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Solving generalized fuzzy data envelopment analysis model: a parametric approach

Ali Asghar Foroughi[1] · Roohollah Abbasi Shureshjani[1]

Abstract Data envelopment analysis (DEA) is a non-parametric technique to assess the performance of a set of homogeneous decision making units (DMUs) with common crisp inputs and outputs. Regarding the problems that are modelled out of the real world, the data cannot constantly be precise and sometimes they are vague or fluctuating. So in the modelling of such data, one of the best approaches is using the fuzzy numbers. Substituting the fuzzy numbers for the crisp numbers in DEA, the traditional DEA problem transforms into a fuzzy data envelopment analysis (FDEA) problem. Different methods have been suggested to compute the efficiency of DMUs in FDEA models so far but the most of them have limitations such as complexity in calculation, non-contribution of decision maker in decision making process, utilizable for a specific model of FDEA and using specific group of fuzzy numbers. In the present paper, to overcome the mentioned limitations, a new approach is proposed. In this approach, the generalized FDEA problem is transformed into a parametric programming, in which, parameter selection depends on the decision maker's ideas. Two numerical examples are used to illustrate the approach and to compare it with some other approaches.

Keywords Data envelopment analysis · Fuzzy numbers · GFDEA model · Parametric programming

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1 Introduction

Data envelopment analysis (DEA) is a non-parametric technique for evaluating the relative efficiency of homogeneous decision making units (DMUs) on the basis of multiple inputs and multiple outputs. DEA, first introduced by Charnes et al. (1978), is in fact the generalization of Farrell's (1957) single-input single-output ratio. The advantage of the DEA approach over other approaches is the possibility of examining the complex and often unknown relations among multiple inputs and multiple outputs. The huge number of published papers and books in DEA field and the extensive applications of it demonstrate the superiority of this approach.

The prerequisite of using traditional DEA approaches is to measure the inputs and outputs precisely. In practice, however, the data occasionally are imprecise or vague. In order to address this problem, we can use fuzzy numbers to model such data and in this case, we have a fuzzy data envelopment analysis (FDEA) problem. By introducing the fuzzy logic into DEA, it has been used more extensively. For instance, FDEA has been applied in preprinting and packaging (Triantis and Girod 1998), in information technology industry (Kuo and Wang 2007), in determining fuzzy efficiency scores of machinery companies (Kao and Liu 2005), in investment programs (Zhou et al. 2008), and banks (Hatami et al. 2009; Puri and Yadav 2013), in ranking libraries (Kao and Liu 2000), and in addressing NATO enlargement problem (Hatami-Marbini et al. 2013).

The aforementioned examples demonstrate the extensive usage of FDEA. They also emphasize the importance and necessity of working more and more on this field and eventually presenting more efficient approaches. From the first study by Sengupta (1992), great amount of papers have been published in FDEA. Emrouznejad et al. (2014) have presented a taxonomy and review of the published papers in the field of FDEA up to 2013. They have classified the FDEA papers in seven different categories:

A. The tolerance approach
B. The α-level based approach
C. The fuzzy ranking approach
D. The possibility approach
E. The fuzzy arithmetic approach
F. The fuzzy random/type-2 fuzzy set approach
G. Other developments in FDEA.

Among them, the most popular approach is the α-level based approach. This is evident by the number of α-level based papers published in the fuzzy DEA literature (Emrouznejad et al. 2014). In this approach, the fuzzy DEA model is solved by parametric programming using α-cuts (Lertworasirikul et al. 2003b).

However, most of the FDEA methods have limitations such as: utilizable for a specific model of FDEA, using specific group of fuzzy numbers, complexity in calculation and non-contribution of decision maker in decision making process.

To overcome the mentioned limitations, the present paper proposes a new method for solving FDEA problems. This method transforms the generalized FDEA problem into a parametric programming problem by using a transformation function. This function was first proposed by Shureshjani and Darehmiraki (2013). It was defined

and used to rank fuzzy numbers. The approach is explained through two examples and is compared with some other approaches.

The present paper is organized as follows: the fuzzy numbers ranking method introduced by Shureshjani and Darehmiraki (2013) is briefly explained in the second section. In the third section, the generalized DEA model is introduced in the crisp and fuzzy environments and the details of our proposed approach in FDEA are mentioned. In the fourth section, two examples will be given to illustrate the new approach and the results will be compared with some other approaches. The materials will be concluded in the last section.

2 The ranking fuzzy numbers approach (Shureshjani and Darehmiraki 2013)

In the next section, we will use a transformation function that is first proposed by Shureshjani and Darehmiraki (2013). They applied it to rank fuzzy numbers. Their method is briefly explained in this section.

