Calcolo (2022) 59:18
https://doi.org/10.1007/s10092-022-00459-4

Virtual element analysis of nonlocal coupled parabolic problems on polygonal meshes

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Received: 18 June 2020 / Revised: 27 December 2021 / Accepted: 11 February 2022 /
Published online: 17 March 2022

Abstract

In this article, we consider the discretization of nonlocal coupled parabolic problem within the framework of the virtual element method. The presence of nonlocal coefficients not only makes the computation of the Jacobian more expensive in Newton's method, but also destroys the sparsity of the Jacobian. In order to resolve this problem, an equivalent formulation that has very simple Jacobian is proposed. We derive the error estimates in the L^2 and H^1 norms. To further reduce the computational complexity, a linearized scheme without compromising the rate of convergence in different norms is proposed. Finally, the theoretical results are justified through numerical experiments over arbitrary polygonal meshes.

Keywords Arbitrary polygonal mesh · Error estimates · Nonlocal parabolic equation · Non-linear equation · Virtual element method

Mathematics subject classification 65N30 · 65N12

1 Introduction

In this work, we present a virtual element framework for the nonlocal coupled parabolic problem. Such problems find Nitsche applications in many fields of applied science and engineering, for example in modelling epidemics [1–3], polymerization

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[4], tumor growth modeling [5], to name a few. In [6], the authors proved the existence and the uniqueness of the analytical solution of the nonlocal coupled parabolic problem. Numerical solutions based on the finite element method (FEM) and the virtual element method have been attempted in [7, 8]. In [7], author employed the conforming linear finite element method for the discretization of the non-local coupled parabolic problems.

In the last decade, there is a growing interest in numerical methods that can accommodate elements with arbitrary shapes and sizes. This has led to the development of a variety of numerical techniques, such as, the Mimetic Finite Difference Method [9–11], Weak Galerkin Method [12, 13], Polygonal Finite Element Method (PFEM) [14–16], Scaled boundary finite element method [17, 18] and the Virtual Element Method (VEM) [19–22]. These methods are very similar to each other that they require suitable discrete formulation of the model problem avoiding traditional approach. Both the polygonal finite element and the virtual element method can accommodate elements with arbitrary shapes and sizes, however, one distinct feature of the VEM when compared to the PFEM is that the later requires an explicit form of the basis functions to compute the bilinear and the linear forms. The basis functions over arbitrary polytopes are rational polynomials, which requires higher order numerical quadrature rules. To the best of author's knowledge, conventional polygonal finite elements are restricted to quadratic elements [23, 24]. Whilst in case of the VEM, no such explicit form of the basis functions is required, moreover, higher order elements even in higher dimensions can easily be constructed. This salient feature of the VEM has attracted researchers to employ VEM for wider variety of problems in science and engineering [25–36].

In this article, we employ the VEM to discretize the nonlocal coupled parabolic problem. The VEM is a generalization of the finite element method over arbitrary polytopes satisfying the Galerkin type orthogonalization over the polynomial space. The basis functions are implicitly known and can be approximated using the degrees of freedom (DoFs) over the general polygonal and polyhedral elements. The discrete variational formulation is computed by avoiding the cumbersome numerical integration schemes. Since the basis functions are constructed virtually, suitable projection operators are introduced on the virtual element space locally that can be computed using the DoFs associated to the polytope. In contrast to the FEM, the direct discretization of the nonlocal term will not be computable. Using the projection operator, the nonlocal term is discretized which is computable from the DoFs. However, the presence of the nonlocal coefficients in the system reduces the sparsity of the jacobian and consequently increases the computation cost. Following [37], an analogous approach is employed to rewrite the nonlinear system, such that the sparsity of the Jacobian is retained. Moreover, a linearized scheme for the coupled nonlocal parabolic problem is introduced that yields optimal order of convergence in both the space and the time variables. The nonlocal coefficients and the load terms can be computed from the previous steps and hence the fully discrete system reduces to a system of linear equations which can be computed easily.

The rest of the paper is organised as follows: In Sect. 2, the model problem and the associated continuous weak formulation are defined. The basic settings of the functional analysis and the assumptions required to develop the theory are also highlighted in the same section. In the next section, the discrete virtual element space in two and three dimensions are constructed and the operators associated with the discrete space are discussed. A priori error estimates for the semi-discrete and the fully discrete schemes are investigated in Sects. 4 and 5, respectively. The error estimates for the linearised scheme are studied in Sect. 6. The theoretical convergence rates are justified with two numerical examples in Sect. 7, followed by concluding remark in the last section.

2 Preliminaries and the continuous problem

Consider a convex polygonal domain $\Omega \subset \mathbb{R}^d$ where $d = 2, 3$ represents the dimension of the domain, with Lipschitz boundary $\partial\Omega$. We define the final time T and the time interval $I = [0, T]$. Further, we denote $L^2(\Omega)$, the space of square integrable functions with standard inner-product $(\phi, \psi)_\Omega := \int_\Omega \phi\, \psi \, d\Omega$. For each positive integer $s \in \mathbb{N}$, we define $H^s(\Omega)$, the Sobolev space with standard norm $\|\phi\|_{s,\Omega} := \Big(\sum_{0\le\alpha\le s} \|D^\alpha \phi\|^2_{0,\Omega} \Big)^{1/2}$, where $D^\alpha\phi$ denotes αth partial derivative of ϕ. Moreover, the function space $L^2(0, T;\ H^s(\Omega))$ consists of function ϕ such that for almost all $t \in [0, T]$, $\phi(\cdot, t) \in H^s(\Omega)$ with the norm

$$\|\phi\|_{L^2(0,T;\,H^s(\Omega))} := \Big(\int_0^T \|\phi(t)\|^2_{s,\Omega} \Big)^{1/2}; \quad \|\phi\|_{L^\infty(0,T;\,H^s(\Omega))} = \operatorname*{ess\,sup}_{0\le t\le T} \|\phi(t)\|_s.$$

In addition, we define $\mathbb{P}_k(E)$, the space of all polynomials of degree less than or equal to k on E and for a function v, the first and the double derivatives with respect to t are denoted by $D_t v$, $D_{tt} v$ respectively.

2.1 Model problem

Let $f_i(u, v) \in L^2(\Omega, I)$ be the force function for $i \in \{1, 2\}$, and (u_0, v_0) be the initial guess for the solution $(u,\ v)$. The continuous problem is then given by: find $(u,\ v)$ such that for $t \in [0, T]$, we have:

$$D_t u - \mathcal{A}_1(g_1(u), g_2(v))\, \Delta u = f_1(u, v) \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$D_t v - \mathcal{A}_2(g_1(u), g_2(v))\, \Delta v = f_2(u, v) \quad \text{in } \Omega \times (0, T), \tag{2}$$

$$u(x, t) = v(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{3}$$

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \tag{4}$$

$$v(x, 0) = v_0(x) \quad \text{on } \Omega, \tag{5}$$

where $g_i(\omega) := \int_\Omega l_i(x)\,\omega\,d\Omega$ for $\omega(\cdot,t) \in L^2(\Omega)$ for almost all $t \in [0,T]$ and $l_i(x) \in L^2(\Omega)$. Further, we define $D_t u := \frac{du}{dt}$. Since the diffusive coefficients $\mathcal{A}_i' s$ depend on the global behaviour of the solution, the problem is termed nonlocal.

Further, we will make some assumptions on the model problem in order to derive the theoretical estimates in the later section.

Assumption 1

- For $i \in \{1,2\}$, $\mathcal{A}_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is bounded, i.e., $0 < m_0 < \mathcal{A}_i(\cdot,\cdot) < M$, where m_0 and M are positive constants.
- $\mathcal{A}_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous, i.e.,

$$|\mathcal{A}_i(r_1,s_1) - \mathcal{A}_i(r_2,s_2)| \le L_A(|r_1 - r_2| + |s_1 - s_2|) \quad \forall (r_i,s_i) \in \mathbb{R}\times\mathbb{R}. \tag{6}$$

- For $i \in \{1,2\}$, the right hand side force function, f_i are Lipschitz continuous w.r.t. u and v. i.e.,

$$|f_i(u_1,v_1) - f_i(u_2,v_2)| \le L_F(|u_1 - u_2| + |v_1 - v_2|) \quad \forall (u_1,v_1),(u_2,v_2) \in \mathbb{R}\times\mathbb{R}. \tag{7}$$

Multiplying Eq. (1) by φ and (2) by ψ and employing Greens' theorem, we derive the continuous weak formulation: Find $(u,v) \in L^2(0,T;H_0^1(\Omega)\cap C(0,T;L^2(\Omega)) \times L^2(0,T;H_0^1(\Omega)\cap C(0,T;L^2(\Omega))$ and $(D_t u, D_t v) \in L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H^{-1}(\Omega))$ for almost all $t \in [0,T]$ such that

$$\frac{d}{dt}(u,\varphi) + \mathcal{A}_1(g_1(u),g_2(v))(\nabla u,\nabla\varphi) = \langle f_1(u,v),\varphi\rangle \text{ in } \mathcal{D}'(0,T) \quad \forall \varphi \in H_0^1(\Omega), \tag{8}$$

$$\frac{d}{dt}(v,\psi) + \mathcal{A}_2(g_1(u),g_2(v))(\nabla v,\nabla\psi) = \langle f_2(u,v),\psi\rangle \text{ in } \mathcal{D}'(0,T) \quad \forall \psi \in H_0^1(\Omega), \tag{9}$$

$$u(\mathbf{x},t) = v(\mathbf{x},t) = 0 \quad \forall (\mathbf{x},t) \in \partial\Omega \times (0,T), \tag{10}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}) \quad \text{and} \quad v(\mathbf{x},0) = v_0(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega, \tag{11}$$

where $\mathcal{D}'(0,T)$ is the space of distributions on [0,T] and $\langle\cdot,\cdot\rangle$ denotes the $H_0^1(\Omega)', H_0^1(\Omega)-$ duality bracket. The existence and the uniqueness of the weak solution satisfying Eqs. (8)–(11) can be easily proved using Schauder fixed point argument [38].

Theorem 2.1 *Under Assumption 1, there exists a unique solution $(u,v) \in H_0^1(\Omega)\times H_0^1(\Omega)$ of the problem (8)–(11).*

Using Assumption 1, Schauder fixed point theorem and proceeding analogously as in [38, Theorem 2.1], we get the desired result.

3 Virtual element methods

In this section, we consider a few regularity assumptions on the family of mesh decompositions $\{\Omega_h\}_h$ and discuss the construction of two and three dimensional virtual element spaces which were originally introduced in [31, 39, 40]. Unlike the finite element space, the virtual element space consists of both polynomial function and implicitly defined non-polynomial function. Non-polynomial parts of the discrete bilinear forms are approximated by suitable projection operators which are computable from known degrees of freedom (DoFs) associated with the VEM space.

3.1 Background material

Let $\{\Omega_h\}_h$ consists of non-overlapping, bounded polygonal/polyhedral elements E or P such that $\bar{\Omega} = \cup_{P\in\Omega_h}\bar{P}$ or $\bar{\Omega} = \cup_{E\in\Omega_h}\bar{E}$ and let h_E/h_P be the diameter of an element $E/P \in \Omega_h$; $h := \max_{E\in\Omega_h} h_E$ and for polyhedron, $h := \max_{P\in\Omega_h} h_P$. For $d = 2$, E has non-intersection *polygonal* boundary ∂E which is assembled from $\mathcal{N}_E^{\varepsilon}$ straight edges e joining $\mathcal{N}_E^V$ vertices. For $d = 3$, each element P has a *polyhedral* boundary ∂P which is formed by $\mathcal{N}_P^F$ planar faces F joining vertices $(x_i, y_i, z_i) \in \mathbb{R}^3, 1 \le i \le \mathcal{N}_P^V$. For an element E/P, we define the measure of E/P by $|E|/|P|$ and barycenter (center of gravity) by $\mathbf{x}_E/\mathbf{x}_P$. To use the interpolation and theory of polynomial approximation of a function, we require some regularity assumptions on the domain decomposition $\{\Omega_h\}_h$.

Assumption 2 (*Mesh regularity*) For polygonal element $E \subset \mathbb{R}^2$, there exists a positive constant γ independent of diameter h such that every polygonal element E satisfies these conditions

- (T_1^{2d}) $E \in \Omega_h$ is star-shaped with respect to every point of a disk of radius greater than $\gamma\, h_E$.
- (T_2^{2d}) for every element E, and for every $e \subset \partial E$ satisfies $h_e > \gamma h_E$.

For polyhedral element $P \subset \mathbb{R}^3$, there exists a positive constant γ independent of diameter h such that every P and each $F \subset \partial P$ satisfy these conditions

- (T_1^{3d}) $P \in \Omega_h$ is star-shaped with respect to every point of a ball of radius greater than $\gamma\, h_P$.
- (T_1^{3d}) for every $e \subset \partial F$ and for every face F, it satisfies $h_e \ge \gamma h_F \ge \gamma^2 h_P$.

***Remark* 3.1** Assumption $(T_1^{2d})$and (T_1^{3d}) ensure elements and the mesh faces are *simply connected* subset of $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$ respectively. Assumption $(T_2^{2d})$and (T_2^{3d}) confirm that there exists a positive number K_0 independent of mesh family $\{\Omega_h\}_h$ such that

$$\mathcal{N}_E^{\varepsilon}/\mathcal{N}_P^{F} \le K_0 \quad \forall E/P \in \Omega_h, 0 < h \le 1.$$

The following canonical convention of the multi-dimensional space is exploited. Let $\mathbf{s} = (s_1, s_2, \dots, s_d)$ and define $|\mathbf{s}| = s_1 + s_2 + \cdots + s_d$. We denote an element $\mathbf{x}^{\mathbf{s}} \in \mathbb{R}^d$, $d = 1,2,3$ by, $\mathbf{x}^{\mathbf{s}} := (x_1^{s_1}\, x_2^{s_2} \dots x_d^{s_d})$. In what follows, $\mathcal{M}_k^d(E) := \left\{ \left(\frac{\mathbf{x}-\mathbf{x}_E}{h_E} \right)^s, |s| \le k \right\}$, $d = 1,2,3$ is the set of scaled monomials with the notational convention $\mathcal{M}_{-1}^d(E) = \{0\}$.

For each element $E \in \Omega_h$, we outline the L^2 projection operator $\Pi_{k,E}^0 : L^2(E) \to \mathbb{P}_k(E)$ defined as

$$\left((\Pi_{k,E}^0 - I)u,\, q \right)_E = 0 \qquad \forall q \in \mathbb{P}_k(E),$$

and define the elliptic projection operator $\Pi_{k,E}^\nabla : H^1(E) \to \mathbb{P}_k(E)$ satisfying,

$$\left(\nabla(\Pi_{k,E}^\nabla - I)u,\, \nabla q \right)_E = 0 \quad \forall q \in \mathbb{P}_k(E) \setminus \mathbb{P}_0(E) \quad \text{and} \quad \int_{\partial E} (\Pi_{k,E}^\nabla u - u)\, ds = 0.$$

The global operator Π_k^0 is defined on $L^2(\Omega)$ such that it is same as $\Pi_{k,E}^0$ on each element E, i.e., $\Pi_k^0|_E = \Pi_{k,E}^0$.

