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Full discretization of time dependent convection–diffusion–reaction equation coupled with the Darcy system

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Abstract

In this article, we study the time dependent convection–diffusion–reaction equation coupled with the Darcy equation. We propose and analyze two numerical schemes based on finite element methods for the discretization in space and the implicit Euler method for the discretization in time. An optimal a priori error estimate is then derived for each numerical scheme. Finally, we present some numerical experiments that confirm the theoretical accuracy of the discretization.

Keywords Darcy's equations · Convection–diffusion–reaction equation · Finite element method · A priori error estimates

Mathematics Subject Classification 74S05 · 65M60 · 35K05 · 65N15

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1 Introduction

Let Ω be a connected bounded open set in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$, and let $[0, T]$ be an interval of $\mathbb{R}$. In this work, we study the concentration distribution of a fluid in a porous medium modelled by a time dependent convection–diffusion–reaction equation coupled with Darcy's law. The system of equations is

$$
\text{(P)} \begin{cases}
\nu(C(\mathbf{x},t))\mathbf{u}(\mathbf{x},t) + \nabla p(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t,C(\mathbf{x},t)) & \text{in } \Omega\times]0,T[,\\
(\operatorname{div}\mathbf{u})(\mathbf{x},t) = 0 & \text{in } \Omega\times]0,T[,\\
\dfrac{\partial C}{\partial t}(\mathbf{x},t) - \alpha\Delta C(\mathbf{x},t) + (\mathbf{u}(\mathbf{x},t)\cdot\nabla C)(\mathbf{x},t) + r_0 C(\mathbf{x},t) = g(\mathbf{x},t) & \text{in } \Omega\times]0,T[,\\
(\mathbf{u}\cdot\mathbf{n})(\mathbf{x},t) = 0 & \text{on } \Gamma\times[0,T],\\
C(\mathbf{x},t) = 0 & \text{on } \Gamma\times[0,T],\\
C(\mathbf{x},0) = 0 & \text{in } \Omega,
\end{cases}
$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. The unknowns are the velocity $\mathbf{u}$, the pressure p and the concentration C of the fluid. The function $\mathbf{f}$ represents a force density that depends on the concentration C and the function g represents an external concentration source. The viscosity ν also depends on the concentration C but the diffusion coefficient α and the parameter r_0 are positive constants. To simplify, a homogeneous Dirichlet boundary condition is prescribed on the concentration C, but the present analysis easily extends to a non homogeneous boundary condition.

The existence of a weak solution of (P) is established in [8,12]. As far as its numerical approximation is concerned, problem (P) was treated for example in [16,17]. In these works, the authors used the semi-implicit Euler method for the time discretization and the Raviart–Thomas $H(\text{div}, \Omega)$ finite element method for the space discretization of the velocity/pressure unknowns. They established an a priori error estimate that is valid in 2-dimensions and place themselves in a somewhat restrictive framework, where Ω is a square, discretized by square cells. Moreover, in order to derive the estimate, the authors imposed multiple conditions on the space and time steps, which lead to a time step $\Delta t = o(h)$ for the lowest-order Raviart–Thomas scheme. The heat equation coupled with the Navier–Stokes system has been treated by many works (see for instance Bernardi, Métivet and Pernaud-Thomas [4], Deteix, Jendoubi and Yakoubi [11], or Gaultier and Lezaun [13]). The stationary coupling of Darcy's system with the heat equation where the viscosity is constant but the exterior force depends on the temperature (like in the model proposed by Boussinesq [7]) was analyzed by Bernardi, Yacoubi and Maarouf [6] and discretized with a spectral method. The same stationary system but coupled by a nonlinear viscosity depending on the temperature is studied by Bernadi et al. in [5], where they propose and analyze two numerical schemes based on finite element methods. In [19,20], the authors discretize a problem similar to (P) using Raviart–Thomas elements methods for the discretization in space. For the discretization in time, Rivière and Walkington in [20] used the Discontinuous Galerkin (DG) method and Li *et al.* in [19] used the Interior Penalty Discontinuous Galerkin

(IPDG) method. These two works do not establish *a priori* error estimates but rather prove the convergence of the schemes using compactness results for functions that may be discontinuous in time. Vassilev and Yotov coupled in [23] the non-stationary Stokes-Darcy equation with the time dependent Transport equation, and established an a priori error estimate.

In this work, we study two types of discrete schemes for the full discretization of Problem (P) in time and space and for both, we prove existence and uniqueness and derive optimal a priori error estimates for the solutions. The first scheme is the lowest order Raviart–Thomas scheme for the velocity/pressure unknowns, for which we extend and improve the results of [16,17], first by considering general domains covered by simplicial meshes, then by extending the proof to the three-dimensional case.

The second scheme uses the $\mathbb{P}_1$-bubble / $\mathbb{P}_1$ scheme for the velocity/pressure unknowns, which is known as the "mini-element" in the Stokes context [2]. Both schemes use the $\mathbb{P}_1$ scheme for the concentration unknown. Finally, we perform several numerical tests to validate the theoretical results. They show that, in certain circumstances, the second scheme, although of higher numerical complexity, may have a better accuracy/complexity ratio.

The outline of the paper is as follows:

- In Sect. 2, we introduce some notations and functional spaces that are useful for the study of the problem.
- In Sect. 3, we introduce two variational formulations.
- Section 4 is devoted to the study of two numerical schemes and the establishment of an a priori error estimation under regularity assumptions of the exact solutions.
- Some numerical experiments are presented in Sect. 5.

