Original paper

Clean Techn Environ Policy 4 (2002) 72–78
DOI 10.1007/s10098-002-0150-2

Formulation of sensitivity analysis in life cycle assessment using a perturbation method

Shinsuke Sakai, Koji Yokoyama

Abstract In this paper, a new method for the efficient sensitivity analysis in life cycle assessment is proposed. Introduction of a perturbation method to matrix-based life cycle assessment will enable one to evaluate the degree of influence of each element on the total sum of environmental loads. The mathematical background is described in detail together with its formulation. Special emphasis is placed on the advantage that the sensitivity matrix, in which each entity denotes the sensitivity of the corresponding element, is derived as a result of matrix operation. Therefore, even if the number of processes becomes larger, all sensitivities can be calculated easily. Furthermore, the process sensitivity is also formulated. The proposed method is compared with another sensitivity analysis by Heijungs [Heijungs R (1994) Ecol Econ 10:69–81] and the validity of the current method is examined using the example given by Heijungs.

Introduction

By standardizing the evaluation procedure of life cycle assessment (LCA) in ISO, this method has become widely utilized in all over the world. With the advance in the construction of the inventory database, the LCA procedure seems to become a practical tool. Although the main feature of LCA is to evaluate environmental loads by summing up every contribution from each stage from procuring raw materials to usage, manufacturing, transportation and disposal, one of its important applications is as a decision-making tool (Smet and Stalmans 1996; Maurice et al. 2000). In particular, it has been required that each country makes efforts to reduce CO_2 emission, since a goal of CO_2 reduction was set in the Kyoto protocol. It is becoming more and more important that LCA is utilized in order to determine the priority of improvements for the reduction of CO_2. For an analysis centered on decision-making, the sensitivity analysis is extremely important in addition to the evaluation of the total amount of environmental load, which is evaluated in conventional LCA. The importance of the sensitivity in LCA has been widely recognized (Steen 1997). The sensitivity has been widely used to investigate the relation between input data quality and model output value (Kennedy et al. 1996). Heijungs pointed out that the identification of key issues in an iterative procedure is also important to achieve a certain level of reliability in LCA (Heijungs and Kleijn 1996). Sensitivity analysis is also expected to play an important role at the interpretation stage, which is the last phase of LCA in the ISO standard (Heijungs and Kleijn 2001). For example, sensitivity and uncertainty analyses are recommended and sometimes requested in ISO 14040, LCA standard (Steen 1997).

It is generally understood that the sensitivity means the degree of the effect of each element in the system on the evaluation quantity. As a simple procedure, the sensitivity can be evaluated by estimating the influence of the minute fluctuation of the element of interest on the summed value of the environmental load. Such a method is also available for the sensitivity analysis in LCA. However, a one by one treatment is required for the sensitivity analysis of all elements since it is rather difficult to generalize the procedure. In such a method, it may be time consuming and laborious work to prioritize the elements if there are many elements in the system. Therefore, it is extremely important to formulate the procedure of the sensitivity analysis for general purposes.

From this viewpoint Heijungs proposed the sensitivity analysis for the LCA using process analysis and formulated the procedure so it could be used for general purposes (Heijungs 1994; Heijungs and Kleijn 1996). In his method, the sensitivity analysis is formulated on the basis of the matrix-based LCA. Since the method is expressed in mathematical form, it is easy to integrate the method to LCA. However, the derived equation is slightly complicated and somewhat difficult for practical use.

In this paper, the perturbation method is applied to formulate the sensitivity analysis in LCA. It will be shown that this method will become advantageous in many aspects for practical use. For example, the derived equation is expressed explicitly in matrix form and so its application is extremely simple. Therefore, not only is its application simple, but the method does not require much calculation time. It will also be shown that several other important properties related to sensitivity can be derived through this formulation. In this paper, the fundamental formulation will be described and its validity is examined

Received: 29 November 2001 / Accepted: 1 April 2002
Published online: 28 June 2002

S. Sakai (✉), K. Yokoyama
The University of Tokyo, School of Engineering,
Department of Mechanical Engineering,
7-3-1 Hongo Bunkyo-ku, Tokyo 113-8656, Japan
e-mail: sakai@fml.t.u-tokyo.ac.jp
Tel.: +81-3-58416436
Fax: +81-3-38180835

We greatly appreciate the valuable comments from Halada K. of NIMS and member of the MLCA committee in the Society of Non-Traditional Technology.

by applying the method to the example that Heijungs set out in his paper and comparing the results.

