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ORIGINAL PAPER

Linear fractional programming and duality

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Abstract This paper presents a dual of a general linear fractional functionals programming problem. Dual is shown to be a linear programming problem. Along with other duality theorems, complementary slackness theorem is also proved. A simple numerical example illustrates the result.

Keywords Linear fractional program · Linear program · Duality · Complementary slackness

1 Introduction

Duality in linear fractional programming has been studied by many, Chadha (1971), Swarup (1968) and Kaska (1969). Swarup has presented the dual of a linear fractional programming problem whose objective function is linear fractional but the constraints are non-linear in nature. Kaska has given the dual of a linear fractional program, which is constrained as the variable of the primal program. Chadha, has studied the duality for a restricted type of linear fractional functionals programming problem.

This present work considers a general maximization linear fractional functionals programming problem as the primal problem $(P\text{–}P)$. A linear programming problem is proposed as its dual problem $(D\text{–}P)$. The primal and the dual problems along with some assumptions are stated in section two. Four theorems, of section three, establish the desired duality relationship between the two problems. The complementary slackness theorem ensures that from the solution of one problem we can recover the

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solution of the other problem. An illustrative numerical example is given in section four of the paper.

2 Primal and dual programs

The primal problem with assumptions is given below

$$\begin{aligned}&\text{maximize } f(x) = \frac{cx+\alpha}{dx+\beta}\\&\text{subject to } x \in S,\\&\text{where } S = [x : Ax \leq b, \quad x \geq 0].\end{aligned} \tag{P-P}$$

Here A is m by n matrix, x and b are column vectors with n and m components respectively, c and d are row vectors with n components, α and β are constants.

It is assumed that $dx + \beta > 0$ for all x in S; objective function is continuously differentiable, and that the set S is regular (non-empty and bounded).

We propose the following problem as our dual problem:

$$\begin{aligned}&\text{minimize } g(y,z) = z\\&\text{subject to}\\&\quad A^T y + d^T z \geq c^T\\&\quad -b^T y + \beta z = \alpha\\&\quad y \geq 0.\end{aligned} \tag{D-P}$$

T over a matrix is used to denote the transpose of the matrix. The constraint set of the dual problem is denoted by L, i.e.

$$\text{L} = [y, z : A^T y + d^T z \geq c^T; \quad -b^T y + \beta z = \alpha; \quad y \geq 0].$$

Here y is a column vector with m components, and z is a scalar.

3 Duality theorems

Theorem 1 (Weak duality theorem) *For all x in S and for all (y,z) in L, $f(x) \leq g(y,z)$.*

Proof To prove this, we observe that

$$\begin{aligned}&Ax \leq b\\ \Rightarrow \quad &x^T A^T y \leq b^T y;\end{aligned}$$

and

$$\begin{aligned} & A^T y + d^T z \geq c^T \\ \Rightarrow \quad & cx \leq y^T Ax + z.dx \\ & \leq b^T y + z.dx \\ & = \beta z - \alpha + z.dx. \end{aligned}$$

Above is same as

$$cx + \alpha \leq z(dx + \beta)$$

or

$$\frac{cx + \alpha}{dx + \beta} \leq z,$$

which is the same as $f(x) \leq g(y, z)$.

Theorem 2 *If $\hat{x}$ is a feasible solution for the $(P\text{–}P)$ and $(\hat{y}, \hat{z})$ is a feasible solution for the $(D\text{–}P)$ such that $f(\hat{x}) = g(\hat{y}, \hat{z})$, then $\hat{x}$ is an optimal solution to the $(P\text{–}P)$, and $(\hat{y}, \hat{z})$ is an optimal solution to the $(D\text{–}P)$.*

Proof By assumption,

$$f(\hat{x}) = g(\hat{\mathrm{y}}, \hat{\mathrm{z}}).$$

But for any x in S, Theorem 1 holds. From this it follows that

$$f(x) \leq g(\hat{\mathrm{y}}, \hat{\mathrm{z}}) = f(\hat{x}),$$

This shows that $\hat{x}$ is an optimal solution for the $(P\text{–}P)$. Similarly, for any (y, z) of L,

$$g(\hat{\mathrm{y}}, \hat{\mathrm{z}}) = f(\hat{x}) \leq g(\mathrm{y}, \mathrm{z}).$$

And therefore, $(\hat{y}, \hat{z})$ is an optimal solution for the $D\text{–}P$.