2 Preliminaries

In this section, we recall the main notations and results which we use later on. We introduce the Sobolev space

$$W^{m,r}(\Omega)^d = \left\{ \mathbf{v} \in L^r(\Omega)^d;\ \partial^k \mathbf{v} \in L^r(\Omega)^d,\ \forall |k| \le m \right\},$$

where $k = \{k_1, \dots, k_d\}$ is a vector of non negative integers, such that $|k| = k_1 + \cdots + k_d$ and

$$\partial^k \mathbf{v} = \frac{\partial^{|k|} \mathbf{v}}{\partial^{k_1} x_1 \dots \partial^{k_d} x_d}.$$

This space is equipped with the semi-norm

$$|\mathbf{v}|_{W^{m,r}(\Omega)^d} = \left(\sum_{|k|=m} \int_\Omega |\partial^k \mathbf{v}|^r d\mathbf{x} \right)^{\frac{1}{r}},$$

and is a Banach space for the norm

$$\parallel \mathbf{v} \parallel_{W^{m,r}(\Omega)^d} = \left(\sum_{l=0}^{m} \int_{\Omega} |\mathbf{v}|^r_{W^{l,r}(\Omega)^d} d\mathbf{x} \right)^{\frac{1}{r}}.$$

When $r = 2$, this space is the Hilbert space $H^m(\Omega)^d$. In particular, we consider the following spaces

$$H^1_0(\Omega)^d = \left\{ \mathbf{v} \in H^1(\Omega)^d;\ \mathbf{v}_{|\partial\Omega} = 0 \right\},$$

and its dual $H^{-1}(\Omega)^d$.

We shall also introduce

$$L^2_0(\Omega) = \left\{ q \in L^2(\Omega);\ \int_{\Omega} q(\mathbf{x}) d\mathbf{x} = 0 \right\}.$$

We define the following scalar product in $L^2(\Omega)$:

$$(v, w) = \int_{\Omega} v(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}, \quad \forall v, w \in L^2(\Omega).$$

We recall the following Poincaré and Sobolev inequalities:

Lemma 2.1 *For any $p \geq 1$ when $d = 1$ or 2, or $1 \leq p \leq \dfrac{2d}{d-2}$ when $d \geq 3$, there exist two positive constants S_p and S^0_p such that*

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad \parallel \boldsymbol{v} \parallel_{L^p(\Omega)^d} \leq S^0_p |\boldsymbol{v}|_{H^1_0(\Omega)^d},$$

and

$$\forall \boldsymbol{v} \in H^1(\Omega)^d\ , \parallel \boldsymbol{v} \parallel_{L^p(\Omega)^d} \leq S_p \parallel \boldsymbol{v} \parallel_{H^1(\Omega)^d}.$$

We shall also use the following continuous embedding:

$$\forall q > d, \quad W^{1,q}(\Omega) \hookrightarrow L^{\infty}(\Omega). \tag{2.1}$$

We recall the standard spaces for Darcy's equations

$$H(\text{div}, \Omega) = \{ \mathbf{v} \in L^2(\Omega)^d;\ \text{div}\, \mathbf{v} \in L^2(\Omega) \}, \tag{2.2}$$

$$H_0(\text{div}, \Omega) = \{ \mathbf{v} \in H(\text{div}, \Omega);\ (\mathbf{v} \cdot \mathbf{n})|_{\Gamma} = 0 \}, \tag{2.3}$$

and

$$\mathcal{V} = \{ \mathbf{v} \in H_0(\text{div}, \Omega);\ \text{div}\, \mathbf{v} = 0 \}, \tag{2.4}$$

equipped with the norm

$$\|\mathbf{v}\|^2_{H(\text{div},\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\text{div}\, \mathbf{v}\|^2_{L^2(\Omega)}. \tag{2.5}$$

Finally, we recall the inf-sup condition between $L^2_0(\Omega)$ and $H_0(\text{div}, \Omega)$,

$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in H_0(\text{div},\Omega)} \frac{\displaystyle\int_\Omega (\text{div}\, \mathbf{v}) q \, d\mathbf{x}}{\|\mathbf{v}\|_{H(\text{div},\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta, \tag{2.6}$$

with a constant $\beta > 0$, and the inf-sup condition between $H^1(\Omega) \cap L^2_0(\Omega)$ and $L^2(\Omega)^d$,

$$\inf_{q \in H^1(\Omega) \cap L^2_0(\Omega)} \sup_{\mathbf{v} \in L^2(\Omega)^d} \frac{\displaystyle\int_\Omega \mathbf{v}.\nabla q \, d\mathbf{x}}{\|\mathbf{v}\|_{L^2(\Omega)^d} |q|_{H^1(\Omega)}} \geq 1. \tag{2.7}$$

Condition (2.6) follows immediately by solving a Laplace equation in Ω with a Neumann boundary condition on Γ, and condition (2.7) by choosing $\mathbf{v} = \nabla q$. As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a separable functional space W equipped with a norm $\| \, . \, \|_W$. Then, for any $r \geq 1$, we introduce the space

$$L^r(a, b; W) = \left\{ f \text{ mesurable on }]a, b[; \int_a^b \| f(t) \|_W^r \, dt < \infty \right\};$$

equipped with the norm

$$\| f \|_{L^r(a,b;W)} = \left(\int_a^b \| f(t) \|_W^r \, dt \right)^{\frac{1}{r}}.$$

If $r = \infty$, then

$$L^\infty(a, b; W) = \left\{ f \text{ mesurable on }]a, b[; \sup_{t \in [a,b]} \| f(t) \|_W < \infty \right\}.$$

Remark 2.2 $L^r(0, T; W)$ is a Banach space if W is a Banach space.

In addition, we define $C^j(0, T; W)$ as the space of functions C^j in time with values in W.

Remark 2.3 Let a and b be two real numbers.