Although the procedure presented in this paper is applicable in the stage of inventory analysis, this method is also available for overall LCA.

Inventory analysis using the matrix method

In the formulation of the sensitivity analysis in this paper, the inventory analysis is in essence based upon the matrix method. The details of this method are described in the paper of Heijungs (1994). A summary of this method is given here briefly since this method is closely related to the discussion in the subsequent section. The fundamental unit to be considered is a process in which data are composed of input, output, and environmental loads. Examples of the input and the output data are materials, products, and energy. All of these are hereafter called "materials" in this paper. On the other hand, environmental loads include emissions to the environment such as CO_2, SO_x, NO_x, and solid waste. These data are usually normalized per standard unit and the property of the process can be expressed as

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ a_n \end{pmatrix} \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_s \end{pmatrix}$$

where the bold face of lower case letters indicates their vector properties. a_i (i=1,....,n) stand for entities for input and output properties and b_i (i=1,...,s) represent entities for environmental load properties. It is assumed that there are n materials and s environmental loads in the current process tree. A sign convention will be applied: inputs will be expressed by negative coefficients and outputs by positive coefficients. By multiplying the quantitative occurrence of the process by **a** and **b**, the total amount of materials and that of environmental loads in each process will be determined, respectively. By assembling each process into a column of a coefficient matrix, the following matrix will be derived:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1q} \\ \vdots & & & & \vdots \\ a_{i1} & & a_{ij} & & a_{iq} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nq} \end{bmatrix}$$

where the ith material of the jth process is described as a_{ij} and q is the number of processes. Similarly, the environmental load matrix will be derived as

$$\mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1q} \\ \vdots & & & & \vdots \\ b_{i1} & & b_{ij} & & b_{iq} \\ \vdots & & & & \vdots \\ b_{s1} & \cdots & b_{sj} & \cdots & b_{sq} \end{bmatrix}$$

where the ith environmental load entity of the jth process is described as b_{ij}. The bold face of capital letters indicates they are matrix quantities. In every matrix, the column j represents the index for the process. In the meantime, the boundary condition for materials at the system boundary will be given in the vector form as

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{pmatrix}$$

If the ith material is closed within the system, α_i=0. The total amount of environmental loads will be expressed in the vector form as

$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \vdots \\ \beta_s \end{pmatrix}$$

Hereafter β is called as the environmental vector. Since the total amount of materials must be balanced within the system, the following equation has to be satisfied:

$$\sum_{j=1}^{q} a_{ij}p_j = \alpha_i \tag{1}$$

Since the above relation holds true for all materials (i=1...n), the following linear equations can be derived:

$$\mathbf{Ap} = \alpha \tag{2}$$

For the uniqueness of the solution, the number of processes should coincide with the number of materials and so **A** is a square matrix. Therefore, the unknown vector **p** is obtained by solving the following equation:

$$\mathbf{p} = \mathbf{A}^{-1}\alpha \tag{3}$$

where $\mathbf{A}^{-1}$ is the inverse matrix of **A** and the ijth entry of $\mathbf{A}^{-1}$ is denoted as a_{ij}^{-1} hereafter. The environmental load vector β can be obtained by using the environmental load matrix **B** as

$$\beta = \mathbf{Bp} \tag{4}$$

Evaluation of small variations of process occurrence using a perturbation method

In the sensitivity analysis, the degree of variation of the evaluation parameter due to the minute variation of an element in the system is investigated. The contribution to the improvement of environmental loads can be identified with the sensitivity value. In the inventory analysis stage, one of the most important properties to be investigated is the variation $\Delta\beta$ due to the minute variation Δa_{ij}. This will help the analyst to specify the most effective part for the improvement in the system. $\Delta\beta$ is calculated from $\Delta\mathbf{p}$, which is the variation of process vector due to Δa_{ij} and is expressed as

$$\Delta\beta = \mathbf{B}\Delta\mathbf{p} \tag{5}$$

Thus, it is necessary to derive $\Delta\mathbf{p}$ due to Δa_{ij} first. The parameter that shows the original value before the variation is expressed by putting a bar on its symbol. When the small variation Δa_{ij} is applied to the original coefficient matrix $\bar{\mathbf{A}}$, the process vector $\mathbf{p}$ in Eq. (2) is also varied. This variation $\Delta\mathbf{p}$ is derived using the concept of a perturbation method here. Let the small variation of $\mathbf{A}$ due to Δa_{ij} be $\Delta\mathbf{A}$; it is given as

$$\Delta\mathbf{A} = \mathbf{A}^{\mathrm{I}}\Delta a_{ij} \tag{6}$$

where

$$\mathbf{A}^{\mathrm{I}} = \frac{\partial}{\partial a_{ij}}\mathbf{A} = [a_{lm}],$$

$$a_{lm} = \begin{cases} 1 & \text{if} \quad l = i, m = j \\ 0 & \text{otherwise} \end{cases}$$