Two dimensional virtual element space For every $E \in \Omega_h$, consider the auxiliary space W_E^k (see [39]) defined by,

$$W_E^k = \left\{ v \in H^1(E) \cap C^0(\partial E) \,:\, v_{|e} \in \mathbb{P}_k(e)\, \forall e \subset \partial E,\ \Delta v \in \mathbb{P}_k(E) \right\}.$$

Upon restricting the functions, we introduce the local virtual element space in two dimension as below

$$\mathcal{H}^k(E) := \left\{ v \in W_E^k \ :\ \int_E (\Pi_{k,E}^\nabla v - v)\, q = 0\ \forall\, q \in \mathbb{P}_k \setminus \mathbb{P}_{k-2}(E) \right\}, \tag{12}$$

where $\mathbb{P}_k \setminus \mathbb{P}_{k-2}(E)$ denotes the set of polynomials of degrees exactly equal to $k-1$ and k. Further, we define a set of DoFs associated with an element $\mathcal{H}^k(E)$ which uniquely characterize the function $\xi_h \in \mathcal{H}^k(E)$ as follows.

- (d_1) The values of ξ_h at the vertices of the element E.
- (d_2) On each edge $e \subset \partial E$, the moments of ξ_h up to order $k-2$, i.e.

$$\frac{1}{|e|} \int_e \xi_h\, \omega \,\mathrm{d}e \quad \forall \omega \in \mathcal{M}_{k-2}^1(e).$$

- (d_3) The moments up to order $k-2$ of ξ_h on E, i.e.,

$$\frac{1}{|E|} \int_E \xi_h\, \omega \,\mathrm{d}E, \quad \forall \omega \in \mathcal{M}_{k-2}^2(E).$$

We deduce that $\mathcal{H}^k(E)$ is unisolvent with respect to the above set of functionals (d_1)–(d_3) (see [40–42] for detailed proof). The global conforming virtual element space is defined as follows

$$\mathcal{H}_h^k := \{ v \in H_0^1(\Omega) \,|\, v|_E \in \mathcal{H}^k(E)\ \forall E \in \Omega_h \}$$

[40–42]. The construction of the conforming virtual element space for $d = 3$ follows an analogous idea as $d = 2$. Hereafter, we will not make any difference between E and P and we will try to be dimension independent if not otherwise specified. For better readability, we append the following remark.

***Remark* 3.2** For each polyhedral $P \in \Omega_h \subset \mathbb{R}^3$, the local VEM space is defined same as two-dimension VEM space. For each face $F \subset \partial P \subset \mathbb{R}^2$, $\mathcal{H}^k(F)$ is a two-dimensional VEM space. Interested reader can refer [30, 39, 41] for detail demonstration of three dimensional VEM space. Also, it can be observed that the local virtual element space $\mathcal{H}^k(E)$ has the same number of DoFs as [39] with an added advantage that the L^2 projection operator $\Pi^0_{k,E}$ is computable on $\mathcal{H}^k(E)$ [41]. The L^2 projection operator is used to discretize the nonlocal term and the non-stationary part of the model problem that will be discussed in the later part of this article.

On the virtual element space $\mathcal{H}^k(E)$, we consider the discrete bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$ corresponding to the continuous forms $a(\cdot,\cdot)$ and $m(\cdot,\cdot)$ respectively. Since, the discrete functions $v_h \in \mathcal{H}^k(E)$ are not available in a closed forms, we employ the projection operators, $\Pi^0_{k,E}$ and $\Pi^\nabla_{k,E}$ to discretize the bilinear forms. The local discrete bilinear form $a^E_h(\cdot,\cdot) : \mathcal{H}^k(E) \times \mathcal{H}^k(E) \to \mathbb{R}$ and $m_h(\cdot,\cdot) : \mathcal{H}^k(E) \times \mathcal{H}^k(E) \to \mathbb{R}$ corresponding to continuous bilinear forms $a^E(\cdot,\cdot)$ and $(\cdot,\cdot)_E$ respectively, are defined as follows:

$$
\begin{aligned}
a^E_h(w,v) &:= a^E(\Pi^\nabla_{k,E}w, \Pi^\nabla_{k,E}v) \\
&\quad + S^E_a((I-\Pi^\nabla_{k,E})w, (I-\Pi^\nabla_{k,E})v) \quad \forall w,v \in \mathcal{H}^k(E), \\
m^E_h(w,v) &:= (\Pi^0_{k,E}w, \Pi^0_{k,E}v)_E \\
&\quad + S^E_m((I-\Pi^0_{k,E})w, (I-\Pi^0_{k,E})v) \qquad \forall w,v \in \mathcal{H}^k(E).
\end{aligned} \tag{13}
$$

The last terms on the right of (13),viz. $S^E_a(\cdot,\cdot)$ and $S^E_m(\cdot,\cdot)$ are the stabilization terms. $S^E_a(\cdot,\cdot)$ is symmetric and positive semi-definite and $S^E_m(\cdot,\cdot)$ is symmetric and positive definite on $\mathcal{H}^k_h \times \mathcal{H}^k_h$. Moreover, the stabilization terms $S^E_a(\cdot,\cdot)$ or $S^E_m(\cdot,\cdot)$ reduce to zero when one of the entries is a polynomial function. Symmetric and positive definite bilinear forms that scale like $(\cdot,\cdot)_E$ can be used as stabilization $S^E_m(\cdot,\cdot)$ and symmetric positive semi-definite bilinear forms that scale like $a^E(\cdot,\cdot)$ can be used as the stabilization $S^E_a(\cdot,\cdot)$. Further, we assume that there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$
\begin{aligned}
&\alpha_1 \; a^E(v,v) \le S^E_a(v,v) \le \alpha_2 \; a^E(v,v) \quad \forall v \in \mathcal{H}^k(E) \cap \mathrm{Ker}(\Pi^\nabla_{k,E}) \\
&\beta_1 \; (w,w)_E \le S^E_m(w,w) \le \beta_2 \; (w,w)_E \quad \forall w \in \mathcal{H}^k(E) \cap \mathrm{Ker}(\Pi^0_{k,E}),
\end{aligned}
$$

where $\mathrm{Ker}(T)$ denotes the *nullspace* of the operator T. The above mentioned assumption implies that $S^E_a(\cdot,\cdot)$ and $S^E_m(\cdot,\cdot)$ are spectrally equivalent to $a^E(\cdot,\cdot)$ and $(\cdot,\cdot)_E$ respectively. Amongst the different computable forms of the projection operators available in the literature [43], we choose the following representation:

$$S_m^E(\phi,\psi) := h_E^d \sum_{z=1}^{N_E^{\rm dof}} {\rm dof}_z(\phi)\,{\rm dof}_z(\psi), \quad \text{and} \quad S_a^E(\phi,\psi) := h_E^{d-2} \sum_{z=1}^{N_E^{\rm dof}} {\rm dof}_z(\phi)\,{\rm dof}_z(\psi).$$

$N_E^{\rm dof}$ denotes dimension of the local space $\mathcal{H}^k(E)$. The local forms $a_h^E(\cdot,\cdot)$ and $m_h^E(\cdot,\cdot)$ satisfy the following two properties :

Polynomial consistency For an element $E \in \Omega_h$, $0 < h \le 1$, the bilinear forms $a_h^E(\cdot,\cdot)$ and $m_h^E(\cdot,\cdot)$ defined in (13), satisfy the following consistency properties:

$$\begin{aligned} a_h^E(p,v) &= a^E(p,v) \quad \forall p \in \mathbb{P}_k(E), \quad \forall v \in \mathcal{H}^k(E) \\ m_h^E(p,v) &= (p,v)_E \quad \forall p \in \mathbb{P}_k(E), \quad \forall v \in \mathcal{H}^k(E). \end{aligned} \tag{14}$$

Stability There exist four mesh independent positive constants, $\alpha^*, \alpha_*, \beta^*, \beta_*$ independent of the element E such that for all $v \in \mathcal{H}^k(E)$, $a_h^E(v,v)$, and $m_h^E(v,v)$ are bounded by $a^E(v,v)$ and $(v,v)_E$, respectively, i.e.,

$$\begin{aligned} \alpha_*\, a^E(v,v) \le a_h^E(v,v) \le \alpha^*\, a^E(v,v); \\ \beta_*\, (v,v)_E \le m_h^E(v,v) \le \beta^*\, (v,v)_E \end{aligned} \tag{15}$$

hold. Condition (15) ensures that the non-polynomial parts $S_a^E(\cdot,\cdot)$ and $S_m^E(\cdot,\cdot)$ scale same as polynomial parts of $a_h^E(\cdot,\cdot)$ and $m_h^E(\cdot,\cdot)$ respectively. Adding the local contributions, the global forms $a_h(\cdot,\cdot) : \mathcal{H}_h^k \times \mathcal{H}_h^k \to \mathbb{R}$ and $m_h(\cdot,\cdot) : \mathcal{H}_h^k \times \mathcal{H}_h^k \to \mathbb{R}$ are defined as

$$a_h(w,v) := \sum_{E\in\Omega_h} a_h^E(w,v) \quad \text{and} \quad m_h(w,v) := \sum_{E\in\Omega_h} m_h^E(w,v) \qquad \forall w,v \in \mathcal{H}_h^k.$$

***Remark* 3.3** To discretize the bilinear form $a^E(\cdot,\cdot)$, we have employed $\Pi_{k,E}^{\nabla}$ operator. However, the term $a^E(\cdot,\cdot)$ can be discretized by employing the external projection operator $\Pi_{k-1,E}^0$ [43].

***Remark* 3.4** In this work, we use the projection operators' matrix representation to evaluate the matrices corresponding to the bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$ respectively. These matrix representation depend on the order of the space and shape of the element E, but is independent of the size of the element. Therefore, the matrices remain unchanged for any transformations that preserve the shape of E. However, this inspection is not true for higher order virtual element space. [22, Remark 3.5]. We compute the matrices following the procedure highlighted in [22].

3.2 Semi-discrete formulation

By using the discrete bilinear form, the semi discrete formulation of (8)–(11) is defined as: *Find* $(u_h(t),\ v_h(t)) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *for all most all* $t \in [0,T]$ *such that*

$$m_h(D_t u_h, \varphi_h) + \mathcal{A}_1(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\, a_h(u_h,\varphi_h) = \langle f_{1h}(u_h,v_h), \varphi_h\rangle \quad \forall \varphi_h \in \mathcal{H}_h^k, \tag{16}$$

$$m_h(D_t v_h, \psi_h) + \mathcal{A}_2(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\, a_h(v_h, \psi_h) = \langle f_{2h}(u_h, v_h), \psi_h \rangle \quad \forall \psi_h \in \mathcal{H}_h^k, \tag{17}$$

where

$$\begin{aligned} \langle f_{1h}(u_h, v_h), \varphi_h \rangle &= \sum_{E \in \Omega_h} \int_E f_1(\Pi_{k,E}^0 u_h,\ \Pi_{k,E}^0 v_h)\, \Pi_{k,E}^0 \varphi_h \, \mathrm{d}E, \\ \text{and} \quad \langle f_{2h}(u_h, v_h),\ \psi_h \rangle &= \sum_{E \in \Omega_h} \int_E f_2(\Pi_{k,E}^0 u_h,\ \Pi_{k,E}^0 v_h)\, \Pi_{k,E}^0 \psi_h \, \mathrm{d}E. \end{aligned} \tag{18}$$

The scheme (16) and (17) constitute a system of differential equations. Since the model problem (1) and (2) satisfy Assumption 1, we deduce that the nonlinear system of equations (16) and (17) have a unique solution for $t \in [0, T_1]$, where $T_1 < T$. Such a solution can be extended to $[0, T]$ following the boundedness property of the discrete solutions. Let C be a generic positive constant that is independent of mesh diameter h and element E, which takes different values at different instances.

Theorem 3.1 *Let the discrete solutions* $(u_h^0, v_h^0) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and the two force functions* $f_1(u, v), f_2(u, v) \in L^2(0, T, L^2(\Omega))$, *then, the solution of* (16) *and* (17) (u_h, v_h) *satisfies the following boundedness property*

$$\begin{aligned} &\|v_h\|_{L^\infty(0,T;\,L^2(\Omega))} \le C, \quad \|u_h\|_{L^\infty(0,T;\,L^2(\Omega))} \le C, \\ &\|D_t v_h\|_{L^2(0,T;\,L^2(\Omega))} \le C, \quad \|D_t u_h\|_{L^2(0,T;\,L^2(\Omega))} \le C. \end{aligned}$$

Proof We consider the semi-discrete formulation (16) and (17). Upon choosing $\varphi_h = u_h$ in (16), we obtain

$$\frac{1}{2}\frac{d}{dt} m_h(u_h, u_h) + \mathcal{A}_1(g_1(\Pi_k^0 u_h), g_2(\Pi_k^0 v_h))\, a_h(u_h, u_h) = \langle f_{1h}(u_h, v_h), u_h \rangle. \tag{19}$$

Using Assumption 1, triangle inequality and continuity of the operator Π_k^0, we obtain

$$\begin{aligned} \|f_1(\Pi_k^0 u_h, \Pi_k^0 v_h)\|_{0,\Omega} &= \|f_1(\Pi_k^0 u_h, \Pi_k^0 v_h) - f_1(0,0) + f_1(0,0)\|_{0,\Omega} \\ &\le L_F\, (\, \|u_h\|_{0,\Omega} + \|v_h\|_{0,\Omega}\,) + \|f_1(0,0)\|_{0,\Omega}. \end{aligned} \tag{20}$$

An application of Cauchy–Schwarz inequality, boundedness of the operator Π_k^0, Young's inequality and (20), we obtain

$$\begin{aligned} |\langle f_{1h}(u_h, v_h), u_h \rangle| &\le \frac{1}{2}\Big(\|f_1(\Pi_k^0 u_h, \Pi_k^0 v_h)\|_{0,\Omega}^2 + \|u_h\|_{0,\Omega}^2 \Big) \\ &\le C\Big(\|u_h\|_{0,\Omega}^2 + \|v_h\|_{0,\Omega}^2 + \|f_1(0,0)\|_{0,\Omega}^2 \Big). \end{aligned} \tag{21}$$