(1) For any positive real number ε, we have

$$ab \leq \frac{1}{2\varepsilon} a^2 + \frac{1}{2} \varepsilon b^2. \tag{2.8}$$

(2) We also have

$$a(a - b) = \frac{1}{2} a^2 - \frac{1}{2} b^2 + \frac{1}{2} (a - b)^2. \tag{2.9}$$

3 Variational formulations

In this section, we start by writing a variational formulation of problem (P). Next, we prove the existence and the uniqueness of the solution. We assume that the data of the problem verify the following assumptions:

Assumption 3.1 *We assume that the data* $\mathbf{f}$, g *and* ν *verify:*

(1) $\mathbf{f}$ *can be written as follows:*

$$\mathbf{f}(\mathbf{x}, t, C) = \mathbf{f}_0(\mathbf{x}, t) + \mathbf{f}_1(\mathbf{x}, C), \tag{3.1}$$

where $\mathbf{f}_0 \in L^\infty(0, T; L^2(\Omega)^d)$ *and* $\mathbf{f}_1(C)$ *is (uniformly in* $\mathbf{x}$*)* $c^*_{\mathbf{f}_1}$*-Lipschitz with respect to its second variable with values in* $\mathbb{R}^d$*. In addition, we suppose that*

$$\forall \mathbf{x} \in \Omega, \forall \xi \in \mathbb{R}, \ |\mathbf{f}_1(\mathbf{x}, \xi)| \le c_{\mathbf{f}_1} |\xi|, \tag{3.2}$$

where $c_{\mathbf{f}_1}$ *is a positive constant.*

(2) $g \in L^2\big(0, T, L^2(\Omega)\big)$.

(3) ν *is* λ*-Lipschitz on* $\mathbb{R}$ *and there exist two strictly positive constants* ν_1 *and* ν_2 *such that, for any* $\theta \in \mathbb{R}$

$$\nu_1 \le \nu(\theta) \le \nu_2. \tag{3.3}$$

There are two possible choices of spaces for Darcy's velocity and pressure $(\mathbf{u}, p)$. The first choice is $L^\infty(0, T; H_0(\text{div}, \Omega)) \times L^\infty(0, T; L^2_0(\Omega))$; it corresponds to a mixed formulation and is analyzed in this section. The second choice is $L^\infty(0, T; L^2(\Omega)^d) \times L^\infty(0, T; (H^1(\Omega)) \cap L^2_0(\Omega))$; it leads to an alternative formulation equivalent to the first. In both cases, the concentration C is in $L^2(0, T; H^1_0(\Omega))$. Then, whereas there is no difficulty in setting Darcy's system in variational form, a variational formulation of the concentration equation is not that obvious. Indeed, the convection term $\mathbf{u} \cdot \nabla C$ cannot be tested by an $H^1(\Omega)$ function, since it is only in $L^1(\Omega)$. Therefore, we choose the test functions in $H^1_0(\Omega) \cap L^\infty(\Omega)$. Thus, we propose the following variational problem: for all $t \in [0, T]$,

$$(V) \begin{cases} \text{Find } (\mathbf{u}(t), p(t), C(t)) \in H_0(\text{div}, \Omega) \times L^2_0(\Omega) \times H^1_0(\Omega) \text{ such that, } C(0) = 0 \text{ and} \\ \forall\, \mathbf{v} \in H_0(\text{div}, \Omega), \quad \displaystyle\int_\Omega \nu(C(t))\mathbf{u}(t) \cdot \mathbf{v}\, d\mathbf{x} - \int_\Omega p(t)(\text{div}\, \mathbf{v})\, d\mathbf{x} = \int_\Omega \mathbf{f}(., t, C(t)) \cdot \mathbf{v}\, d\mathbf{x}, \\ \forall\, q \in L^2_0(\Omega), \quad \displaystyle\int_\Omega q(\text{div}\, \mathbf{u}(t))\, d\mathbf{x} = 0, \\ \forall\, S \in H^1_0(\Omega) \cap L^\infty(\Omega), \quad \displaystyle\int_\Omega \frac{\partial C}{\partial t}(t) S d\mathbf{x} + \alpha \int_\Omega \nabla C(t) \cdot \nabla S\, d\mathbf{x} + \int_\Omega (\mathbf{u}(t) \cdot \nabla C(t)) S\, d\mathbf{x} \\ \qquad\qquad + r_0 \displaystyle\int_\Omega C(t)\, S\, d\mathbf{x} = \int_\Omega g(t)\, S\, d\mathbf{x}. \end{cases}$$

A straightforward argument shows that any triplet of functions $(\mathbf{u}(t), p(t), C(t))$ in $H_0(\text{div}, \Omega) \times L^2_0(\Omega) \times H^1_0(\Omega)$ that solves the first three lines of problem (P) in the sense of distributions in Ω, and the last two lines in the sense of traces in $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ respectively, is a solution of (V). Conversely, any solution $(\mathbf{u}(t), p(t), C(t))$ of problem (V) solves problem (P) in the above sense.

For the a priori bound on the concentration C, we have the following theorem:

Theorem 3.2 *Every solution of (V) such that* $C \in L^\infty([0,T]\times\Omega)$ *verifies the bounds:*

$$\|C\|^2_{L^\infty(0,T;L^2(\Omega))} + \alpha\|C\|^2_{L^2(0,T;H^1_0(\Omega))} + 2r_0\|C\|^2_{L^2(0,T;L^2(\Omega))} \le 2\frac{(S_2^0)^2}{\alpha}\|g\|^2_{L^2(0,T;L^2(\Omega))} \tag{3.4}$$

and

$$\|\mathbf{u}(t)\|_{L^2(\Omega)^d} \le \frac{1}{\nu_1}\big(\|\mathbf{f}_0(t)\|_{L^2(\Omega)^d} + c_{\mathbf{f}_1}\|C\|_{L^\infty(0,T;L^2(\Omega))}\big). \tag{3.5}$$

