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An adaptive finite element discretisation for a simplified Signorini problem

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Abstract. Adaptive mesh design based on *a posteriori* error control is studied for finite element discretisations for variational problems of Signorini type. The techniques to derive residual based error estimators developed, e.g., in ([2, 10, 20]) are extended to variational inequalities employing a suitable adaptation of the duality argument [17]. By use of this variational argument weighted *a posteriori* estimates for controlling arbitrary functionals of the error are derived here for model situations for contact problems. All arguments are based on Hilbert space methods and can be carried over to the more general situation of linear elasticity. Numerical examples demonstrate that this approach leads to effective strategies for designing economical meshes and to bounds for the error which are useful in practice.

1 Introduction

A fundamental model situation for contact problems in elasticity is Signorini's problem describing the deformation of an elastic body which is unilaterally supported by a frictionless rigid foundation. We intend to derive efficient *a posteriori* error control techniques for this equation with special emphasis on local error phenomena, e.g., the error for stresses in the contact zone. In order to demonstrate the concept for our method for *a posteriori* error estimation, we first consider the simplified case

$$
\begin{aligned}
-\Delta u = f \quad &\text{in } \Omega \subset \mathbb{R}^2, \\
u = 0 \quad &\text{on } \Gamma_D, \\
u \geq 0, \quad \partial_n u \geq 0, \quad u\,\partial_n u = 0 \quad &\text{on } \Gamma_C,
\end{aligned}
\tag{1.1}
$$

where $\Gamma_C = \partial\Omega \setminus \Gamma_D$ and $\partial_n u = \nabla u \cdot n$.

Problem (1.1) is to be solved by the finite element Galerkin method on adaptively optimised meshes. By variational arguments, we derive weighted *a posteriori* error estimates for controlling arbitrary linear functionals of the error. This approach leads to effective strategies for designing economical meshes and to bounds for the error which are useful in practice. The extension to Signorini's problem is illustrated in the last section.

The basis for applying the finite element method to (1.1) is the formulation as a variational inequality, i.e., a solution $u \in K$ is sought which satisfies

$$
(\nabla u, \nabla(\varphi - u)) \geq (f, \varphi - u) \quad \forall \varphi \in K, \tag{1.2}
$$

where we set $V = \{v \in H^1 \mid v = 0 \text{ on } \Gamma_D\}$ and $K = \{v \in V \mid v \geq 0 \text{ on } \Gamma_C\}$. Here, and in what follows, $H^m = H^m(\Omega)$ denotes the standard Sobolev space of L^2-functions with derivatives in $L^2(\Omega)$ up to order m.

Equation (1.2) is uniquely solvable (cf. Lions and Stampacchia [14]) and, under appropriate smoothness conditions on the boundary and data, the solution is known to satisfy the regularity result $u \in H^2(\Omega)$ (see Brézis [3]).

In the following, we apply the finite element method on decompositions $\mathbb{T}_h = \{T\}$ of Ω consisting of quadrilaterals T satisfying the usual condition of shape regularity. Simplifying notation, we assume the domain Ω to be polyhedral in order to ease the approximation of the boundary. More general situations may be treated by the usual modifications. For ease of mesh refinement and coarsening, hanging nodes are allowed in our implementation. The width of the mesh $\mathbb{T}_h$ is characterised in terms of a piecewise constant mesh size function $h = h(x) > 0$, where $h_T := h_{|T} = \operatorname{diam}(T)$. We use standard bilinear finite elements to construct the spaces $V_h \subset V$ and assume that $K_h = K \cap V_h$.

Eventually, the finite element approximation u_h of u in (1.2) is determined by

$$
(\nabla u_h, \nabla(\varphi - u_h)) \geq (f, \varphi - u_h) \quad \forall \varphi \in K_h. \tag{1.3}
$$

This finite dimensional problem can be shown to be uniquely solvable following the same line of arguments as in the continuous case. Optimal order *a priori* error estimates in the energy norm have been given, for example, in Falk [7] and Brezzi et al. [4]. Dobrowolski and Staib [6] show $\mathcal{O}(h)$-convergence in the energy norm without additional assumptions on the structure of the free boundary. Error estimates with respect to the L^∞-norm have been obtained, e.g., by Nitsche [16] based on a discrete maximum principle.