Substituting the estimation (21) into (19), we derive

$$\begin{aligned}&\frac{1}{2}\,\beta_*\,\frac{d}{dt}\,\|u_h\|^2_{0,\Omega}+m_0\,\alpha_*\|\nabla u_h\|^2_{0,\Omega}\\&\quad\le C\Big(\|u_h\|^2_{0,\Omega}+\|v_h\|^2_{0,\Omega}+\|f_1(0,0)\|^2_{0,\Omega}\Big).\end{aligned}\tag{22}$$

In the analogous way, we obtain

$$\begin{aligned}&\frac{1}{2}\beta_*\frac{d}{dt}\|v_h\|^2_{0,\Omega}+m_0\,\alpha_*\|\nabla v_h\|^2_{0,\Omega}\\&\quad\le C\Big(\|u_h\|^2_{0,\Omega}+\|v_h\|^2_{0,\Omega}+\|f_2(0,0)\|^2_{0,\Omega}\Big).\end{aligned}\tag{23}$$

By adding (22) and (23), we have

$$\begin{aligned}&\frac{1}{2}\,\beta_*\,\frac{d}{dt}\Big(\|u_h\|^2_{0,\Omega}+\|v_h\|^2_{0,\Omega}\Big)+m_0\,\alpha_*\Big(\|\nabla u_h\|^2_{0,\Omega}+\|\nabla v_h\|^2_{0,\Omega}\Big)\\&\quad\le C\Big(\|u_h\|^2_{0,\Omega}+\|v_h\|^2_{0,\Omega}+\|f_1(0,0)\|^2_{0,\Omega}+\|f_2(0,0)\|^2_{0,\Omega}\Big).\end{aligned}\tag{24}$$

Integrating both sides of (24), and an application of Gronwall inequality, we obtain:

$$\begin{aligned}&\Big(\|u_h\|^2_{0,\Omega}+\|v_h\|^2_{0,\Omega}\Big)+C(m_0,\beta_*,\alpha_*)\int_0^t\Big(\|\nabla u_h\|^2_{0,\Omega}+\|\nabla v_h\|^2_{0,\Omega}\Big)\mathrm{d}t\\&\quad\le C\Big(\|u_h(0)\|^2_{0,\Omega}+\|v_h(0)\|^2_{0,\Omega}+\|f_1(0,0)\|^2_{0,\Omega}+\|f_2(0,0)\|^2_{0,\Omega}\Big).\end{aligned}\tag{25}$$

for all $t\in[0,T]$ which implies that $\|u_h\|_{L^\infty(0,T:L^2(\Omega))}$ and $\|v_h\|_{L^\infty(0,T:L^2(\Omega))}$ are bounded. In order to prove the terms $\|D_t u\|_{L^2(0,T;L^2(\Omega))}<\infty$ and $\|D_t v\|_{L^2(0,T;L^2(\Omega))}<\infty$, we choose $\varphi_h=D_t u_h$ in (16) and $\psi_h=D_t v_h$ in (17), followed by applying analogous arguments as the proof of $\|u_h\|_{L^2(0,T:L^2(\Omega))}<\infty$ and $\|v_h\|_{L^2(0,T:L^2(\Omega))}<\infty$.

3.3 Fully discrete scheme

We employ VEM and backward Euler method for discretizing the space variable and the time variable, respectively. To this end, we consider a partition of non-overlapping sub interval $[t_{n-1},t_n]$ of $[0,T]$, where $n=0,1,2,\dots,N_T$ with time-step $\Delta t^n:=t_n-t_{n-1}$ such that $T=\sum_{n=0}^{N_T}\Delta t^n$. To reduce the computational complexity, let us assume that $\Delta t^n=\Delta t$ for all n, i.e., equal time steps. Thus, the fully discrete virtual element scheme of (8)–(11) is defined as: *For each* $n=1,2,3,\dots,N_T$, *Find* $(U^n,\ V^n)\in\mathcal{H}_h^k\times\mathcal{H}_h^k$ *such that*

$$\begin{aligned}&m_h\Big(\frac{U^n-U^{n-1}}{\Delta t},\,\varphi_h\Big)+\mathcal{A}_1(g_1(\Pi_k^0U^n),g_2(\Pi_k^0V^n))\,a_h(U^n,\varphi_h)\\&\quad=\langle f_{1h}(U^n,V^n),\varphi_h\rangle\quad\forall\varphi_h\in\mathcal{H}_h^k,\end{aligned}\tag{26}$$

$$m_h\left(\frac{V^n - V^{n-1}}{\Delta t}, \psi_h\right) + \mathcal{A}_2(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(V^n, \psi_h) \\ = \langle f_{2h}(U^n, V^n), \psi_h\rangle \quad \forall \psi_h \in \mathcal{H}_h^k, \tag{27}$$

$$U^0 = I_h(u_0) \quad \text{and} \quad V^0 = I_h(v_0), \tag{28}$$

where U^0 and V^0 are initial approximation of u and v at time $t = 0$, respectively. The discrete scheme (26) and (27) reduces to a system of nonlinear equations which can be solved by employing iterative methods. To reduce the computation cost, we incorporate the technique introduced in [37]. A detailed implementation procedure will be discussed in Sect. 3.5. In addition, we would like to introduce a linearized scheme for the weak formulation, (8)–(11). where the unknowns are computed at time t_n and the nonlocal diffusive coefficients and the load terms are computed at the previous time-step, i.e., at $t = t_{n-1}$. We present the linearized scheme as follows:

For each $n = 1, 2, 3, \dots, N_T$, Find $(\widetilde{U}^n, \widetilde{V}^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ such that

$$m_h\left(\frac{\widetilde{U}^n - \widetilde{U}^{n-1}}{\Delta t}, \varphi_h\right) + \mathcal{A}_1(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1}))\, a_h(\widetilde{U}^n, \varphi_h) \\ = \left\langle f_{1h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \varphi_h\right\rangle \quad \forall \varphi_h \in \mathcal{H}_h^k, \tag{29}$$

$$m_h\left(\frac{\widetilde{V}^n - \widetilde{V}^{n-1}}{\Delta t}, \psi_h\right) + \mathcal{A}_2(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1}))\, a_h(\widetilde{V}^n, \psi_h) \\ = \left\langle f_{2h}(\widetilde{U}^{n-1}, \widetilde{V}^{n-1}), \psi_h\right\rangle \quad \forall \psi_h \in \mathcal{H}_h^k, \tag{30}$$

$$U^0 = I_h(u_0) \quad \text{and} \quad V^0 = I_h(v_0). \tag{31}$$

The discrete formulation (29) and (30) reduces to system of linear equations that can be solved by a linear solver directly. Let **A** and **B** be the matrix representation of the bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, which are positive semi-definite and positive definite respectively. For better representation, we introduce

$$\Xi_1 := \mathcal{A}_1(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1})) \quad \text{and} \quad \Xi_2 := \mathcal{A}_2(g_1(\Pi_k^0 \widetilde{U}^{n-1}), g_2(\Pi_k^0 \widetilde{V}^{n-1})).$$

Thus, both the matrices $\mathbf{B} + \Delta t\, \Xi_1\, \mathbf{A}$ and $\mathbf{B} + \Delta t\, \Xi_2\, \mathbf{A}$ are invertible that ensures unique solution to the system (29)–(31). Further, in Sect. 6, we will show that the approximation $(\widetilde{U}^n, \widetilde{V}^n)$ converges to the analytical solution with an optimal order in both the space and time variables. The rate of convergence depends on the initial approximation of the solution, i.e., (U^0, V^0). Therefore, the initial guess could be chosen as an interpolation of the analytical solution at $t = 0$.

3.4 Existence and uniqueness of the solution for the fully discrete scheme

In this section, we shall use the following variant of Brouwer fixed point theorem [44, Lemma 4.3] to ensure the existence of a solution for the discrete problem (26)–(28).

Theorem 3.2 (Brouwer theorem) *Let $\mathcal{K}$ be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{K}}$. Let $g : \mathcal{K} \to \mathcal{K}$ be a continuous function. If there exists a constant, $R > 0$ such that $(g(z), z)_{\mathcal{K}} > 0$ for all z with $\|z\|_{\mathcal{K}} = R$, then, there exists a $z^* \in \mathcal{K}$, such that $\|z^*\|_{\mathcal{K}} < R$ and $g(z^*) = 0$.*

The existence result of the fully discrete scheme (26)–(28) is given in the following result [8, Proposition 4.1].

Theorem 3.3 *Let $1 \le n \le N_T$ and $(U^J, V^J) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ be the given unique solution of the system (26)–(28) for $1 \le J \le n-1$. Then the system (26)–(28) has a unique solution $(U^n, V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ at time t_n.*

Proof We prove that the discrete system (26)–(28) has a solution (U^n, V^n) and that the solution is unique at time $t = t_n$, where $n = 1, \dots, N_T$. We will use mathematical induction method to prove the theorem. Let, for $n = 0$, the solution is (U^0, V^0) and assume that (U^{n-1}, V^{n-1}) be the solution of (26)–(28) at time $t = t_{n-1}$. We define a map

$$\mathcal{L} : \mathcal{H}_h^k \times \mathcal{H}_h^k \to \mathcal{H}_h^k \times \mathcal{H}_h^k, \tag{32}$$

such that

$$\begin{aligned}
[\mathcal{L}(U^n, V^n), \Phi] := m_h(U^n, \varphi_h) &+ \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(U^n, \varphi_h) \\
&- \Delta t\, \langle f_{1h}(U^n, V^n), \varphi_h \rangle - m_h(U^{n-1}, \varphi_h) \\
&+ m_h(V^n, \psi_h) + \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(V^n, \psi_h) \\
&- \Delta t\, \langle f_{2h}(U^n, V^n), \psi_h \rangle - m_h(V^{n-1}, \psi_h),
\end{aligned} \tag{33}$$

where $\Phi = (\varphi_h, \psi_h) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$. Since, the bilinear forms $m_h(\cdot,\cdot)$, $a_h(\cdot,\cdot)$ are bounded and the nonlocal coefficients $\mathcal{A}_1(\cdot,\cdot)$, and $\mathcal{A}_2(\cdot,\cdot)$, and the discrete force functions f_{1h}, f_{2h} are Lipschitz continuous, therefore $\mathcal{L}$ is continuous. Further, using boundedness of the bilinear forms $m_h(\cdot,\cdot)$, $a_h(\cdot,\cdot)$ and Assumption 1, we have

$$\begin{aligned}
&m_h(U^n, \varphi_h) + \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(U^n, \varphi_h) - \Delta t\, \langle f_{1h}(U^n, V^n), \varphi_h \rangle \\
&\quad - m_h(U^{n-1}, \varphi_h) \le \beta^* \left(\|U^n\|_{0,\Omega}\, \|\varphi_h\|_{0,\Omega} \right) \\
&\quad + \Delta t\, \alpha^*\, M \Big(\|\nabla U^n\|_{0,\Omega} + \|\nabla \varphi_h\|_{0,\Omega} \Big) \\
&\quad + C(L_A)\Delta t \left(\|U^n\|_{0,\Omega} + \|V^n\|_{0,\Omega} + |f_1(0,0)| \right) \|\varphi_h\|_{0,\Omega} + \beta^* \|U^{n-1}\|_{0,\Omega} \|\varphi_h\|_{0,\Omega}.
\end{aligned} \tag{34}$$

Applying analogous arguments as (34), we obtain

$$
\begin{aligned}
&m_h(V^n,\psi_h) + \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n)) a_h(V^n,\psi_h) - \Delta t\, \langle f_{2h}(U^n,V^n),\psi_h\rangle \\
&\quad - m_h(V^{n-1},\psi_h) \le \beta^*(\|V^n\|_{0,\Omega}\, \|\psi_h\|_{0,\Omega}) \\
&\quad + \Delta t\, M\alpha^*\Big(\|\nabla V^n\|_{0,\Omega} + \|\nabla \psi_h\|_{0,\Omega}\Big) \\
&\quad + C(L_A)\,\Delta t\,\Big(\|U^n\|_{0,\Omega} + \|V^n\|_{0,\Omega} + |f_2(0,0)|\Big)\, \|\psi_h\|_{0,\Omega} + \beta^* \|V^{n-1}\|_{0,\Omega}\, \|\psi_h\|_{0,\Omega}.
\end{aligned} \tag{35}
$$

Adding inequalities (34) and (35), we derive

$$
\begin{aligned}
[\mathcal{L}(U^n,V^n),\Phi] \le C(\beta^*,\alpha^*,M)\,&\Big(\|U^n\|_{1,\Omega} + \|V^n\|_{1,\Omega} + |f_1(0,0)| + |f_2(0,0)| + \|U^{n-1}\|_{0,\Omega} \\
&+ \|V^{n-1}\|_{0,\Omega}\Big)\Big(\|\psi_h\|_{1,\Omega} + \|\varphi_h\|_{1,\Omega}\Big),
\end{aligned}
$$

which implies the map $\mathcal{L}$ is bounded on $\mathcal{H}_h^k \times \mathcal{H}_h^k$. Next, we derive that

$$
[\mathcal{L}(U^n,V^n),(U^n,V^n)] > 0, \tag{36}
$$

for sufficiently large values of norm of (U^n,V^n). By choosing $\Phi = (U^n,V^n)$ in (33), we obtain:

$$
\begin{aligned}
&m_h(U^n,U^n) + \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(U^n,U^n) - \Delta t\, \langle f_{1h}(U^n,V^n),U^n\rangle \\
&\quad - m_h(U^{n-1},U^n) \ge \beta_* \|U^n\|_{0,\Omega}^2 + \Delta t\, m_0\, \alpha_*\, \|\nabla U^n\|_{0,\Omega}^2 \\
&\quad - C(L_F)\Delta t\,\Big(\|U^n\|_{0,\Omega} + \|V^n\|_{0,\Omega} \\
&\quad + |f_1(0,0)|\Big)\|U^n\|_{0,\Omega} - \beta^*\, \|U^{n-1}\|_{0,\Omega}\, \|U^n\|_{0,\Omega}.
\end{aligned} \tag{37}
$$

Similarly, we derive

$$
\begin{aligned}
&m_h(V^n,V^n) + \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(V^n,V^n) - \Delta t\, \langle f_{2h}(U^n,V^n),V^n\rangle \\
&\quad - m_h(V^{n-1},V^n) \ge \beta_* \|V^n\|_{0,\Omega}^2 + \Delta t\, m_0\, \alpha_*\, \|\nabla V^n\|_{0,\Omega}^2 \\
&\quad - \Delta t\, C(L_F)\Big(\|V^n\|_{0,\Omega} + \|U^n\|_{0,\Omega} \\
&\quad + |f_2(0,0)|\Big)\, \|V^n\|_{0,\Omega} - \beta^*\, \|V^{n-1}\|_{0,\Omega}\, \|V^n\|_{0,\Omega}.
\end{aligned} \tag{38}
$$