In other word, all the entries of $\mathbf{A}^{\mathrm{I}}$ are 0 except entry (i, j) which is 1. Therefore, the solution after the small variation can be given by solving the following equation:

$$\bar{\mathbf{A}}\mathrm{p} + \mathbf{A}^{\mathrm{I}}\Delta a_{ij}\mathbf{p} = \alpha \tag{7}$$

In the perturbation method, it is assumed that the solution of Eq. (7) is expressed by a series in Δa_{ij} as

$$\mathbf{p} = \bar{\mathbf{p}} + \mathbf{p}^{\mathrm{I}}\Delta a_{ij} + \mathbf{p}^{\mathrm{II}}\Delta a_{ij}^2 + \cdots \tag{8}$$

The unknown coefficients of $\mathbf{p}^{\mathrm{I}}$, $\mathbf{p}^{\mathrm{II}}$, ... can be determined by substituting Eq. (8) into Eq. (7) and comparing the terms of Δa_{ij} of the same order on both sides. For example, comparing the 0th order terms yields the following equation:

$$\bar{\mathbf{A}}\bar{\mathrm{p}} = \alpha \tag{9}$$

This equation must be solved initially for conventional LCA anyway. Similarly, comparison of 1st order terms yields the following equation:

$$\mathbf{p}^{\mathrm{I}} = -\bar{\mathbf{A}}^{-1}\mathbf{A}^{\mathrm{I}}\bar{\mathbf{p}} \tag{10}$$

After obtaining $\bar{\mathbf{p}}$ by solving Eq. (9), $\mathbf{p}^{\mathrm{I}}$ can be obtained by substituting $\bar{\mathbf{p}}$ into Eq. (10). Similarly, coefficient vectors for higher order terms can be determined in order by comparing corresponding terms on both sides. If we can determine the higher order terms in Eq. (8), the precision of $\mathbf{p}$ may be improved. For the sensitivity analysis, the quantity of interest is a first order variation and so $\Delta\mathbf{p}$ is here evaluated up to the first order term in Eq. (8). Later, the treatment is extended to the higher order terms. Thus, $\Delta\mathbf{p}$ is approximated as

$$\Delta\mathbf{p} \approx \mathbf{p}^{\mathrm{I}}\Delta a_{ij} = -\bar{\mathbf{A}}^{-1}\mathbf{A}^{\mathrm{I}}\bar{\mathbf{p}}\Delta a_{ij} \tag{11}$$

It should be noted that the inverse matrix $\mathbf{A}^{-1}$ is common regardless of the location of entry (i, j). Thus, once $\mathbf{A}^{-1}$ has been calculated, it does not need to be calculated again for the evaluation of $\Delta\mathbf{p}$ due to Δa_{ij}. The expression of $\mathbf{A}^{\mathrm{I}}$ is simple as was already shown in Eq. (6) and so the calculations of $\Delta\mathbf{p}$ for various Δa_{ij} are extremely simple.

Derivation of sensitivity

Sensitivity to processes

Based upon Eq. (11) which is formulated using the perturbation method, the sensitivity of a_{ij} to p_l will be derived. By expanding Eq. (11), the following equation is derived.

$$\Delta\mathbf{p} = -\begin{bmatrix} a_{1i}^{-1} \\ \vdots \\ \vdots \\ a_{ni}^{-1} \end{bmatrix}\bar{p}_j\Delta a_{ij} \tag{12}$$

where, a_{ij}^{-1} means ijth entry of $\mathbf{A}^{-1}$.