Below, we shall demonstrate how functionals $J(u - u_h)$ of the error can be controlled in an *a posteriori* manner, i.e., we estimate the error in terms of quantities at the element level containing only the discrete solution and the data of the problem.

2 A posteriori error estimate

For elliptic variational equalities, i.e., in the case $K = V$, many different, but related, approaches for *a posteriori* error control have been developed in the last two decades; see, e.g., Verfürth [21] for a survey. Most estimators are designed to control the error in the energy norm. A general concept for estimating $e = u - u_h$ for more general error measures given in terms of a linear functional $J(\cdot)$ has been proposed in Becker and Rannacher [2] and further developed, e.g., in Kanschat [10] and Suttmeier [20]. One main ingredient in deriving such residual based *a posteriori* estimates is a duality argument known as the "Aubin–Nitsche trick" from *a priori* analysis. In principle, such techniques can be carried over to variational inequalities when, for example, penalty techniques are employed to avoid the explicit treatment of the constraints. This again leads to variational equalities of the form mentioned above.

In the present paper, we attack the original unpenalised problem. Since we are mainly interested in local phenomena like the normal stress on the contact surface, we intend to adopt local control techniques to estimate a functional $J(e)$.

In Natterer [17], there is described a generalisation of Nitsche's trick for variational inequalities, which we employ to derive an *a posteriori* error estimate for the scheme (1.3). To this end, we consider the dual solution $z \in G$ of

$$(\nabla(\varphi - z), \nabla z) \geq J(\varphi - z) \quad \forall \varphi \in G, \tag{2.1}$$

where $G = \{v \in V \mid v \geq 0 \text{ on } B_h \text{ and } \int_{\Gamma_C} \partial_n u(v + u_h) \leq 0\}$ and $B_h = \{x \in \Gamma_C \mid u_h(x) = 0\}$.

In order to show that $z + u - u_h \in G$, we observe that $z + u - u_h \geq 0$ on B_h, since on $B_h \subset \Gamma_C$ we have $u \geq 0$, $u_h = 0$. Furthermore,

$$\int_{\Gamma_C} \partial_n u((z + u - u_h) + u_h) = \int_{\Gamma_C} \partial_n u z \leq - \int_{\Gamma_C} \partial_n u u_h \leq 0.$$

Now, we can choose $\varphi = z + u - u_h$ as a test function in (2.1) and obtain

$$J(e) \le (\nabla(u - u_h), \nabla z).$$

Next, we use the solution $\mathcal{U} \in V$ of the nonrestricted problem

$$(\nabla \mathcal{U}, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in V,$$

to rewrite (1.2) and (1.3) in the form

$$(\nabla(\mathcal{U} - u), \nabla(\varphi - u)) \le 0 \quad \forall \varphi \in K, \tag{2.2}$$

$$(\nabla(\mathcal{U} - u_h), \nabla(\varphi - u_h)) \le 0 \quad \forall \varphi \in K_h. \tag{2.3}$$

It is easily seen that $u_h \in W_h = \{v \in V \mid v \ge 0 \text{ on } B_h\} \cap V_h$, i.e., u_h coincides with the solution $\tilde{u}_h \in W_h$ of the discrete variational inequality

$$(\nabla(\mathcal{U} - \tilde{u}_h), \nabla(\varphi - \tilde{u}_h)) \le 0 \quad \forall \varphi \in W_h. \tag{2.4}$$

With $z_h \in W_h$ and choosing $\varphi = u_h + z_h$ in (2.4) we see that the first term on the right-hand side of the identity

$$\begin{aligned}(\nabla(u - u_h), \nabla z_h) = (\nabla(\mathcal{U} - u_h), \nabla z_h) + (\nabla(u - \mathcal{U}), \nabla(z_h + u_h - u))\\ + (\nabla(u - \mathcal{U}), \nabla(u - u_h)) \qquad \forall z_h \in W_h,\end{aligned} \tag{2.5}$$

is negative. So also is the last term by taking $\varphi = u_h$ in (2.2). To sum up, we have shown the inequality

$$(\nabla(u - u_h), \nabla z_h) \le (\nabla(u - \mathcal{U}), \nabla(z_h + u_h - u)) \quad \forall z_h \in W_h. \tag{2.6}$$