Adding (37) and (38), and using Young's inequality, we have

$$
\begin{aligned}
&[\mathcal{L}(U^n,V^n),(U^n,V^n)] \ge \Big(\beta_* - \Delta t\, C_u(L_F,\beta^*)\Big)\|U^n\|_{0,\Omega}^2 \\
&\quad + \Big(\beta_* - \Delta t\, C_v(L_F,\beta^*)\Big)\|V^n\|_{0,\Omega}^2 \\
&\quad + \Delta t\, m_0\, \alpha_*\Big(\|\nabla U^n\|_{0,\Omega}^2 + \|\nabla V^n\|_{0,\Omega}^2\Big) - \Delta t\, C(L_F)\Big(|f_1(0,0)|^2 + |f_2(0,0)|^2\Big) \\
&\quad - \frac{1}{2}\Big(\|U^{n-1}\|_{0,\Omega}^2 + \|V^{n-1}\|_{0,\Omega}^2\Big).
\end{aligned} \tag{39}
$$

Further, we choose the time-step Δt sufficiently small such that the coefficients of $\|U^n\|_{0,\Omega}$ and $\|V^n\|_{0,\Omega}$ are positive, i.e. $\min\Big\{(\beta_* - \Delta t\, C_u(L_F, \beta^*)), (\beta_* - \Delta t\, C_v(L_F, \beta^*))\Big\} > 0$. We rewrite (39) as [8, Proposition 4.1]

$$[\mathcal{L}(U^n, V^n), (U^n, V^n)] \geq \widehat{C}\Big(\|U^n\|^2_{1,\Omega} + \|V^n\|^2_{1,\Omega}\Big) - C(L_F)\Big(|f_1(0,0)|^2 + |f_2(0,0)|^2 \Big) - \frac{1}{2}\Big(\|U^{n-1}\|^2_{1,\Omega} + \|V^{n-1}\|^2_{1,\Omega}\Big). \tag{40}$$

We define

$$\mathfrak{R} := \frac{1}{\widehat{C}}\Big(2C(L_F)\Big(|f_1(0,0)|^2 + |f_2(0,0)|^2 \Big) + \Big(\|U^{n-1}\|^2_{1,\Omega} + \|V^{n-1}\|^2_{1,\Omega}\Big)\Big).$$

Therefore, $[\mathcal{L}(U^n, V^n), (U^n, V^n)] \geq 0$ for $\|U^n\|^2_{1,\Omega} + \|V^n\|^2_{1,\Omega} = \mathfrak{R}$. By Brouwer fixed point theorem, we can assure the existence of the solution $(U^n, V^n) \in \mathcal{B}_{\mathfrak{R}} := \Big\{ (\mathcal{U}^n, \mathcal{V}^n) \in \mathcal{H}^k_h \times \mathcal{H}^k_h \ : \ \|\mathcal{U}^n\|^2_{1,\Omega} + \|\mathcal{V}^n\|^2_{1,\Omega} \leq \mathfrak{R} \Big\}$. Now, we will prove that the discrete solution (U^n, V^n) of (26) and (27) is unique. Let (U^n_1, V^n_1) and $(U^n_2, V^n_2) \in \mathcal{H}^k_h \times \mathcal{H}^k_h$ be two solutions of (26) and (27). Then, from (26), we have

$$\begin{aligned} &m_h(U^n_1 - U^n_2, \varphi_h) + \Delta t\, \mathcal{A}_1\Big(g_1(\Pi^0_k U^n_1), g_2(\Pi^0_k V^n_1)\Big)\, a_h(U^n_1, \varphi_h) \\ &\quad - \Delta t\, \mathcal{A}_1\Big(g_1(\Pi^0_k U^n_2), g_2(\Pi^0_k V^n_2)\Big)\, a_h(U^n_2, \varphi_h) - \Delta t\, \langle f_{1h}(U^n_1, V^n_1), \varphi_h\rangle \\ &\quad + \Delta t\, \langle f_{1h}(U^n_2, V^n_2), \varphi_h\rangle = 0. \end{aligned} \tag{41}$$

Applying analogous arguments, we derive

$$\begin{aligned} &m_h(V^n_1 - V^n_2, \psi_h) + \Delta t\, \mathcal{A}_2\Big(g_1(\Pi^0_k U^n_1), g_2(\Pi^0_k V^n_1)\Big)\, a_h(V^n_1, \psi_h) \\ &\quad - \Delta t\, \mathcal{A}_2\Big(g_1(\Pi^0_k U^n_2), g_2(\Pi^0_k V^n_2)\Big)\, a_h(V^n_2, \psi_h) - \Delta t\, \langle f_{2h}(U^n_1, V^n_1), \psi_h\rangle \\ &\quad + \Delta t\, \langle f_{2h}(U^n_2, V^n_2), \psi_h\rangle = 0. \end{aligned} \tag{42}$$

For better readability, we introduce the following notations: $\tau := U^n_1 - U^n_2$ and $\chi := V^n_1 - V^n_2$. Further, we choose the test function $\varphi_h = \tau$ and substituting in (41), we have

$$\begin{aligned} &m_h(\tau, \tau) + \Delta t\, \mathcal{A}_1\Big(g_1(\Pi^0_k U^n_1), g_2(\Pi^0_k V^n_1)\Big)\, a_h(U^n_1, \tau) \\ &\quad - \Delta t\, \mathcal{A}_1\Big(g_1(\Pi^0_k U^n_2), g_2(\Pi^0_k V^n_2)\Big)\, a_h(U^n_2, \tau) - \Delta t\, \langle f_{1h}(U^n_1, V^n_1), \tau\rangle \\ &\quad + \Delta t\, \langle f_{1h}(U^n_2, V^n_2), \tau\rangle = 0. \end{aligned} \tag{43}$$

An application of Lipschitz continuity of the force function f_1 (Assumption 1) and the boundedness of the projection operator Π^0_k yield

$$\Big| \langle f_{1h}(U^n_1, V^n_1), \tau\rangle - \langle f_{1h}(U^n_2, V^n_2), \tau\rangle \Big| \leq C(L_F)\, \Big(\|\tau\|_{0,\Omega} + \|\chi\|_{0,\Omega}\Big)\, \|\tau\|_{0,\Omega}. \tag{44}$$

Adding and subtracting $\mathcal{A}_1\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big)\, a_h(U_2^n,\tau)$, we rewrite the difference of the nonlocal terms in the following way:

$$\begin{aligned}\mathcal{T}_1 :=& \mathcal{A}_1\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big)\, a_h(U_1^n,\tau) - \mathcal{A}_1\Big(g_1(\Pi_k^0 U_2^n), g_2(\Pi_k^0 V_2^n)\Big)\, a_h(U_2^n,\tau)\\ =& \mathcal{A}_1\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big)\, a_h(U_1^n - U_2^n,\tau)\\ &+ \Big(\mathcal{A}_1\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big) - \mathcal{A}_1\Big(g_1(\Pi_k^0 U_2^n), g_2(\Pi_k^0 V_2^n)\Big)\Big)\, a_h(U_2^n,\tau).\end{aligned} \tag{45}$$

Using the boundedness of the projection operator Π_k^0 and the assumption on the nonlocal coefficient (Assumption 1) $\mathcal{A}_1(\cdot,\cdot)$, the second term of the right hand side of (45) can be bounded as

$$\begin{aligned}&\Big|\Big(\mathcal{A}_1\Big(g_1(\Pi_k^0 U_1^n), g_2(\Pi_k^0 V_1^n)\Big) - \mathcal{A}_1\Big(g_1(\Pi_k^0 U_2^n), g_2(\Pi_k^0 V_2^n)\Big)\Big)\, a_h(U_2^n,\tau)\Big|\\ &\le C(L_A)\Big(\|\tau\|_{0,\Omega} + \|\chi\|_{0,\Omega}\Big)\, \|\nabla U_2^n\|_{0,\Omega}\, \|\nabla\tau\|_{0,\Omega}.\end{aligned} \tag{46}$$

Substituting (44) and (46) into (43), we derive the following result:

$$\begin{aligned}&m_h(\tau,\tau) + \Delta t\, m_0\, \alpha_*\, \|\nabla\tau\|_{0,\Omega}^2\\ &\quad\le\; C(L_A)\,\Delta t\Big(\|\tau\|_{0,\Omega} + \|\chi\|_{0,\Omega}\Big)\Big(\|\tau\|_{0,\Omega} + \|\nabla\tau\|_{0,\Omega}\Big).\end{aligned} \tag{47}$$

Using analogous techniques as (47), we derive from Eq. (42),

$$\begin{aligned}&m_h(\chi,\chi) + \Delta t\, m_0\, \alpha_*\, \|\nabla\chi\|_{0,\Omega}^2\\ &\quad\le C(L_A)\,\Delta t\,\Big(\|\tau\|_{0,\Omega} + \|\chi\|_{0,\Omega}\Big)\Big(\|\chi\|_{0,\Omega} + \|\nabla\chi\|_{0,\Omega}\Big).\end{aligned} \tag{48}$$

Upon adding (47) and (48) and an application of Young's inequality and the stability of the discrete bilinear forms $m_h(\cdot,\cdot)$ yield

$$\begin{aligned}&\Big(\beta_* - \widetilde{C_u}(L_F,\alpha_*,m_0)\Delta t\Big)\|\tau\|_{0,\Omega}^2 + \Big(\beta_* - \widetilde{C_v}(L_F,\alpha_*,m_0)\Delta t\Big)\|\chi\|_{0,\Omega}^2\\ &\quad+ \frac{\Delta t\, m_0\, \alpha_*}{2}(\|\nabla\tau\|_{0,\Omega}^2 + \|\nabla\chi\|_{0,\Omega}^2) \le 0.\end{aligned} \tag{49}$$

By choosing Δt sufficiently small, we derive

$$\|\tau\|_{1,\Omega} + \|\chi\|_{1,\Omega} \le 0, \tag{50}$$

which implies $\tau = 0$ and $\chi = 0$.

***Remark* 3.5** In the proof of Theorem 3.3, we have exploited Brouwer theorem to prove that the fully discrete scheme has a unique solution. In the proof, we have assumed that the time-step $\Delta t > 0$ is sufficiently small such that

$$\min\Big\{(\beta_* - \Delta t\, C_u(L_F,\beta^*)),\ (\beta_* - \Delta t\, C_v(L_F,\beta^*)),\ (\beta_* - \widetilde{C_u}(L_F,\alpha_*,m_0)\,\Delta t),$$
$$(\beta_* - \widetilde{C_v}(L_F,\alpha_*,m_0)\Delta t)\Big\} > 0.$$

By using Brouwer theorem, we have deduced that the discrete solution $(U^n, V^n) \in \mathcal{B}_{\mathfrak{R}}$, where $\mathfrak{R}$ is the radius of the ball $\mathcal{B}_{\mathfrak{R}}$ that depends on values of force functions and discrete solution at previous time step. In particular, the radius depends on values of force functions f_1,f_2 and norms of solution at time $t = 0$, i.e., $C\Big(|f_1(0,0)| + |f_2(0,0)| + \|U^0\|_{1,\Omega} + \|V^0\|_{1,\Omega}\Big)$, where C is a positive constant. The radius of the ball at time t_n is greater than the radius of ball at time t_{n-1} [2, 8].

3.5 Implementation of the scheme

The fully discrete formulation (26)–(28) can be solved by employing Newton's method. However, the presence of the nonlocal coefficient reduces the sparse structure of the Jacobian of the nonlinear system, thereby increasing the computational cost. Since our model problem contains a coupled system, the computational cost is twice. In order to avoid this difficulty, we incorporate the idea provided in [37]. The fully discrete scheme (26)–(28) can be rewritten as

$$\begin{aligned} m_h(U^n,\varphi_h) + \Delta t\, \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(U^n,\varphi_h) &= \Delta t\langle f_{1h}(U^n,V^n),\varphi_h\rangle \\ &\quad + m_h(U^{n-1},\varphi_h),\\ m_h(V^n,\psi_h) + \Delta t\, \mathcal{A}_2(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(V^n,\psi_h) &= \Delta t\langle f_{2h}(U^n,V^n),\psi_h\rangle \\ &\quad + m_h(V^{n-1},\psi_h). \end{aligned}$$

We introduce two new independent variables such as $d_1 = g_1(\Pi_k^0 U^n)$ and $d_2 = g_2(\Pi_k^0 V^n)$. Then, the above system reduce to the following non-linear system,

$$\begin{aligned} & m_h(U^n,\varphi_h) + \Delta t\, \mathcal{A}_1(d_1,d_2)\, a_h(U^n,\varphi_h) \\ &\quad = \Delta t\, \langle f_{1h}(U^n,V^n),\varphi_h\rangle + m_h(U^{n-1},\varphi_h),\\ & m_h(V^n,\psi_h) + \Delta t\, \mathcal{A}_2(d_1,d_2)\, a_h(V^n,\psi_h) \\ &\quad = \Delta t\langle f_{2h}(U^n,V^n),\psi_h\rangle + m_h(V^{n-1},\psi_h),\\ & d_1 = g_1(\Pi_k^0 U^n),\\ & d_2 = g_2(\Pi_k^0 V^n). \end{aligned} \tag{51}$$

The Jacobian of the system (51) will be of the form

$$J = \begin{bmatrix} A_1 & 0 & C_1 & D_1 \\ 0 & B_2 & C_2 & D_2 \\ A_3 & 0 & C_3 & 0 \\ 0 & B_4 & 0 & D_4 \end{bmatrix}_{2N^{\mathrm{dof}}+2\times 2N^{\mathrm{dof}}+2}$$

where, N^{dof} represents the total number of degrees of freedom of the global virtual element space $\mathcal{H}_h^k$. In what follows, we define the residual of the fully discrete system (51) as

$$\begin{aligned} F_{1j} &:= m_h(U^n,\psi_j) + \Delta t\, \mathcal{A}_1(d_1,d_2)a_h(U^n,\psi_j) \\ &\quad - \Delta t\, \langle f_{1h}(U^n,V^n),\psi_j\rangle - m_h(U^{n-1},\psi_j) = 0, \quad 1 \le j \le N^{\mathrm{dof}}, \\ F_{2j} &:= m_h(V^n,\psi_j) + \Delta t\, \mathcal{A}_2(d_1,d_2)a_h(V^n,\psi_j) \\ &\quad - \Delta t\, \langle f_{2h}(U^n,V^n),\psi_j\rangle - m_h(V^{n-1},\psi_j) = 0, \quad 1 \le j \le N^{\mathrm{dof}}, \\ F_{1N^{\mathrm{dof}}+1} &:= g_1(\Pi_k^0 U^n) - d_1 = 0, \\ F_{2N^{\mathrm{dof}}+1} &:= g_2(\Pi_k^0 V^n) - d_2 = 0. \end{aligned} \tag{52}$$