Theorem 3 (Strong direct duality theorem) *If $\hat{x}$ solves the $(P\text{–}P)$, then there exists $(\hat{y}, \hat{z})$ which solves the $(D\text{–}P)$.*

Proof Associated with the $(P\text{–}P)$, we define the Lagrange function

$$\phi(x, \Pi) = \left[-\frac{cx + \alpha}{dx + \beta} + \Pi^T \left(A\hat{x} - b \right) \right]$$

Π is a column vector (Lagrange vector) with m components.

It is well known that, Kuhn and Tucker (1951) and Tucker (1957), if $\hat{x}$ solves the $(P\text{–}P)$; then it is necessary that there exists $\hat{\Pi} \geq 0$ such that

$$A\hat{x} - b \leq 0 \tag{1}$$

$$\hat{\Pi}^T \left(A\hat{x} - b\right) = 0 \tag{2}$$

$$\frac{-c\left(d\hat{x} + \beta\right) + d\left(c\hat{x} + \alpha\right)}{(d\hat{x} + \beta)^2} + \hat{\Pi}^T A \geq 0 \tag{3}$$

$$\frac{-c\hat{x}\left(d\hat{x} + \beta\right) + d\hat{x}\left(c\hat{x} + \alpha\right)}{(d\hat{x} + \beta)^2} + \hat{\Pi}^T A\hat{x} = 0 \tag{4}$$

(4) on using (2), simplifies to

$$\frac{-\beta.c\hat{x} + \alpha.d\hat{x}}{(d\hat{x} + \beta)^2} + \hat{\Pi}^T b = 0$$

$$\Rightarrow \quad \frac{b^T\hat{\Pi}\left(d\hat{x} + \beta\right)}{\left(d\hat{x} + \beta\right)} + \frac{\alpha.d\hat{x} + \alpha\beta - \alpha\beta - \beta.c\hat{x}}{(d\hat{x} + \beta)^2} = 0$$

$$\Rightarrow \quad \frac{b^T\hat{\Pi}\left(d\hat{x} + \beta\right)}{\left(d\hat{x} + \beta\right)} + \frac{\alpha(d\hat{x} + \beta) - \beta(c\hat{x} + \alpha)}{(d\hat{x} + \beta)^2} = 0$$

$$\Rightarrow \quad b^T\hat{\Pi}\left(d\hat{x} + \beta\right) + \alpha - \beta\frac{c\hat{x} + \alpha}{d\hat{x} + \beta} = 0. \tag{5}$$

By letting

$$\hat{\Pi}\left(d\hat{x} + \beta\right) = \hat{y}$$

and

$$\frac{c\hat{x} + \alpha}{d\hat{x} + \beta} = \hat{z} \tag{6}$$

Eq. (5) reads as

$$b^T\hat{y} + \alpha - \beta\hat{z} = 0$$

or

$$-b^T\hat{y} + \beta\hat{z} = \alpha. \tag{7}$$

Again, let us observe (3), which is the same as

$$\frac{-c}{d\hat{x} + \beta} + \frac{c\hat{x} + \alpha}{d\hat{x} + \beta}.\frac{d}{(d\hat{x} + \beta)} + \hat{\Pi}^T A \geq 0$$

$$\Rightarrow \quad -c + \frac{c\hat{x} + \alpha}{d\hat{x} + \beta}.d + \hat{\Pi}^T A.\left(d\hat{x} + \beta\right) \geq 0.$$

using (6), above reduces to

$$A^T \hat{y} + d^T \hat{z} \geq c^T \tag{8}$$

From (7) and (8) it follows that $(\hat{y}, \hat{z})$, as defined in (6), is a feasible solution for the $(D\text{–}P)$.

$\hat{z} = \frac{c\hat{x}+\alpha}{d\hat{x}+\beta}$, along with Theorem 1, proves this theorem.

Theorem 4 (Complementary slackness theorem). *Let u and v be slack and surplus column vectors associated with the $(P\text{–}P)$ and $(D\text{–}P)$, respectively. $(\hat{x},\ \hat{u})$ solves the primal problem and $(\hat{y},\ \hat{z},\ \hat{v})$ the dual problem if and only if*

$$\hat{v}^t \hat{x} + \hat{u}^t \hat{y} = 0$$

or

$$\begin{aligned} \hat{x}_j \hat{v}_j &= 0, \quad j = 1, 2, \ldots n. \\ \hat{y}_i \hat{u}_i &= 0, \quad i = 1, 2, \ldots m. \end{aligned}$$

Proof Using u as slack vector, $Ax \leq b$ is the same as

$$\begin{aligned} & Ax + u = b \\ \Rightarrow \quad & x^T A^T y + u^T y = b^T y. \end{aligned} \tag{9}$$