We now proceed with estimating $J(e)$ by

$$\begin{aligned} J(e) &\le (\nabla(u - u_h), \nabla(z - z_h)) + (\nabla(u - u_h), \nabla z_h)\\ &\le (\nabla(u - u_h), \nabla(z - z_h)) + (\nabla(u - \mathcal{U}), \nabla(z_h + u_h - u))\\ &= (\nabla(u - u_h), \nabla(z - z_h)) + (\nabla(u - \mathcal{U}), \nabla(z + u_h - u))\\ &\quad + (\nabla(u - \mathcal{U}), \nabla(z_h - z)). \end{aligned}$$

Due to $u \partial_n u = 0$ on Γ_C, we have, for $z \in G$,

$$(\nabla(u - \mathcal{U}), \nabla(z + u_h - u)) = \int_{\Gamma_C} \partial_n u (z + u_h) \le 0.$$

Eventually, we obtain the *a posteriori* error estimate

$$J(e) \le (\nabla(\mathcal{U} - u_h), \nabla(z - z_h)). \tag{2.7}$$

With standard techniques this can be exploited as follows. Element-wise integration by parts yields

$$J(e) \le \sum_{T \in \mathbb{T}_h} \left\{ (f + \Delta u_h, z - z_h)_T - \tfrac{1}{2}([\partial_n u_h], z - z_h)_{\partial T} \right\}, \tag{2.8}$$

where, for interior interelement boundaries, $[\partial_n u_h]$ denotes the jump of the normal derivative $\partial_n u_h$. Furthermore we set $[\partial_n u_h] = 0$ and $[\partial_n u_h] = \partial_n u_h$ on edges belonging to Γ_D and Γ_C respectively.

From (2.8), we deduce the *a posteriori* error bound

$$|J(e)| \leq \sum_{T \in \mathbb{T}_h} \omega_T \rho_T \; =: \; \eta_{\text{weight}}, \tag{2.9}$$

with the local *residuals* ρ_T and *weights* ω_T defined by

$$\begin{aligned} \rho_T &:= h_T \|f + \Delta u_h\|_T + \tfrac{1}{2} h_T^{1/2} \|n \cdot [\nabla u_h]\|_{\partial T}, \\ \omega_T &:= \max\left\{ h_T^{-1} \|z - z_h\|_T, h_T^{-1/2} \|z - z_h\|_{\partial T} \right\}. \end{aligned}$$

In general, the weights ω_T cannot be determined analytically, but have to be computed by solving the dual problem numerically on the available mesh. To this end, interpreting z_h as a suitable interpolant of z, one uses the interpolation estimate

$$\omega_T \leq C_{i,T} h_T \|\nabla^2 z\|_T, \tag{2.10}$$

for $z \in H^2(T)$. For less regular z an estimate similar to (2.10) could be used involving a lower power of a local mesh size, which typically corresponds to higher values of ω_T. To evaluate the right-hand side in (2.10) one may simply take second order difference quotients of the approximate dual solution $\tilde{z}_h$,

$$\omega_T \approx \tilde{\omega}_T := \tilde{C}_{i,T} h_T^2 |\nabla_h^2 \tilde{z}_h(x_T)|, \tag{2.11}$$

where x_T is the midpoint of element T. This results in approximate *a posteriori* error bounds using

$$\eta_{\text{weight}} \approx \sum_{T \in \mathbb{T}_h} \tilde{\omega}_T \rho_T. \tag{2.12}$$

It has been demonstrated in Becker and Rannacher [2] that this approximation has only minor effects on the quality of the resulting meshes. The *interpolation constant* C_i may be set equal to one for mesh designing.

In the following, we compare this *weighted* estimator against the traditional approach of Zienkiewicz and Zhu [22]. This error indicator for finite element models in structural mechanics is based on the idea of higher–order stress recovery by local averaging. The element-wise error $\|\sigma - \sigma_h\|_T$ is thought to be well represented by the auxiliary quantity $\|\mathcal{M}_h \sigma_h - \sigma_h\|_T$, where $\mathcal{M}_h \sigma_h$ is a local (super-convergent) approximation of σ. The corresponding (heuristic) global error estimator reads

$$\|\sigma - \sigma_h\| \; \approx \; \eta_{ZZ} \; := \; \Big(\sum_{T \in \mathbb{T}_h} \|\mathcal{M}_h \sigma_h - \sigma_h\|_T^2 \Big)^{1/2}, \tag{2.13}$$

with $\sigma = \nabla u$ and $\sigma_h = \nabla u_h$.