Dealing with fuzzy numbers, we often need an appropriate ranking method in order to compare these numbers with each other and rank them. Different ranking methods, with specific advantages and limitations, have been presented so far. A new and logical approach in this field is the one presented by Shureshjani and Darehmiraki (2013). This approach has the following advantages that distinguishes it from other approaches in this field:

A. The fuzzy numbers can be shown by the arbitrary membership functions.
B. The fuzzy numbers can be normal or abnormal.
C. The fuzzy numbers can be intersected.
D. The fuzzy numbers ranking depends on the decision maker's (DM) ideas and on the basis of α-cuts.
E. The approach is easily computable.

We will also introduce the approach in the following paragraphs:

Definition 2.1 A fuzzy number $\tilde{A}$ in parametric form is an ordered pair $(\underline{A}(r), \bar{A}(r))$ of functions $\underline{A}(r)$ and $\bar{A}(r)$, $0 \leq r \leq \omega$, which satisfy the following requirements:

1. $\underline{A}(r)$ is a bounded monotonic increasing left continuous function over $[0, \omega]$,
2. $\bar{A}(r)$ is a bounded monotonic decreasing left continuous function over $[0, \omega]$,
3. $\underline{A}(r) \leq \bar{A}(r), 0 \leq r \leq \omega$.

ω is an arbitrary constant such that $0 < \omega \leq 1$.

If, we set $\omega = 1$ in the above mentioned definition, then $\tilde{A}$ is a normal fuzzy number. A crisp (non-fuzzy) number K is simply represented by $\underline{A}(r) = \bar{A}(r) = K$, $0 \leq r \leq 1$.

Definition 2.2 If $\tilde{A}$ is an arbitrary fuzzy number then the α-cut of $\tilde{A}$ is defined as $[\tilde{A}]_\alpha = [\underline{A}(\alpha), \bar{A}(\alpha)]$, $0 \leq \alpha \leq \omega$.

Here $\tilde{A}_\omega$ represents a fuzzy number in which 'ω' is the maximum membership value that a fuzzy number takes on.
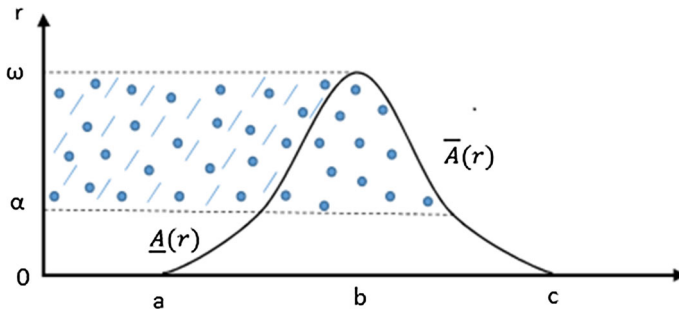
Fig. 1 $\tilde{A}_\omega = \left(\underline{A}(r), \bar{A}(r) \right)$

Let $\tilde{A}_\omega = (\underline{A}(r), \bar{A}(r)), 0 \leq r \leq \omega$ be a fuzzy number. The following transformation function $Q_\alpha(\tilde{A}_\omega)$ is assigned to $\tilde{A}_\omega$ which is calculated as follows:

$$Q_\alpha\left(\tilde{A}_\omega\right) = \int_\alpha^\omega \left(\underline{A}(r) + \bar{A}(r)\right) dr, \quad 0 \leq \alpha \leq \omega$$

and, it is supposed that if $\alpha \geq \omega$, then $Q_\alpha(\tilde{A}_\omega) = 0$.

This function will be used as a basis for comparing fuzzy numbers.

It has been graphically shown in Fig. 1 that the quantity of $Q_\alpha(\tilde{A})$ is the summation of the cross-hatched area and the dotted area from α to ω. Hence, in identifying the quantity of $Q_\alpha(\tilde{A})$, only those elements of the fuzzy number $\tilde{A}$ are computed which have the membership quantities that are equal with or higher than α. The selection of the quantity of α depends on the decision maker's ideas. If the quantity of α is chosen close to 1, that is, while comparing the fuzzy numbers, the elements with high membership quantities are only of importance and the decision maker looks for a low risk decision. In contrary, if the decision maker chooses the quantity of α close to zero, it means that the elements with low membership quantities are important as well and the decision maker compares the fuzzy numbers with high risk. Therefore, if the decision maker chooses a quantity for α close to 1, then the decision made is called "high level decision" and if α is given a quantity close to zero, then the decision made is called "low level decision".

Definition 2.3 If $\tilde{A}_\omega$ and $\tilde{B}_{\omega'}$ are two arbitrary fuzzy numbers and $\omega, \omega' \in (0, 1]$, then we have:

$$\tilde{A}_\omega \preccurlyeq \tilde{B}_{\omega'} \leftrightarrow \forall \alpha \in [0, 1], \quad Q_\alpha\left(\tilde{A}_\omega\right) \leq Q_\alpha\left(\tilde{B}_{\omega'}\right)$$

$$\tilde{A}_\omega = \tilde{B}_{\omega'} \leftrightarrow \forall \alpha \in [0, 1], \quad Q_\alpha\left(\tilde{A}_\omega\right) = Q_\alpha\left(\tilde{B}_{\omega'}\right)$$

$$\tilde{A}_\omega \succcurlyeq \tilde{B}_{\omega'} \leftrightarrow \forall \alpha \in [0, 1], \quad Q_\alpha\left(\tilde{A}_\omega\right) \geq Q_\alpha\left(\tilde{B}_{\omega'}\right)$$

