

A linear time algorithm for inverse obnoxious center location problems on networks

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Abstract For an inverse *obnoxious* center location problem, the edge lengths of the underlying network have to be changed within given bounds at minimum total cost such that a predetermined point of the network becomes an *obnoxious* center location under the new edge lengths. The cost is proportional to the increase or decrease, resp., of the edge length. The total cost is defined as sum of all cost incurred by length changes. For solving this problem on a network with m edges an algorithm with running time $\mathcal{O}(m)$ is developed.

Keywords Obnoxious center location · Combinatorial optimization · Inverse optimization · Computational complexity

1 Introduction

Obnoxious facility location problems are basic models in location theory in which customers no longer consider the facilities desirable, but attempt to have them as far away as possible from their own locations. Examples of such facilities include nuclear reactors, military installations, stadiums, garbage dump sites, mega-airports, oil plants and chemical plants. Although these facilities may pose certain risks or disturbances to the public, their significance cannot be ignored because they provide essential services

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to the society. Two well-known models in obnoxious location optimization are the *obnoxious center* and the *obnoxious median* problems. Whereas in the obnoxious median problem the task is to find the best location of one or more obnoxious facilities such that the sum of the (weighted) distances from customers to the nearest facility becomes maximum, the obnoxious center problem seeks to determine the best location of these facilities such that the minimum (weighted) distance between customers and the closest facility is maximized. For detailed surveys on obnoxious location problems the reader is referred to [Cappanera et al. \(2003\)](#), [Carrizosa and Plastria \(1999\)](#), [Plastria \(1996\)](#) and [Zanjirani and Hekmatfar \(2009\)](#).

In contrast to obnoxious location problems, the goal for an *inverse* obnoxious location problem is to modify specific parameters (like edge lengths) of a given obnoxious location problem in the cheapest possible way subject to certain modification bounds such that one or more prespecified locations become optimal under the new parameter values.

In 2007, [Gassner \(2007\)](#) considered the inverse obnoxious 1-median (or 1-maxian) problem with *edge length* modifications and proved that this problem is strongly \mathcal{NP} -hard on general graphs and weakly \mathcal{NP} -hard on series-parallel graphs. Through a transformation to a minimum cost circulation problem she solved the original problem in $\mathcal{O}(m \log m)$ -time on a tree with m edges. Later, [Galavii \(2008\)](#) investigated the inverse 1-maxian problem with *vertex weight* variations on a path and proposed an $\mathcal{O}(m)$ -time solution algorithm. Within the context of *desirable* models of inverse median problems, however, see, e.g. [Baroughi Bonab et al. \(2010, 2011\)](#), [Burkard et al. \(2004, 2007, 2010\)](#) and [Gassner \(2012\)](#).

To the best of our knowledge, inverse obnoxious center location problems have not been investigated until now. Within the context of *desirable* models, however, the \mathcal{NP} -hardness of the uniform-cost inverse center location problem with edge length modification on directed graphs was proved by [Cai et al. \(1999\)](#) in 1999. Later, [Alizadeh and Burkard \(2011a,b\)](#), [Alizadeh et al. \(2009\)](#) and [Yang and Zhang \(2008\)](#) developed exact algorithms with different solution strategies for variants of the inverse absolute and vertex 1-center location problems on tree networks.

In this paper, we consider the inverse obnoxious center location problem with edge length modification on unweighted networks with m edges and design an algorithm with time complexity of $\mathcal{O}(m)$ for its solution. The paper is organized as follows: In Sect. 2 we state the obnoxious center location problem and its inverse version on an unweighted network. Applying the optimality condition for the underlying obnoxious center problem we derive a generic solution idea for solving the inverse obnoxious center location problem. In Sect. 3, we develop a linear time solution approach for the problem under investigation. We close with an outline for further research.