Thus the physical sensitivity of a_{ij} to p_i is given as

$$\frac{\Delta p_l}{\Delta a_{ij}} = -a_{li}^{-1}\bar{p}_j \tag{13}$$

On the other hand, Heijungs derived the corresponding sensitivity as

$$\frac{\partial p_l}{\partial a_{ij}} = -\frac{\bar{p}_l}{\det(\mathbf{A})}(-1)^{i+j}\det\left(\mathbf{A}_{ij}\right) + \begin{cases} 0(j = l) \\ \frac{(-1)^{i+j}\det\left(\mathbf{A}_{ij}^l\right)}{\det(\mathbf{A})}(\text{otherwise}) \end{cases} \tag{14}$$

where det($\mathbf{A}$) denotes the *determinant* of the matrix $\mathbf{A}$, $\mathbf{A}_{ij}$ denotes the matrix $\mathbf{A}$ with the ith row and the jth column deleted (the so-called *minor*) and $\mathbf{A}_{ij}^l$ denotes the minor of the matrix $\mathbf{A}^l$. $\mathbf{A}^l$ is equal to the matrix $\mathbf{A}$ with the lth column replaced by the vector α. It is seen that Eq. (13), which is formulated with the perturbation method, is much simpler than Eq. (14) and thus the calculation time is expected to be short. In particular, it is advantageous when sensitivities for many a_{ij} are to be calculated. When one would like to know the relative relation within entities, the important property is the rate sensitivity, which is given as

$$\frac{\Delta p_l/\bar{p}_l}{\Delta a_{ij}/\bar{a}_{ij}} = -\frac{\bar{p}_j}{\bar{p}_l}\bar{a}_{ij}a_{li}^{-1} \tag{15}$$

Hereafter, the property given by Eq. (15) is denoted by c_{ij}^l. The confirmation of this derivation will be given later.

Sensitivity to environmental loads

Let the sensitivity of a_{ij} to the kth environmental load item be denoted as d_{ij}^k. This d_{ij}^k will be derived in the same manner as Eq. (15) by substituting Eq. (11) into Eq. (5) as

$$d_{ij}^k = \frac{\Delta\beta_k/\bar{\beta}_k}{\Delta a_{ij}/\bar{a}_{ij}} = -\frac{\bar{a}_{ij}}{\bar{\beta}_k}\bar{p}_j\sum_l b_{kl}a_{li}^{-1} \tag{16}$$

This sensitivity parameter is extremely important to the improvement of the analysis for the environmental load. On the other hand, Heijungs (1994) derived the sensitivity corresponding to Eq. (16) as

$$\frac{\Delta\beta_k/\bar{\beta}_k}{\Delta a_{ij}/\bar{a}_{ij}} = -\,\bar{a}_{ij}\frac{(-1)^{i+j}\det(\mathbf{A}_{ij})}{\det(\mathbf{A})} + \frac{\bar{a}_{ij}(-1)^{i+j}}{\bar{\beta}_k\det(\mathbf{A})}\sum_{\substack{l=1\\ l\neq j}}^{n}\left(b_{kl}\det\left(\mathbf{A}_{ij}^{l}\right)\right) \tag{17}$$

By comparing Eq. (17) with Eq. (16), it is clear that the formulation derived by the perturbation method is much simpler. Furthermore, once $\mathbf{A}^{-1}$ and β are obtained, the calculation of Eq. (16) is not a time-consuming process. The difference becomes remarkable when the number of processes and materials becomes larger in the practical LCI.

Sensitivity matrix

The process sensitivity matrix $\mathbf{S}_p^l$ of size $n\times n$ is defined here so that its ijth entry coincides with c_{ij}^l defined in Eq. (15). Since all sensitivities of c_{ij}^l are listed in this matrix, the relation between the sensitivities can be understood easily. Similarly, the environmental load sensitivity matrix $\mathbf{S}_\beta^k$ of size $n\times n$ can be defined so that its ijth entry coincides with d_{ij}^l defined in Eq. (16). Since this matrix includes all sensitivities of d_{ij}^l, we can determine the priorities of the entry from this matrix in the improvement analysis. As is shown in the Appendix in detail, the process sensitivity matrix is derived from Eq. (15) as

$$\mathbf{S}_p^l = -\frac{1}{\bar{p}_l}\left(\mathbf{A}^{-1}\right)^{\mathrm{t}}\mathbf{e}_l\bar{\mathbf{p}}^{\mathrm{t}}\otimes\mathbf{A} \tag{18}$$

where $(\mathbf{A})^{\mathrm{t}}$ means the transpose matrix of $\mathbf{A}$ and $\otimes$ is the operation between matrices of size $n\times n$ which is defined as