Writing explicitly discrete solution in term of basis functions, we have

$$U^n = \sum_{i=1}^{N^{\mathrm{dof}}} \alpha_i^n\, \psi_i, \quad \text{and} \quad V^n = \sum_{i=1}^{N^{\mathrm{dof}}} \beta_i^n\, \psi_i,$$

where $\mathcal{B} := \{\psi_1,\ldots,\psi_{N^{\mathrm{dof}}}\}$ forms the canonical basis of the finite dimensional space $\mathcal{H}_h^k$, and α_i^n, and β_i^n are unknowns. Further, the entries of the Jacobian matrix are given by:

$$\begin{aligned} (A_1)_{ij} &= \frac{\partial F_{1j}}{\partial \alpha_i^n} = m_h(\psi_i,\psi_j) + \Delta t\, \mathcal{A}_1(d_1,d_2)\, a_h(\psi_i,\psi_j), \quad 1 \le i,j \le N^{\mathrm{dof}}, \\ (C_1)_{1j} &= \frac{\partial F_{1j}}{\partial d_1} = \Delta t\, \frac{\partial \mathcal{A}_1(d_1,d_2)}{\partial d_1}\, a_h(U^n,\psi_j), \quad 1 \le j \le N^{\mathrm{dof}}, \\ (D_1)_{1j} &= \frac{\partial F_{1j}}{\partial d_2} = \Delta t\, \frac{\partial \mathcal{A}_1(d_1,d_2)}{\partial d_2}\, a_h(U^n,\psi_j), \quad 1 \le j \le N^{\mathrm{dof}}, \\ (B_2)_{ij} &= \frac{\partial F_{2j}}{\partial \beta_i^n} = m_h(\psi_i,\psi_j) + \Delta t\, \mathcal{A}_2(d_1,d_2)a_h(\psi_i,\psi_j), \quad 1 \le i,j \le N^{\mathrm{dof}}, \\ (C_2)_{1j} &= \frac{\partial F_{2j}}{\partial d_1} = \Delta t\, \frac{\partial \mathcal{A}_2(d_1,d_2)}{\partial d_1} a_h(V^n,\psi_j), \quad 1 \le j \le N^{\mathrm{dof}}, \\ (D_2)_{1j} &= \frac{\partial F_{2j}}{\partial d_2} = \Delta t\, \frac{\partial \mathcal{A}_2(d_1,d_2)}{\partial d_2} a_h(V^n,\psi_j), \quad 1 \le j \le N^{\mathrm{dof}}, \\ (A_3)_{1i} &= \frac{\partial F_{1N^{\mathrm{dof}}+1}}{\partial \alpha_i^n} = \frac{\partial g_1(\Pi_k^0 U^n)}{\partial \alpha_i^n}, \quad 1 \le i \le N^{\mathrm{dof}}, \\ (C_3)_{11} &= \frac{\partial F_{1N^{\mathrm{dof}}+1}}{\partial d_1} = -1, \\ (B_4)_{1i} &= \frac{\partial F_{2N^{\mathrm{dof}}+1}}{\partial \beta_i^n} = \frac{\partial g_2(\Pi_k^0 V^n)}{\partial \beta_i^n}, \quad 1 \le i \le N^{\mathrm{dof}}, \\ (D_4)_{11} &= \frac{\partial F_{2N^{\mathrm{dof}}+1}}{\partial d_2} = -1. \end{aligned} \tag{53}$$

Theorem 3.4 *Let Assumptions 1 and 2 hold. Also assume that* $(U^n, V^n, d_1, d_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k \times \mathbb{R} \times \mathbb{R}$ *be the solution of the system* (51), *then* $(U^n, V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the solution of* (26) and (27). *Conversely, let* $(U^n, V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the solution of the system of equations* (26) and (27), *then* $(U^n, V^n, d_1, d_2) \in \mathcal{H}_h^k \times \mathcal{H}_h^k \times \mathbb{R} \times \mathbb{R}$ *be the solution of the system* (51).

Proof Proceeding similar to proof of Theorem 4.1 in [38], the desired result can be obtained.

4 A priori error estimate for semi-discrete scheme

In this section, we establish *a priori* error estimate for the semi discrete scheme in the L^2 and H^1 norms. It is observed that the direct bound of the error $\|u(t) - u_h(t)\|_0 + \|v(t) - v_h(t)\|_0$ may not be straightforward. To achieve the goal, we introduce the Ritz projection operator $\mathcal{R}_h : H^1(\Omega) \rightarrow \mathcal{H}_h^k$ that is defined as

$$a_h(\mathcal{R}_h u, \omega) = a(u, \omega) \quad \forall \omega \in H^1(\Omega). \tag{54}$$

The well-posedness of the Ritz projection operator $\mathcal{R}_h$, directly follows from the coercivity and boundedness of the bilinear form $a_h(\cdot,\cdot)$ and the continuity of the function $a(u,\cdot)$ on $\mathcal{H}_h^k$. Employing the projection operator $\mathcal{R}_h$, we divide the errors $u(\cdot,t) - u_h(\cdot,t)$ and $v(\cdot,t) - v_h(\cdot,t)$ into two parts as

$$u(\cdot,t) - u_h(\cdot,t) = \underbrace{u(\cdot,t) - \mathcal{R}_h u(\cdot,t)}_{=:\rho_1} - \underbrace{(-\mathcal{R}_h u(\cdot,t) + u_h(\cdot,t))}_{=:\rho_2}, \tag{55}$$

$$v(\cdot,t) - v_h(\cdot,t) = \underbrace{v(\cdot,t) - \mathcal{R}_h v(\cdot,t)}_{=:\mu_1} - \underbrace{(-\mathcal{R}_h v(\cdot,t) + v_h(\cdot,t))}_{=:\mu_2}. \tag{56}$$

Using the approximation properties of $\mathcal{R}_h$, we bound the term ρ_1, μ_1. To bound the right hand side terms of (55) and (56), i.e., ρ_2, μ_2, we use the semi-discrete formulation (16) and (17) and the approximation properties of the projection operators on the polynomial space that will be discussed in forthcoming theorems. Next, we introduce the approximation properties of the polynomial projection operator (refer [45]).

Lemma 4.1 *Consider Assumption 2 holds on the discretized domain. Then, for all* $E \in \Omega_h$, *where* $0 < h \leq 1$, *and* $v \in H^s(E)$, *where* $1 \leq s \leq k+1$, *there exists a polynomial* $v_\pi \in \mathbb{P}_k(E)$ *such that*:

$$\|v - v_\pi\|_{0,E} + h_E \|\nabla v - \nabla v_\pi\|_{0,E} \leq C\, h_E^s\, |v|_{s,E}, \tag{57}$$

where, the positive generic constant C *depends on the mesh regularity parameter* γ, *order* k *of the polynomial space* $\mathbb{P}_k(E)$, *but is independent of the mesh size* h_E.

Let I_h^E be the nodal interpolation operator on the virtual element space $\mathcal{H}^k(E)$. For each element $E \in \Omega_h$, and for $v \in H^1(\Omega)$, there exists an element $I_h^E v \in \mathcal{H}^k(E)$ such that:

$$\mathrm{dof}_i(v) = \mathrm{dof}_i(I_h^E v) \quad 1 \le i \le N_E^{\mathrm{dof}},$$

where, N_E^{dof} denotes the total numbers of DoFs in $\mathcal{H}^k(E)$. The global interpolation operator I_h is defined such that it is reduced to I_h^E when restricted to an element E, i.e., $I_h|_E = I_h^E$. The approximation properties of the global interpolation operator is now presented below (see [40]).

Lemma 4.2 *Let Assumption 2 holds on the discretization of the computational domain* Ω. *Further, we assume that* $v \in H^s(\Omega)$. *Then, for* $1 \le s \le k+1$, *the following approximation property holds*

$$\|v - I_h v\|_{0,\Omega} + h\,\|\nabla v - \nabla I_h v\|_{0,\Omega} \le \; Ch^s |v|_{s,\Omega}, \tag{58}$$

where the generic constant C depends on mesh regularity parameter γ *but independent of mesh size h.*

Using the interpolation operator I_h, we can prove that the Ritz projection operator that approximates optimally .

Lemma 4.3 *Let* $u \in H^k(\Omega)$. *Then, there exists an unique functions* $\mathcal{R}_h u \in \mathcal{H}_h^k$ *such that*

$$\|u - \mathcal{R}_h u\|_{\alpha,\Omega} \le \; Ch^{\beta-\alpha} |u|_{\beta,\Omega}, \quad \alpha = 0, 1 \text{ and } \alpha \le \beta \le k+1. \tag{59}$$

For interested reader, we refer to [20, Lemma 3.1] for a detailed discussion. Now we prove optimal order convergence for the semi-discrete approximation (16) and (17), in L^2 norm and H^1 semi-norm.

Theorem 4.4 *Let* $(u(t), v(t)) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be the solution of the system* (8)–(11) *and let* $(u_h(t), v_h(t)) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the discrete solution of the problem* (16) *and* (17). *Further, assume that* $\|u\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, $\|v\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, $\|D_t u\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, $\|D_t v\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$, *and* $\|f_i(u, v)\|_{L^2(0,T;H^{k+1}(\Omega))} < \infty$ *for* $i = 1, 2$. *Then, for almost all* $t \in (0, T]$, *there exists a positive constant C which depends on the mesh regularity parameter* γ, *the order of the virtual element space k, the stability parameter of the discrete bilinear forms* $a_h(\cdot,\cdot)$ *and* $m_h(\cdot,\cdot)$, *but independent of the mesh size h such that the following bound holds*

$$\|u_h(t)-u(t)\|_{0,\Omega}+\|v_h(t)-v(t)\|_{0,\Omega}\leq C\Big(\|u_h(0)-u(0)\|_{0,\Omega}+\|v_h(0)-v(0)\|_{0,\Omega}\Big)$$
$$+Ch^{k+1}\Big(|u(0)|_{k+1,\Omega}+|v(0)|_{k+1,\Omega}+\|u\|_{L^2(0,T;H^{k+1}(\Omega))}+\|v\|_{L^2(0,T;H^{k+1}(\Omega))}$$
$$+\|D_tu\|_{L^2(0,T;H^{k+1}(\Omega))}+\|D_tv\|_{L^2(0,T;H^{k+1}(\Omega))}+\|f_1(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}$$
$$+\|f_2(u,v)\|_{L^2(0,T;H^{k+1}(\Omega))}\Big),$$

where the initial guess $u_h(0)$ and $v_h(0)$ *are chosen as* $u_h(0):=I_hu_0$ *and* $v_h(0):=I_hv_0$.

Proof Using the semi discrete scheme (16) and (17) and the definition of Ritz projection operator $\mathcal{R}_h$, we have

$$m_h(\rho_2,\varphi_h)+\Delta t\,\mathcal{A}_1\Big(g_1(\Pi_k^0u_h),g_2(\Pi_k^0v_h)\Big)\,a_h(\rho_2,\varphi_h)=\langle f_{1h}(u_h,v_h),\varphi_h\rangle$$
$$-\langle f_1(u,v),\varphi_h\rangle-m_h(D_t\mathcal{R}_hu(t),\varphi_h)+(D_tu(t),\varphi_h)+\Big(\mathcal{A}_1(g_1(u),g_2(v))\quad(60)$$
$$-\mathcal{A}_1(g_1(\Pi_k^0u_h),g_2(\Pi_k^0v_h))\Big)\,a(u(t),\varphi_h).$$

Using the approximation property of the L^2 projection operator Π_k^0 and Assumption 1, we have [46, Theorem 4.2]

$$|\langle f_{1h}(u_h,v_h),\varphi_h\rangle-\langle f_1(u,v),\varphi_h\rangle|\leq C(L_F)\Big(h^{k+1}\,|u|_{k+1,\Omega}+h^{k+1}\,|v|_{k+1,\Omega}$$
$$+h^{k+1}|f_1(u,v)|_{k+1,\Omega}+\|u-u_h\|_{0,\Omega}+\|v-v_h\|_{0,\Omega}\Big)\,\|\varphi_h\|_{0,\Omega}.\quad(61)$$

Moreover, since the nonlocal function $\mathcal{A}_1(\cdot,\cdot)$ satisfies Assumption 1, and using the approximation properties of the L^2 projection operator Π_k^0, we derive the estimation

$$|\mathcal{A}_1(g_1(u),g_2(v))-\mathcal{A}_1(g_1(\Pi_k^0u_h),g_2(\Pi_k^0v_h))|\leq C(L_A)\Big(h^{k+1}\,|u|_{k+1,\Omega}+h^{k+1}\,|v|_{k+1,\Omega}$$
$$+\|u-u_h\|_{0,\Omega}+\|v-v_h\|_{0,\Omega}\Big).\quad(62)$$

Using the polynomial consistency property of the bilinear form $m_h(\cdot,\cdot)$ and approximation properties of the L^2 projection operator and the Ritz projection operator, we derive [20]

$$|-m_h(D_t\mathcal{R}_hu(t),\varphi_h)+(D_tu(t),\varphi_h)|\leq C\,h^{k+1}\,|D_tu|_{k+1,\Omega}\|\varphi_h\|_{0,\Omega}.\quad(63)$$

Substituting $\varphi_h=\rho_2(t)$ in (60) and using the estimations (61)–(63), and the stability property of $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\beta_*\|\rho_2(t)\|^2_{0,\Omega} + C\, m_0\, \alpha_*\|\nabla\rho_2(t)\|^2_{0,\Omega} \le C\,\Big(h^{k+1}|u|_{k+1,\Omega} + h^{k+1}|v|_{k+1,\Omega}\\
&\quad + h^{k+1}|f_1(u,v)|_{k+1,\Omega} + \|u-u_h\|_{0,\Omega} + \|v-v_h\|_{0,\Omega}\Big)\,\|\rho_2(t)\|_{0,\Omega} + C\Big(h^{k+1}\,|u|_{k+1,\Omega}\\
&\quad + h^{k+1}\,|v|_{k+1,\Omega} + \|u-u_h\|_{0,\Omega} + \|v-v_h\|_{0,\Omega}\Big)\,\|\Delta u(t)\|_{0,\Omega}\,\|\rho_2(t)\|_{0,\Omega}\\
&\quad + C\,h^{k+1}\,|D_t u|_{k+1,\Omega}\,\|\rho_2(t)\|_{0,\Omega}.
\end{aligned}
\tag{64}
$$

