ \sum_p \bar{z}_{pj} - \sum_i \bar{x}_{ij} \le 0, \quad \forall j, & \\ \sum_r \bar{y}_{rj} - \sum_p \bar{z}_{pj} \le 0, \quad \forall j, & \\ \left(\sum_p \bar{z}'_{pj} - \bar{g}_o\right) - \sum_i \bar{x}_{ij} \le 0, \quad \forall j, & \\ \bar{x}_{io} = 1, & \\ v_i(\alpha x_{ij}^m + (1-\alpha)x_{ij}^l) \le \bar{x}_{ij} \le v_i(\alpha x_{ij}^m + (1-\alpha)x_{ij}^u), & \forall i, j, \\ u_r(\alpha y_{rj}^m + (1-\alpha)y_{rj}^l) \le \bar{y}_{rj} \le u_r(\alpha y_{rj}^m + (1-\alpha)y_{rj}^u), & \forall r, j, \\ w_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^l) \le \bar{z}_{pj} \le w_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^u), & \forall p, j, \\ w'_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^l) \le \bar{z}'_{pj} \le w'_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^u), & \forall p, j, \\ \alpha g^m + (1-\alpha)g^l \le \bar{g}_o \le \alpha g^m + (1-\alpha)g^u, & \\ u \ge 0, v \ge 0, w \ge 0, w' \ge 0. & \end{array} \right\} \tag{16}$$

$$B_o^{2\alpha} = \max\left\{ \sum_r \bar{y}_{ro} \;\middle|\; \begin{array}{ll} \sum_r \bar{y}_{ro} = E_o^{\alpha*} \sum_i \bar{x}_{io}, & \\ \sum_p \bar{z}_{po} = E_o^{1\alpha*} \sum_i \bar{x}_{io}, & \\ \sum_p \bar{z}_{pj} - \sum_i \bar{x}_{ij} \le 0, \quad \forall j, & \\ \sum_r \bar{y}_{rj} - \sum_p \bar{z}_{pj} \le 0, \quad \forall j, & \\ \sum_r \bar{y}_{rj} - (\sum_p \bar{z}'_{pj} - \bar{g}_o) \le 0, \quad \forall j, & \\ \sum_p \bar{z}'_{po} - \bar{g}_o = 1, & \\ v_i(\alpha x_{ij}^m + (1-\alpha)x_{ij}^l) \le \bar{x}_{ij} \le v_i(\alpha x_{ij}^m + (1-\alpha)x_{ij}^u), & \forall i, j, \\ u_r(\alpha y_{rj}^m + (1-\alpha)y_{rj}^l) \le \bar{y}_{rj} \le u_r(\alpha y_{rj}^m + (1-\alpha)y_{rj}^u), & \forall r, j, \\ w_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^l) \le \bar{z}_{pj} \le w_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^u), & \forall p, j \\ w'_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^l) \le \bar{z}'_{pj} \le w'_p(\alpha z_{pj}^m + (1-\alpha)z_{pj}^u), & \forall p, j, \\ \alpha g^m + (1-\alpha)g^l \le \bar{g}_o \le \alpha g^m + (1-\alpha)g^u & \\ u \ge 0, v \ge 0, w \ge 0, w' \ge 0. & \end{array} \right\} \tag{17}$$

On the one hand, models (16) and (17) yield the pure input and output efficiency measures for a given $\alpha \in [0,\ 1]$. On the other hand, the optimal values of $\tilde{B}_o^{1*}$ and $B_o^{2\alpha*}$ alongside $E_o^{1\alpha*}$ and $E_o^{2\alpha*}$ can be utilised to define three efficiency decompositions with respect to the internal structure with the aim of identifying the source of inefficiency in the fuzzy environment. In this regard, the *(input) fuzzy SE* for Stage 1 and *(output) fuzzy scale efficiency* (SE) for Stage 2 apropos of a certain α level can be computed as $S_o^{1\alpha} = E_o^{1\alpha*} \big/ B_o^{1\alpha*}$ and $S_o^{2\alpha} = E_o^{2\alpha*} \big/ B_o^{2\alpha*}$, respectively. In addition, the *fuzzy SE of the system* can be the product of the *fuzzy SE* for Stage 1 and Stage 2, i.e.,$S_o^{\alpha} = S_o^{1\alpha} \times S_o^{2\alpha}$. Stage 1 or Stage 2 is recognised as the *most productive scale size* for a given α level if its *fuzzy SE* measure is one, that is, the stage is efficient in both the CRS and VRS models. It is obvious that a system is up and running in the *most productive scale size* for a given α level as soon as both Stage 1 and Stage 2 are the *most productive scale size.* For a given α level, if Stage 1 (Stage 2) is efficient under the VRS but inefficient under the CRS, then it is up and running locally efficiently but not globally efficiently thanks to the scale size of Stage 1 (Stage 2). We finally define a decomposition of efficiency for Stage 1 and Stage 2 as $E_o^{1\alpha} = S_o^{1\alpha} \times B_o^{1\alpha}$ and $E_o^{2\alpha} = S_o^{2\alpha} \times B_o^{2\alpha}$, respectively, that can be deployed to pinpoint the sources of inefficiency for a given α level. That is, the inefficiency of Stage 1 (i.e.,$E_o^{1\alpha} < 1$) bespeaks the inefficient operation (i.e.,$B_o^{1\alpha} < 1$), scale inefficiency (i.e.,$S_o^{1\alpha} < 1$) or both operation and scale inefficiency (i.e. $B_o^{1\alpha} < 1$ and $S_o^{1\alpha} < 1$). Clearly, the same can be applied to identify the sources of inefficiency for Stage 2. Interestingly, we can decompose the efficiency measure of the system as $E_o^{\alpha} = S_o^{1\alpha} \times S_o^{2\alpha} \times B_o^{1\alpha} \times B_o^{2\alpha}$. This decomposition including four components from the internal processes is informative for evaluating the sources of inefficiency.