$$\mathbf{X} = \mathbf{Y}\otimes\mathbf{Z}, x_{ij} = y_{ij} * z_{ij} \tag{19}$$

$\mathbf{e}_l$ is a vector of size n and is defined as

$$\mathbf{e}_l = \begin{bmatrix} e_1 \\ \vdots \\ e_i \\ \vdots \\ e_n \end{bmatrix}, \quad e_i\begin{cases} 1 & \text{if} \quad i = l \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Similarly, the environmental load matrix is derived from Eq. (16) as

$$\mathbf{S}_\beta^k = \mathbf{E}^{\mathrm{t}}\mathbf{e}_k\bar{\mathbf{p}}^{\mathrm{t}}\otimes\mathbf{A} \tag{21}$$

where

$$\mathbf{E} = -\beta_{diag}\mathbf{B}\mathbf{A}^{-1}$$

$$\beta_{diag} = \begin{bmatrix} 1/\beta_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1/\beta_s \end{bmatrix}$$

Both Eq. (18) and Eq. (21) have the following advantages.

1. The sensitivity matrix is expressed by an explicit operation of matrices and vectors.
2. Once $\mathbf{A}^{-1}$ and $\bar{\mathbf{p}}$ are calculated, only one vector modification is sufficient for the calculation of the sensitivity corresponding to p_l or β_k: only $\mathbf{e}_l$ needs to be modified in Eq. (18) and $\mathbf{e}_k$ in Eq. (21).
3. Thus, it is extremely easy to incorporate the algorithm into computer programs.

Verification of the proposed method

Applicability

At first, the applicability of the proposed sensitivity analysis method is examined by applying the method to a practical example. Fig. 1 shows a schematic drawing of the steel making process given in a previous paper (Sakai 1998). This system includes a large number of systems together with material loops and so is very complicated. Performing sensitivity analysis for CO_2 by applying the proposed method shown above, it is confirmed that the sensitivity for all parameters can be calculated practically. In Table 1, 10 calculated sensitivities are shown in descending order of the absolute values. If the sensitivity is negative, it means that the total sum of CO_2 decreases in accordance with the increase of the parameter. If the sensitivity is positive, however, it means that the total sum of CO_2 increases in accordance with the increase of the

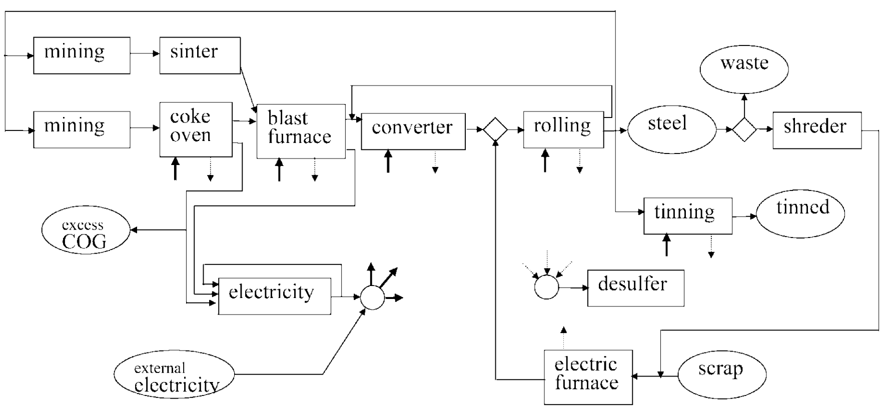

Fig. 1. Process tree for steel-making process

Table 1. Results of sensitivity analysis for steel-making process

Order	Process	Material	I/O	Sensitivity
1	Rolling	Steel	O	-1.0648
2	Rolling	Steel ingot	I	0.996102
3	Converter	Crude steel	O	-0.997843
4	Converter	Pig iron	I	0.954844
5	Blast furnace	Pig iron	O	-0.89579
6	Blast furnace	BFG	O	0.70339
7	Power generation	BFG	I	-0.70339
8	Plating	Tinplate	O	-0.18247
9	Plating	Steel	I	0.179626
10	Blast furnace	Sintered steel	I	0.119452

parameter. It is easy to specify the parameter that contributes to the reduction of total CO_2. Thus the applicability of the method is confirmed for a practical example.

Sensitivity matrix

The verification of the sensitivity matrices presented in the previous section is done using a simple example set out in Heijungs's reference (Heijungs 1994). The process data used are shown in Table 2.