By decomposing the error $u(t) - u_h(t)$ on the right hand side of (64) into $\rho_1(t)$ and $\rho_2(t)$, and $v(t) - v_h(t)$ into $\mu_1(t)$ and $\mu_2(t)$ and using Lemma 4.3, we derive

$$
\begin{aligned}
&\frac{1}{2}\,\beta_*\,\frac{d}{dt}\|\rho_2(t)\|^2_{0,\Omega} + C\,\alpha_*\,m_0\,\|\nabla\rho_2(t)\|^2_{0,\Omega} \le\; C\Big(\|\rho_2(t)\|_{0,\Omega} + \|\mu_2(t)\|_{0,\Omega}\\
&\quad + h^{k+1}\,|u|_{k+1,\Omega} + h^{k+1}|v|_{k+1,\Omega} + h^{k+1}\,|f_1(u,v)|_{k+1,\Omega} + h^{k+1}|D_t u|_{k+1,\Omega}\Big)\|\rho_2(t)\|_{0,\Omega}.
\end{aligned}
$$

Using Young's inequality and integrating both sides from 0 to t, we have

$$
\begin{aligned}
&\|\rho_2(t)\|^2_{0,\Omega} - \|\rho_2(0)\|^2_{0,\Omega} + C(\alpha_*,\beta_*,m_0)\int_0^t \|\nabla\rho_2(s)\|^2_{0,\Omega}\,\mathrm{ds} \le\; C(\beta_*)\Big(\int_0^t (\|\rho_2(s)\|^2_{0,\Omega}\\
&\quad + \|\mu_2(s)\|^2_{0,\Omega})\,\mathrm{ds}\Big) + C(\beta_*)h^{2k+2}\Big(\|u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\\
&\quad + \|f_1(u,v)\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|D_t u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\Big).
\end{aligned}
\tag{65}
$$

Using analogous arguments as (65), we obtain from (17)

$$
\begin{aligned}
&\|\mu_2(t)\|^2_{0,\Omega} - \|\mu_2(0)\|^2_{0,\Omega} + C(\alpha_*,\beta_*,m_0)\int_0^t \|\nabla\mu_2(s)\|^2_{0,\Omega}\,\mathrm{ds} \le\; C\Big(\int_0^t (\|\rho_2(s)\|^2_{0,\Omega}\\
&\quad + \|\mu_2(s)\|^2_{0,\Omega})\,\mathrm{ds}\Big) + C(\beta_*)h^{2k+2}\,\Big(\|u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\\
&\quad + \|f_2(u,v)\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|D_t v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\Big).
\end{aligned}
\tag{66}
$$

Upon adding (65) and (66), and neglecting the term $\int_0^t(\|\nabla\mu_2(s)\|^2_{0,\Omega} + \|\nabla\rho_2(s)\|^2_{0,\Omega})\,\mathrm{ds}$, we obtain

$$
\begin{aligned}
&\|\mu_2(t)\|^2_{0,\Omega} - \|\mu_2(0)\|^2_{0,\Omega} + \|\rho_2(t)\|^2_{0,\Omega} - \|\rho_2(0)\|^2_{0,\Omega} \le\; C\Big(\int_0^t (\|\rho_2(s)\|^2_{0,\Omega} + \|\mu_2(s)\|^2_{0,\Omega})\,\mathrm{ds}\Big)\\
&\quad + C\,h^{2k+2}\,\Big(\|u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|f_2(u,v)\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\\
&\quad + \|f_1(u,v)\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|D_t v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|D_t u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\Big).
\end{aligned}
$$

An application of Gronwall inequality yields

$$
\begin{aligned}
\|\mu_2(t)\|^2_{0,\Omega} + \|\rho_2(t)\|^2_{0,\Omega} &\le \|\mu_2(0)\|^2_{0,\Omega} + \|\rho_2(0)\|^2_{0,\Omega} + C\, h^{2k+2}\Big(\|u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} \\
&+ \|v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|f_2(u,v)\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|D_t v\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} \\
&+ \|f_1(u,v)\|^2_{L^1(0,t;\,H^{k+1}(\Omega))} + \|D_t u\|^2_{L^1(0,t;\,H^{k+1}(\Omega))}\Big).
\end{aligned}
$$

By using the definition of μ_2 and ρ_2 [(55) and (56)], the approximation property of the projection operator $\mathcal{R}_h$ in Lemma 4.3, we obtain:

$$
\begin{aligned}
\|u(t) - u_h(t)\|_{0,\Omega} + \|v(t) - v_h(t)\|_{0,\Omega} &\le C\Big(\|u(0) - u_h(0)\|_{0,\Omega} + \|v(0) - v_h(0)\|_{0,\Omega}\Big) \\
&+ C\, h^{k+1}\Big(|u(0)|_{k+1,\Omega} + |v(0)|_{k+1,\Omega} + \|u\|_{L^1(0,T;\,H^{k+1}(\Omega))} + \|v\|_{L^1(0,T;\,H^{k+1}(\Omega))} \\
&+ \|D_t u\|_{L^1(0,T;\,H^{k+1}(\Omega))} + \|D_t v\|_{L^1(0,T;\,H^{k+1}(\Omega))} + \|f_1(u,v)\|_{L^1(0,t;\,H^{k+1}(\Omega))} \\
&+ \|f_2(u,v)\|_{L^1(0,t;\,H^{k+1}(\Omega))}\Big).
\end{aligned}
$$

Next, we proceed to bound the error in H^1 semi-norm.

Theorem 4.5 *Let* $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ *be the solution of the system* (8)–(11) *and let* $(u_h(t), v_h(t)) \in \mathcal{H}^k_h \times \mathcal{H}^k_h$ *be the discrete solution of the problem* (16) *and* (17). *Then, under the assumptions of Theorem* 4.4 *and for almost all* $t \in (0,T]$, *we have*

$$
\begin{aligned}
\|\nabla u_h(t) - \nabla u(t)\|_{0,\Omega} + \|\nabla v_h(t) - \nabla v(t)\|_{0,\Omega} &\le C\Big(\|\nabla u_h(0) - \nabla u(0)\|_{0,\Omega} \\
&+ \|\nabla v_h(0) - \nabla v(0)\|_{0,\Omega}\Big) + Ch^k\Big(|u(0)|_{k+1,\Omega} + |v(0)|_{k+1,\Omega} + \|u\|_{L^2(0,T;\,H^{k+1}(\Omega))} \\
&+ \|v\|_{L^2(0,T;\,H^{k+1}(\Omega))} + \|D_t u\|_{L^2(0,T;\,H^{k+1}(\Omega))} + \|D_t v\|_{L^2(0,T;\,H^{k+1}(\Omega))} \\
&+ \|f_1(u,v)\|_{L^2(0,T;\,H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^2(0,T;\,H^{k+1}(\Omega))}\Big),
\end{aligned}
\tag{67}
$$

where C is the positive constant independent of h, but depends on the mesh regularity parameter, stability parameter of the bilinear forms $a_h(\cdot,\cdot)$ *and* $m_h(\cdot,\cdot)$, *order of the polynomial space* $\mathbb{P}_k(E)$, *and regularity of the Sobolev space.*

Proof Recollecting (60)–(63), and substituting $\varphi_h = D_t\rho_2(t)$ in (60) and using the stability property of $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, we obtain

$$
\begin{aligned}
\frac{1}{2}\beta_*\|D_t\rho_2(t)\|^2_{0,\Omega} + \frac{1}{2}C(m_0,\alpha_*)\frac{d}{dt}\|\nabla\rho_2(t)\|^2_{0,\Omega} &\le C\,\Big(h^{k+1}|u(t)|_{k+1,\Omega} + h^{k+1}|v(t)|_{k+1,\Omega} \\
&+ h^{k+1}|f_1(u,v)|_{k+1,\Omega} + \|u - u_h\|_{0,\Omega} + \|v - v_h\|_{0,\Omega}\Big)\, \|D_t\rho_2(t)\|_{0,\Omega} + C\Big(h^{k+1}\, |u(t)|_{k+1,\Omega} \\
&+ h^{k+1}\, |v(t)|_{k+1,\Omega} + \|u - u_h\|_{0,\Omega} + \|v - v_h\|_{0,\Omega}\Big)\, \|\Delta u(t)\|_{0,\Omega}\, \|D_t\rho_2(t)\|_{0,\Omega} \\
&+ C\, h^{k+1}\, |D_t u(t)|_{k+1,\Omega}\, \|D_t\rho_2(t)\|_{0,\Omega}.
\end{aligned}
$$

By using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{4}\beta_*\|D_t\rho_2(t)\|^2_{0,\Omega} + \frac{1}{2}C(m_0,\alpha_*)\frac{d}{dt}\|\nabla\rho_2(t)\|^2_{0,\Omega}\\
&\quad \le C\left(h^{2(k+1)}|u(t)|^2_{k+1,\Omega} + h^{2(k+1)}|v(t)|^2_{k+1,\Omega}\right.\\
&\qquad \left. + h^{2(k+1)}|f_1(u,v)|^2_{k+1,\Omega} + h^{2(k+1)}\ |D_tu(t)|^2_{k+1,\Omega} + \|u-u_h\|^2_{0,\Omega} + \|v-v_h\|^2_{0,\Omega}\right).
\end{aligned} \tag{68}$$

Applying analogous arguments as (68) to (17), we derive

$$\begin{aligned}
&\frac{1}{4}\beta_*\|D_t\mu_2(t)\|^2_{0,\Omega} + \frac{1}{2}C(m_0,\alpha_*)\frac{d}{dt}\|\nabla\mu_2(t)\|^2_{0,\Omega} \le C\left(h^{2(k+1)}|u(t)|^2_{k+1,\Omega}\right.\\
&\quad + h^{2(k+1)}|v(t)|^2_{k+1,\Omega} + h^{2(k+1)}|f_2(u,v)|^2_{k+1,\Omega} + h^{2(k+1)}\ |D_tv(t)|^2_{k+1,\Omega}\\
&\quad \left. + \|u-u_h\|^2_{0,\Omega} + \|v-v_h\|^2_{0,\Omega}\right).
\end{aligned} \tag{69}$$

Adding (68) and (69), and neglecting the positive term $\frac{1}{4}\beta_*\left(\|D_t\rho_2(t)\|^2_{0,\Omega} + D_t\mu_2(t)\|^2_{0,\Omega}\right)$ and using Theorem 4.4 (for bounding the errors in L^2 norm, i.e., $\|u-u_h\|^2_{0,\Omega} + \|v-v_h\|^2_{0,\Omega}$), we obtain,

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\rho_2(t)\|^2_{0,\Omega} + \frac{d}{dt}\|\nabla\mu_2(t)\|^2_{0,\Omega} \le C\Big(\|u_h(0)-u(0)\|^2_{0,\Omega} + \|v_h(0)-v(0)\|^2_{0,\Omega}\Big)\\
&\quad + Ch^{2(k+1)}\Big(|u(0)|^2_{k+1,\Omega} + |v(0)|^2_{k+1,\Omega} + \|u\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \|v\|^2_{L^2(0,T;H^{k+1}(\Omega))}\\
&\quad + \|D_tu\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \|D_tv\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \|f_1(u,v)\|^2_{L^2(0,T;H^{k+1}(\Omega))}\\
&\quad + \|f_2(u,v)\|^2_{L^2(0,T;H^{k+1}(\Omega))}\Big).
\end{aligned}$$

Integrating the above equation on both sides from 0 to t, we get

$$\begin{aligned}
&\|\nabla\rho_2(t)\|^2_{0,\Omega} + \|\nabla\mu_2(t)\|^2_{0,\Omega} \le C\Big(\|\nabla\rho_2(0)\|^2_{0,\Omega} + \|\nabla\mu_2(0)\|^2_{0,\Omega} + \|u_h(0)-u(0)\|^2_{0,\Omega}\\
&\quad + \|v_h(0)-v(0)\|^2_{0,\Omega}\Big) + Ch^{2(k+1)}\Big(|u(0)|^2_{k+1,\Omega} + |v(0)|^2_{k+1,\Omega} + \|u\|^2_{L^2(0,T;H^{k+1}(\Omega))}\\
&\quad + \|v\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \|D_tu\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \|D_tv\|^2_{L^2(0,T;H^{k+1}(\Omega))}\\
&\quad + \|f_1(u,v)\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \|f_2(u,v)\|^2_{L^2(0,T;H^{k+1}(\Omega))}\Big).
\end{aligned} \tag{70}$$

Using the definition of ρ_2, and μ_2 [(55) and (56)], approximation property of $\mathcal{R}_h$ (Lemma 4.3), we obtain the desired estimate (67).

5 Error estimation for fully discrete scheme

In this section, we would like to derive *a priori* error estimates assuring optimal order of convergence of the fully discrete approximation (26) and (27) in the L^2 norm and H^1 semi-norm. In what follows, we split the errors for fully discrete approximation as follows

$$u(t_n) - U^n = \underbrace{u(t_n) - \mathcal{R}_h u(t_n)}_{=:\rho_1^n} - \underbrace{(-\mathcal{R}_h u(t_n) + U^n)}_{=:\rho_2^n},$$
$$v(t_n) - V^n = \underbrace{v(t_n) - \mathcal{R}_h v(t_n)}_{=:\mu_1^n} - \underbrace{(-\mathcal{R}_h v(t_n) + V^n)}_{=:\mu_2^n},$$

and for a function $z \in \mathcal{H}_h^k$, we define $\partial z^n := \frac{z(t_n)-z(t_{n-1})}{\Delta t}$.