Remark The choice of (2.1) is not uniquely determined. Other approaches in *a priori* analysis in similar situations can be found, e.g., in Mosco [15]. Here separate dual problems for the negative and positive part of the error are considered, but it seems to be difficult to exploit these techniques for *a posteriori* analysis, since the data of the problem do not enter the estimate directly.

3 Numerical results

The implementation is based on the tools of the object-oriented FE package DEAL [1]. The solution process is simply done by an iteration of Gauss-Seidel-type (cf. Glowinski et al. [9]). The solutions on very fine (adaptive) meshes with about 200,000 cells are taken as *reference* solutions u_{ref} for determining the relative errors

$$E^{\text{rel}} := |J(u_h) - J(u_{\text{ref}})|/|J(u_{\text{ref}})|$$

on coarser meshes, while

$$\text{Ratio} := \frac{\eta(u_h)}{|J(u_{\text{ref}}) - J(u_h)|}$$

are the overestimation factors of the error estimators.

Let an error tolerance TOL or a maximal number of cells $N_{\max}$ be given. Starting from some initial coarse mesh the refinement criteria are chosen in terms of the *local error indicators* $\eta_T := \omega_T \rho_T$. Then, for the mesh refinement, we use the following *fixed fraction* strategy: in each refinement cycle, the elements are ordered according to the size of η_T and then a fixed portion (say 30%) of the elements with largest η_T is refined which results in about a doubling of the number N of cells. This process is repeated until the stopping criterion $\eta(u_h) \leq$ TOL is satisfied, or $N_{\max}$ is exceeded. For the numerical tests given below, we confined ourselves to 8 adaptive cycles. The corresponding values for $N_{\max}$ can be taken from the tables below.

For determining $J(u_{\text{ref}})$, we employ an adaptive algorithm based on (2.12), where in every third adaptive step we also do a global refinement.

The approximation of the dual problem (2.1)

$$(\nabla(\varphi - z), \nabla z) \geq J(\varphi - z) \quad \forall \varphi \in G,$$

is realised as follows. Assuming $\partial_n u > 0$ on B_h and $\partial_n u = 0$ on $\Gamma_C \setminus B_h$ suggests approximating G by $\tilde{G} = \{v \in V | v = 0 \text{ on } B_h\}$. Therefore, we only have to solve a *linear Dirichlet problem* with zero boundary conditions on $\Gamma_D + B_h$.

Examples

As a test example, we consider (1.1) on $\Omega = (0,1)^2$, $\Gamma_D = \{(x_1, x_2) \in \partial\Omega | x_1 = 0\}$ and right-hand side $f = 1000 \sin(2\pi x_1)$. The contact set $B = \{x \in \Gamma_C | u(x) = 0\}$ in this case is determined by $B = \{(x_1, x_2) \in \Gamma_C | x_1 \geq b\}$ with $b \approx 0.609374$ taken from u_{ref}. The structure of the solution is sketched in Fig. 1 (left).

Applying an adaptive algorithm on the basis of the indicator η_{ZZ} yields locally refined grids with a structure shown in Fig. 1, which can be compared with the grids based on η_{weight} for the following examples (Figs. 2, 3 and 4).

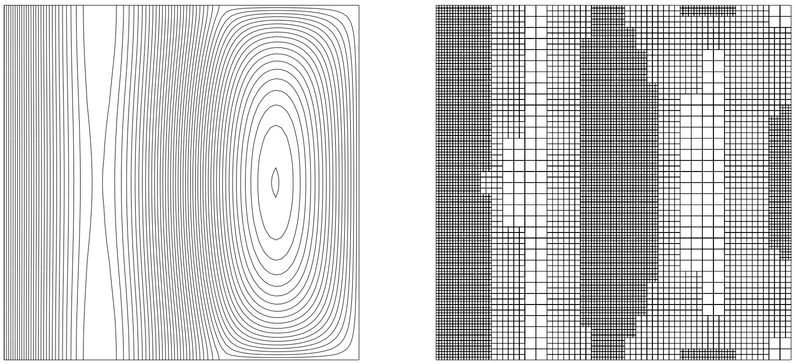

Fig. 1. Isolines of the solution (left) and structure of grids produced on the basis of η_{ZZ} (right)