3.3 Efficiency ranking in fuzzy TsDEA

For a given α, we report three set of efficiencies which are associated to the whole production system, Stage 1 and Stage 2 using the proposed fuzzy TsDEA models. Although it is possible that there is no efficient DMU with an efficiency score of 1, one may observe several efficient units in each efficiency set which reveal the lack of sufficient discriminatory power. In certain occasions, the manager will seek to obtain a fully efficiency ranking and to be able to discriminate among all DMUs and processes. To improve the discriminatory power of TsDEA, we adapt the idea of super-efficiency approach introduced by Andersen and Petersen (1993) to provide a full ranking where the observations are characterised by the triangular fuzzy numbers. Let us first focus on the dual model (5) in the fuzzy environment which is here of interest to rank efficient systems. The fuzzy dual model for the two-stage structure is given in (18).

$$E_o^{\alpha} = \min\left\{\theta \;\middle|\; \begin{array}{l} \theta\tilde{x}_o - \sum_j \lambda_j \tilde{x}_j \geq 0, \\ \sum_j (\lambda_j - \mu_j)\tilde{z}_j \geq 0, \\ \sum_j \mu_j \tilde{y}_j \geq \tilde{y}_o, \\ \lambda_j, \mu_j \geq 0, \quad \forall j. \end{array}\right\} \tag{18}$$

where $\tilde{x}_j = (x_j^l, x_j^m, x_j^u)$, $\tilde{z}_j = (z_j^l, z_j^m, z_j^u)$, and $\tilde{y}_j = (y_j^l, y_j^m, y_j^u)$ represent the fuzzy input, intermediate and output values of the jth DMU. The α-level concept is similarly adapted to transform the fuzzy data to the intervals where α lies within [0,1].

$$E_o^{\alpha} = \min\left\{\theta \;\middle|\; \begin{array}{l} \theta\widehat{x}_o - \sum_j \lambda_j \widehat{x}_j \geq 0, \\ \sum_j (\lambda_j - \mu_j)\widehat{z}_j \geq 0, \\ \sum_j \mu_j \widehat{y}_j \geq \widehat{y}_o, \\ \lambda_j, \mu_j \geq 0, \quad \forall j. \end{array}\right\} \tag{19}$$

where

$$\widehat{x}_{ij} \in [(\alpha x_{ij}^m + (1-\alpha)x_{ij}^l), (\alpha x_{ij}^m + (1-\alpha)x_{ij}^u)], \quad \forall i, j,$$
$$\widehat{z}_{pj} \in [\alpha z_{pj}^m + (1-\alpha)z_{pj}^l, \alpha z_{pj}^m + (1-\alpha)z_{pj}^u], \quad \forall p, j,$$
$$\widehat{y}_{rj} \in [\alpha y_{rj}^m + (1-\alpha)y_{rj}^l, \alpha y_{rj}^m + (1-\alpha)y_{rj}^u], \quad \forall r, j.$$

We propose an alternative method for solving model (19) according to the concept of super-efficiency DEA model for determining a piece-wise linear empirical best practice frontier in the absence of the DMU under evaluation and constituting the basis for comparing the DMUs. Given the interval observations in model (19), the best point associated with a DMU_o under evaluation which is the lower bound of inputs and upper bound of outputs is deployed to compare to its other points of DMU_o and remaining $\mathrm{DMU}_j, j=1,\ldots,n$ (i.e., the inner region of production frontier). In other words, if the best point of the DMU_o is laid in a location out of the production frontier, its efficiency score is greater than one. More importantly, we need to scrutinise the intermediate constraint in the above model where the shadow prices μ_j and λ_j in the constraint $\sum_j (\lambda_j - \mu_j)\widehat{z}_j \geq 0$ correspond to the two different constraints of its multiplier model which literally lead to two input and output roles for $\widehat{z}_j$ as an intermediate measure. We hence re-write this constraint as $\sum_j \lambda_j \widehat{z}_j - \sum_j \mu_j \widehat{z}_j \geq 0$ where the first component $\sum_j \lambda_j z_j$ plays as an output role and the second one $\sum_j \mu_j z_j$ plays as an input role. This idea allows us to evaluate the efficiency of DMU_o by taking its best part, i.e., $(x_o, y_o) = [\alpha x_o^m + (1-\alpha)x_o^l, \alpha y_o^m + (1-\alpha)y_o^u]$ and the worst part for other DMUs, i.e., $(x_j, y_j) = [\alpha x_j^m + (1-\alpha)x_j^u, \alpha y_j^m + (1-\alpha)y_{rj}^l]$. In addition, z_j is divided into $(\alpha z_j^m + (1-\alpha)z_j^l)$ and $(\alpha z_j^m + (1-\alpha)z_j^u)$ with respect to its output and input roles. The resulting LP problem based on the modified constraints and comparison of intervals is presented as follows:

$$E_o = \min \left\{ \theta \;\middle|\; \begin{array}{l} \theta(\alpha x_o^m + (1-\alpha)x_o^l) - \sum\limits_j \lambda_j(\alpha x_j^m + (1-\alpha)x_j^u) \geq 0, \\ \sum\limits_j \lambda_j(\alpha z_j^m + (1-\alpha)z_j^l) - \sum\limits_j \mu_j(\alpha z_j^m + (1-\alpha)z_j^u) \geq 0, \\ \sum\limits_j \mu_j(\alpha y_j^m + (1-\alpha)y_j^l) \geq (\alpha y_o^m + (1-\alpha)y_o^u), \\ \lambda_j, \mu_j \geq 0, \quad \forall j. \end{array} \right\} \tag{20}$$