Theorem 5.1 *Let $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of (8) and (9) and let $(U^n, V^n) \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ be the solution of (26)–(28) at time $t_n \in [0, T]$. Further, consider the initial guess for the independent variables u, v as $U^0 = I_h(u_0)$ and $V^0 = I_h(v_0)$. Then, there exists a positive constant C that is independent of the mesh diameter h and the time increment Δt, but depends on the Sobolev regularity, the mesh regularity parameter γ (Assumption 2), the final time step T and the stability parameters of the discrete bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, such that the following estimation holds*

$$\begin{aligned}
&\|U^n - u(t_n)\|_{0,\Omega} + \|V^n - v(t_n)\|_{0,\Omega} \le C\Big(\|U^0 - u(0)\|_{0,\Omega} + \|V^0 - v(0)\|_{0,\Omega}\Big)\\
&\quad + C\, h^{k+1}\Big(|u(0)|_{k+1,\Omega} + |v(0)|_{k+1,\Omega} + \|u\|_{L^\infty(0,t_n,H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\\
&\quad + \|D_t u\|_{L^1(0,t_n,H^{k+1}(\Omega))} + \|D_t v\|_{L^1(0,t_n,H^{k+1}(\Omega))} + \|f_1(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\\
&\quad + \|f_2(u,v)\|_{L^\infty(0,t_n,H^{k+1}(\Omega))}\Big) + C\,\Delta t\,\Big(\|D_{tt}u\|_{L^1(0,t_n,L^2(\Omega))} + \|D_{tt}v\|_{L^1(0,t_n,L^2(\Omega))}\Big).
\end{aligned}$$

Proof To prove the fully discrete estimation, we employ (26), the definition of the Ritz projection operator, the continuous weak formulation (8) and deduce that

$$\begin{aligned}
&m_h\Big(\frac{\rho_2^n - \rho_2^{n-1}}{\Delta t}, \varphi_h\Big) + \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(\rho_2^n, \varphi_h) = \langle f_{1h}(U^n, V^n), \varphi_h\rangle\\
&\quad - \langle f_1(u(t_n), v(t_n), \varphi_h\rangle - m_h\Big(\frac{\mathcal{R}_h u(t_n) - \mathcal{R}_h u(t_{n-1})}{\Delta t}, \varphi_h\Big) + (D_t u(t_n), \varphi_h)\\
&\quad + \Big(\mathcal{A}_1\Big(g_1(u(t_n)), g_2(v(t_n))\Big) - \mathcal{A}_1\Big(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n)\Big)\Big)\, a(u(t_n), \varphi_h).
\end{aligned} \tag{71}$$

Upon choosing $\varphi_h = \rho_2^n$ and $\varphi_h = \mu_2^n$ in (71) and proceeding same as (65), and (66), we bound $\|\rho_2^n\|_{0,\Omega}$, and $\|\mu_2^n\|_{0,\Omega}$ as

$$\|\rho_2^n\|_{0,\Omega} \le C\|\rho_2^{n-1}\|_{0,\Omega} + C\Delta t\Big(\|\rho_2^n\|_{0,\Omega} + \|\mu_2^n\|_{0,\Omega}\Big) + C\Delta t\, h^{k+1}\Big(|u(t_n)|_{k+1,\Omega} + |v(t_n)|_{k+1,\Omega} + |f_1(u(t_n),v(t_n))|_{k+1,\Omega}\Big) + C\,\Big(\eta_1^n + \eta_2^n\Big), \tag{72}$$

and

$$\|\mu_2^n\|_{0,\Omega} \le C\|\mu_2^{n-1}\|_{0,\Omega} + C\Delta t\Big(\|\rho_2^n\|_{0,\Omega} + \|\mu_2^n\|_{0,\Omega}\Big) + C\Delta t\, h^{k+1}\Big(|u(t_n)|_{k+1,\Omega} + |v(t_n)|_{k+1,\Omega} + |f_2(u(t_n),v(t_n))|_{k+1,\Omega}\Big) + C\,\Big(\xi_1^n + \xi_2^n\Big). \tag{73}$$

Adding (72) and (73) and proceeding same as in [46, Theorem 4.4], we obtain

$$\begin{aligned}\|\rho_2^n\|_{0,\Omega} + \|\mu_2^n\|_{0,\Omega} \le\ & C(\|\rho_2^0\|_{0,\Omega} + \|\mu_2^0\|_{0,\Omega}) + C\,h^{k+1}\Big(\|f_1(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\\ &+ \|u\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,t_n;H^{k+1}(\Omega))} + \|D_t v\|_{L^1(0,t_n;H^{k+1}(\Omega))}\\ &+ \|f_2(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\Big) + C\,\Delta t\,\Big(\|D_{tt}u\|_{L^1(0,t_n;L^2(\Omega))} + \|D_{tt}v\|_{L^1(0,t_n;L^2(\Omega))}\Big).\end{aligned}$$

By using the estimations of $\|\rho_1^n\|_{0,\Omega}$ and $\|\mu_1^n\|_{0,\Omega}$ from Lemma 4.3, we obtain the desired result.

Theorem 5.2 *Let $(u,v) \in H_0^1(\Omega)\times H_0^1(\Omega)$ be the solution of the weak formulation* (8) *and* (9) *and $(U^n,V^n) \in \mathcal{H}_h^k(\Omega)\times\mathcal{H}_h^k(\Omega)$ be the solution of the discrete scheme* (26)–(28). *Then, under the assumption of Theorem* 5.1, *the following error estimation holds*

$$\begin{aligned}\|\nabla(U^n - u(t_n))\|_{0,\Omega} + \|\nabla(V^n - v(t_n))\|_{0,\Omega} \le\ & C\,\Big(\|\nabla U^0 - \nabla u(0)\|_{0,\Omega} + \|\nabla V^0 - \nabla v(0)\|_{0,\Omega}\Big)\\ &+ C\,h^{k+1}\,\Big(|u(0)|_{k+1,\Omega} + |v(0)|_{k+1,\Omega} + \|u\|_{L^\infty(0,t_n,H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\\ &+ \|f_1(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|f_2(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))} + \|D_t u\|_{L^2(0,t_n;H^{k+1}(\Omega))}\\ &+ \|D_t v\|_{L^2(0,t_n;H^{k+1}(\Omega))}\Big) + C\Delta t\Big(\|D_{tt}u\|_{L^2(0,t_n;L^2(\Omega))} + \|D_{tt}v\|_{L^2(0,t_n;L^2(\Omega))}\Big).\end{aligned}$$

Proof Using the fully discrete scheme (26) and (27), and the definition of the Ritz projection operator, we write an equation consists of ρ_2^n as follows

$$\begin{aligned}m_h\Big(\frac{\rho_2^n - \rho_2^{n-1}}{\Delta t},\varphi_h\Big) + \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\, a_h(\rho_2^n,\varphi_h) &= \langle f_{1h}(U^n,V^n),\varphi_h\rangle\\ - \langle f_1(u(t_n),v(t_n)),\varphi_h\rangle - m_h(\partial\mathcal{R}_h u(t_n),\varphi_h) &+ (D_t u(t_n),\varphi_h)\\ + \Big(\mathcal{A}_1(g_1(u(t_n)), g_2(v(t_n))) - \mathcal{A}_1(g_1(\Pi_k^0 U^n), g_2(\Pi_k^0 V^n))\Big)\, a(u(t_n),\varphi_h).\end{aligned} \tag{74}$$

Upon substituting $\varphi_h = \partial\rho_2^n$, and $\varphi_h = \partial\mu_2^n$ in (74) and borrowing arguments from [20, 38], we deduce that

$$\|\nabla \rho_2^n\|_{0,\Omega}^2 \le \|\nabla \rho_2^{n-1}\|_{0,\Omega}^2 + C\, h^{2k+2}\,\Delta t\Big(|u(t_n)|_{k+1,\Omega}^2 + |f_1(u(t_n), v(t_n))|_{k+1,\Omega}^2 + |v(t_n)|_{k+1,\Omega}^2 \Big) + C\frac{1}{\Delta t}\Big(\|\eta_1^n\|_{0,\Omega}^2 + \|\eta_2^n\|_{0,\Omega}^2 \Big) + \Delta t\Big(\|\rho_2^n\|_{0,\Omega}^2 + \|\mu_2^n\|_{0,\Omega}^2 \Big), \tag{75}$$

and

$$\|\nabla \mu_2^n\|_{0,\Omega}^2 \le \|\nabla \mu_2^{n-1}\|_{0,\Omega}^2 + C\, h^{2k+2}\Delta t\Big(|u(t_n)|_{k+1,\Omega}^2 + |f_2(u(t_n), v(t_n))|_{k+1,\Omega}^2 + |v(t_n)|_{k+1,\Omega}^2 \Big) + C\frac{1}{\Delta t}\Big(\|\xi_1^n\|_{0,\Omega}^2 + \|\xi_2^n\|_{0,\Omega}^2 \Big) + \Delta t\Big(\|\rho_2^n\|_{0,\Omega}^2 + \|\mu_2^n\|_{0,\Omega}^2 \Big). \tag{76}$$

Upon summing (75) and (76) and letting the sum for $\nu = 1, \dots, n$, and using the estimation of $\sum_{\nu=1}^{n} \Big(\|\rho_2^\nu\|_0 + \|\mu_2^\nu\|_0 \Big)$ from Theorem 5.1, we obtain the desired result.

6 Error estimation for linearized scheme

In this section, we estimate the rate of convergence in the space and the time variables for the approximation $(\widetilde{U}^n, \widetilde{V}^n)$ satisfying (29)–(30). Employing the Ritz projection operator $\mathcal{R}_h$ (see (54)), we split the terms $u(t_n) - \widetilde{U}^n$ and $v(t_n) - \widetilde{V}^n$ as follows

$$u(t_n) - \widetilde{U}^n := \rho_1^n + \widetilde{\rho_2}^n; \quad v(t_n) - \widetilde{V}^n := \mu_1^n + \widetilde{\mu_2}^n.$$

Theorem 6.1 *Let* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *be the solution of* (8)–(11) *and* $\{(\widetilde{U}^n, \widetilde{V}^n)\}_n \in \mathcal{H}_h^k \times \mathcal{H}_h^k$ *be the sequence of solutions of* (26)–(28) *for different time steps* $t_1, t_2, \dots, t_n \in [0, T]$. *Further, assume that the exact solution* $(u,\ v)$, *and the force function* $f_i(u, v)$, $i \in \{1, 2\}$ *satisfy the regularity assumption, i.e.,* $\|u\|_{L^\infty(0,t_n; H^{k+1}(\Omega))} < \infty$, $\|v\|_{L^\infty(0,t_n; H^{k+1}(\Omega))} < \infty$, $\|D_t u\|_{L^1(0,t_n; H^{k+1}(\Omega))} < \infty$, $\|D_t v\|_{L^1(0,t_n; H^{k+1}(\Omega))} < \infty$, $\|D_{tt} u\|_{L^1(0,t_n; H^{k+1}(\Omega))} < \infty$, $\|D_{tt} v\|_{L^1(0,t_n; H^{k+1}(\Omega))} < \infty$, $\|f_i(u, v)\|_{L^1(0,t_n; H^{k+1}(\Omega))} < \infty$. *Then the following error estimation holds*

$$\begin{aligned} \|\widetilde{U}^n - u(t_n)\|_{0,\Omega} + \|\widetilde{V}^n - v(t_n)\|_{0,\Omega} &\le C\Big(\|U^0 - u(0)\|_{0,\Omega} + \|V^0 - v(0)\|_{0,\Omega} \Big) \\ &+ C\, h^{k+1}\Big(\|u\|_{L^\infty(0,t_n; H^{k+1}(\Omega))} + \|v\|_{L^\infty(0,t_n; H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,t_n; H^{k+1}(\Omega))} \\ &+ \|D_t v\|_{L^1(0,t_n; H^{k+1}(\Omega))} + \|f_1(u, v)\|_{L^\infty(0,t_n; H^{k+1}(\Omega))} + \|f_2(u, v)\|_{L^\infty(0,t_n; H^{k+1}(\Omega))} \Big) \\ &+ C\, \Delta t\Big(\|D_{tt} u\|_{L^1(0,t_n; H^{k+1}(\Omega))} + \|D_t u\|_{L^1(0,t_n; H^{k+1}(\Omega))} + \|D_{tt} v\|_{L^1(0,t_n; H^{k+1}(\Omega))} \\ &+ \|D_t v\|_{L^1(0,t_n; H^{k+1}(\Omega))} \Big). \end{aligned} \tag{77}$$

The positive generic constant C depends on mesh regularity γ, stability parameters of the discrete bilinear forms $a_h(\cdot,\cdot)$ and $m_h(\cdot,\cdot)$, *but is independent of the mesh parameter h and time step Δt.*

Proof The estimations of ρ_1^n and μ_1^n are known from the approximation properties of $\mathcal{R}_h$. In order to estimate $\widetilde{\rho}_2^n$ and $\widetilde{\mu}_2^n$, we proceed as follows. By considering (29), we obtain

$$\begin{aligned} m_h\Big(\frac{\widetilde{\rho}_2^n-\widetilde{\rho}_2^{n-1}}{\Delta t},\varphi_h\Big)&+\mathcal{A}_1\Big(g_1(\Pi_k^0\widetilde{U}^{n-1}),g_2(\Pi_k^0\widetilde{V}^{n-1})\Big)\,a_h(\widetilde{\rho}_2^n,\varphi_h)\\ &=\langle f_{1h}(\widetilde{U}^{n-1},\widetilde{V}^{n-1}),\varphi_h\rangle\\ &\quad-\langle f_1(u(t_n),v(t_n)),\varphi_h\rangle-m_h\Big(\frac{\mathcal{R}_hu(t_n)-\mathcal{R}_hu(t_{n-1})}{\Delta t},\varphi_h\Big)+(D_tu(t_n),\varphi_h)\\ &\quad+\Big(\mathcal{A}_1\Big(g_1(u(t_n)),g_2(v(t_n))\Big)-\mathcal{A}_1\Big(g_1(\Pi_k^0\widetilde{U}^{n-1}),g_2(\Pi_k^0\widetilde{V}^{n-1})\Big)\Big)\,a(u(t_n),\varphi_h). \end{aligned} \tag{78}$$

The load term in the right hand side can be split as follows

$$\begin{aligned} &|\langle f_{1h}(\widetilde{U}^{n-1},\widetilde{V}^{n-1}),\varphi_h\rangle-\langle f_1(u(t_n),v(t_n)),\varphi_h\rangle|\\ &\quad\le|\langle f_1(\Pi_k^0\widetilde{U}^{n-1},\Pi_k^0\widetilde{V}^{n-1}),\Pi_k^0\varphi_h\rangle-\langle f_1(\Pi_k^0u(t_n),\Pi_k^0v(t_n)),\Pi_k^0\varphi_h\rangle|\\ &\qquad+|\langle f_1(\Pi_k^0u(t_n),\Pi_k^0v(t_n)),\Pi_k^0\varphi_h\rangle-\langle f_1(u(t_n),v(t_n)),\Pi_k^0\varphi_h\rangle|\\ &\qquad+|\langle f_1(u(t_n),v(t_n)),\Pi_k^0\varphi_h\rangle-\langle f_1(u(t_n),v(t_n)),\varphi_h\rangle|. \end{aligned} \tag{79}$$

Using Assumption 1, the approximation property and boundedness of the L^2 projection operator Π_k^0, we derive

$$\begin{aligned} |\langle f_{1h}(\widetilde{U}^{n-1},\widetilde{V}^{n-1}),\varphi_h\rangle-\langle f_1(u(t_n),v(t_n)),\varphi_h\rangle|&\le C\Big(\|\widetilde{U}^{n-1}-u(t_n)\|_{0,\Omega}\\ +\|\widetilde{V}^{n-1}-v(t_n)\|_{0,\Omega}\Big)\,\|\varphi_h\|_{0,\Omega}+C\,h^{k+1}\,&\Big(|u(t_n)|_{k+1,\Omega}+|v(t_n)|_{k+1,\Omega}\\ +|f_1(u(t_n),v(t_n))|_{k+1,\Omega}\Big)\,\|\varphi_h\|_{0,\Omega}.& \end{aligned} \tag{80}$$