1) Point value. For the first test, we choose

$$J(\varphi) = \varphi(x_0), \quad x_0 = (0.25, 0.25),$$

to control the point-error in x_0. The computational results are shown in Table 1. Evaluating *Ratio* shows the constant relation between *true* error and the corresponding estimation, and consequently it is demonstrated that the proposed approach to *a posteriori* error control gives useful error bounds. Fig. 2 (left) shows that η_{weight} produces a (monotonically) converging scheme with respect to the point value in contrast to the ZZ-approach. Fig. 2 (right) shows the structure of grids produced on the basis of η_{weight}.

2) Mean value. As error functional for the second test, we choose

$$J(\varphi) = \int_B \partial_n \varphi,$$

Table 1. Numerical results for the first test example: functional value $J(u_h)$, relative error E^{rel} and over-estimation factor Ratio

Cells	$J(u_h)$	E^{rel}	Ratio
484	2.928820e+01	1.254902e-03	4.25
928	2.928258e+01	1.446547e-03	2.07
1720	2.929928e+01	8.770673e-04	2.64
3148	2.930866e+01	5.572038e-04	2.97
5572	2.931476e+01	3.491901e-04	3.14
9604	2.931715e+01	2.676897e-04	3.40
16468	2.931918e+01	1.984655e-04	3.96
27724	2.932013e+01	1.660699e-04	2.99

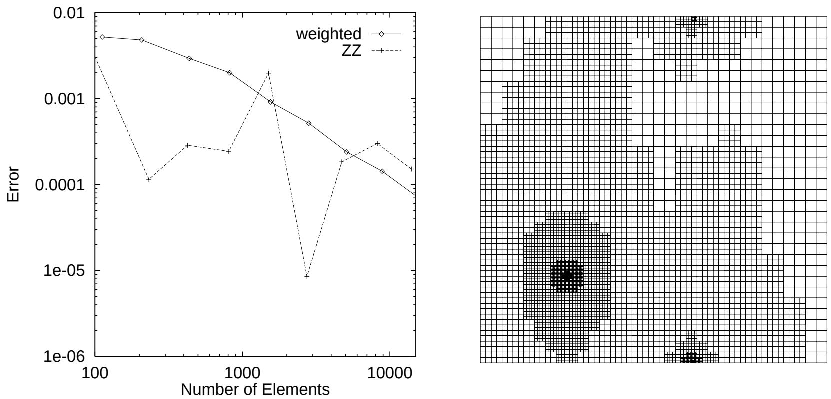

Fig. 2. Relative error for the first example on adaptive grids according to the *weighted* estimate and the ZZ-indicator (left) showing that η_{weight} produces a (monotonically) converging scheme with respect to the point value. Structure of grids produced on the basis of η_{weight} (right)

to control the mean value of the normal derivative along the contact set B. In this case, the treatment of J, which is determined by derivatives of u, requires some additional care since in this case the functional is *singular*, i.e., the dual solution is not properly defined on G. The remedy (cf. Becker and Rannacher [2] and see Rannacher and Suttmeier [18] for an application in linear elasticity) is to work with a regularised functional $J^{\varepsilon}(.)$. In the present case, we set

$$J^{\varepsilon}(\varphi) \;=\; |B_{\varepsilon}|^{-1} \int_{B_{\varepsilon}} \partial_n \varphi \, dx,$$

Table 2. Numerical results for the second test example: functional value $J(u_h)$, relative error E^{rel} and over-estimation factor Ratio

Cells	$J(u_h)$	E^{rel}	Ratio
1840	-1.730673e+02	1.273645e-02	1.51
3256	-1.739847e+02	7.503137e-03	1.96
5980	-1.745723e+02	4.151169e-03	2.50
10528	-1.748522e+02	2.554478e-03	2.81
19204	-1.750084e+02	1.663434e-03	2.47
34540	-1.750833e+02	1.236167e-03	3.90
65212	-1.751289e+02	9.760411e-04	3.85
122284	-1.751526e+02	8.408443e-04	2.67