Definition 2.4 If we compare two arbitrary fuzzy numbers including $\tilde{A}_\omega$ and $\tilde{B}_{\omega'}$ at decision levels higher than "α" and $\alpha, \omega, \omega' \in (0, 1]$, then we have:
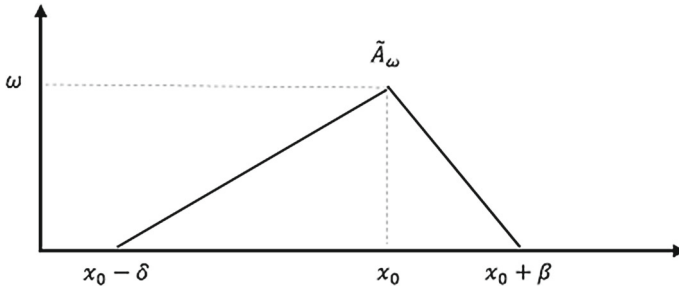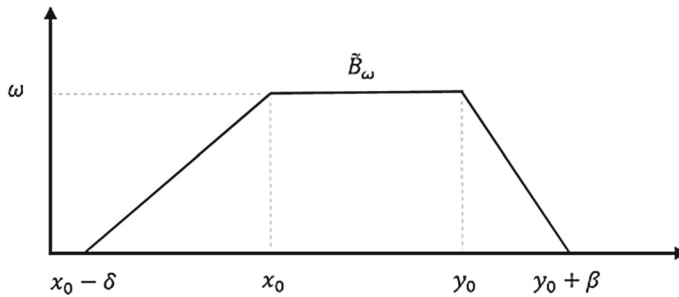
Fig. 2 A triangular fuzzy number

Fig. 3 A trapezoidal fuzzy number

$$\tilde{A}_\omega \preccurlyeq_\alpha \tilde{B}_{\omega'} \leftrightarrow Q_\alpha\left(\tilde{A}_\omega\right) \leq Q_\alpha\left(\tilde{B}_{\omega'}\right)$$

$$\tilde{A}_\omega =_\alpha \tilde{B}_{\omega'} \leftrightarrow Q_\alpha\left(\tilde{A}_\omega\right) = Q_\alpha\left(\tilde{B}_{\omega'}\right)$$

$$\tilde{A}_\omega \succcurlyeq_\alpha \tilde{B}_{\omega'} \leftrightarrow Q_\alpha\left(\tilde{A}_\omega\right) \geq Q_\alpha\left(\tilde{B}_{\omega'}\right)$$

where $\tilde{A}_\omega \preccurlyeq_\alpha \tilde{B}_{\omega'}$, i.e., at decision levels higher than α, $\tilde{B}_{\omega'}$ is greater than or equal to $\tilde{A}_\omega$.

Two well-known kinds of fuzzy numbers are the triangular fuzzy numbers and the trapezoidal fuzzy numbers. In Figs. 2 and 3, $\tilde{A}_\omega = \left(\underline{A}(r), \bar{A}(r)\right) = \left(x_0 - \delta + \frac{\delta}{\omega}r, x_0 + \beta - \frac{\beta}{\omega}r\right)$ and $\tilde{B}_\omega = \left(\underline{B}(r), \bar{B}(r)\right) = \left(x_0 - \delta + \frac{\delta}{\omega}r, y_0 + \beta - \frac{\beta}{\omega}r\right)$ are triangular and trapezoidal fuzzy numbers, respectively, and by computing the quantity of Q_α, we have:

$$Q_\alpha^{Tri}\left(\tilde{A}_\omega\right) = \int_\alpha^\omega \{\underline{A}(r) + \bar{A}(r)\}dr = 2x_0\left(\omega - \alpha\right) + \frac{(\beta - \delta)}{2\omega}(\omega - \alpha)^2$$

$$Q_\alpha^{Tra}\left(\tilde{B}_\omega\right) = \int_\alpha^\omega \{\underline{B}(r) + \bar{B}(r)\}dr = (x_0 + y_0)\left(\omega - \alpha\right) + \frac{(\beta - \delta)}{2\omega}(\omega - \alpha)^2$$

It has been obviously observed that if the triangular fuzzy numbers and trapezoidal fuzzy numbers are symmetric ($\delta = \beta$), then the above mentioned formulas will be simpler. Besides, for the crisp numbers like K, we have:

$$Q_\alpha (K) = 2K (1 - \alpha)$$

3 The generalized FDEA model and the proposed approach

Lots of various models have been presented in DEA. The most popular and useful models in DEA are the CCR model (Charnes et al. 1978), the BCC model (Banker et al. 1984), the FG model (Färe and Grosskopf 1985) and the ST model (Seiford and Thrall 1990). In these models, the DMU is allowed to evaluate its efficiency in the most favorable way. The significant difference between above models are: the CCR model is constant returns to scale, the BCC model is variable returns to scale, the FG model is non-increasing returns to scale, and the ST model is non-decreasing returns to scale.

