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Vector space theory of sustainability assessment of industrial processes

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Abstract In this work, a sustainability analysis based on concepts of normed linear vector space is presented. The processes of production are vectors whose components are the manufacturing factors. The sustainability assessments are measured by the magnitude of the vectors, that is, by their norms. The properties of the space are simple, and the applications are straightforward. The formalism applies equally well whether the indicators are positive or negative, without the recourse of shifting variables. Whenever available, values of the sustainability threshold limits can be handled by the formalism.

Keywords Industrial sustainability · Sustainability assessment · Index of sustainability

Introduction

Sustainability assessment is nowadays a key factor of the industrial production line. Manufactured goods are usually made by a sequence of different factors of processes that involve each a distinct level of sustainability. A comprehensive evaluation of such systems must comprise the economical, environmental, and societal building blocks of the modern concept of sustainability. To this effect, Sikdar (2003) has introduced the idea of a three-dimensional space to the sustainability assessment: one dimension for each domain. In order to compare different manufacturing processes of the same product, Sikdar (2009) defined an aggregated metric measuring the relative distance of each process to a predetermined reference process. This distance was defined by the geometric mean of the factor indicators normalized by those of the reference process. Subsequently, Sikdar et al. (2012) have adapted the geometric mean and also a Euclidean distance in order to incorporate negative indicators that represent benefits to the production process. This was accomplished by introducing shifting variables.

The above procedure toward an aggregated sustainability metric involves the following steps. Since factors of production have in general different physical units, and values that may vary by several orders of magnitude, one must perform a *normalization* that transforms the physical dataset into dimensionless indicators within a coherent scale. A *reference point* is chosen from which a *distance* of each process to this point is determined. This distance represents the aggregated sustainability metric of the corresponding process. The set of distances provides an *inequality ordering* of aggregated metrics where the lower the metric the more sustainable (or less unsustainable) is the process.

The present work considers processes as vectors whose components are the indicators of the production factors. The processes are then vectors in a normed linear space spanned by the factors. The aggregated metric of a process P is measured by its length, i.e., to its *norm,* denoted by $\|P\|$. Any normed linear vector space is a metric space whose induced distance between two vectors, P and V, is a function of the norm: $d(P,V) = \|P - V\|$. Hence, the sustainability metric has no distance to a predetermined reference point other than the origin ($V = 0$). As a consequence of this property, the results of different processes assessed separately and defined by similar indicators are comparable.

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Positive and negative indicators represent distinct categories, the former the ordinary production factors and the latter the added benefits that increase sustainability of the processes. Hence, positive indicators are defined such that a smaller indicator is more favorable than a larger one, whereas negative indicators imply a benefit to the process of production so that the greater the absolute value (or the smaller the negative value), the greater are their contributions to sustainability. Therefore, they represent two distinct categories that have opposed functions in terms of sustainability assessment. The present formalism treats both categories in a consistent and coherent way without requiring the use of shifting variables or the introduction of extraneous parameters.

The norms of processes provide an inequality ordering that stipulates a ranking of aggregated sustainability metrics. Since process alternatives are determined by inequalities, ambiguities may arise in the interpretation of the results. For instance, two formalisms can both yield the metric of P_1 smaller than that of P_2, but the difference between the two metrics may be small or large depending on the formalism. Clearly, the right formalism is the one that is more representative of the physical system. From the inequality ordering, one may choose the smallest, or the largest, aggregated metric as a reference in order to determine the relative magnitude among the various process alternatives. This procedure can only be reliable if the value of each aggregated metric in the inequality ranking represents faithfully the corresponding physical system.

The theory

We assume that products can be manufactured by m different processes and each one of these by n different factors. In general, the majority of factors are common to the processes. This dataset can be arranged in a $m \times n$ array, $A[a_{ij}]$, whose rows correspond to processes, columns to factors, and the element a_{ij} is the physical dimensional indicator from the j-factor to the i-process. The i-process and the j-factor are, respectively, given by a of set n-tuples and m-tuples, namely, i-process row = $\{a_{i1,}a_{i2},\ldots, a_{in}\}$ and j-factor column = $\{a_{1,j}a_{2j},\ldots, a_{mj}\}$. Since each indicator has its own metric unit, the components of the processes have different units. The normalization to dimensionless indicators is performed by the transformation T on the j-factor of the array A as follows:

$$T(a_{ij}) = a_{ij}/\left|a_{ij}^{>} + a_{ij}^{<}\right| \equiv x_{ij} \tag{1}$$

where the normalization parameter is twice the absolute value of the median of the interval $[a_{lj}^{<}, a_{kj}^{>}]$ of the j-factor indicator, where

$$a_{kj}^{>} = \max\{a_{ij}|a_{ij} \neq 0\}, \tag{2}$$

$$a_{lj}^{<} = \min\{a_{ij}|a_{ij} \neq 0\}. \tag{3}$$

Equations (2) and (3) are applied separately for positive and negative indicators due to the distinct categories of these data. Zero indicators in A are excluded from these equations because they mean that the respective factor is absent from the corresponding process and do not contribute to the median.

The transformation T maps the array $A[a_{ij}]$ into a $m \times n$ matrix $M[x_{ij}]$ of dimensionless indicators that belong to the interval $(-1,1)$. An important property of T consists in the invariance of all ratios between indicators of the same factor with equal sign in both A and M, which ensures that M is a faithful representation of the physical system A, i.e.,

$$a_{ij}/a_{kj} = x_{ij}/x_{kj}, \ (a_{ij}a_{kj} > 0) \tag{4}$$

The rows and columns of M display processes and factors as n- and m-tuples,

$$P_i = (x_{i1}, x_{i2}, \ldots, x_{in}), F_j = (x_{1j}, x_{2j}, \ldots, x_{mj}) \tag{5}$$

Since the factors are linearly independent, the set of processes $\{P_i\}$ can be interpreted as vectors in n-dimensional Euclidean space, E^n, spanned by the set of factors $\{F_j\}$. The E^n space is endowed with an inner product, $\langle P_i|P_j\rangle$, and a norm, $\|P_i\|$, defined by

$$\|P_i\| = (\langle P_i|P_i\rangle)^{1/2} = \left(\sum_j |x_{ij}|^2\right)^{1/2,} \quad (j = 1, \ldots, n). \tag{6}$$

The aggregated metric of a process is to be determined by its norm, Eq. (6). A key feature of the present formalism is that the sustainability assessment is not measured relative to a reference point.

In order to handle negative indicators, it is convenient to split the processes into positive and negative indicators, i.e.,

$$P_i[x_{ij}] = P_i'[x_{ij} \geq 0] + P_i''[x_{ij} \leq 0]. \tag{7}$$

Equation (7) implies that whenever $x_{ij} > 0$ (<0), the corresponding component of P_i'' (P_i') is replaced by zero. In regard to calculations, a more useful way of splitting the process vectors consists in rewriting (7) in the equivalent form,

$$P_i[x_{ij}] = P_i'[x_{ij} \geq 0] - P_i''[x_{ij} \geq 0] \tag{8}$$

In Eq. (8), all indicators are now non-negative and vary within the interval [0,1). The matrix M is likewise split, with $P_i' \in M'$ and $P_i'' \in M''$, namely,

$$M\left[x_{ij}\right] = M'\left[x_{ij} \geq 0\right] - M''\left[x_{ij} \geq 0\right] \tag{9}$$

The split vectors of each process have an important property: they are orthogonal for their inner product vanishes, $\langle P_i' | P_i'' \rangle = 0$. As a result, the norm of P_i reads

$$\|P_i\| = \|P_i' + P_i''\| = \|P_i' - P_i''\| \tag{10}$$

This is consistent with the equivalence between Eqs. (7) and (8). On the other hand, Eq. (10) shows that the aggregated metric with negative indicators would be ambiguous if determined by $\|P_i\|$. This apparent ambiguity can be resolved by the basic properties of normed vector space.