It should be noted that the rankings of inefficient DMUs do not alter by means of the proposed model (20). Likewise, we develop the following programming model for a given α based upon the dual model (6) as the occurrence of multiple efficient sub-processes is highly probable:

$$E_o^1 = \min\left\{\theta + hE_o \left| \begin{array}{l} \theta(\alpha x_o^m + (1-\alpha)x_o^l) - \sum_j \lambda_j(\alpha x_j^m + (1-\alpha)x_j^u) \geq 0, \\ \sum_j \lambda_j(\alpha z_j^m + (1-\alpha)z_j^l) - \sum_j \mu_j(\alpha z_j^m + (1-\alpha)z_j^u) \geq (\alpha z_o^m + (1-\alpha)z_o^u), \\ \sum_j \mu_j(\alpha y_j^m + (1-\alpha)y_j^l) + h(\alpha y_o^m + (1-\alpha)y_o^u) \geq 0, \\ \lambda_j, \mu_j \geq 0 \quad \forall j. \end{array} \right.\right\} \tag{21}$$

Due to priority given to Stage 1, we evaluate the efficiency of the first stage, and consequently z_o of DMU under evaluation is treated as the output. This explains why we consider $(\alpha z_o^m + (1-\alpha)z_o^u)$ in the second constraint of the above model. We can proceed in a similar way to rank the second sub-processes when giving priority to Stage 2 as formulated below.

$$E_o^2 = \min\left\{\theta \left| \begin{array}{l} \sum_j \lambda_j(\alpha x_j^m + (1-\alpha)x_j^u) + hE_o(\alpha x_o^m + (1-\alpha)x_o^l) \leq 0, \\ \sum_j \lambda_j(\alpha z_j^m + (1-\alpha)z_j^l) - \sum_j \mu_j(\alpha z_j^m + (1-\alpha)z_j^u) \geq 0, \\ \sum_j \mu_j(\alpha y_j^m + (1-\alpha)y_j^l) \geq (\alpha y_o^m + (1-\alpha)y_o^u) - h, \\ \lambda_j, \mu_j \geq 0, \quad \forall j. \end{array} \right.\right\} \tag{22}$$

Thought the above discussion for the ranking of the efficient DMUs and stages can be generalised to the models under the VRS assumption, the models can suffer from infeasibilities (see e.g. Hatami-Marbini et al. 2017).

4 Non-life insurance companies in Taiwan: an illustrative example

The operation in a non-life insurance company includes *premium acquisition* as Stage 1 and *profit generation* as Stage 2 (Kao and Hwang 2008). Stage 1 calls particular attention to clients who pay direct written premiums and to other insurance companies which pay reinsurance premiums. Stage 2 utilises premiums as capital for the purpose of investment to make a profit.

The inputs for Stage 1 (*premium acquisition*) are *operating expenses* (x_1) and *insurance expenses* (x_2) to be consumed to generate *direct written premiums* (z_1) and *reinsurance premiums* (z_2) as two intermediate measures which are in fact the inputs of the Stage 2 (*profit generation*) to produce *underwriting profit* (y_1) and *investment profit* (y_2). The data set involving 24 non-life insurance companies is taken from Kao and Liu (2011) where triangular fuzzy numbers are utilised to deal with uncertainty. The data set is reported in Table 1. Let us now analyse this problem using the models proposed in this study. First, models (11) and (13) are used for five different possibility levels (0, 0.25, 0.5, 0.75, 1) to calculate the technical efficiency of 24 insurance companies from input and output orientations, which are given in Table 2. The possibility level $\alpha = 0$ shows the efficiency score that is less likely to occur while the possibility level $\alpha = 1$ shows the efficiency score that is most likely to happen. For two α levels (0 and 0.25), DMU_5 is 100% efficient, i.e.,$E_5^0 = E_5^{0.25} = 1$, but it is no longer to be