Proof By testing the last line of (V) with $S = C(t)$, and by noticing that $\int_\Omega (\mathbf{u}\cdot\nabla C)C = 0$, we use the Cauchy–Schwarz inequality and get:

$$\frac{1}{2}\frac{d}{dt}\|C(t)\|^2_{L^2(\Omega)} + \alpha\|\nabla C(t)\|^2_{L^2(\Omega)} + r_0\|C(t)\|^2_{L^2(\Omega)} \le \|g(t)\|_{L^2(\Omega)}\|C(t)\|_{L^2(\Omega)}.$$

We use Relation (2.8) with $\varepsilon = \dfrac{\alpha}{(S_2^0)^2}$ and Lemma 2.1 and we integrate between 0 and t to obtain

$$\|C(t)\|^2_{L^2(\Omega)} + \alpha\|C\|^2_{L^2(0,t;H^1_0(\Omega))} + 2r_0\|C\|^2_{L^2(0,t;L^2(\Omega))} \le \frac{(S_2^0)^2}{\alpha}\|g\|^2_{L^2(0,t;L^2(\Omega))}.$$

The last relation leads to the following one:

$$\|C\|^2_{L^\infty(0,T;L^2(\Omega))} + \alpha\|C\|^2_{L^2(0,T;H^1_0(\Omega))} + 2r_0\|C\|^2_{L^2(0,T;L^2(\Omega))} \le 2\frac{(S_2^0)^2}{\alpha}\|g\|^2_{L^2(0,T;L^2(\Omega))}.$$

Next, by testing the first line of (V) with $\mathbf{v} = \mathbf{u}(t)$ and using the second line, we immediately derive from (3.1), (3.2) and (3.3) the following a priori bound:

$$\|\mathbf{u}(t)\|_{L^2(\Omega)^d} \le \frac{1}{\nu_1}\big(\|\mathbf{f}_0(t)\|_{L^2(\Omega)^d} + c_{\mathbf{f}_1}\|C\|_{L^\infty(0,T;L^2(\Omega))}\big). \tag{3.6}$$

□

Alternative variational formulation

The variational problem (V) is well adapted to locally conservative discrete schemes. However, the numerical implementation of such schemes is not straightforward and can be simplified by eliminating the divergence term from the first two equations of (V) by means of Green's formula, thus reducing the regularity of $\mathbf{u}$. This leads to the following alternative formulation: for all $t \in [0, T]$,

$$(V_a)\begin{cases}\text{Find } (\mathbf{u}(t), p(t), C(t)) \in L^2(\Omega)^d \times (H^1(\Omega) \cap L^2_0(\Omega)) \times H^1_0(\Omega) \text{ such that}\\ \forall\, \mathbf{v} \in L^2(\Omega)^d, \quad \int_\Omega \nu(C(t))\mathbf{u}(t)\cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega \nabla p(t)\cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f}(.,t,C(t))\cdot \mathbf{v}\, d\mathbf{x},\\ \forall\, q \in H^1(\Omega)\cap L^2_0(\Omega), \quad \int_\Omega \nabla q \cdot \mathbf{u}(t)\, d\mathbf{x} = 0,\\ \forall\, S \in H^1_0(\Omega)\cap L^\infty(\Omega), \quad \int_\Omega \frac{\partial C}{\partial t}(t) S d\mathbf{x} + \alpha \int_\Omega \nabla C(t)\cdot \nabla S\, d\mathbf{x} + \int_\Omega (\mathbf{u}(t)\cdot \nabla C(t)) S\, d\mathbf{x}\\ \qquad\qquad + r_0 \int_\Omega C(t)\, S\, d\mathbf{x} = \int_\Omega g(t)\, S\, d\mathbf{x}.\end{cases}$$

This variational formulation is obviously equivalent to (V). It leads to numerical schemes that are simpler to implement.

4 Discretization

In this section, we propose a space-time discretization of the problem (P), derive and prove an a priori error estimation. We use the semi-implicit Euler method for the time discretization and the finite element method for the space discretization. For the time discretization, we introduce a partition of the interval $[0, T]$ into N subintervals $[t_{n-1}, t_n]$ of length τ (the time step). For the space discretization, we assume that Ω is a polygon when $d = 2$ or polyhedron when $d = 3$, so it can be completely meshed. Now, we describe the discretization space. A regular family of triangulations (see Ciarlet [9]) $(\mathcal{T}_h)_h$ of Ω, is a set of closed non degenerate triangles or tetrahedra, called elements, satisfying,

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- the intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge or face;
- the ratio of the diameter of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle or ball is bounded by a constant independent of h.

As usual, h denotes the maximal diameter of all elements of $\mathcal{T}_h$. For each K in $\mathcal{T}_h$, we denote by $\mathbb{P}_1(K)$ the space of restrictions to K of polynomials in d variables and total degree at most one.

In what follows, $c, c', C, C', c_1, \ldots$ stand for generic constants which may vary from line to line but are always independent of h. For a given triangulation $\mathcal{T}_h$, we define the following finite dimensional spaces:

$$Z_h = \{S_h \in \mathcal{C}^0(\bar{\Omega});\ \forall\, K \in \mathcal{T}_h,\ S_h|_K \in \mathbb{P}_1(K)\} \quad \text{and} \quad X_h = Z_h \cap H^1_0(\Omega). \tag{4.1}$$

We shall use the following result: There exists an approximation operator (when $d = 2$, see Bernardi and Girault [3] or Clément [10]; when $d = 2$ or $d = 3$, see Scott and Zhang [22]), R_h in $\mathcal{L}(W^{1,p}(\Omega); Z_h)$ and in $\mathcal{L}(W^{1,p}(\Omega) \cap H^1_0(\Omega); X_h)$ such that for all K in $\mathcal{T}_h$, $m = 0, 1$, $l = 0, 1$, and all $p \geq 2$,

$$\forall\, S \in W^{l+1,p}(\Omega),\ \ |S - R_h(S)|_{W^{m,p}(K)} \le c(p,m,l)\, h^{l+1-m} |S|_{W^{l+1,p}(\Delta_K)}, \tag{4.2}$$

where Δ_K is the macro element containing the values of S used in defining $R_h(S)$.