We consider the following generalized DEA model (GDEA) (Yu et al. 1996a, b; Hadi-Vencheh et al. 2008).

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Generalized DEA model (GDEA)

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Multiplier form	Envelopment form
$Z = \max \sum_{r=1}^{s} u_r y_{ro} - \delta_1 u_0$	$Z = \min \theta$
$s.t. \quad \sum_{i=1}^{m} v_i x_{io} = 1$	$s.t. \quad \theta x_{io} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0, \forall i$
$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \delta_1 u_0 \leq 0, \forall j$	$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \forall r$
$\delta_1 \delta_2 (-1)^{\delta_3} u_0 \geq 0, v_i, u_r \geq 0, \forall i, r$	$\lambda \in \Lambda$

where

$$\Lambda = \left\{ \lambda | \lambda = (\lambda_1, \dots, \lambda_n), \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} v \right) \right.$$

$$\left. = \delta_1, v \geq 0, \lambda_j \geq 0, \quad j = 1, \dots, n \right\}$$

and δ_1, δ_2, δ_3 are binary parameters and we can see that

(i)   If $\delta_1 = 0$ then the GDEA model is reduced to CCR model.
(ii) If $\delta_1 = 1$ and $\delta_2 = 0$ then the GDEA model is reduced to BCC model.
(iii) If $\delta_1 = \delta_2 = 1$ and $\delta_3 = 0$ then the GDEA model is reduced to FG model.
(iv) If $\delta_1 = \delta_2 = \delta_3 = 1$ then the GDEA model is reduced to ST model.

Regarding the fact that the data are not always precise or deterministic in the real world, using the conventional DEA models can cause some problems. In order to overcome these problems, the crisp inputs and outputs can be replaced by the

fuzzy numbers. In this case, the GDEA model is converted into the following form:

Generalized Fuzzy DEA model (GFDEA)

Multiplier form

$$Z = \max \sum_{r=1}^{s} u_r \tilde{y}_{ro} - \delta_1 u_0$$

$$s.t. \quad \sum_{i=1}^{m} v_i \tilde{x}_{io} = 1$$

$$\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} - \delta_1 u_0 \leq 0, \forall j$$

$$\delta_1 \delta_2 (-1)^{\delta_3} u_0 \geq 0, v_i, u_r \geq 0, \forall i, r$$

Envelopment form

$$Z = \min \theta$$

$$s.t. \quad \theta \tilde{x}_{io} - \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \geq 0, \forall i$$

$$\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \geq \tilde{y}_{ro}, \forall r$$

$$\lambda \in \Lambda$$

where

$$\Lambda = \left\{ \lambda | \lambda = (\lambda_1, \ldots, \lambda_n), \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} v \right) \right.$$

$$\left. = \delta_1, v \geq 0, \lambda_j \geq 0, \quad j = 1, \ldots, n \right\}$$

$\delta_1, \delta_2, \delta_3$ are binary parameters and $(\tilde{x}_{ij} : i = 1, \ldots, m)$ and $(\tilde{y}_{rj} : r = 1, \ldots, s)$ are fuzzy input and fuzzy output vectors of DMU$_j$, $j = 1, \ldots, n$, respectively.

Different methods have been presented to solve the FDEA models, which in fact are a fuzzy linear programming. Herein, we use the Q_α function proposed by Shureshjani and Darehmiraki (2013) and present a new approach to solve the FDEA models.

As can be seen from Sect. 2, the Q_α transformation function is dependent on α-cuts, considers both left and right parts of the fuzzy numbers, and is sensitive to any change in the left and right part of fuzzy numbers. So, it is a good representative of the fuzzy inputs and outputs in fuzzy DEA models. By substituting the fuzzy inputs and outputs with their assigned Q_α functions, the generalized fuzzy DEA model is transformed into the following parametric programming which is dependent on α-cuts.

Transformed GFDEA model

Multiplier form

$$Z = \max \sum_{r=1}^{s} u_r Q_\alpha (\tilde{y}_{ro}) - \delta_1 u_0$$

$$s.t. \quad \sum_{i=1}^{m} v_i Q_\alpha (\tilde{x}_{io}) = 1$$

$$\sum_{r=1}^{s} u_r Q_\alpha (\tilde{y}_{rj}) - \sum_{i=1}^{m} v_i Q_\alpha (\tilde{x}_{ij}) - \delta_1 u_0 \leq 0,$$

$$\delta_1 \delta_2 (-1)^{\delta_3} u_0 \geq 0, v_i, u_r \geq 0,$$

Envelopment form

$$Z = \min \theta$$

$$s.t. \quad \theta Q_\alpha (\tilde{x}_{io}) - \sum_{j=1}^{n} \lambda_j Q_\alpha (\tilde{x}_{ij}) \geq 0$$

$$\sum_{j=1}^{n} \lambda_j Q_\alpha (\tilde{y}_{rj}) \geq Q_\alpha (\tilde{y}_{ro})$$

$$\lambda \in \Lambda$$

where

$$\Lambda = \left\{ \lambda | \lambda = (\lambda_1, \ldots, \lambda_n), \delta_1 \left(\sum_{j=1}^{n} \lambda_j + \delta_2 (-1)^{\delta_3} v \right) \right.$$
$$\left. = \delta_1, v \geq 0, \lambda_j \geq 0, \quad j = 1, \ldots, n \right\}$$