Table 1 Triangular fuzzy numbers of 24 insurance companies in Taiwan

Co.	x_1	x_2	z_1	z_2	y_1	y_2
1	(1113, 1178, 1256)	(636, 673, 717)	(7041, 7451, 7943)	(809, 856, 912)	(930, 984, 1049)	(644, 681, 726)
2	(1305, 1381, 1472)	(1278, 1352, 1441)	(9469, 10020, 10681)	(1712, 1812, 1932)	(1160, 1228, 1309)	(788, 834, 889)
3	(1112, 1117, 1255)	(559, 592, 63)	(4513, 4776, 5091)	(529, 560, 597)	(277, 293, 312)	(622, 658, 701)
4	(568, 601, 641)	(561, 594, 633)	(2999, 3174, 3383)	(351, 371, 395)	(234, 248, 264)	(167, 177, 189)
5	(6331, 6699, 7141)	(3167, 3351, 3572)	(35335, 37362, 39680)	(1657, 1753, 1869)	(7419, 7851, 8369)	(3709, 3925, 4184)
6	(2483, 2627, 2800)	(631, 668, 712)	(9211, 9747, 10,390)	(900, 952, 1015)	(1619, 1713, 1826)	(392, 415, 442)
7	(1853, 1942, 2047)	(1377, 1443, 1521)	(10193, 10685, 11262)	(613, 643, 678)	(2136, 2239, 2360)	(419, 439, 463)
8	(3615, 3789, 3994)	(1787, 1873, 1974)	(16473, 17267, 18199)	(1082, 1134, 1195)	(3720, 3899, 4110)	(593, 622, 656)
9	(1495, 1567, 1652)	(906, 950, 1001)	(10945, 11473, 12093)	(521, 546, 575)	(995, 1043, 1099)	(252, 264, 278)
10	(1243, 1303, 1373)	(1238, 1298, 1368)	(7832, 8210, 8653)	(481, 504, 531)	(1619, 1697, 1789)	(529, 554, 584)
11	(1872, 1962, 2068)	(641, 672, 708)	(6890, 7222, 7612)	(613, 643, 678)	(1418, 1486, 1566)	(17, 18, 19)
12	(2473, 2592, 2732)	(620, 650, 685)	(9000, 9434, 9943)	(1067, 1118, 1178)	(1502, 1574, 1652)	(867, 909, 958)
13	(2481, 2609, 2739)	(1301, 1368, 1436)	(13239, 13921, 14617)	(771, 811, 852)	(3432, 3609, 3789)	(212, 223, 234)
14	(1328, 1369, 1466)	(940, 988, 1037)	(7034, 7396, 7766)	(442, 465, 488)	(1332, 1401, 1471)	(316, 332, 349)
15	(2077, 2184, 2293)	(619, 651, 684)	(9911, 10,422, 10943)	(712, 749, 786)	(3191, 3355, 3523)	(528, 555, 583)
16	(1152, 1211, 1272)	(395, 415, 436)	(5331, 5606, 5886)	(382, 402, 422)	(812, 854, 897)	(187, 197, 207)
17	(1382, 1453, 1526)	(1032, 1085, 1139)	(7318, 7695, 8080)	(325, 342, 359)	(2990, 3144, 3301)	(353, 371, 390)
18	(720, 757, 795)	(520, 547, 574)	(3453, 3631, 3813)	(947, 995, 1045)	(658, 692, 727)	(155, 163, 171)
19	(151, 159, 167)	(173, 182, 191)	(1083, 1141, 1196)	(458, 483, 506)	(493, 519, 544)	(44, 46, 48)
20	(138, 145, 152)	(50, 53, 56)	(300, 316, 331)	(124, 131, 137)	(337, 355, 372)	(25, 26, 27)
21	(80, 84, 88)	(25, 26, 27)	(214, 225, 236)	(38, 40, 42)	(48, 51, 53)	(6, 6, 6)
22	(14, 15, 16)	(9, 10, 10)	(49, 52, 54)	(13, 14, 15)	(78, 82, 86)	(4, 4, 4)
23	(51, 54, 57)	(27, 28, 29)	(233, 245, 257)	(47, 49, 51)	(1, 1, 1)	(17, 18, 19)
24	(155, 163, 171)	(223, 235, 246)	(452, 476, 499)	(611, 644, 675)	(135, 142, 149)	(15, 16, 17)

100% efficient by increasing α level. Table 2 bespeaks that as the value of α increases, the efficiency score becomes equal or smaller.

The inherent property of the conventional DEA model under the CRS with precise data is the direct relationship between the optimal solution of the input- and output- oriented models. Analogously, Table 2 allows us to show that the optimal solution of the proposed output-oriented model (13) is the inverse of the proposed input-oriented model (11) for a specific α level. For instance, the efficiency of DMU_4 is 0.392 when $\alpha = 0.25$ and its inverse is 2.551 (= 1/0.392) that is exactly derived from the output-oriented model (13). In addition, looking at the ranking of the companies for the different α levels can be informative and useful as shown in Table 2. DMU_5 is possibilistically superior all the time, followed by DMU_{12} and DMU_1 while DMU_{24} stands at the end of the ranking list. The ranking of DMUs for all possibility levels are exactly identical with the exception of DMUs {6, 19, 23}. In spite of possibility levels, DMU_{11} and DMU_{24} are classified as two companies with the weakest performance that may emerge from the ineffective functioning of two stages. It is assumed that Stage 1 (*premium acquisition*) has a higher priority than Stage 2 (*profit generation*) since profit generation is not obtainable unless providing an efficient marketing process. We thereby apply model (12) to measure the efficiency of the first stage measures while preserving the system efficiency at $E_o^{\alpha *}$. Afterwards, the efficiency of the second stage measures are calculated via $E_j^{2\alpha} = E_j^{\alpha *}/E_j^{1\alpha *}$. The resulting stage efficiencies for $\alpha = 0$, 0.25, 0.5, 0.75, 1 are shown in Table 3, where E1 and E2 represent the efficiency of Stage 1 and Stage 2, respectively. DMUs {9, 12, 15, 19} perform efficiently in the premium acquisition process (Stage 1) for all predefined possibility levels, and DMUs {3, 22} are efficient for in the profit generation process (Stage 2) for all predefined possibility levels.