2 The obnoxious center location problem and its inverse model on networks

Let $G = (V, E)$ be an undirected network with vertex set V and edge set E , $|E| = m$. Every edge $e \in E$ has a positive length $\ell(e)$. Let $d_\ell(u, v)$ denote the shortest path distance between two vertices u and v under the edge lengths ℓ . It is said that point p lies in network G , $p \in G$, if p coincides with a vertex or lies on an edge $e = uv$

with endpoints $u, v \in V$. In the latter case point p is fixed by selecting a parameter $\theta \in (0, 1)$ such that

$$d_\ell(u, p) = \theta \ell(e).$$

The *obnoxious center location problem* on network G asks for an optimal solution to

$$\begin{aligned} &\text{maximize} && \min_{v \in V} d_\ell(v, p) && (1) \\ &\text{subject to} && p \in G. \end{aligned}$$

An optimal solution p^* of problem (1) is called an *obnoxious center location* on the given network G . The obnoxious centers of network G can be obtained in $\mathcal{O}(m)$ -time according to the following basic lemma which is the immediate consequence of the definition of the obnoxious center of a graph.

Lemma 2.1 (optimality criterion)

For an unweighted network G , the midpoint of a longest edge is an obnoxious center location.

In contrast to the classical obnoxious center problem (1), the inverse obnoxious center location problem on a network is stated as follows: Let a network $G = (V, E)$, $|E| = m$, with positive edge lengths $\ell(e)$, $e \in E$, be given. Let s be a prespecified *interior point* (i.e., $s \notin V$) on a specific edge e^s of G which divides e^s into two edge-segments e_1^s and e_2^s satisfying

$$\begin{aligned} \ell(e_1^s) + \ell(e_2^s) &= \ell(e^s), \\ e_1^s \cap e_2^s &= \{s\}. \end{aligned}$$

Without loss of generality, assume that

$$\ell(e_2^s) \leq \ell(e_1^s).$$

We want to modify the edge (and edge-segment) lengths in the cheapest possible way such that the prespecified point s becomes an obnoxious center location under the modified edge lengths. Let

$$\hat{E} = \{e_1^s, e_2^s\} \cup E \setminus \{e^s\}.$$

Suppose that we incur the nonnegative cost $c^+(e)$ if length $\ell(e)$, $e \in \hat{E}$, is increased by one unit and we incur the nonnegative cost $c^-(e)$ if length $\ell(e)$ is reduced by one unit. Moreover, assume that we are not allowed to modify the edge lengths arbitrarily. Therefore, let $u^+(e)$ and $u^-(e)$ be the maximum permissible amounts by which length $\ell(e)$, $e \in \hat{E}$, can be increased and reduced, respectively. We can now formally state the *inverse obnoxious center location problem* (IOCP for short) on network G as follows:

Modify the lengths $\ell(e)$, $e \in E \cup \{e_1^s, e_2^s\}$, to $\tilde{\ell}(e)$ such that the following three statements (i), (ii) and (iii) are satisfied:

- (i) The prespecified point s becomes an obnoxious center location on network G with respect to new lengths $\tilde{\ell}$.
- (ii) The cost function

$$\sum_{e \in \hat{E}} \left(c^+(e) \max\{0, \tilde{\ell}(e) - \ell(e)\} + c^-(e) \max\{0, \ell(e) - \tilde{\ell}(e)\} \right)$$

for changing the edge lengths on G is minimized.

- (iii) The new edge (or edge-segment) lengths lie within the given modification bounds

$$-u^-(e) \leq \tilde{\ell}(e) - \ell(e) \leq u^+(e) \quad \text{for all } e \in \hat{E}.$$

According to Lemma 2.1, the *generic solution idea* for solving IOCP is as follows: Either increase or reduce the lengths $\ell(e)$, $e \in \hat{E}$, at minimum total cost subject to the given modification bounds $u^-(e)$ and $u^+(e)$ such that the equalities

$$\tilde{\ell}(e^s) = \max\{\tilde{\ell}(e) : e \in \hat{E}\}, \tag{2}$$

$$\tilde{\ell}(e_1^s) = \tilde{\ell}(e_2^s) \tag{3}$$

are satisfied with respect to the new edge lengths $\tilde{\ell}(e)$. In the sequel we denote the amounts by which the edge length $\ell(e)$ is increased and reduced by $x(e)$ and $y(e)$, respectively.