Herein, α relies on the decision maker's opinions. If α is considered close to 1, then the achieved result will be a result which is obtained from a risk-averse decision making and if the decision maker adopts an amount close to zero for α, then the result is a risk-prone result.

By proper selection of $\delta_1, \delta_2, \delta_3$ in GFDEA model, we define:

Definition 3.1 DMU_o is FCCR-efficient $\leftrightarrow \forall \alpha \in [0, 1)$, DMU_o is efficient in transformed FCCR model.

Definition 3.2 DMU_o is FCCR α-efficient $\leftrightarrow \exists \alpha \in [0, 1)$, DMU_o is efficient in transformed FCCR model.

Similarity, we have above definitions for fuzzy BCC, FG and ST models.

For conventional BCC model, Cooper et al. (2007) proved that if DMU_o has a minimum input value for any input item, or a maximum output value for any output item then DMU_o is BCC-efficient.

The following two theorems generalize the above mentioned theorem to provide a tool to check the accuracy of the obtained results from our proposed method for the fuzzy BCC model.

Theorem 3.1 *By substituting the fuzzy inputs and outputs with the Q_α function, if for all α,α ∈ [0, 1), DMU_o has a minimum input value for any input item or a maximum output value for any output item then DMU_o is FBCC-efficient.*

Proof Choosing an arbitrary α, α ∈ [0, 1), the transformed FBCC model is converted to a traditional BCC model. Now, if DMU_o has a minimum input value for any input item or a maximum output value for any output item, then DMU_o is efficient (Cooper et al. 2007, p. 93). As α was arbitrary, so according to the Definition 3.2, DMU_o is FBCC-efficient.   □

Theorem 3.2 *By substituting the fuzzy inputs and outputs with the Q_α function, if for one α,α ∈ [0, 1), DMU_o has a minimum input value for any input item or a maximum output value for any output item then DMU_o is FBCC α-efficient.*

Proof The proof of this theorem is the same as the former.   □

Although a variety of techniques are available to solve fuzzy DEA models, each method has its own limitations. Some of the techniques are only developed for using triangular or trapezoidal fuzzy inputs and outputs; however they either do not work

for general forms of fuzzy inputs and outputs with arbitrary membership functions or need to solve a nonlinear programing with complex calculations. For example, Guo and Tanaka (2001) method is applicable only for symmetric triangular fuzzy numbers. Moreover, proposed models by Lertworasirikul et al. (2003a, b), Wen and Li (2009) and Hatami-Marbini et al. (2013) are nonlinear programs which can be converted to linear programs only for trapezoidal fuzzy numbers.

However, in real-world problems, numerous fuzzy inputs and outputs with arbitrary membership functions are involved. So, simple yet effective methods need to be developed to solve these problems.

As mentioned earlier, in our proposed method fuzzy inputs and outputs have arbitrary membership functions (not necessarily triangular or trapezoidal form). In this method after determining the amount of alpha, which depends on the decision maker's opinion, a simple linear program needs to be solved to compute the efficiency of DMUs. So, it is easily possible to obtain the efficiency measures of DMUs for different amounts of alpha and plot their efficiency functions.

In addition to normal fuzzy numbers, non-normal ones are also defined in fuzzy logic (Cheng 1998; Wang et al. 2006; Shureshjani and Darehmiraki 2013; Hatami-Marbini et al. 2013). Almost all available methods for solving fuzzy DEA models are applicable just for normal fuzzy numbers but according to the structure of Q_α transformation function, our proposed method is applicable for both normal and non-normal fuzzy numbers.

4 Numerical examples

In this section, two numerical examples are provided to illustrate the proposed approach in fuzzy DEA. First, we will explain the proposed approach through a simple numerical example, with a fuzzy input and a fuzzy output. Then, in another example, we will compare the obtained results of our approach with the results of some other approaches in this field.

Example 1 Consider 3 DMUs with a fuzzy input and a fuzzy output. All the data are normal and triangular except $\tilde{B}$ (Table 1).

$\tilde{B} = \left(\underline{B}(r), \bar{B}(r) \right) = \left(2 - \sqrt{1-r^2}, 2 + 2\sqrt{1-r^2} \right)$, $0 \le r \le 1$ is a general fuzzy number (Fig. 4).