Furthermore, we introduce the following inverse inequalities: for any number $p \ge 2$, for any dimension d, and for any non negative integer r, there exist constants $c_I^0(p)$ such that for any polynomial function v_h of degree r on K,

$$\|v_h\|_{L^p(K)} \le c_I^0(p) h_K^{\frac{d}{p}-\frac{d}{2}} \|v_h\|_{L^2(K)}. \tag{4.3}$$

4.1 First discrete scheme

The velocity and pressure are discretized in space by the Raviart–Thomas RT_0 elements. More precisely, the discrete spaces $(\mathcal{W}_{h,1}, M_{h,1})$ are defined as follows:

$$\begin{aligned} \mathcal{W}_h &= \{\mathbf{v}_h \in H(\mathrm{div},\Omega);\ \ \mathbf{v}_h(\mathbf{x})|_K = a_K \mathbf{x} + \mathbf{b}_K, a_K \in \mathbf{R}, \mathbf{b}_K \in \mathbf{R}^d,\ \forall\, K \in \mathcal{T}_h\}, \\ \mathcal{W}_{h,1} &= \mathcal{W}_h \cap H_0(\mathrm{div},\Omega), \end{aligned} \tag{4.4}$$

$$M_h = \{q_h \in L^2(\Omega);\ \forall\, K \in \mathcal{T}_h,\ \ q_h|_K \text{ is constant}\} \quad \text{and} \quad M_{h,1} = M_h \cap L_0^2(\Omega). \tag{4.5}$$

The kernel of the divergence in $\mathcal{W}_{h,1}$ is denoted by $\mathcal{V}_{h,1}$,

$$\mathcal{V}_{h,1} = \{\mathbf{v}_h \in \mathcal{W}_{h,1};\ \ \mathrm{div}\,\mathbf{v}_h = 0 \text{ in } \ \Omega\}. \tag{4.6}$$

There exists an approximation operator ξ_h^1 belonging to $\mathcal{L}(H^1(\Omega); \mathcal{W}_h)$ and to $\mathcal{L}(H^1(\Omega) \cap H_0(\mathrm{div},\Omega); \mathcal{W}_{h,1})$ such that for all K in $\mathcal{T}_h$ (Roberts and Thomas [21]):

$$\forall\, \mathbf{v} \in H^1(\Omega)^d,\ \ \|\mathbf{v} - \xi_h^1(\mathbf{v})\|_{L^2(K)^d} \le c_1\, h |\mathbf{v}|_{H^1(K)^d}, \tag{4.7}$$

and

$$\forall\, \mathbf{v} \in H^1(\Omega)^d \text{ s.t. } \mathrm{div}\,\mathbf{v} \in H^1(\Omega),\ \ \|\mathrm{div}(\mathbf{v} - \xi_h^1(\mathbf{v}))\|_{L^2(K)} \le c_2\, h |\mathrm{div}\,\mathbf{v}|_{H^1(K)}. \tag{4.8}$$

Furthermore, if $\mathrm{div}\,\mathbf{u} = 0$ then $\mathrm{div}(\xi_h^1(\mathbf{u})) = 0$. In addition, we shall use the operator ρ_h that belongs to $\mathcal{L}(L^2(\Omega); M_h) \cap \mathcal{L}(L_0^2(\Omega); M_{h,1})$, defined by

$$\rho_h(q)|_K = \frac{1}{|K|} \int_K q\, d\mathbf{x},\ \forall\, K \in \mathcal{T}_h. \tag{4.9}$$

This operator satisfies the following result

$$\forall q \in H^1(\Omega), \quad \|q - \rho_h(q)\|_{L^2(K)} \le c\, h\, |q|_{H^1(K)}. \tag{4.10}$$

The following discrete inf-sup condition holds (see Roberts and Thomas [21]):

$$\forall\, q_h \in M_{h,1}, \quad \sup_{\mathbf{v}_h \in \mathcal{W}_{h,1}} \frac{\displaystyle\int_\Omega q_h (\operatorname{div} \mathbf{v}_h)\, d\mathbf{x}}{\|\mathbf{v}_h\|_{H(\operatorname{div},\Omega)}} \geq \beta_1 \|q_h\|_{L^2(\Omega)}, \tag{4.11}$$

with a constant $\beta_1 > 0$ independent of h.

We then consider the straightforward discretization of Problem (V):

$$(V_{h,1}) \begin{cases} \text{Having } C_h^{n-1} \in X_h, \text{ Find } (\mathbf{u}_h^n, p_h^n) \in \mathcal{W}_{h,1} \times M_{h,1} \text{ such that} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_{h,1}, \ \displaystyle\int_\Omega \nu(C_h^{n-1})\mathbf{u}_h^n \cdot \mathbf{v}_h\, d\mathbf{x} - \int_\Omega p_h^n (\operatorname{div} \mathbf{v}_h)\, d\mathbf{x} = \int_\Omega \mathbf{f}^n(C_h^{n-1}) \cdot \mathbf{v}_h\, d\mathbf{x}, \\ \forall\, q_h \in M_{h,1}, \ \displaystyle\int_\Omega q_h (\operatorname{div} \mathbf{u}_h^n)\, d\mathbf{x} = 0, \\ \\ \text{Having } C_h^{n-1} \in X_h, \text{ Find } C_h^n \in X_h \text{ such that} \\ \forall\, S_h \in X_h, \ \displaystyle\int_\Omega \frac{C_h^n - C_h^{n-1}}{\tau} S_h\, d\mathbf{x} + \alpha \int_\Omega \nabla C_h^n \cdot \nabla S_h\, d\mathbf{x} + \int_\Omega (\mathbf{u}_h^n \cdot \nabla C_h^n) S_h\, d\mathbf{x} \\ \qquad\qquad + r_0 \displaystyle\int_\Omega C_h^n\, S_h\, d\mathbf{x} = \int_\Omega g^n\, S_h\, d\mathbf{x}, \end{cases}$$