Vectors in normed linear spaces must satisfy three axioms: (i) $\|P_i\| \geq 0$, $\|P_i\| = 0$ if and only if $P_i = 0$; (ii) $\|\alpha P_i\| = |\alpha| \cdot \|P_i\|$ for all scalars α; and (iii) the triangle inequality, namely,

$$\|P_i + P_k\| \leq \|P_i\| + \|P_k\| \tag{11}$$

From this inequality and $\|P_i\| - \|P_k\| = \|P_i - P_k + P_k\| - \|P_k\|$, it follows that

$$\|P_i - P_k\| \geq \|P_i\| - \|P_k\| \tag{12}$$

Equations (10) to (12) then yield

$$\|P_i'\| - \|P_i''\| \leq \|P_i\| \leq \|P_i'\| + \|P_i''\| \tag{13}$$

The contribution of the benefits, $\|P_i''\|$, has the role of decreasing the overall aggregated metric of the production processes. From the inequalities (13), we are thus led to define an index of aggregated sustainability metric by

$$S_i \equiv \left|\|P_i'\|\right| - \left|\|P_i''\|\right| \tag{14}$$

This metric is indeed a natural choice since $S_i = \|P_i\|$ in the absence of benefit factors and in the presence of these factors the index decreases as it should. The smaller the index, the more favorable is the process of production. S_i can be positive or negative (or, by chance, zero). If negative, it means that the benefits overwhelm the positive indicators of production. A hierarchy of processes follows from the ordering $S_k \leq S_l \leq \ldots$, where the smallest S_k is more favorable in terms of sustainability.

Applications

The preceding formalism is applied in three cases of industrial production. The first case treats three processes for making chlorine where the indicators have a small variation among them. The second case deals with the sustainability management of automotive shredder residues where negative indicators appear. The third case compares five alternatives of automobile fender production. Each case has its own peculiarity. In Case 1, the factor indicators for each process are very close to each other. Case 2 has negative indicators, namely, benefit factors. In Case 3, the indicators among factors vary up to four orders of magnitude. The three analyses follow the same pattern. Physical processes and factors of the original arrays A and corresponding matrices M are listed in Tables 8, 9 and 10 in the Appendix.

Case 1: three processes for making chlorine

Martins et al. (2007) and Sikdar (2009) have compared three methods for the manufacturing of chlorine. The dataset of the A array is displayed in Table 1, and applying the transformation T on A one obtains the matrix M in Table 2. The S-column shows the sustainability indices S_i for each process P_i. The R-column displays the S_i indices normalized with respect to the smallest S_i. The indicators in Table 2 are nearly centered to 0.5 due to the closeness of the indicators of each factor in Table 1. The R-column serves the purpose of providing the relative magnitude of the sustainability indices among the processes. From this column one may write, in an obvious notation,

$$S_1 : S_2 : S_3 = 1 : 1.004 : 1.009 \tag{15}$$

Thus, the sustainability indices of processes P_2 and P_3 are, respectively, only 0.4 and 0.9 % greater than process P_1, and P_3 is 0.5 % greater than P_2.

The three alternatives of chlorine production were comprehensively studied by Martins et al. (2007). These authors came to the conclusion that the processes are not

Table 1 Dataset of processes and factors for chlorine manufacturing

	F_1	F_2	F_3	F_4
P_1	10.80	2.15	202.114	116.102
P_2	10.98	2.15	202.116	116.103
P_3	11.70	2.15	200.104	112.002

Source: Sikdar (2009)

Table 2 Indicators of Table 1 normalized by the transformation Eq. (1)

	F_1	F_2	F_3	F_4	S	R
P_1	0.480	0.5	0.503	0.509	0.996	1
P_2	0.488	0.5	0.503	0.509	1.000	1.0039
P_3	0.520	0.5	0.498	0.491	1.005	1.0085

very different. By using a geometric mean, Sikdar (2009) found for the ratios in the R-column, Table 2, the values 1:1.004:1.008, giving an increment of 0.4 and 0.8 % of P_2 and P_3 over P_1. Martins et al. argue that processes P_1 and P_2 ought to be substituted by process P_3 from health and environmental concerns, although the latter being costly and operationally less efficient. In terms of sustainability, however, this substitution is quite feasible due to the small variations of the indices S_i among the processes. In fact, as pointed out by Sikdar et al. (2012), the substitution of the mercury cell by the membrane process is nowadays the usual trend.