3 The solution algorithm

In this section we develop a solution algorithm with linear time complexity for IOCP on the underlying network G .

If e^s is a longest edge of network G and further the equality $\ell(e_1^s) = \ell(e_2^s)$ holds, then the prespecified point s is the wanted obnoxious center location on the given network G according to Lemma 2.1 and the problem has been solved. Otherwise, we have to modify the edge lengths at minimum total cost such that (2) and (3) hold.

Since all cost coefficients for modifying edge lengths are positive and e^s should become a longest edge, it does not make sense to decrease the length of e_2^s as this would incur additional cost decreasing the length of other edges. The same argument shows that increasing the length of any edge $e \in E \setminus \{e^s\}$ would imply an additional cost. This means that for solving IOCP on the given network G , the length $\ell(e_2^s)$ has to be increased (or stays as it is) and lengths $\ell(e)$, $e \neq e^s$, have to be reduced (or stay as they are). But an optimal modification of the problem may either reduce or increase the length $\ell(e_1^s)$. Hence we have to take into consideration both of these cases.

- **Case 1** Assume that length $\ell(e_1^s)$ is increased in an optimal solution of IOCP.

In this case, both lengths $\ell(e_1^s)$ and $\ell(e_2^s)$ are to be increased and the other lengths $\ell(e)$, $e \neq e^s$, may be reduced until e^s becomes a longest edge of network G , both edge-segments e_1^s and e_2^s reach equal lengths, and the sum of modification cost is minimum.

If the inequality

$$\ell(e_1^s) - \ell(e_2^s) \leq u^+(e_2^s) \tag{4}$$

is violated, then the problem is infeasible. Hence, we assume that inequality (4) holds in the current Case 1. Let

$$\frac{1}{2}z = (\ell(e_1^s) + x(e_1^s))$$

be the modified length of e_1^s . It is obvious that z must satisfy the inequality

$$\ell(e_1^s) \leq z \leq \max\{2\ell(e_1^s), \ell(e) \text{ for all } e \in \hat{E}\}.$$

Now we set

$$w^+(e_2^s) = u^+(e_2^s) - \ell(e_1^s) + \ell(e_2^s).$$

As soon as length $\ell(e_2^s)$ is increased to length $\ell(e_1^s)$ in a first step, the solution of IOCP in the current case is reduced to the solution of the nonlinear program

$$\begin{aligned} \text{minimize } & f(z) = \left(\sum_{i=1}^2 c^+(e_i^s)\right) \left(\frac{1}{2}z - \ell(e_1^s)\right) + \sum_{e: \ell(e) \geq z} c^-(e) (\ell(e) - z) \\ \text{subject to } & \frac{1}{2}z - \ell(e_1^s) \leq \min\{u^+(e_1^s), u^+(e_2^s)\}, \\ & \ell(e) - z \leq u^-(e) \quad \text{for all } e \in \hat{E} \text{ with } \ell(e) \geq z, \\ & 2\ell(e_1^s) \leq z \leq \max\{2\ell(e_1^s), \ell(e); e \in \hat{E}\}. \end{aligned} \tag{P_1}$$

Let z^* be the optimal solution of the nonlinear program (P₁). Then the optimal solution to IOCP is obtained by

$$\begin{aligned} x^*(e) &= \begin{cases} \frac{1}{2}z^* - \ell(e) & e = e_1^s, e_2^s, \\ 0 & \text{otherwise,} \end{cases} \\ y^*(e) &= \begin{cases} \ell(e) - z^* & \text{for all } e \text{ with } \ell(e) \geq z^*, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where the total cost incurred for changing the edge lengths of network G is

$$C_1 = f(z^*) + c^+(e_2^s) (\ell(e_1^s) - \ell(e_2^s)).$$

In the following we show that an optimal solution z^* of the nonlinear programming problem (P₁) can be obtained in $\mathcal{O}(m)$ -time. Observe that the objective function $f(z)$ can be rewritten as

$$f(z) = z \left(A - \sum_{e: \ell(e) \geq z} c^-(e) \right) + \sum_{e: \ell(e) \geq z} c^-(e) \ell(e) - B.$$

with

$$A = \frac{1}{2} \sum_{i=1}^2 c^+(e_i^s) \quad \text{and} \quad B = \left(\sum_{i=1}^2 c^+(e_i^s) \right) \ell(e_1^s).$$