This produces the following related matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -50 & -1 & 0 \\ -0.01 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & -5 & 0 & 0 \\ -0.5 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 2 & 10 & 0 & 1 \end{bmatrix} \tag{22}$$

$$\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}$$

By substituting these values into Eqs. (3) and (4), the environmental load vector can be obtained as

$$\beta = \begin{bmatrix} -1.01 \\ -5.1 \\ 30.6 \\ 22.52 \end{bmatrix} \tag{23}$$

These values agree with those in Heijungs's results. On the other hand, the process sensitivity factor matrix for the process of production of aluminum can be calculated using Eq. (18) and is obtained as

$$\mathbf{S}_p^2 = \begin{bmatrix} -1.010 & 1.000 & 0.010 & 0 \\ 1.010 & -2.000 & 0.990 & 0 \\ 0 & 0 & -1.000 & 1.000 \\ 0 & 0 & 0 & -1.000 \end{bmatrix} \tag{24}$$

Furthermore, the environmental load sensitivity matrix for the solid waste can be calculated using Eq. (20) as

$$\mathbf{S}_\beta^4 = \begin{bmatrix} -1.902 & 1.884 & 0.019 & 0 \\ -0.996 & 1.973 & 0.977 & 0 \\ 0 & 0 & -0.996 & -0.996 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{25}$$

These solutions agree completely with those given by Heijungs and so the applicability of Eq. (20) is confirmed numerically. The proof in the equation level is investigated in the subsequent section.

Proof in the equation level

The agreement of the perturbation method and Heijungs's method is examined using a simple property of $\partial p_i/\partial \alpha_j$ here. He derived the following expression in the appendix to his paper.

$$\frac{\partial p_i}{\partial \alpha_j} = \frac{1}{\det(\mathbf{A})}(-1)^{j+i}\det(\mathbf{A}_{ji}) \tag{26}$$

On the other hand, the property is derived using the perturbation method as follows. If a perturbation is given to α_j, the equation of material balance should be modified as

$$\mathbf{A}\mathbf{p} = \overline{\alpha} + \frac{\partial \alpha}{\partial \alpha_j}\Delta\alpha_j = \overline{\alpha} + \mathbf{e}_j\Delta\alpha_j \tag{27}$$

Table 2. List of process data (Heijungs 1994)

Process material	Production of electricity	Production of aluminum	Production of aluminum foil	Usage of aluminum foil	Total sum
Electricity	1	-50	-1	0	0
Aluminum	-0.01	1	-1	0	0
Aluminum foil	0	0	1	-1	0
Sandwich packages	0	0	0	1	0.1
Env. load					
Bauxite	0	-5	0	0	
Crude oil	-0.5	0	0	0	
CO_2	3	0	0	0	
Solid waste	2	10	0	1	

where $\mathbf{e}_j$ is the vector defined in Eq. (20). If the solution of $\mathbf{p}$ in Eq. (27) is expressed as

$$\mathbf{p} = \bar{\mathbf{p}} + \mathbf{p}^{\mathrm{I}}\Delta\alpha_j \tag{28}$$

substituting Eq. (28) into Eq. (27) and comparing on both sides the first order terms of $\Delta\alpha_j$, we obtain the following equation:

$$\mathbf{A}\mathbf{p}^{\mathrm{I}} = \mathbf{e}_j$$

Thus $\mathbf{p}^{\mathrm{I}}$ can be derived as

$$\mathbf{p}^{\mathrm{I}} = \mathbf{A}^{-1}\mathbf{e}_j = \begin{bmatrix} a_{1j}^{-1} \\ \vdots \\ \vdots \\ a_{nj}^{-1} \end{bmatrix} \tag{29}$$

By substituting Eq. (29) into Eq. (28) and differentiating $\mathbf{p}$ with respect to α_j, the following equation is obtained.

$$\frac{\partial p_i}{\partial \alpha_j} = a_{ij}^{-1} \tag{30}$$

By the way, the inverse matrix of $\mathbf{A}$ satisfies the following fundamental relationship:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} \mathbf{A}'_{11} & \cdots & \cdots & \mathbf{A}'_{n1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \mathbf{A}'_{1n} & \cdots & \cdots & \mathbf{A}'_{nn} \end{bmatrix} \tag{31}$$

where $\mathbf{A}'_{ij}$ is a so called cofactor determinant which is defined as

$$\mathbf{A}'_{ij} = (-1)^{i+j} \det(\mathbf{A}_{ij})$$

Since a_{ij}^{-1} is the (i, j)th entry of $\mathbf{A}^{-1}$, a_{ij}^{-1} is derived as

$$a_{ij}^{-1} = \frac{\mathbf{A}'_{ji}}{\det(\mathbf{A})} = \frac{1}{\det(\mathbf{A})}(-1)^{j+i}\det(\mathbf{A}_{ji}) \tag{32}$$

Substituting this equation into Eq. (30) yields Eq. (26). Thus it is proved that Eq. (30), which is derived by the perturbation method, agrees with Eq. (26), which was derived by Heijungs.