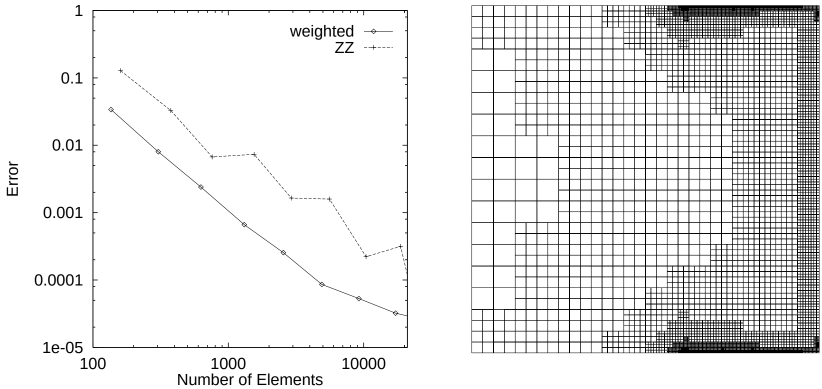

Fig. 3. Relative error for the second example on adaptive grids according to the *weighted* estimate and the ZZ-indicator (left) demonstrating η_{weight} to be most economical. Structure of grids produced on the basis of η_{weight} (right)

where $B_\varepsilon := \{x \in \Omega,\ \mathrm{dist}(x, B) < \varepsilon\}$. For each adaptive computation, the regularisation is done with the choice $\varepsilon = 0.5\eta_{\mathrm{weight}}(u_h)$, where u_h is taken from the previous step.

The numerical results are presented in Table 2. Again, it is demonstrated that the proposed approach to *a posteriori* error control gives useful error bounds. In Fig. 3 (left) the relative errors on adaptive grids according to the *weighted* estimate and the ZZ-indicator are depicted, demonstrating η_{weight} to be most economical. Figure 3 (right) shows the structure of grids produced on the basis of η_{weight}.

3) Normal derivative. For the third test, we choose

$$J(\varphi) = \partial_n \varphi(x_0), \quad x_0 = (1.00, 0.25),$$

to control the point error of the normal derivative in x_0. This example is chosen to indicate the applicability of the proposed techniques for our final goal of *a posteriori* error estimation of contact stresses in elasticity problems.

Again the treatment of J has to be done by regularisation as in the second example. Again the results presented in Table 3 and Fig. 4 demonstrate η_{weight} to be reliable and efficient.

Table 3. Numerical results for the third test example: functional value $J(u_h)$, relative error E^{rel} and over-estimation factor Ratio

Cells	$J(u_h)$	E^{rel}	Ratio
304	1.140179e+02	8.022446e-03	1.77
628	1.146645e+02	2.396903e-03	1.59
1312	1.148638e+02	6.629546e-04	2.17
2548	1.149107e+02	2.549156e-04	2.17
4912	1.149301e+02	8.613189e-05	3.27
9208	1.149339e+02	5.307117e-05	2.29
17200	1.149363e+02	3.219071e-05	2.08
31468	1.149372e+02	2.436054e-05	1.71

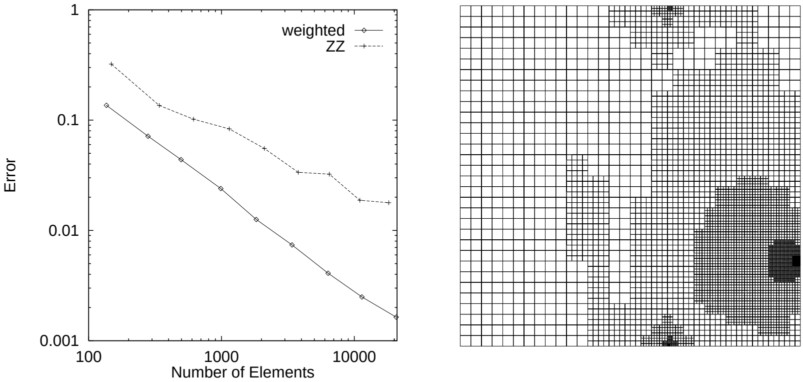

Fig. 4. Relative error for the third example on adaptive grids according to the *weighted* estimate and the ZZ-indicator (left) demonstrating η_{weight} to be most economical. Structure of grids produced on the basis of η_{weight} (right)