Again, using v as surplus vector, $A^T\ y + d^T z \geq c^T$, is the same as

$$\begin{aligned} & A^T\ y + d^T z - v = c^T \\ \Rightarrow \quad & y^T\ Ax + zdx - v^T x = cx. \end{aligned} \tag{10}$$

From (9) and (10) it follows that

$$b^T y - u^T y = cx - zdx + v^T x.$$

On using the fact that

$$b^T y = \beta z - \alpha,$$

above reads as

$$\beta z - \alpha - u^T y = cx - zdx + v^T x \tag{11}$$

(11) holds for all feasible (x, u) for the $(P\text{–}P)$ and for all feasible $(y,\ z,\ v)$ for the $(D\text{–}P)$. As $\left(\hat{x},\ \hat{u}\right)$ solves the $(P\text{–}P)$ and $\left(\hat{y},\ \hat{z},\ \hat{v}\right)$ solves the $(D\text{–}P)$, it follows that

$$\begin{aligned} &\frac{c\hat{x}+\alpha}{d\hat{x}+\beta}=\hat{z}\\ \Rightarrow\quad & c\hat{x}+\alpha=\beta\hat{z}+\hat{z}\,d\hat{x}. \end{aligned} \tag{12}$$

Because of (12), (11) reduces to

$$\hat{v}^T\hat{x}+\hat{u}^T\hat{y}=0.$$

Conversely, let

$$\hat{v}^T\hat{x}+\hat{u}^T\hat{y}=0,$$

Because of this, (11) reads as

$$\begin{aligned} &\hat{z}\left(d\hat{x}+\beta\right)=\left(c\hat{x}+\alpha\right)\\ \Rightarrow\quad & \frac{c\hat{x}+\alpha}{d\hat{x}+\beta}=\hat{z}. \end{aligned}$$

Thus $\hat{x}$ solves the $(P\text{–}P)$ and $(\hat{y},\ \hat{z})$ solves the $(D\text{–}P)$.
Result

$$\hat{x}_j\hat{v}_j=0 \text{ for } \quad j=1,2,\ldots n$$

and

$$\hat{y}_i\hat{u}_i=0 \text{ for } \quad i=1,2,\ldots m$$

follows from the fact that $\hat{u}_j,\ \hat{v}_j,\ \hat{y}_i,\ \hat{u}_i\geq 0$.

4 Numerical example

Let $(P\text{–}P)$ be

$$\begin{aligned} &\text{maximize } f(x)=\frac{3x_1+5x_2}{x_1+x_2+2}\\ &\text{subject to}\\ &\quad x_1+x_2\leq 6\\ &\quad 3x_1+8x_2\leq 24\\ &\quad x_1,x_2\geq 0 \end{aligned}$$

The $(D\text{–}P)$ will be

$$\begin{aligned}&\text{minimize } g(y,z)=z\\&\text{subject to}\\&\quad y_1+3y_2+z\geq 3\\&\quad y_1+8y_2+z\geq 5\\&\quad 6y_1+24y_2=2z\\&\quad y_1,y_2\geq 0.\end{aligned}$$

Solution of dual linear programming problem, with v_1, and v_2 as surplus variables, is

$$\hat{y}_1=0,\ \hat{y}_2=\frac{1}{4},\ \hat{z}=3,\ \hat{v}_1=\frac{3}{4},\ \hat{v}_2=0.$$

Constraints of the $(P\text{–}P)$ can be expressed as

$$\begin{aligned}x_1+x_2+u_1&=6\\3x_1+8x_2+u_2&=24\\x_1,x_2,u_1,u_2&\geq 0.\end{aligned}\tag{13}$$

Because of complementary slackness theorem we should have

$$\hat{x}_1\hat{v}_1=\hat{x}_2\hat{v}_2=\hat{y}_1\hat{u}_1=\hat{y}_2\hat{\hat{u}}_2=0.$$

Now $\hat{y}_2=\frac{1}{4}\Rightarrow\hat{u}_2=0$; and $\hat{v}_1=\frac{3}{4}\Rightarrow\hat{x}_1=0$

With $\hat{x}_1=0$, and $\hat{u}_2=0$ from (13) we find $\hat{x}_2=3$, and $\hat{u}_3=3$.

Thus $\hat{x}_1=0$, and $\hat{x}_2=3$ solves the $(P\text{–}P)$ with objective function value, $f\left(\hat{x}\right)=3$.

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