By using the Q_α function for fuzzy numbers in Table 1, we will have Table 2.

Table 1 Three DMUs with a fuzzy input and a fuzzy output

DMUs	Input	Output
1	(1,2,5)	(2,3,4)
2	$\tilde{B}$	(1,3,5)
3	(5,8,11)	(3,4,6)

Table 2 Transformation functions of fuzzy numbers

DMUs	Input	Output
1	$4(1-\alpha)+(1-\alpha)^2$	$6(1-\alpha)$
2	$4+\frac{\pi}{4}-\left(4\alpha+\frac{\alpha}{2}\sqrt{1-\alpha^2}+\frac{1}{2}\sin^{-1}\alpha\right)$	$6(1-\alpha)$
3	$16(1-\alpha)$	$8(1-\alpha)+0.5(1-\alpha)^2$

Table 3 FCCR efficiency scores

DMUs	$\alpha=0.1$	0.3	0.5	0.7	0.9
1	0.971	0.999	1	1	1
2	1	1	0.975	0.957	0.954
3	0.419	0.408	0.386	0.365	0.343

Table 4 FBCC efficiency scores

DMUs	$\alpha=0.1$	0.3	0.5	0.7	0.9
1	0.971	0.999	1	1	1
2	1	1	0.975	0.957	0.954
3	1	1	1	1	1

Table 5 FFG efficiency scores

DMUs	$\alpha=0.1$	0.3	0.5	0.7	0.9
1	0.971	0.999	1	1	1
2	1	1	0.975	0.957	0.954
3	1	1	1	1	1

By computing the fuzzy efficiency of DMUs in four models of transformed GFDEA, for different decision levels made by decision maker, for example $\alpha=0.1, 0.3, 0.5, 0.7, 0.9$, we will have: Tables 3, 4, 5 and 6

As it was shown in Fig. 4, the inputs of first and second DMUs are two intersected fuzzy numbers. From their Q_α function, different rankings are achieved for them in different amounts of α. For example, if $\alpha=0.1$, $\tilde{A} \succ \tilde{B}$ and when $\alpha=0.7$, $\tilde{A} \prec \tilde{B}$. As it is shown in the FCCR efficiency results (Table 3), DMUs 1 and 2 are FCCR α-efficient but they are not FCCR-efficient (Definitions 3.1 and 3.2).

Also, by using Q_α function, the output data of the first and second DMUs are equal. From Table 2 we can see that for all amounts of α, DMU3 has the biggest output among all DMUs, so according to the Theorem 3.1 DMU3 should be FBCC-efficient. Inputs of DMUs 1 and 2 are two intersected fuzzy numbers and from Table 2 we can see that for $\alpha \leq 0.3$ DMU2 has the smallest input among all DMUs. So, from Theorem 3.2 for $\alpha \leq 0.3$ DMU2 should be FBCC α-efficient. Similarly we can see that for $\alpha \geq 0.5$ DMU1 should be FBCC α-efficient. Table 4 shows that the obtained results are coincident with Theorems 3.1 and 3.2.

The obtained results for FFG and FST models are shown in Tables 5 and 6.

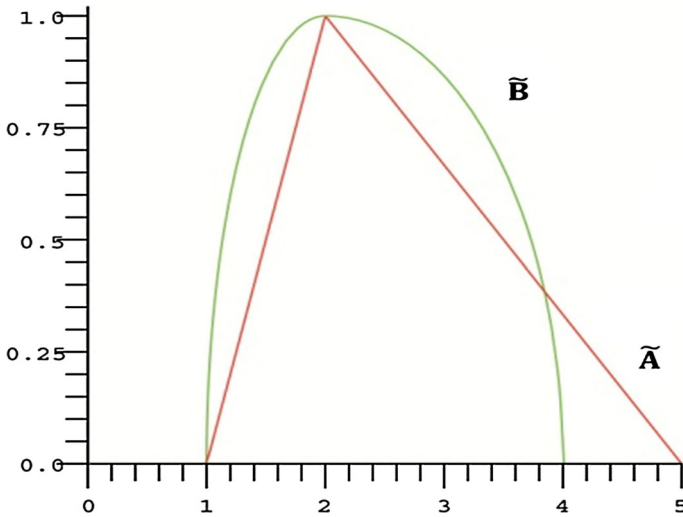| Table 6 FST efficiency scores	DMUs	$\alpha = 0.1$	0.3	0.5	0.7	0.9
	1	0.971	0.999	1	1	1
	2	1	1	0.975	0.957	0.954
	3	0.419	0.408	0.386	0.365	0.343

Fig. 4 The inputs of DMUs 1 and 2 (two intersected fuzzy numbers)

The efficiency diagram of DMUs for FCCR, FBCC, FFG, and FST models are presented in Figs. 5, 6, 7, and 8.

Example 2 In this example, we use the numerical example proposed by Guo and Tanaka (2001) which is also used by Saati et al. (2002), Lertworasirikul et al. (2003a, b), Wen and Li (2009), and Hatami-Marbini et al. (2013).