As can be seen in Table 3, the stage efficiency scores of all DMUs do not exceed as the value of α increases with an exception of DMU_8 for Stage 1 from $\alpha = 0.75$ to $\alpha = 1$, that is, the number of efficient companies, particularly in the first stage, significantly decreases as the value of α increases. For example, at $\alpha = 0$ half of companies in their first process are classified into the efficient set while it turns into about 17% at $\alpha = 1$. Also, the ranking of the companies as per their efficiencies are presented in the parentheses in Table 3.

Generally, the results are almost identical and the big difference especially occurs when an efficient company at a lower value of α is transformed to inefficient one at a greater value of α. Put differently, a greater value of α gives rise to the enhancement of the discrimination power of the model. It should be noted that the same as the result calculated from efficiency assessment for the whole systems (see Table 2), DMU_5 is only 100% efficient in the first and second stages at $\alpha = \{0, 0.25\}$. In addition, the weak performance of DMU_{11} and DMU_{24} is observable for two stages that can be the sources of relative inefficiency of their systems. It can be observed that due to the relationship between the system and processes efficiencies, the overall efficiency is always less than or equal to the efficiencies of the first stage and the second stage for a given possibility level. The efficiency of Stage 2 is no greater than Stage 1 for all α levels, apart from companies 3, 5 and 22.

Let us take account of the Wilcoxon Signed-Rank test as a non-parametric statistical hypothesis test to pinpoint whether the difference between the efficiency scores of

Table 2 Input- and output- oriented CCR models for measuring 24 insurance companies

Co.	α = 0			α = 0.25			α = 0.5			α = 0.75			α = 1		
	Inp.	Out.	Rank	Inp.	Out.	Rank	Inp.	Out.	Rank	Inp.	Out.	Rank	Inp.	Out.	Rank
1	0.904	1.106	3	0.851	1.174	3	0.799	1.251	3	0.750	1.333	3	0.700	1.430	3
2	0.795	1.257	5	0.750	1.333	5	0.707	1.414	5	0.667	1.500	5	0.626	1.598	5
3	0.861	1.162	4	0.815	1.228	4	0.771	1.297	4	0.729	1.371	4	0.690	1.449	4
4	0.426	2.345	15	0.392	2.551	15	0.360	2.775	15	0.331	3.020	15	0.304	3.286	15
5	1	1	1	1	1	1	0.936	1.068	1	0.861	1.161	1	0.792	1.262	1
6	0.510	1.963	11	0.481	2.077	11	0.455	2.199	12	0.423	2.366	12	0.389	2.568	12
7	0.375	2.666	17	0.348	2.875	17	0.323	3.100	17	0.299	3.343	17	0.277	3.607	17
8	0.371	2.692	18	0.345	2.902	18	0.320	3.128	18	0.296	3.373	18	0.275	3.637	18
9	0.294	3.402	20	0.274	3.643	20	0.256	3.901	20	0.239	4.176	20	0.224	4.469	20
10	0.636	1.572	9	0.589	1.698	9	0.545	1.833	9	0.505	1.980	9	0.468	2.139	9
11	0.217	4.599	23	0.201	4.969	23	0.186	5.370	23	0.172	5.803	23	0.159	6.272	23
12	0.942	1.062	2	0.893	1.120	2	0.846	1.182	2	0.802	1.247	2	0.760	1.316	2
13	0.276	3.625	21	0.257	3.895	21	0.239	4.187	21	0.222	4.500	21	0.207	4.838	21
14	0.392	2.550	16	0.363	2.751	16	0.337	2.969	16	0.312	3.204	16	0.289	3.458	16
15	0.794	1.259	6	0.744	1.345	6	0.696	1.436	6	0.653	1.532	6	0.612	1.635	6
16	0.433	2.307	14	0.401	2.491	14	0.372	2.689	14	0.344	2.904	14	0.319	3.135	14
17	0.484	2.068	13	0.450	2.225	13	0.418	2.394	13	0.388	2.576	13	0.361	2.772	13
18	0.350	2.855	19	0.325	3.080	19	0.301	3.323	19	0.279	3.585	19	0.259	3.868	19
19	0.497	2.011	12	0.476	2.100	12	0.456	2.193	11	0.436	2.291	11	0.413	2.420	10
20	0.725	1.380	8	0.675	1.481	8	0.627	1.594	8	0.582	1.719	8	0.539	1.855	8
21	0.247	4.048	22	0.232	4.316	22	0.217	4.602	22	0.204	4.908	22	0.191	5.235	22
22	0.783	1.277	7	0.734	1.362	7	0.688	1.453	7	0.646	1.549	7	0.606	1.650	7
23	0.556	1.799	10	0.513	1.948	10	0.474	2.109	10	0.438	2.284	10	0.404	2.474	11
24	0.176	5.698	24	0.163	6.131	24	0.152	6.599	24	0.141	7.104	24	0.131	7.648	24

Table 3 Input-oriented models for measuring the efficiencies and ranks (in parentheses) of 24 insurance companies for Stage 1 and Stage 2 (Priority to Stage 1)