where $C_h^0 = 0$, g^n and $\mathbf{f}^n(C_h^{n-1})$ are given as

$$g^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} g(s) ds,$$

$$\mathbf{f}^n(C_h^{n-1}) = \mathbf{f}_0^n + \mathbf{f}_1(C_h^{n-1}), \quad \text{where} \quad \mathbf{f}_0^n = \mathbf{f}_0(t_n). \tag{4.12}$$

It is easy to see that the second equation of the above system implies that $\operatorname{div} \mathbf{u}_h^n = 0$ in Ω, since $\mathbf{u}_h^n \in \mathcal{W}_{h,1}$ implies that $\operatorname{div} \mathbf{u}_h^n \in M_{h,1}$. Hence this scheme preserves the zero divergence condition. This and the conformity of C_h imply in turn that $\int_\Omega (\mathbf{u}_h^n \cdot \nabla C_h) C_h = 0$ for all $C_h \in X_h$.

For the existence and uniqueness of the solution of $(V_{h,1})$, we have the following theorem:

Theorem 4.1 *(Existence and uniqueness of the solution of $(V_{h,1})$) At each time step n and for a given $C_h^{n-1} \in X_h$, Problem $(V_{h,1})$ has a unique solution $(\mathbf{u}_h^n, p_h^n, C_h^n) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h$ which verifies, for $m = 1, \ldots, N$, the following bounds*

$$\|\mathbf{u}_h^m\|_{L^2(\Omega)^d} \leq \frac{1}{\nu_1} \big(\|\mathbf{f}_0\|_{L^\infty(0,T;L^2(\Omega)^d)} + c_{\mathbf{f}_1} \|C_h^{m-1}\|_{L^2(\Omega)} \big) \tag{4.13}$$

and

$$\|C_h^m\|^2_{L^2(\Omega)} + \alpha \sum_{n=1}^m \tau |C_h^n|^2_{H^1(\Omega)} + 2r_0 \sum_{n=1}^m \tau \|C_h^n\|^2_{L^2(\Omega)} \leq \frac{(S_2^0)^2}{\alpha} \|g\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.14}$$

where c is positive constant independent of h and m.

Proof It is clear that the first equation of Problem $(V_{h,1})$ has a unique solution $(\mathbf{u}_h^n, p_h^n)$ as a consequence of the coerciveness of the corresponding bilinear form on $\mathcal{W}_{h,1} \times \mathcal{W}_{h,1}$ and the inf-sup condition (4.11). Thus, knowing $\mathbf{u}_h^n \in \mathcal{W}_{h,1}$ and $C_h^{n-1} \in X_h$, the third equation of Problem $(V_{h,1})$ also admits a unique solution $C_h^n \in X_h$. Therefore, by taking $\mathbf{v}_h = \mathbf{u}_h^n$ in the first equation we get (4.13), and $S_h = C_h^n$ in the third equation of Problem $(V_{h,1})$ we get, using the Cauchy–Schwarz inequality, Lemma 2.1 and Remark 2.3

$$\frac{1}{2}(\|C_h^n\|^2_{L^2(\Omega)} - \|C_h^{n-1}\|^2_{L^2(\Omega)}) + \frac{\alpha}{2}\tau|C_h^n|^2_{H^1(\Omega)} + \tau r_0\|C_h^n\|^2_{L^2(\Omega)} \leq \frac{(S_2^0)^2}{2\alpha}\tau\|g^n\|^2_{L^2(\Omega)}. \tag{4.15}$$

We sum over $n = 1, \ldots, m$ and we obtain (4.14). □

4.2 Second discrete scheme

Let K be an element of $\mathcal{T}_h$ with vertices a_i, $1 \leq i \leq d+1$, and corresponding barycentric coordinates λ_i. We denote by $b_K \in \mathbb{P}_{d+1}(K)$ the basic bubble function

$$b_K(\mathbf{x}) = \lambda_1(\mathbf{x}) \ldots \lambda_{d+1}(\mathbf{x}). \tag{4.16}$$

We observe that $b_K(\mathbf{x}) = 0$ on ∂K and that $b_K(\mathbf{x}) > 0$ in the interior of K.

Let $(\mathcal{W}_{h,2}, M_{h,2})$ be a pair of discrete spaces approximating $L^2(\Omega)^d \times \big(H^1(\Omega) \cap L_0^2(\Omega)\big)$ defined by

$$\mathcal{W}_{h,2} = \{\mathbf{v}_h \in (\mathcal{C}^0(\bar{\Omega}))^d;\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_h|_K \in \mathcal{P}(K)^d\}, \tag{4.17}$$

$$\tilde{M}_h = \{q_h \in \mathcal{C}^0(\bar{\Omega});\ \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathbb{P}_1(K)\} \quad \text{and} \quad M_{h,2} = \tilde{M}_h \cap L_0^2(\Omega), \tag{4.18}$$

where

$$\mathcal{P}(K) = \mathbb{P}_1(K) \oplus \text{Vect}\{b_K\},$$

is the space associated to the discretisation in space by the “mini-élément” introduced by Arnold *et al.* in [2].