Consider five DMUs with two fuzzy inputs and two fuzzy outputs presented in Table 7.

All the applied fuzzy numbers in inputs and outputs are normal fuzzy numbers which are triangular and symmetric.

According to many approaches in ranking fuzzy numbers, for example, Cheng (1998), Yao and Wu (2000), Detyniecki and Yager (2001), Chu and Tsao (2002), Wang and Lee (2008), Abbasbandy and Hajjari (2009), Chen and Sanguansat (2011), and Shureshjani and Darehmiraki (2013), when two normal triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ have the same modes and symmetric spreads, $\tilde{A}$ is equal to $\tilde{B}$ and is equal with the crisp number which is in the mode of $\tilde{A}$ and $\tilde{B}$. So, using the presented approach, when we use the data of Table 7 in the GFDEA model, the efficiency quantities don't change for different amounts of α, $\alpha \in [0, 1)$.

For FCCR model, regarding the achieved results from approaches compared in this example (Table 8), it can be seen that:
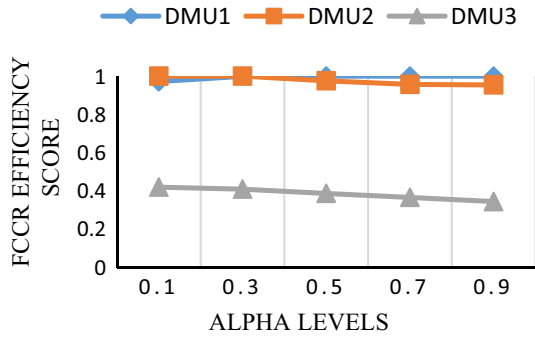
Fig. 5 FCCR efficiency diagram
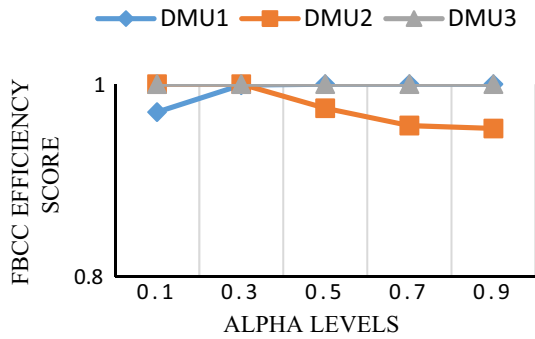

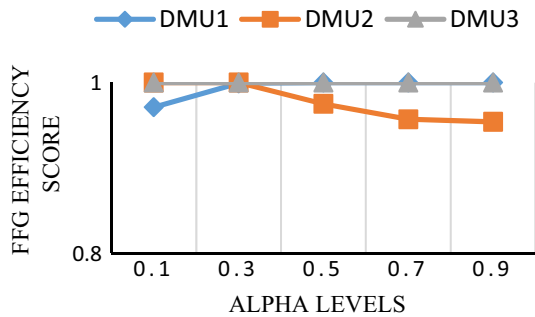**Fig. 6** FBCC efficiency diagram

Fig. 7 FFG efficiency diagram

Fig. 8 FST efficiency diagram
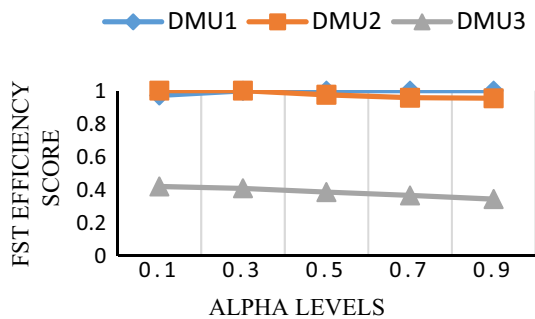
Table 7 The numerical example of Guo and Tanaka (2001)

DMU	Input 1	Input 2	Output 1	Output 2
A	(3.5,4,4.5)	(1.9,2.1,2.3)	(2.4,2.6,2.8)	(3.8,4.1,4.4)
B	(2.9,2.9,2.9)	(1.4,1.5,1.6)	(2.2,2.2,2.2)	(3.3,3.5,3.7)
C	(4.4,4.9,5.4)	(2.2,2.6,3)	(2.7,3.2,3.7)	(4.3,5.1,5.9)
D	(3.4,4.1,4.8)	(2.2,2.3,2.4)	(2.5,2.9,3.3)	(5.5,5.7,5.9)
E	(5.9,6.5,7.1)	(3.6,4.1,4.6)	(4.4,5.1,5.8)	(6.5,7.4,8.3)

DMU A is always inefficient, in Guo and Tanaka (2001), Wen and Li (2009), and the proposed approach in this paper. Also, in Saati et al. (2002) DMU A is always inefficient except for α = 0 but in Lertworasirikul et al. (2003a) different α leads to different results.

For DMU B, all the approaches agree on its efficiency except for Wen and Li (2009) at credibility level 0.1 (Table 9).