Case 2: automotive shredder residue

Vermeulen et al. (2012) and Sikdar et al. (2012) have studied the sustainability assessment of the automotive shredder residue. The dataset is shown in Table 3. The transformation T is applied separately on the positive and the negative indicators displayed, respectively, in Tables 4 and 5. From the last column of these Tables, the determination of the overall sustainability indices given by Eq. (14) reads,

$$S_1 = 1.789,\ S_2 = -0.827,\ S_3 = -0.241,\ S_4 = -1.313 \tag{16}$$

The R-column with positive and negative metrics is difficult to be interpreted. Any shift of the metrics would distort the relative magnitude among them. The inequality ordering $S_4 < S_2 < S_3 < S_1$ coincides with Sikdar et al. (2012). It corresponds in Table 3 to the respective processes that have eight, seven, three, and none negative indicators. The overwhelming presence of negative indicators of P_4, P_2, and P_3 in the A matrix is an accurate representation of the physical system. The negative values of these processes are a clear indication of the importance of recycling and energy recovery in industrial production.

Case 3: automobile fender production

Sikdar et al. (2012) analyzed automobile fender manufactured by aluminum, steel, and three types of plastic materials studied by Sauer et al. (2000). The analysis of these five processes involved twelve factors of production as specified in the Appendix. The data are shown in Table 6, and the results of applying the transformation T are depicted in Table 7. In contrast to Case 1, the indicators cover the whole interval (0,1), for those in Table 6 differ by orders of magnitude. As all indicators are positive, the sustainability indices are simply $S_i = \|P_i\|$, and the result shown in Table 7 together with normalized values with respect to the maximum index. The inequality ordering $S_4 < S_3 < S_2 < S_5 < S_1$ agrees with the geometric mean results found by Sikdar et al. (2012). In fact, the numerical values in the R-column are not very different from that

Table 3 Dataset of processes and factors for automotive shredder residue

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
P_1	1.8	3.6	1.7	8.7	637	3,844	472	533	106
P_2	−13.1	−408	−4.3	−3.6	−641	1,614	−675	−2,617	161
P_3	−24.6	−48.2	−5.2	−11.5	841	841	12	−383	133
P_4	−26	−438	−7.8	−14.6	−325	−325	−812	−3,000	177

Source: Sikdar et al. (2012)

Table 4 Positive indicators of Table 3 normalized by Eq. (1)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	$\|P'\|$
P_1'	0.5	0.5	0.5	0.5	0.431	0.821	0.975	0.5	0.375	0.789
P_2'	0	0	0	0	0	0.345	0	0	0.569	0.665
P_3'	0	0	0	0	0.569	0.180	0.025	0	0.470	0.760
P_4'	0	0	0	0	0	0	0	0	0.626	0.626

Table 5 Negative indicators of Table 3 normalized by Eq. (1)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	$\|P''\|$
P_1''	0	0	0	0	0	0	0	0	0	0
P_2''	0.335	0.839	0.356	0.198	0.664	0	0.454	0.774	0	1.492
P_3''	0.629	0.099	0.430	0.632	0	0	0	0.113	0	1.001
P_4''	0.665	0.901	0.645	0.802	0.337	0.5	0.546	0.887	0	1.939

Table 6 Dataset of processes and factors for automobile fender production

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
P_1	1,290	15	36	104	1	28	4.4	6.7	3.9	0.66	2.9	3.7
P_2	1,120	25	27	105	0.1	19	4.2	9.2	3.7	0.92	3.4	1.2
P_3	1,060	18	22	83	0.4	20	3.9	8.7	2.5	0.99	2.7	1
P_4	810	14	17	62	0.2	16	3.5	8	1.9	0.62	1.9	0.25
P_5	1,080	21	25	115	1.2	20	7.2	9.1	2.5	0.74	2.4	0.25

Source: Sikdar et al. (2012)

Table 7 Indicators of Table 6 normalized by Eq. (1)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	S	R
P_1	0.615	0.385	0.680	0.588	0.770	0.637	0.411	0.422	0.673	0.410	0.547	0.937	2.100	1
P_2	0.534	0.641	0.510	0.593	0.077	0.432	0.393	0.579	0.638	0.572	0.642	0.304	1.796	0.855
P_3	0.505	0.462	0.415	0.469	0.308	0.455	0.365	0.547	0.431	0.615	0.510	0.253	1.575	0.750
P_4	0.386	0.359	0.321	0.351	0.154	0.364	0.327	0.503	0.328	0.355	0.359	0.064	1.186	0.565
P_5	0.515	0.539	0.472	0.650	0.923	0.455	0.673	0.573	0.431	0.460	0.453	0.064	1.909	0.909

work. The reciprocal of the R-column values gives the relative sustainability indices referred to the smallest one, i.e.,

$$S_4 : S_3 : S_2 : S_5 : S_1 = 1 : 1.327 : 1.513 : 1.609 : 1.770 \qquad (17)$$

In contrast to Case 1, the relations (17) show a large variation of the sustainability indices among the processes. For example, the process P_1 is 77 % greater than P_4.