Co.	α = 0		α = 0.25		α = 0.5		α = 0.75		α = 1	
	E1	E2	E1	E2	E1	E2	E1	E2	E1	E2
1	1 (1, 1.237)*	0.904 (5)	1 (1, 1.171)	0.851 (5)	1 (1, 1.108)	0.799 (5)	1 (1, 1.049)	0.750 (5)	0.993 (6)	0.705 (5)
2	1 (1, 1.245)	0.795 (6)	1 (1, 1.178)	0.750 (6)	1 (1, 1.115)	0.707 (6)	1 (1, 1.055)	0.667 (6)	0.998 (5)	0.627 (6)
3	0.861 (16)	1 (1)	0.815 (16)	1 (1)	0.771 (16)	1 (1)	0.729 (16)	1 (1)	0.69 (17)	1 (1)
4	0.903 (14)	0.472 (15)	0.854 (14)	0.459 (15)	0.809 (14)	0.445 (15)	0.765 (15)	0.433 (14)	0.724 (16)	0.420 (13)
5	1 (1, 1.063)	1 (1)	1 (1, 1.007)	1 (1)	0.954 (10)	0.981 (3)	0.904 (11)	0.952 (3)	0.856 (1, 1)	0.925 (3)
6	1 (1, 1.196)	0.510 (12)	1 (1, 1.132)	0.481 (12)	1 (1, 1.072)	0.455 (14)	1 (1, 1.014)	0.423 (17)	0.96 (7)	0.405 (16)
7	0.818 (18)	0.458 (17)	0.778 (18)	0.447 (17)	0.74 (19)	0.436 (17)	0.703 (19)	0.425 (15)	0.669 (20)	0.414 (14)
8	0.807 (20)	0.460 (16)	0.769 (20)	0.449 (16)	0.732 (20)	0.437 (16)	0.697 (20)	0.425 (16)	0.725 (15)	0.379 (17)
9	1 (1, 1.221)	0.294 (22)	1 (1, 1.161)	0.274 (23)	1 (1, 1.105)	0.256 (24)	1 (1, 1.051)	0.239 (24)	1 (1, 1)	0.224 (24)
10	1 (1, 1.052)	0.636 (9)	1 (1, 1)	0.589 (10)	0.952 (11)	0.572 (10)	0.905 (10)	0.558 (10)	0.861 (10)	0.544 (10)
11	0.788 (21)	0.275 (24)	0.75 (21)	0.268 (24)	0.714 (21)	0.261 (23)	0.68 (21)	0.253 (23)	0.647 (21)	0.246 (23)
12	1 (1, 1.220)	0.942 (4)	1 (1, 1.161)	0.893 (4)	1 (1, 1.105)	0.846 (4)	1 (1, 1.051)	0.802 (4)	1 (1, 1)	0.760 (4)
13	0.822 (17)	0.336 (21)	0.782 (17)	0.329 (20)	0.744 (17)	0.321 (21)	0.708 (17)	0.314 (21)	0.673 (18)	0.308 (20)
14	0.817 (19)	0.480 (14)	0.777 (19)	0.467 (14)	0.74 (18)	0.455 (13)	0.704 (18)	0.443 (12)	0.67 (19)	0.431 (12)
15	1 (1, 1.220)	0.794 (7)	1 (1, 1.161)	0.744 (7)	1 (1, 1.104)	0.696 (7)	1 (1, 1.051)	0.653 (7)	1 (1, 1)	0.612 (7)
16	1 (1, 1.078)	0.433 (18)	1 (1, 1.027)	0.401 (18)	0.977 (9)	0.381 (18)	0.93 (9)	0.370 (18)	0.886 (9)	0.360 (18)
17	0.765 (23)	0.633 (10)	0.728 (23)	0.618 (9)	0.693 (22)	0.603 (9)	0.659 (22)	0.589 (9)	0.627 (22)	0.576 (9)
18	0.969 (13)	0.361 (19)	0.922 (13)	0.352 (19)	0.877 (13)	0.343 (19)	0.834 (13)	0.335 (19)	0.794 (13)	0.326 (19)
19	1 (1, 1.221)	0.497 (13)	1 (1, 1.162)	0.476 (13)	1 (1, 1.105)	0.456 (12)	1 (1, 1.051)	0.436 (13)	1 (1, 1)	0.413 (15)
20	1 (1, 1.148)	0.725 (8)	1 (1, 1.090)	0.675 (8)	1 (1, 1.036)	0.627 (8)	0.984 (8)	0.591 (8)	0.936 (8)	0.576 (8)
21	0.883 (15)	0.280 (23)	0.842 (15)	0.276 (22)	0.803 (15)	0.270 (22)	0.766 (14)	0.266 (22)	0.731 (14)	0.261 (22)
22	0.783 (22)	1 (1)	0.734 (22)	1 (1)	0.688 (23)	1 (1)	0.646 (23)	1 (1)	0.606 (23)	1 (1)
23	1 (1, 1.026)	0.556 (11)	0.978 (12)	0.525 (11)	0.932 (12)	0.509 (11)	0.889 (12)	0.493 (11)	0.848 (12)	0.476 (11)
24	0.522 (24)	0.337 (20)	0.497 (24)	0.328 (21)	0.472 (24)	0.322 (20)	0.449 (24)	0.314 (20)	0.427 (24)	0.307 (21)

*There are two numbers in parentheses when a DMU takes the first place in the ranking. The first number “1” represents its rank and the other one is calculated from model (21) each pair of models is significant. We perform this test for assessing the discrepancy between the efficiencies of Stage 2 and Stage 1 for the various α levels. The resulting z-statistic and p value for $\alpha = 0$, 0.25, 0.5, 0.75 and 1 are (− 1.5968, 0.1096), (− 1.4751, 0.13888), (− 0.9286, 0.35238), (− 0.5844, 0.56192) and (− 0.2922, 0.77182), respectively. Resultantly, the null hypothesis is rejected at the 5% and 1% levels for all α levels, meaning that there is statistically difference between the efficiency scores of two internal processes. This finding bespeaks that the performance of the whole production system is highly dependent on the first stage (profit generation).