In particular, the function

$$\sum_{e: \ell(e) \geq z} c^-(e) (\ell(e) - z)$$

is continuous as in every break point the left limit equals to the right limit. This yields immediately:

Lemma 3.1 *The objective function $f(z)$ is piecewise linear and convex.*

Now define

$$\alpha_1 = \max\{\ell(e) - \ell^-(e) : e \in E, e \neq e^s\}$$

and

$$\alpha_2 = \sum_{i=1}^2 (\ell(e_i^s) + \ell^+(e_i^s)).$$

Let $\Delta = [a, b]$ denote the feasible solution set of (P_1) if we replace the constraints

$$\ell(e) - z \leq u^-(e) \quad \text{for all } e \in \hat{E} \text{ with } \ell(e) \geq z$$

in (P_1) by $\alpha_1 \leq z \leq \alpha_2$. It can easily be observed that the problem of minimizing function $f(z)$ subject to $z \in \Delta$ and the optimization model (P_1) have the same optimal solutions. Therefore, according to Lemma 3.1, function $f(z)$ admits an optimal solution either at one of the *boundary points* a and b of Δ or at a *break point* $z = \ell(e^*)$ such that

$$\ell(e^*) \geq 2\ell(e_1^s),$$

and

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^2 c^+(e_i^s) - \sum_{\substack{e: \\ \ell(e) \geq \ell(e^*)}} c^-(e) &\leq 0, \\ \frac{1}{2} \sum_{i=1}^2 c^+(e_i^s) - \sum_{\substack{e: \\ \ell(e) > \ell(e^*)}} c^-(e) &\geq 0. \end{aligned}$$

The break point $\ell(e^*)$ can be determined by a combination of the linear time algorithm for finding the median of a finite set with a binary search approach (Procedure B-P).

Procedure B-P (Finds the break point $z = \ell(e^*)$ with minimum value $f(z)$)

1. Set $H = 0$ and let

$$\mathcal{I} = \{e \in \hat{E} : \ell(e) > 2\ell(e_1^s)\}.$$

2. If all elements $e \in \mathcal{I}$ have the same length $\ell(e)$, then an optimal solution of (P_1) is attained either for $z = a$ or $z = b$. Terminate the procedure.
3. Find (recursively) the median μ of set $\{\ell(e) : e \in \mathcal{I}\}$. Moreover, let

$$\begin{aligned} Q &= \{e \in \mathcal{I} : \ell(e) = \mu\}, \\ R &= \{e \in \mathcal{I} : \ell(e) > \mu\}, \\ L &= \{e \in \mathcal{I} : \ell(e) < \mu\}. \end{aligned}$$

4. Compute

$$\begin{aligned} \gamma_1 &= \frac{1}{2} \sum_{i=1}^2 c^+(e_i^s) - \sum_{e \in R \cup Q} c^-(e) - H, \\ \gamma_2 &= \frac{1}{2} \sum_{i=1}^2 c^+(e_i^s) - \sum_{e \in R} c^-(e) - H, \end{aligned}$$

If the inequalities $\gamma_1 \leq 0$ and $\gamma_2 \geq 0$ are satisfied, then the break point $\ell(e^*) = \mu$ is obtained and stop. Otherwise select an edge $e_0 \in Q$. If $\gamma_1 \leq 0$ and $\gamma_2 \leq 0$, then set $\mathcal{I} = R \cup \{e_0\}$ and go to Step 2, else update

$$H = H + \sum_{e \in R \cup Q \setminus \{e_0\}} c^-(e),$$

set $\mathcal{I} = L \cup \{e_0\}$ and go to Step 2.