Using the analogous techniques as [46, Theorem 4.5], and proceeding same as Theorem 5.1, we derive

$$\begin{aligned} \|\widetilde{\rho}_2^n\|_{0,\Omega}+\|\widetilde{\mu}_2^n\|_{0,\Omega}&\le C\,\Big(\|U^0-u_0\|_{0,\Omega}+\|V^0-v_0\|_{0,\Omega}\Big)+C\,h^{k+1}\,\Big(\|u\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\\ &+\|v\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}+\|f_1(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}+\|f_2(u,v)\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\\ &+\|D_tu\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}+\|D_tv\|_{L^\infty(0,t_n;H^{k+1}(\Omega))}\Big)+C\,\Delta t\Big(\|D_{tt}u\|_{L^1(0,t_n;L^2(\Omega))}\\ &+\|D_{tt}v\|_{L^1(0,t_n;L^2(\Omega))}+\|D_tu\|_{L^1(0,t_n;L^2(\Omega))}+\|D_tv\|_{L^1(0,t_n;L^2(\Omega))}\Big). \end{aligned} \tag{81}$$

Together with (81) and an application of the estimations $\|\rho_1^n\|_{0,\Omega}$ and $\|\mu_1^n\|_{0,\Omega}$ (using Lemma 4.3), we obtain the desired result. □

7 Numerical experiments

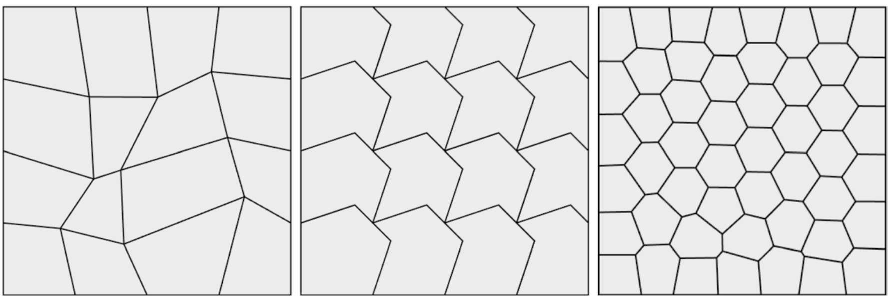
In this section, we study the convergence and the accuracy of the virtual element method by solving a nonlocal parabolic problem for a manufactured solution. We consider a square domain, $\Omega = [0,1] \times [0,1]$. The computational domain is discretized with different type of elements, viz., distorted square, non-convex mesh and smoothed Voronoi. A few representative meshes are shown in Fig. 1. In this study, for spatial discretization, we have considered the virtual element space of orders, $k = 1$, 2 and 3. For temporal discretization, we have employed the backward Euler time integration scheme. For convergence study, the errors are computed at the final time T in the L^2 and the H^1 norms. Since the discrete solutions are implicitly defined on the virtual space, the errors are computed using the two projection operators as follows:

$$L^2\text{-norm error: } \mathcal{E}_{h,0} := \sqrt{\sum_{E \in \Omega_h} \|u(T) - \Pi^0_{k,E} U^{N_T}\|^2_{0,E}}.$$

$$H^1\text{-norm error: } \mathcal{E}_{h,1} := \sqrt{\sum_{E \in \Omega_h} \|\nabla u(T) - \nabla \Pi^\nabla_{k,E} U^{N_T}\|^2_{0,E}}.$$

Consider the model problem (1)–(5), where the nonlocal coefficients are defined as:

$$\mathcal{A}_1\Big(g_1(u), g_2(v)\Big) := 3 + \cos(g_1(u)) + \sin(g_2(v))$$
$$\mathcal{A}_2\Big(g_1(u), g_2(v)\Big) := 5 - \cos(g_1(u)) + \sin(g_2(u)).$$



(a) Distorted Square **(b)** Non-convex **(c)** Smoothed Voronoi

Fig. 1 A schematic representation of different discretizations employed in this study

The force functions (f_1, f_2) are computed by imposing the following manufactured solutions:

$$u = (x - x^2)\,(y - y^2)\,e^{-t}$$
$$v = 2\,(x - x^2)\,(y - y^2)\,e^{2t}$$

as the exact solutions of (1) and (2) and $g_1(u) = \int_\Omega u \,\mathrm{d}\Omega$, $g_2(v) = \int_\Omega v \,\mathrm{d}\Omega$. To reduce the computational cost, two additional variables are augmented to the nonlinear system and the resulting nonlinear system is solved using the Newton's method with a user specified tolerance as $\mathcal{O}(10^{-10})$. This ensures that the sparsity of the Jacobian is

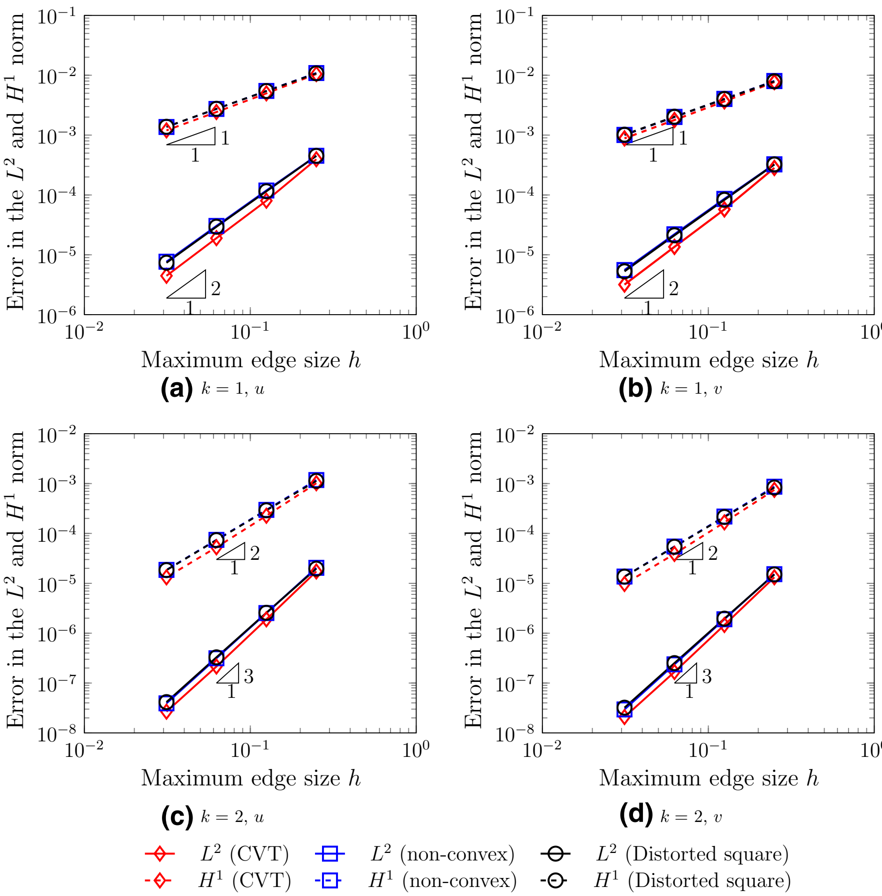


Fig. 2 Convergence of the errors in the L^2 norm and H^1 norm for $k = 1$ and 2 and for the variables, u and v

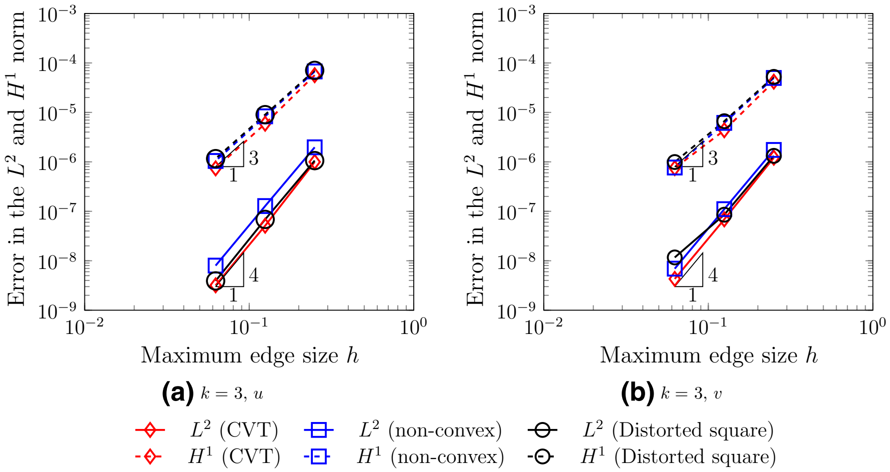


Fig. 3 Convergence of the errors in the L^2 norm and H^1 norm for $k = 3$ and for the variables, u and v

retained. The nonlinear loop takes between two to five iterations for the convergence of the numerical solution. The convergence of the error in the L^2 and H^1 norms for the independent variables, u and v are shown in Figs. 2 and 3 for $k = 1, 2$ and $k = 3$, respectively. It is seen that the numerical scheme converges at an optimal order in the respective norms. In Fig. 4, the convergence behaviour of the numerical solution obtained from the linearized scheme (29) and (30) for the virtual element space of orders $k = 1, 2$ is shown. It is observed that the numerical solution converges optimally to the analytical solution as predicted in Theorem 6.1.

Now, we study the convergence behavior in the temporal variable t. This is done by setting the mesh parameter $h = 1/80$ for all the considered discretization types. The time increment is chosen as $\Delta t = 1/4, 1/8, 1/16, 1/32$. The errors are computed at the end of the each time step t_n for $n = 1, \ldots, N_T$ and added to obtain the cumulative errors up to the final time T and is given by:

$$e_{0,T,h} := \left(\Delta t \sum_{n=1}^{N_T} \left(\sum_{E \in \Omega_h} \| u(t_n) - \Pi^0_{k,E} U^n \|^2_{0,E} \right) \right)^{1/2}. \tag{82}$$

In this case, we only report the results for the lowest order virtual element space, i.e., $k = 1$. Figure 5 shows the convergence of the error in the L^2 norm for both the independent variables. It can be inferred that the numerical scheme yields optimal convergence rate as predicted in Theorem 5.1. Further, it is noted that for higher order virtual element space, the numerical scheme converges at an optimal rate.

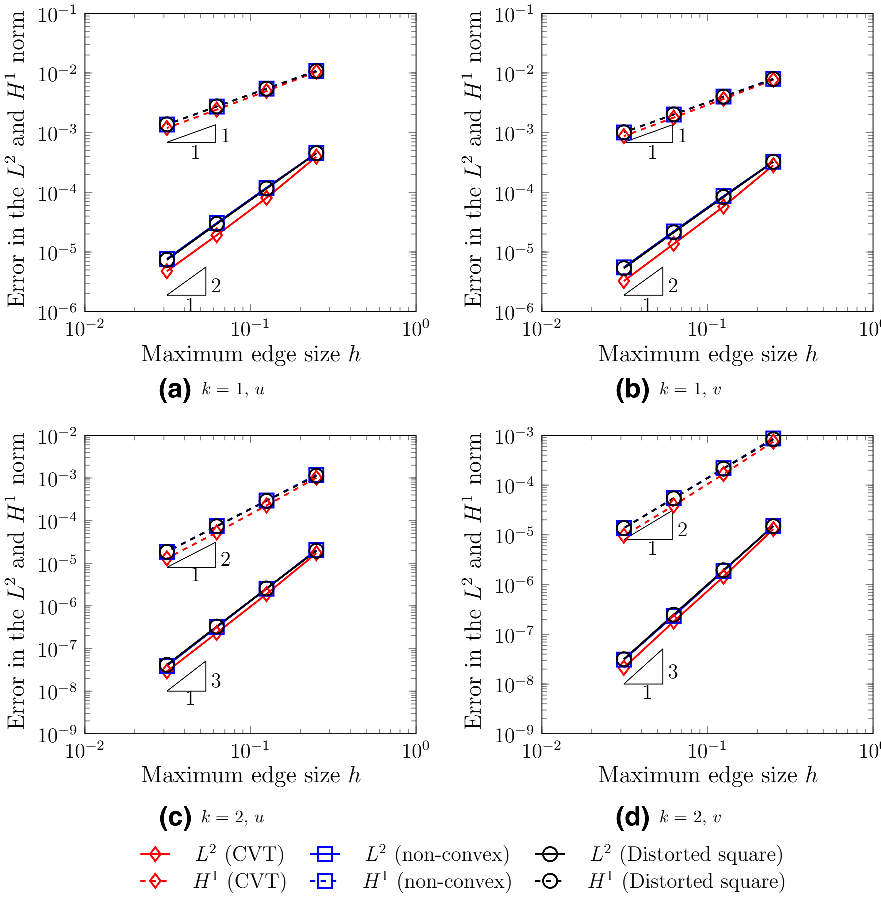


Fig. 4 Convergence of the errors in the L^2 norm and H^1 norm for $k = 1$ and 2 and for the variables, u and v for the linearized scheme

8 Conclusions

In this work, we have employed the virtual element method to solve the coupled nonlocal parabolic equation. The presence of the nonlocal diffusive coefficients reduces the sparsity of the Jacobian of the nonlinear system. To alleviate this problem, we have extended Gudi's approach within the context of the virtual element method. Further, a linearized scheme is proposed which can be solved using a linear solver. Theoretical estimates are derived and are numerically supported by benchmark examples. It is noted that the nonlocal parabolic problem can also be

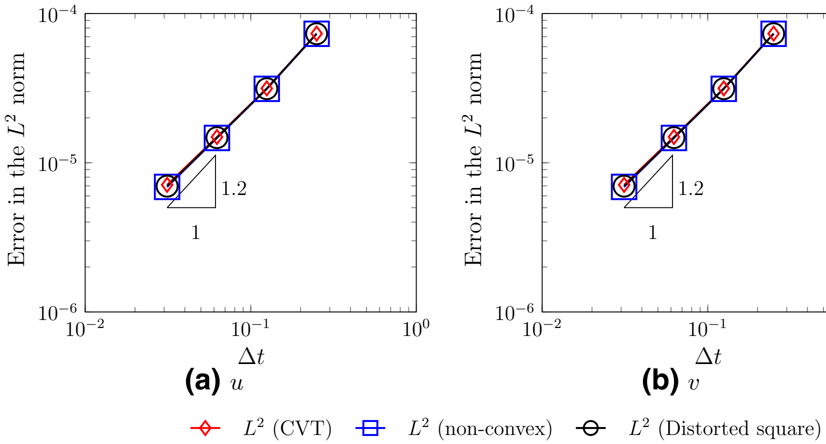


Fig. 5 Convergence of the error in the L^2 norm $k = 1$ and $h = 1/80$ and for the variables, u and v

approximated by a mixed virtual element method approach, which could be a topic for future communication.

Acknowledgements Dibyendu Adak was partially supported by CONICYT-Chile through FONDECYT Postdoctorado project 3200242, Departamento de Matemática, Universidad del Bío-Bío, Chile and Institute Postdoctoral fellowship at Department of Mechanical Engineering, Indian Institute of Technology-Madras.

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