Calculating two component efficiencies as well as the overall efficiency can assist an organization in determining the sources of inefficiency that trigger inefficiency. At present, it also necessitates to focus on the decomposition of scale efficiencies to delve into alternative sources of inefficiencies. To this end, we calculate the pure input and output [technical] efficiencies associated to two stages of all the companies ($B_j^{1\alpha}$ and $B_j^{2\alpha}$) using models (16) and (17), respectively. These results are presented in Table 4. We can observe more efficient companies because the assumption of variable returns to scale is deemed. As such, 34 and 13 per cent of companies in the first and second stage, respectively, are efficient at all levels. Apart from units {3, 17 and 22}, Stage 1 equals or outperforms Stage 2 in terms of efficiency measures reported in Table 4.

Moreover, the efficiency measures of units are not risen by increasing α level, implying that the efficiency measures at $\alpha = 1$ is the lowest possible scores for two stages. By making use of efficiencies calculated and reported in Tables 3 and 4, we obtain the *input fuzzy SE* of Stage 1 and *output fuzzy* SE of Stage 2 for different α levels to identify alternative source of inefficiency.

Furthermore, the *fuzzy SE of the system* as the product of the *fuzzy SE* for Stage 1 and Stage 2 yields further insights about the overall technical and scale efficiency of each insurance company (see Table 5). Companies {12, 15, 22} are identified as the *most productive scale size* for all α levels as their SE scores for Stage 1 and Stage 2 are equal to 1. Based on the decomposition $E_o^{1\alpha} = S_o^{1\alpha} \times B_o^{1\alpha}$ and $E_o^{2\alpha} = S_o^{2\alpha} \times B_o^{2\alpha}$, we enable to identify whether inefficiency of Stages 1 and 2 pertains to the inefficient operation, scale inefficiency or both operation and scale inefficiency for a given α level. For further clarification, consider DMU_8 for $\alpha = 0.25$. The overall system efficiency of this company is 0.345, that is, 65.5% of its performance is far from the best practice. To look into the inefficiency sources and take appropriate actions, the production process is broken down into two processes for measuring its global and pure efficiency, that are 0.769 and 0.933 for the first stage, and 0.449 and 0.483 for the second stage. That is to say that the inefficiency measures for stages 1 and 2 under the CRS condition are 0.231 and 0.551 and, under the VRS condition, are 0.067 and 0.517. On the other hand, the scale efficiency measures of this company that are 0.824 and 0.929 for the 1st and 2nd stage allow us to identify other sources of inefficiency. Since the global and pure efficiency measures of two processes associated with DMU_8 are less than unity, this company is destructively affected by both operation and scale inefficiency. Finally, due to the weak discriminatory power of Stage 1 under CRS condition, we apply model (21) to the efficient units with the aim of ranking these units for a certain α level. The optimal objective values for efficient units which are greater than or equal to one are provided in parenthesis of Table 3.

Table 4 The pure input and output efficiency measures for Stage 1 and Stage 2

Co.	$\alpha = 0$		$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$	
	E1	E2	E1	E2	E1	E2	E1	E2	E1	E2
1	1	0.904	1	0.852	1	0.8	1	0.751	1	0.706
2	1	0.795	1	0.75	1	0.707	1	0.667	1	0.627
3	0.865	1	0.816	1	0.771	1	0.729	1	0.690	1
4	0.903	0.535	0.854	0.504	0.809	0.475	0.765	0.448	0.724	0.422
5	1	1	1	1	1	1	1	1	1	1
6	1	0.51	1	0.481	1	0.455	1	0.429	0.960	0.406
7	0.837	0.54	0.790	0.514	0.746	0.488	0.704	0.464	0.669	0.441
8	0.989	0.509	0.933	0.483	0.880	0.459	0.829	0.436	0.800	0.4
9	1	0.303	1	0.287	1	0.271	1	0.256	1	0.242
10	1	0.789	1	0.748	0.952	0.71	0.905	0.673	0.861	0.638
11	0.851	0.395	0.809	0.375	0.770	0.356	0.733	0.338	0.697	0.321
12	1	0.942	1	0.893	1	0.846	1	0.802	1	0.76
13	0.986	0.371	0.930	0.354	0.876	0.338	0.826	0.322	0.779	0.308
14	0.817	0.564	0.777	0.535	0.740	0.508	0.704	0.482	0.670	0.458
15	1	0.794	1	0.744	1	0.696	1	0.653	1	0.612
16	1	0.444	1	0.421	0.978	0.4	0.931	0.379	0.886	0.36
17	0.769	0.823	0.732	0.782	0.696	0.743	0.663	0.706	0.630	0.67
18	1	0.399	1	0.379	1	0.361	1	0.344	0.960	0.328
19	1	0.497	1	0.476	1	0.456	1	0.436	1	0.417
20	1	0.959	1	0.91	1	0.868	0.984	0.831	0.936	0.795
21	0.883	0.309	0.842	0.296	0.803	0.284	0.766	0.273	0.731	0.262
22	0.783	1	0.734	1	0.688	1	0.646	1	0.606	1
23	1	0.732	0.978	0.692	0.932	0.655	0.889	0.62	0.848	0.586
24	1	0.394	1	0.375	1	0.357	1	0.34	1	0.324