Lemma 3.2 *Procedure B-P runs in $\mathcal{O}(m)$ -time.*

Proof Let us denote by $T(m)$ the worst-case running time for m elements. The median of a set of m elements can be determined in $\mathcal{O}(m)$ -time (see e.g. [Cormen et al. 2001](#)). Moreover, in each iteration of the procedure we drop $\mathcal{O}\left(\left\lceil \frac{|\mathcal{I}|}{2} \right\rceil\right)$ elements of the current set \mathcal{I} at Step 2. Therefore, we obtain

$$T(m) = T\left(\left\lceil \frac{m}{2} \right\rceil\right) + \mathcal{O}(m)$$

which implies the time complexity $T(m) = \mathcal{O}(m)$ for the procedure. □

As the feasible solution interval $\Delta = [a, b]$ is also derived in $\mathcal{O}(m)$ -time, we conclude

Corollary 3.3 *In Case 1 the nonlinear programming problem (P₁) can be solved in $\mathcal{O}(m)$ -time.*

- **Case 2** Assume that length $\ell(e_1^s)$ is reduced in an optimal solution of IOCP.

In this case, length $\ell(e_2^s)$ may be increased and the lengths $\ell(e_1^s)$ as well as $\ell(e)$, $e \neq e^s$, may be reduced until e^s becomes a longest edge of network G , both edge-segments e_1^s and e_2^s reach equal lengths, and the sum of the modification cost is minimum.

Let $G' = (V, E')$ be a network which is obtained from the underlying network $G = (V, E)$ if we replace edge e^s by two new edges e', e'' with corresponding lengths

$$\ell(e') = 2\ell(e_2^s) \quad \text{and} \quad \ell(e'') = 2\ell(e_1^s).$$

Moreover, for these new edges we define

$$u^+(e') = 2u^+(e_2^s), \quad u^-(e'') = 2u^-(e_1^s)$$

and the cost coefficients

$$c^+(e') = \frac{1}{2}c^+(e_2^s), \quad c^-(e'') = \frac{1}{2}c^-(e_1^s)$$

and $c^-(e') = c^+(e'') = 0$.

Now we get

Lemma 3.4 *Under the assumption that length $\ell(e_1^s)$ is reduced in an optimal solution of IOCP, there exists a one-to-one correspondence with the same incurred total cost between the feasible solutions of IOCP on the given network G and the feasible solutions of the problem to modify the edge lengths in G' such that e' becomes a longest edge in G' .*

Now let us consider IOCP on the network G' . Since all cost coefficients for the modification of edge lengths are positive, it suffices to increase $\ell(e')$ and to reduce lengths $\ell(e)$, $e \in E'$, $e \neq e'$, at minimum total cost until e' becomes a longest edge of network G' . Let $z = \ell(e') + x(e')$ be the modified length of edge e' . Thus the solution of IOCP on the network G' is reduced to the solution of the nonlinear programming model

$$\begin{aligned} &\text{minimize} && g(z) = c^+(e') (z - \ell(e')) + \sum_{\substack{e \in E': \\ \ell(e) \geq z}} c^-(e) (\ell(e) - z) \\ &\text{subject to} && z - \ell(e') \leq u^+(e'), && (P_2) \\ &&& \ell(e) - z \leq u^-(e) \quad \text{for all } e \in E' \text{ with } \ell(e) \geq z. \end{aligned}$$

Note that the objective function $g(z)$ is piecewise linear and convex and can be rewritten as

$$g(z) = z \left(c^+(e') - \sum_{e \in E': \ell(e) \geq z} c^-(e) \right) + \sum_{\substack{e \in E': \\ \ell(e) \geq z}} c^-(e)\ell(e) - c^+(e')\ell(e').$$

The structure of the nonlinear program (P₂) is similar to the structure of (P₁). Hence, we can obtain the optimal solution z^* and the optimal objective value $g(z^*)$ of (P₂) in $\mathcal{O}(m)$ -time in an analogous way as for (P₁). From an optimal solution z^* of problem (P₂) we can derive an optimal solution of IOCP on network G by

$$x^*(e) = \begin{cases} \frac{1}{2} (z^* - \ell(e')) & e = e_2^s, \\ 0 & \text{otherwise,} \end{cases}$$

$$y^*(e) = \begin{cases} \frac{1}{2} (\ell(e'') - z^*) & e = e_1^s, \\ \ell(e) - z^* & \text{for all } e \in E \text{ with } \ell(e) \geq z^*, e \neq e^s, \\ 0 & \text{otherwise.} \end{cases}$$