Discussion

In the foregoing work, we attempted to observe the main properties that a sustainability metric must obey, as well summarized by Martins et al. (2007), namely: (i) clear, simple, and unambiguous; (ii) coherence set of quantifiable variables; and (iii) representative of the physical system.

By using basic principles of the Euclidean space together with a coherent set of quantifiable variables, we hope to have contemplated properties (i) and (ii). The transformation T: $A \rightarrow M$ defines the mapping of the actual physical dataset A into the matrix of process vectors M. The normalization of positive and negative factor indicators is unambiguously well defined by the dataset from Eqs. (2) and (3). The norm of a vector is the simplest magnitude attainable in a Euclidean space, and the sustainability index S_i leads to coherent results. Neither the transformation T nor the norm (6) has extraneous parameters.

However, the formalism allows the introduction of heuristic and phenomenological variables. Output of positive indicators may be hazardous to human health and/or to the surrounding environment. If the value of the sustainability threshold limit of a factor F_J is known, then one can substitute the corresponding normalization in Eq. (1) by this threshold limit. Obviously, in this case (positive) indicators would not be restricted to the domain [0,1). Indicators smaller than the threshold limit ($x_{ij} < 1$) would be sustainable and those larger ($x_{ij} > 1$) would not be sustainable.

Another example of heuristic variable consists in the weights that are eventually introduced into the process of aggregation in order to differentiate the importance of the factors of production. This can be trivially accomplished by multiplying the summand of Eq. (6) by such weights.

The last property (iii) that an aggregated metric must satisfy, as already emphasized, is particularly important since process alternatives are determined by inequalities. Equation (4) ensures that the mapping T: $A \rightarrow M$ represents faithfully the physical system given by the array A.

In summary, positive, negative, and zero indicators are treated alike within the strict concepts of normed vector space, without the recourse of shifting variables or extraneous parameters. The physical indicators, whether close together or far apart, when normalized belong to the interval [0,1). The ratio between physical indicators of the same factor is invariant after normalization, and the aggregated metrics are representative of the physical system. In addition, the absence of a reference point allows a comparison among processes with similar factors of production, and whose sustainability assessments are determined separately and independently.

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Appendix

See Tables 8, 9 and 10.

Table 8 Processes and factors

P_1: Mercury cells	F_1: Energy intensity (MJ/kg Cl_2)
P_2: Diaphragm cells	F_2: Material intensity (kg/kg Cl_2)
P_3: Membrane cells	F_3: Potential chemical risk
	F_4: Potential environment impact

Source: Sikdar (2009)

Table 9 Processes and factors

P_1: Landfill	F_5: GW short term (kg CO_2-eq./tASR)
P_2: Recycle landfill	F_6: GW long term (kg CO_2-eq./tASR)
P_3: Energy recovery	F_7: Human tox. Short term (kg $C_6H_4C_{12}$-eq./tASR)
P_4: Recycle energy	–
F_1: Energy intensity (GJ/tASR)	F_8: Human tox. Long term (kg $C_6H_4C_{12}$-eq./tASR)
F_2: Material intensity (kg Fe-eq./tASR)	–
F_3: Water consumption (m^3/tASR)	F_9: Treatment costs (€/tASR)
F_4: Land use (m^2a/tASR)	–

Source: Sikdar et al. (2012)

Table 10 Processes and factors

P_1: Aluminum	F_5: ODP
P_2: Steel	F_6: AP
P_3: PC/PBT	F_7: EP
P_4: PP/EPDM	F_8: POCP
P_5: PPO/PA	F_9: Htox air
F_1: Energy	F_{10}: Htox water
F_2: Resources	F_{11}: EcoTox
F_3: Water	F_{12}: Waste
F_4: GWP	

Source: Sikdar et al. (2012)

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