Table 5 Scale efficiency for Stage 1, Stage 2 and the whole system

Co.	α = 0			α = 0.25			α = 0.5			α = 0.75			α = 1		
	SE1	SE2	SE	SE1	SE2	SE	SE1	SE2	SE	SE1	SE2	SE	SE1	SE2	SE
1	1	1	1	1	0.999	0.999	1	0.999	0.999	1	0.999	0.999	0.993	0.998	0.992
2	1	1	1	1	1	1	1	1	1	1	1	1	0.998	1	0.998
3	0.995	1	0.995	0.999	1	0.999	1	1	1	1	1	1	1	1	1
4	1	0.882	0.882	1	0.911	0.911	1	0.937	0.937	1	0.966	0.966	1	0.995	0.995
5	1	1	1	1	1	1	0.954	0.981	0.936	0.904	0.952	0.861	0.856	0.925	0.792
6	1	1	1	1	1	1	1	1	1	1	0.986	0.986	1	0.998	0.998
7	0.977	0.849	0.830	0.985	0.870	0.857	0.992	0.894	0.887	0.999	0.917	0.915	1	0.939	0.939
8	0.816	0.903	0.737	0.824	0.929	0.766	0.832	0.952	0.792	0.841	0.974	0.819	0.906	0.948	0.859
9	1	0.970	0.970	1	0.955	0.955	1	0.945	0.945	1	0.934	0.934	1	0.926	0.926
10	1	0.806	0.806	1	0.787	0.787	1	0.806	0.806	1	0.829	0.829	1	0.852	0.852
11	0.926	0.697	0.646	0.927	0.715	0.663	0.927	0.732	0.679	0.928	0.748	0.694	0.928	0.766	0.711
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	0.834	0.905	0.754	0.841	0.928	0.781	0.849	0.950	0.807	0.857	0.974	0.835	0.864	1	0.866
14	1	0.851	0.851	1	0.873	0.873	1	0.896	0.896	1	0.919	0.919	1	0.942	0.942
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
16	1	0.975	0.975	1	0.952	0.952	0.999	0.952	0.951	0.999	0.976	0.975	1	1	1
17	0.995	0.769	0.765	0.995	0.790	0.786	0.996	0.812	0.808	0.994	0.834	0.829	0.995	0.859	0.855
18	0.969	0.905	0.877	0.922	0.930	0.858	0.877	0.951	0.834	0.834	0.972	0.811	0.827	0.995	0.823
19	1	1	1	1	1	1	1	1	1	1	1	1	1	0.990	0.990
20	1	0.756	0.756	1	0.742	0.742	1	0.722	0.722	1	0.712	0.712	1	0.724	0.724
21	1	0.905	0.905	1	0.931	0.931	1	0.952	0.952	1	0.976	0.976	1	0.997	0.997
22	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	1	0.760	0.760	1	0.758	0.758	1	0.776	0.776	1	0.795	0.795	1	0.813	0.813
24	0.522	0.856	0.447	0.497	0.875	0.435	0.472	0.902	0.426	0.449	0.924	0.415	0.427	0.947	0.404

5 Conclusions

This paper looks into internal structures of a production system to evaluate its performance against the estimated best practice when the qualitative and imprecise data must be regarded by an evaluator or decision-maker. In the case of some fuzzy data, we present two-stage data envelopment analysis (TsDEA) models to calculate the global and pure technical efficiencies of a system and sub-processes. The developed TsDEA models render the assessment results further informative and useful since we enable to precisely identify sources of inefficiencies throughout the key course of actions of a complicated system under evaluation. In addition, we thoroughly study the efficiency decompositions in terms of scale efficiency to indicate the sources of inefficiency as well as presenting a method for ranking the efficient units. An empirical analysis is also presented to illustrate the applicability of the proposed approach.

The framework explored in this research can potentially lend itself to many practical applications, especially supply chain management. We plan to implement the proposed models in a real-world problem to demonstrate the practical and managerial implications, which are of great interest to managers and practitioners. The proposed framework enables us to study structures with two subsequent processes while there are many systems which consist of more than two sub-processes. The development of a more generalised framework with more complicated series and parallel production systems would be an interesting direction for future research.

Appendix 1. Fuzzy sets theory

This appendix reviews some basic definitions of fuzzy sets theory and fuzzy numbers (Zimmermann 1996).

Let U be a universe set. A fuzzy set $\tilde{M}$ in the universe set U is defined by the membership function $\mu_{\tilde{M}}(x) \rightarrow [0, 1]$ where $\forall x \in U \rightarrow \mu_{\tilde{M}}(x)$ stands for the grade of membership of $\tilde{M}$ in U. A fuzzy subset $\tilde{M}$ is said to be normal and convex if $\sup_{x \in U} \mu_{\tilde{M}}(x) = 1$ and $\mu_{\tilde{M}}(\lambda x + (1-\lambda) y) \geq \left(\mu_{\tilde{M}}(x) \wedge \mu_{\tilde{M}}(y)\right), \forall x, y \in U, \forall \lambda \in [0, 1]$, respectively, where $\wedge$ is the minimum operator. A fuzzy number is a normal and convex fuzzy subset with a given membership whose grade varies between 0 and 1. A triangular fuzzy number, denoted as $\tilde{M} = (l, m, u)$, is the most widely used fuzzy numbers in practice and theory with the following membership function:

$$\mu_{\tilde{M}}(x) = \begin{cases} \frac{x-l}{m-l}, & l \leq \mathrm{x} \leq m \\ \frac{u-x}{u-m}, & m < \mathrm{x} \leq u \\ 0, & \text{otherwise} \end{cases}$$

Note that a crisp number M is a special case of the triangular fuzzy number in which $l = m = u$.

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