Let $\mathcal{V}_{h,2}$ be the kernel of the divergence in $\mathcal{W}_{h,2}$,

$$\mathcal{V}_{h,2} = \{\mathbf{v}_h \in \mathcal{W}_{h,2};\ \forall q_h \in M_{h,2}, \int_\Omega \mathbf{v}_h \cdot \nabla q_h \, d\mathbf{x} = 0\}. \tag{4.19}$$

We shall use a variant of R_h denoted by $\mathcal{F}_h$ which is constructed in [5, p. 336] and has the following properties:

$$\forall\, \mathbf{v} \in H^1(\Omega)^d,\ ||\mathbf{v} - \mathcal{F}_h(\mathbf{v})||_{L^2(K)^d} \leq C\, h||\mathbf{v}||_{H^1(\Delta_K)^d}, \tag{4.20}$$

and $\mathcal{F}_h(\mathbf{v}) \in \mathcal{V}_{h,2}$ when div $\mathbf{v} = 0$.

Regarding the pressure, since Z_h coincides with $\tilde{M}_h$, an easy modification of R_h yields an operator r_h in $\mathcal{L}(H^1(\Omega); \tilde{M}_h)$ and in $\mathcal{L}(H^1(\Omega) \cap L^2_0(\Omega); M_{h,2})$ (see for instance Abboud, Girault and Sayah [1]), satisfying (4.2). We approximate problem (V_a) by the following discrete scheme:

$$(V_{h,2}) \begin{cases} \text{Having } C_h^{n-1} \in X_h, \text{Find } (\mathbf{u}_h^n, p_h^n) \in \mathcal{W}_{h,2} \times M_{h,2} \text{ such as} \\ \\ \forall \mathbf{v}_h \in \mathcal{W}_{h,2}, \quad \displaystyle\int_\Omega \nu(C_h^{n-1}) \mathbf{u}_h^n \cdot \mathbf{v}_h \, d\mathbf{x} + \int_\Omega \nabla p_h^n \cdot \mathbf{v}_h \, d\mathbf{x} = \int_\Omega \mathbf{f}^n(C_h^{n-1}) \cdot \mathbf{v}_h \, d\mathbf{x}, \\ \forall q_h \in M_{h,2}, \quad \displaystyle\int_\Omega \nabla q_h \cdot \mathbf{u}_h^n \, d\mathbf{x} = 0, \\ \\ \text{Having } C_h^{n-1} \in X_h, \text{Find } C_h^n \in X_h \text{ such that} \\ \forall S_h \in X_h, \quad \displaystyle\int_\Omega \frac{C_h^n - C_h^{n-1}}{\tau} S_h \, d\mathbf{x} + \alpha \int_\Omega \nabla C_h^n \cdot \nabla S_h \, d\mathbf{x} + \int_\Omega (\mathbf{u}_h^n \cdot \nabla C_h^n) S_h \, d\mathbf{x} \\ \qquad\qquad \displaystyle + \frac{1}{2} \int_\Omega \operatorname{div}\,(\mathbf{u}_h^n)\, C_h^n \, S_h \, d\mathbf{x} + r_0 \int_\Omega C_h^n \, S_h \, d\mathbf{x} \\ \qquad\qquad \displaystyle = \int_\Omega g^n \, S_h \, d\mathbf{x}. \end{cases}$$

where as usual, the second nonlinear term in the last equation is added to compensate the fact that $\operatorname{div} \mathbf{u}_h^n \neq 0$. It is well-known that Green's formula and the functions regularity imply that

$$\begin{aligned} &\int_\Omega (\mathbf{u}_h^n \cdot \nabla C_h^n) S_h \, d\mathbf{x} + \frac{1}{2} \int_\Omega (\operatorname{div} \mathbf{u}_h^n) C_h^n \, S_h \, d\mathbf{x} \\ &\quad = \frac{1}{2} \left(\int_\Omega (\mathbf{u}_h^n \cdot \nabla C_h^n) S_h \, d\mathbf{x} - \int_\Omega (\mathbf{u}_h^n \cdot \nabla S_h) C_h^n \, d\mathbf{x} \right), \end{aligned} \tag{4.21}$$

so that the nonlinear term is antisymmetric. One of the key points for studying $(V_{h,2})$ is the discrete inf-sup condition satisfied by the pair of spaces $(\mathcal{W}_{h,2}, M_{h,2})$ (see for instance [5]):

$$\forall q_h \in M_{h,2}, \quad \sup_{\mathbf{v}_h \in \mathcal{W}_{h,2}} \frac{\displaystyle\int_\Omega \nabla q_h \cdot \mathbf{v}_h \, d\mathbf{x}}{\|\mathbf{v}_h\|_{L^2(\Omega)^d}} \geq \beta_2 \, |q_h|_{H^1(\Omega)}, \tag{4.22}$$

with a constant $\beta_2 > 0$ independent of h.

By following the same steps of the proof of Theorem (4.1), we deduce the following theorem:

Theorem 4.2 *(Existence and uniqueness of the solution of $(V_{h,2})$) At each time step n and for a given $C_h^{n-1} \in X_h$, Problem $(V_{h,2})$ has a unique solution $(\mathbf{u}_h^n, p_h^n, C_h^n) \in \mathcal{W}_{h,2} \times M_{h,2} \times X_h$ which verifies, for $m = 1, \ldots, N$, the following bounds*

$$\|\mathbf{u}_h^m\|_{L^2(\Omega)^d} \leq \frac{1}{\nu_1}\big(\|\mathbf{f}_0\|_{L^\infty(0,T;L^2(\Omega)^d)} + c_{\mathbf{f}_1}\|C_h^{m-1}\|_{L^2(\Omega)}\big) \tag{4.23}$$

and

$$\|C_h^m\|^2_{L^2(\Omega)} + \alpha\sum_{n=1}^{m}\tau|C_h^n|^2_{H^1(\Omega)} + 2r_0\sum_{n=1}^{m}\tau\|C_h^n\|^2_{L^2(\Omega)} \leq \frac{(S_2^0)^2}{\alpha}\|g\|^2_{L^2(0,T;L^2(\Omega))}, \tag{4.24}$$

where c is positive constant independent of h and m.