For DMU C, different approaches leads to different results, of which some are in contrary to the others. But, in the proposed approach, DMU C is always inefficient.

For DMU D, all the approaches agree on its efficiency.

DMU E is also efficient in all the approaches except for Guo and Tanaka (2001).

As mentioned above, in this example, the efficiency results obtained from our proposed approach in FCCR model are coincident with the efficiency results of most of the other approaches.

Lertworasirikul et al. (2003b) and Hatami-Marbini et al. (2013) have presented new models in order to compute the efficiency in FBCC model. For this example, their results are shown in Table 10, besides, their results have been compared with our presented approach.

According to the Theorem 1, it can be seen that DMU B has a minimum input value for any input item and DMU E has a maximum output value for any output item in the transformed FBCC model. So, the DMUs B and E are efficient. This result has been achieved in all three presented approaches. In addition to this, DMU D is also efficient in all the three approaches. According to the proposed approach, DMUs A and C are always inefficient but in the other approaches, the results of efficiency in different amounts of α are various.

5 Conclusion

As the data are sometimes imprecise or vague in real world, using the fuzzy numbers to show the inputs and outputs in DEA facilitates the computation of practical problems in DEA field. After modeling the problem in FDEA, next step is finding a proper approach to solve it. In the present paper, a new parametric method is proposed to solve the FDEA problems and specifically applied for a generalized FDEA model including FCCR, FBCC, FFG and FST models. Some important characteristics of this approach are, the possibility of using fuzzy numbers with arbitrary membership functions in

Table 8 Evaluating results with different alpha in FCCR model

α	Methods	DMU A	DMU B	DMU C	DMU D	DMU E
0	Guo and Tanaka (2001)	(0.66,0.81,0.99)	(0.88,0.98,1.09)	(0.60,0.82,1.12)	(0.71,0.93,1.25)	(0.61,0.79,1.02)
	Saati et al. (2002)	1	1	1	1	1
	Lertworasirikul et al. (2003a)	1.107	1.238	1.276	1.52	1.296
	Proposed approach	0.855	1	0.861	1	1
0.5	Guo and Tanaka (2001)	(0.75,0.83,0.92)	(0.94,0.97,1.00)	(0.71,0.83,0.97)	(0.85,0.97,1.12)	(0.72,0.82,0.93)
	Saati et al. (2002)	0.954	1	1	1	1
	Lertworasirikul et al. (2003a)	1.032	1.173	1.149	1.386	1.226
	Proposed approach	0.855	1	0.861	1	1
0.75	Guo and Tanaka (2001)	(0.80,0.84,0.88)	(0.96,0.99,1.02)	(0.77,0.83,0.90)	(0.92,0.98,1.00)	(0.78,0.83,0.89)
	Saati et al. (2002)	0.901	1	0.929	1	1
	Lertworasirikul et al. (2003a)	0.904	1.055	0.932	1.131	1.095
	Proposed approach	0.855	1	0.861	1	1
1	Guo and Tanaka (2001)	(0.85,0.85,0.85)	(1.00,1.00,1.00)	(0.86,0.86,0.86)	(1.00,1.00,1.00)	(1.00,1.00,1.00)
	Saati et al. (2002)	0.855	1	0.862	1	1
	Lertworasirikul et al. (2003a)	0.855	1	0.861	1	1
	Proposed approach (near 1)	0.855	1	0.861	1	1

Table 9 Efficiency results for different credibility levels (Wen and Li 2009)

Credibility level	DMU1	DMU2	DMU3	DMU4	DMU5
0.5	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.4	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.3	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.2	Inefficiency	Efficiency	Inefficiency	Efficiency	Efficiency
0.1	Inefficiency	Inefficiency	Efficiency	Efficiency	Efficiency

Table 10 Evaluating results with different α in FBCC model

α	Methods	DMU A	DMU B	DMU C	DMU D	DMU E
0	Lertworasirikul et al. (2003b)	1.299	1.247	1.699	1.692	∞
	Hatami-Marbini et al. (2013)	1	1	1	1	1
	Proposed approach	0.889	1	0.935	1	1
0.5	Lertworasirikul et al. (2003b)	1.062	1.119	1.243	1.300	∞
	Hatami-Marbini et al. (2013)	1	1	1	1	1
	Proposed approach	0.889	1	0.935	1	1
0.75	Lertworasirikul et al. (2003b)	0.969	1.059	1.074	1.142	∞
	Hatami-Marbini et al. (2013)	0.982	1	1	1	1
	Proposed approach	0.889	1	0.935	1	1
1	Lertworasirikul et al. (2003b)	0.889	1.000	0.935	1.000	1.000
	Hatami-Marbini et al. (2013)	0.918	1	0.96	1	1
	Proposed approach (near 1)	0.889	1	0.935	1	1

inputs and outputs, contribution of decision maker in the process of decision making, and simplicity of calculation.

The presented approach in this paper can be easily generalized to other FDEA models.

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