Discussion

In calculating all sensitivities regarding relations between elements and environmental loads, the method proposed above has great advantages. It does not consume much time for the calculation compared with the case when the elements are perturbed one by one. In the matrix based LCI, most of the calculation time is consumed in calculating the inverse matrix. However, it is not required to calculate the inverse matrix every time the sensitivity is calculated using the proposed perturbation method, because once the inverse matrix has been calculated, it can be used for the evaluation of other sensitivities. Since the inverse matrix must be calculated once for conventional LCI anyway, the additional time to the LCI for the evaluation of sensitivities is not so great. Furthermore, the sensitivity matrix is expressed by the explicit operation of given vectors as shown in Eq. (21). Thus it is extremely easy to obtain sensitivities of all elements and so the implementation in computer software is not difficult.

However, it should be noted that the sensitivity shows only the index for the variation due to the input perturbation. If the property of interest is the range of output variation, the evaluation by a first order approximation is not sufficient for the accurate estimation. Considering the higher order term in the perturbation method shown above, we can improve the accuracy of the evaluation of output values. By expressing $\Delta\mathbf{p}$ in Eq. (8) by the series expansion until the nth order term as

$$\Delta\mathbf{p} = \mathbf{p}^{\mathrm{I}}\Delta a_{ij} + \mathbf{p}^{\mathrm{II}}\Delta a_{ij}^2 + \cdots + \mathbf{p}^{n}\Delta a_{ij}^{n} \tag{33}$$

and using a derivation similar to Eq. (8), the coefficient $\mathbf{p}^n$ in this equation can be expressed as follows:

$$\mathbf{p}^n = (-1)^n(\bar{\mathbf{A}}^{-1}\mathbf{A}^{\mathrm{I}})^n\bar{\mathbf{p}} \tag{34}$$

This means that Eq. (33) is a geometric progression with a common ratio $-\mathbf{A}^{-1}\mathbf{A}^{\mathrm{I}}\Delta a_{ij}$. Therefore, using the formulation of the geometric progression, the summation of the series expansion can be described as

$$\Delta\mathbf{p} = \frac{\bar{p}_j}{a_{ji}^{-1}} \begin{bmatrix} a_{1i}^{-1} \\ \vdots \\ \vdots \\ a_{ni}^{-1} \end{bmatrix} \gamma \frac{1-\gamma^{n+1}}{1-\gamma} \tag{35}$$

where $\gamma = -a_{ji}^{-1}\Delta a_{ij}$. If γ is sufficiently small compared with 1, Eq. (35) converges to

$$\Delta\mathbf{p} = -\frac{\Delta a_{ij}}{1 + a_{ji}^{-1}\Delta a_{ij}}\bar{p}_j \begin{bmatrix} a_{1i}^{-1} \\ \vdots \\ \vdots \\ a_{ni}^{-1} \end{bmatrix} \tag{36}$$

since $\gamma^{n+1} \to 0$ as $n \to \infty$. Finally, the variation of the environmental loads can be described as

$$\Delta\beta = \mathbf{B}\Delta\mathbf{p} = -\frac{\Delta a_{ij}}{1 + a_{ji}^{-1}\Delta a_{ij}}\bar{p}_j\mathbf{B} \begin{bmatrix} a_{1i}^{-1} \\ \vdots \\ \vdots \\ a_{ni}^{-1} \end{bmatrix} \tag{37}$$

This result is shown in the curve in Fig. 2 which is calculated using Eq. (37). Solid circles in the figure show the results calculated directly using conventional LCA and are confirmed to be situated on the curve. Thus, Eq. (37) is shown to be valid for the evaluation of the variation of β. The straight line in the figure shows the first order approximation evaluated using Eq. (21) and it is clear that the sensitivity represents the slope at the inventory data point. Thus, for the investigation of the influence of the variation of input value on the output value, Eq. (21) is