4 Outlook: Application to Signorini's problem

We now demonstrate how the above techniques might be extended for *a posteriori* error control to Signorini's problem which, in classical notation, reads (cf. Kikuchi and Oden [11])

$$\begin{aligned} -\operatorname{div}\sigma = f, \quad A\sigma = \varepsilon(u) \quad \text{in } \Omega,\\ u = 0 \text{ on } \Gamma_D, \quad \sigma\cdot n = t \text{ on } \Gamma_N,\\ \left.\begin{aligned}\sigma_T = 0, \quad (u_n - g)\sigma_n = 0\\ u_n - g \le 0, \quad \sigma_n \le 0\end{aligned}\right\} \text{ on } \Gamma_C. \end{aligned} \tag{4.1}$$

This idealised model describes the deformation of an elastic body occupying the domain $\Omega \subset \mathbb{R}^3$, which is unilaterally supported by a frictionless rigid foundation. The displacement u and the corresponding stress tensor σ are caused by a body force f and a surface traction t along Γ_N. Along the portion Γ_D of the boundary the body is fixed and $\Gamma_C \subset \partial\Omega$ denotes the part which is a candidate contact surface. We use the notation $u_n = u \cdot n$, $\sigma_n = \sigma_{ij}n_in_j$ and $\sigma_T = \sigma \cdot n - \sigma_n n$, where n is the outward normal of $\partial\Omega$, and g denotes the gap between Γ_C and the foundation.

Further, the deformation is assumed to be small so that the strain tensor can be written as $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. The compliance tensor A is assumed to be symmetric and positive definite.

The weak solution $u \in K$ of (4.1) is defined by the variational formulation

$$a(u, \varphi - u) \ge F(\varphi - u) \quad \forall \varphi \in K, \tag{4.2}$$

with the definitions

$$\begin{aligned} V = \{v \in H^1 \times H^1 \mid v = 0 \text{ on } \Gamma_D\}, \quad K = \{v \in V \mid v_n - g \le 0\},\\ a(v,\varphi) = \int_\Omega A^{-1}\varepsilon(v)\varepsilon(\varphi) \quad \forall v,\varphi \in V,\\ F(\varphi) = \int_\Omega f\varphi + \int_{\Gamma_N} t\varphi \quad \forall \varphi \in V. \end{aligned}$$

As above, the discrete solution $u_h \in K_h = K \cap V_h \subset V$ is determined by

$$a(u_h, \varphi - u_h) \ge F(\varphi - u_h) \quad \forall \varphi \in K_h. \tag{4.3}$$

Again, for estimating measures defined by $J(.)$ of $e = u - u_h$, we employ $z \in G$ given by

$$a(\varphi - z, z) \ge J(\varphi - z) \quad \forall \varphi \in G, \tag{4.4}$$

where $G = \{v \in V \mid v \geq 0 \text{ on } B_h \text{ and } a(\mathcal{U} - u, v + u_h - u) \geq 0\}$ and $B_h = \{x \in \Gamma_C \mid u_h(x) \cdot n = g(x)\}$. In the above $\mathcal{U}$ denotes the solution of

$$a(\mathcal{U}, \varphi) = F(\varphi) \quad \forall \varphi \in V.$$

Eventually, the techniques used for the model case yield an *a posteriori* error estimate of the form (2.9) $|J(e)| \leq \sum_{T \in \mathbb{T}_h} \omega_T \rho_T$ with

$$\rho_T := h_T \|f + \operatorname{div}(A^{-1}\varepsilon(u_h))\|_T + \tfrac{1}{2} h_T^{1/2} \|[n \cdot A^{-1}\varepsilon(u_h)]\|_{\partial T},$$
$$\omega_T := \max\left\{ h_T^{-1} \|z - z_h\|_T, h_T^{-1/2} \|z - z_h\|_{\partial T} \right\}.$$

The approximation of the dual problem (4.4) may be realised as follows. Assuming B_h to be an appropriate approximation of B suggests replacing G by $\tilde{G} = \{v \in V | v = 0 \text{ on } B_h\}$ and solving a *linear elasticity problem* with Dirichlet boundary conditions on $\Gamma_D + B_h$.