The incurred total cost is given by

$$C_2 = g(z^*).$$

Based on the considerations above, the solution algorithm for IOCP on the given network G can be summarized as follows: obtain the optimal objective values C_1 and C_2 of problems (P₁) and (P₂). The minimum of these values and the corresponding solution yields an optimal solution to IOCP on network G .

Altogether we get

Theorem 3.5 *The inverse obnoxious center location problem can be solved in $\mathcal{O}(m)$ -time on a network G with m edges.*

4 Conclusion and further research

In this paper we derived a linear algorithm for solving obnoxious center problems on graphs. We assumed that the objective function value is proportional to the sum of edge length modifications. In practice the cost for an edge length modification may depend on the specific edge. This more realistic problem would lead to a weighted sum objective function for the inverse problem. Also other types of objective functions like minimizing the maximum edge length modification are conceivable.

A related model is the budget-constraint improvement model: given the obnoxious center of a graph and a budget B , modify the edge lengths such that the objective function value increases as much as possible subject to the constraints that the edge modifications (which incur costs) can be performed within the budget B .

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References

- Alizadeh B, Burkard RE (2011) Combinatorial algorithms for inverse absolute and vertex 1-center location problems on trees. *Networks* 58:190–200
- Alizadeh B, Burkard RE (2011) Uniform-cost inverse absolute and vertex center location problems with edge length variations on trees. *Discret Appl Math* 159:706–716
- Alizadeh B, Burkard RE, Pferschyl U (2009) Inverse 1-center location problems with edge length augmentation on trees. *Computing* 86:331–343
- Baroughi Bonab F, Burkard RE, Alizadeh B (2010) Inverse median location problems with variable coordinates. *Central Eur J Oper Res* 18:365–381
- Baroughi Bonab F, Burkard RE, Gassner E (2011) Inverse p -median problems with variable edge lengths. *Math Methods Oper Res* 73:263–280
- Burkard RE, Galavii M, Gassner E (2010) The inverse Fermat-Weber problem. *Eur J Oper Res* 206:11–17
- Burkard RE, Pleschiutchnig C, Zhang J (2004) Inverse median problems. *Discret Optim* 1:23–39
- Burkard RE, Pleschiutchnig C, Zhang J (2007) The inverse 1-median problem on a cycle. *Discret Optim* 5:242–253
- Cai MC, Yang XG, Zhang JZ (1999) The complexity analysis of the inverse center location problem. *J Glob Optim* 15:213–218
- Cappanera P, Gallo G, Maffioli F (2003) Discrete facility location and routing of obnoxious activities. *Discret Appl Math* 133:3–28
- Carrizosa E, Plastria F (1999) Location of semi-obnoxious facilities. *Stud Locat Anal* 12:1–27
- Cormen TH, Leiserson CE, Rivest RL, Stein C (2001) Introduction to algorithms. 2nd edn. MIT Press, Cambridge
- Galavii M (2008) Inverse 1-median problems, Ph.D. Thesis, Institute of Optimization and Discrete Mathematics, Graz University of Technology, Graz, Austria
- Gassner E (2007) The inverse 1-maxian problem with edge length modification. *J Comb Optim* 16:50–67
- Gassner E (2012) An inverse approach to convex ordered median problems in trees. *J Comb Optim* 23: 261–273
- Plastria F (1996) Optimal location of undesirable facilities: a selective overview. *Belgian J Oper Res Stat Comput Sci* 36:109–127
- Yang X, Zhang J (2008) Inverse center location problem on a tree. *J Syst Sci Complex* 21:651–664
- Zanjirani R, Hekmatfar M (2009) Facility location: concepts, models, algorithms and case studies. Physica-Verlag, Berlin