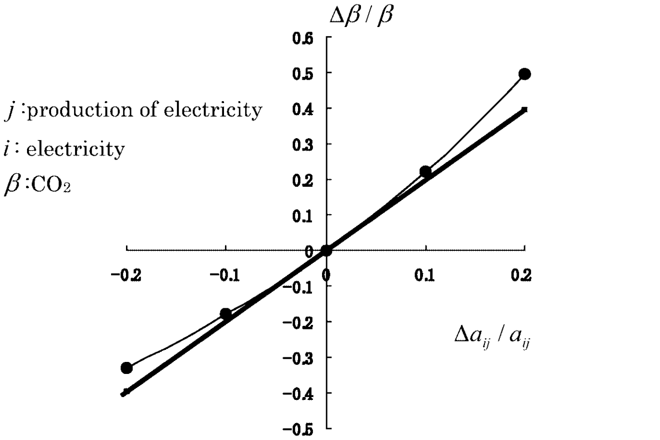

Fig. 2. Relation between variation ratio of input value and that of output value

effective and Eq. (37) should be used for the evaluation of the varied range.

Conclusions

This paper develops a new method for sensitivity analysis in LCA by introducing the perturbation method. After the formulation for sensitivity for each element is derived, the formulation for the sensitivity matrix using matrix operation is also shown. Applying the proposed method to a steel-making process, it is shown that the method is sufficiently applicable to a practical problem. Finally, the method is examined using the example given by Heijungs and its validity is shown.

Appendix: Derivation of process sensitivity matrix

The process sensitivity matrix $\mathbf{S}_p^l$, the ijth entry of which is given by Eq. (15), can be derived as follows. All entries of $\mathbf{S}_p^l$ can be expressed as follows.

$$\mathbf{S}_p^l = -\frac{1}{\bar{p}_l}\begin{bmatrix} \bar{p}_1 a_{11} a_{l1}^{-1} & \cdots & \cdots & \bar{p}_n a_{1n} a_{l1}^{-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \bar{p}_1 a_{n1} a_{ln}^{-1} & \cdots & \cdots & \bar{p}_n a_{nn} a_{ln}^{-1} \end{bmatrix} = -\frac{1}{\bar{p}_l}\begin{bmatrix} a_{l1}^{-1} \\ \vdots \\ \vdots \\ a_{ln}^{-1} \end{bmatrix}\left[\bar{p}_1 \quad \cdots \quad \cdots \quad \bar{p}_n\right] \otimes \mathbf{A} \tag{38}$$

In this equation, the following relation holds.

$$\begin{bmatrix} a_{l1}^{-1} \\ \vdots \\ \vdots \\ a_{ln}^{-1} \end{bmatrix} = \left[a_{l1}^{-1} \quad \cdots \quad \cdots \quad a_{ln}^{-1} \right]^t = \left(\mathbf{A}^{-1}\right)^t \mathbf{e}_l \tag{39}$$

Substituting this equation into Eq. (38) yields

$$\mathbf{S}_p^l = -\frac{1}{\bar{p}_l}\left(\mathbf{A}^{-1}\right)^t \mathbf{e}_l \bar{\mathbf{P}}^t \otimes \mathbf{A}$$

References

Heijungs R (1994) A generic method for the identification of options for cleaner products. Ecol Econ 10:69–81

Heijungs R, Kleijn R (1996) Identification of key issues for further investigation in improving the reliability of life cycle assessment. J Cleaner Prod 4:159–166

Heijungs R, Kleijn R (2001) Numerical approaches towards life cycle interpretation. Int J LCA 6:141–148

Kennedy DJ, Montgomery DC, Quay BH (1996) Data quality: stochastic environmental life cycle assessment modeling. Int J LCA 1:199–207

Maurice B, Frischknech R, Coelho-Schwirtz V, Hungerbuhler K (2000) Uncertainty analysis in life cycle inventory. Application to the production of electricity with French coal power plants. J Cleaner Prod 8:95–108

Smet BD, Stalmans M (1996) LCI data and data quality – thoughts and considerations. Int J LCA 1:96–104

Steen B (1997) On uncertainty and sensitivity of LCA-based priority setting. J Cleaner Prod 5:255–262

Sakai S (1998) Influence of non-linearity of I/O relation in life cycle inventory analysis (in Japanese). Tetsu to Hagane 84:829–836