A similar situation is given for nonlinear variational equalities, where the dependence of the dual operator on u and u_h is in practice simply expressed in terms of the computed u_h alone. The experiences in the case of the stationary Navier–Stokes equations (see Becker and Rannacher [2]) and for nonlinear elasto-plastic material behaviour (see Rannacher and Suttmeier [19]) indicate that for these examples the perturbation of the dual problem is not critical in stable situations. In the present case, investigations of the influence of the approximation of (4.4) on the accuracy of the resulting *a posteriori* estimate in detail have to be done and are the subject of a forthcoming paper.

References

1. Becker, R., Kanschat, G., Suttmeier, F.-T.: DEAL – differential equations analysis library. Available via http://gaia.iwr.uni-heidelberg.de/DEAL.html, 1995
2. Becker, R., Rannacher, R.: A feed-back approach to error control in finite element methods: Basic analysis and examples. East-West J. Numer. Math. **4**, 237–264 (1996)
3. Brézis, H.: Problèmes unilatéraux. (Thèse.) J. Math. Pures Appl. (9) **51**, 1–168 (1972)
4. Brezzi, F., Hager, W.W., Raviart, P.-A.: Error Estimates for the finite element solution of variational inequalities. I. primal theory. Numer. Math. **28**, 431–443 (1977)
5. Carstensen, C., Scherf, O., Wriggers, P.: Adaptive finite elements for elastic bodies in contact. SIAM J. Sci. Comput **20**, 1605–1626 (1999)
6. Dobrowolski, M., Staib, T.: On finite element approximation of a second order unilateral variational inequality. Numer. Funct. Anal. Optim. **13**, 243–247 (1992)
7. Falk, R.S.: Error estimates for the approximation of a class of variational inequalities. Math. Comp. **28**, 963–971 (1974)
8. Glowinski, R.: Numerical methods for nonlinear variational problems. (Springer Series in Computational Physics) New York: Springer 1983
9. Glowinski, R., Lions, J.-L., Trémolières, R.: Numerical analysis of variational inequalities. Amsterdam: North-Holland 1981

10. Kanschat, G.: Parallel and adaptive Galerkin methods for radiative transfer problems. Dissertation. Heidelberg: Univ. Heidelberg, Naturwiss.-Math. Gesamtfax. 1996
11. Kikuchi, N., Oden, J.T.: Contact problems in elasticity: a study of variational inequalities and finite element methods. (SIAM Studies in Applied Mathematics **8**) Philadelphia: SIAM 1988
12. Kinderlehrer, D., Stampacchia, G.: An introduction to variational inequalities and their applications. New York: Academic Press 1980
13. Kornhuber, R.: A posteriori error estimates for elliptic variational inequalities. Comput. Math. Applic. **31**, 49–60 (1996)
14. Lions, J.-L., Stampacchia, G.: Variational inequalities. Comm. Pure Appl. Math. **20**, 493–519 (1967)
15. Mosco, U.: Error estimates for some variational inequalities. In: Galligani, I., Magenes, E. (eds.) Mathematical aspects of finite element methods, (Lecture Notes in Math. **606**) Berlin: Springer 1977, pp. 224–236
16. Nitsche, J.: L^∞-convergence of finite element approximations. In: Galligani, I., Magenes, E. (eds.) Mathematical aspects of finite element methods, (Lecture Notes in Math. **606**) Berlin: Springer 1977, pp. 261–274
17. Natterer, F.: Optimale L^2-Konvergenz finiter Elemente bei Variationsungleichungen. Bonn. Math. Schrift **89**, 1–12 (1976)
18. Rannacher, R., Suttmeier, F.-T.: A feed-back approach to error control in finite element methods: application to linear elasticity. Comput. Mech. **19**, 434–446 (1997)
19. Rannacher, R., Suttmeier, F.-T.: A posteriori error control in finite element methods via duality techniques: application to perfect plasticity. Comput. Mech. **21**, 123–133 (1998)
20. F.-T. Suttmeier: Adaptive finite element approximation of problems in elasto-plasticity theory. Dissertation. Heidelberg: Univ. Heidelberg, Naturwiss.-Math. Gesamtfax. 1997
21. Verfürth, R.: A review of a posteriori error estimation and adaptive mesh-refinement techniques. (Wiley-Teubner Skripten zur Numerik). Stuttgart: Teubner 1996
22. Zienkiewicz, O.C., Zhu, J.Z.: A simple error estimator and adaptive procedure for practical engineering analysis. Internat. J. Numer. Methods Engrg. **24**, 337–357 (1987)