5 A priori error estimate

In this section, we establish the *a priori* estimates corresponding to the proposed numerical schemes. We begin by establishing the error estimates corresponding to the velocity and the pressure, and then we will establish those corresponding to the concentration for both schemes $(V_{h,1})$ and $(V_{h,2})$.

In all the rest of the paper, we denote by $\mathbf{u}^n = \mathbf{u}(t_n)$, $p^n = p(t_n)$ and $C^n = C(t_n)$.

Theorem 5.1 *Let* $(\mathbf{u}, p, C)$ *be the solution of Problem* (V) *and* $(\mathbf{u}_h^n, p_h^n, C_h^n)$ *be the solution of Problem* $(V_{h,1})$. *If* $\mathbf{u} \in L^\infty(0,T;H^1(\Omega)^d) \cap L^\infty(0,T;L^\infty(\Omega)^d)$, $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;L^4(\Omega)^d)$, $p \in L^\infty(0,T;H^1(\Omega))$, $C \in L^\infty(0,T;W^{2,4}(\Omega))$ *and* $\frac{\partial C}{\partial t} \in L^2(0,T;W^{1,4}(\Omega))$ *and under Assumption 3.1, there exist positive constants* c, c_1, c', c'' *depending on* $\mathbf{u}$ *and* α *such that,*

$$\begin{aligned}&\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d}\\ &\quad\leq \frac{1}{\nu_1}\big(ch + (c^*_{\mathbf{f}_1} + \lambda\|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega)^d)})||C^n - C_h^{n-1}||_{L^2(\Omega)}\big),\end{aligned} \tag{5.1}$$

$$\begin{aligned}\|p^n - p_h^n\|_{L^2(\Omega)} &\leq c'h + \frac{\nu_2}{\beta_1}\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d}\\ &\quad+\frac{1}{\beta_1}(c^*_{\mathbf{f}_1} + \lambda\|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega)^d)})||C^n - C_h^{n-1}||_{L^2(\Omega)},\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}&\sup_{0\leq n\leq N}\|C_h^n - C^n\|^2_{L^2(\Omega)} + \alpha\sum_{n=1}^{N}\tau|C_h^n - C^n|^2_{1,\Omega}\\ &\quad+\sum_{n=1}^{N}\|(C_h^n - C^n) - (C_h^{n-1} - C^{n-1})\|^2_{L^2(\Omega)}\\ &\quad+r_0\sum_{n=1}^{N}\tau\|C_h^n - C^n\|^2_{L^2(\Omega)} \leq c''(h^2+\tau^2) + c_1\sum_{n=1}^{N}\tau\|\mathbf{u}_h^n - \mathbf{u}^n\|^2_{L^2(\Omega)^d}.\end{aligned} \tag{5.3}$$

Proof Let $(\mathbf{u}, p, C)$ and $(\mathbf{u}_h^n, p_h^n, C_h^n)$ solve respectively (V) and $(V_{h,1})$. We shall prove first (5.1), next (5.2), and finally (5.3).

We start by estimating the error on the velocity approximation. By taking the difference between the first equations of (V) for $t = t_n$ and $(V_{h,1})$ and testing with $\mathbf{v} = \mathbf{v}_h \in \mathcal{V}_{h,1}$, we obtain

$$\int_\Omega \nu(C_h^{n-1}(\mathbf{x}))(\mathbf{u}^n - \mathbf{u}_h^n)(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x} = \int_\Omega (\mathbf{f}_1(C^n(\mathbf{x})) - \mathbf{f}_1(C_h^{n-1}(\mathbf{x}))) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x}$$
$$+ \int_\Omega (\nu(C_h^{n-1}(\mathbf{x})) - \nu(C^n(\mathbf{x})))\mathbf{u}^n(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x}. \tag{5.4}$$

By inserting $\xi_h^1 \mathbf{u}^n$, taking $\mathbf{v}_h = \xi_h^1 \mathbf{u}^n - \mathbf{u}_h^n$, using the triangle inequality and the properties of ξ_h^1, and using the properties of $\mathbf{f}_1$ and ν, we obtain

$$\nu_1 \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d} \le ch + c^*_{\mathbf{f}_1} ||C^n - C_h^{n-1}||_{L^2(\Omega)} + \lambda \|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega)^d)} ||C^n - C_h^{n-1}||_{L^2(\Omega)}.$$

Hence, we deduce (5.1).

To prove the error estimate on the pressure, we take the difference between the first equations of (V) (for $t = t_n$) and $(V_{h,1})$, insert $\rho_h(p^n)$, test with $\mathbf{v}_h$ in $\mathcal{W}_{h,1}$, and obtain

$$\begin{aligned}\int_\Omega (\rho_h(p^n) - p_h^n)(\mathbf{x}) \text{div}\, \mathbf{v}_h(\mathbf{x})d\mathbf{x} = &\int_\Omega (\rho_h(p^n) - p^n)(\mathbf{x}) \text{div}\, \mathbf{v}_h(\mathbf{x})d\mathbf{x} \\ &+ \int_\Omega \nu(C_h^{n-1})(\mathbf{u}^n - \mathbf{u}_h^n)(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x} \\ &+ \int_\Omega (\nu(C^n(\mathbf{x})) - \nu(C_h^{n-1}(\mathbf{x})))\mathbf{u}^n(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x} \\ &- \int_\Omega (\mathbf{f}_1(C^n(\mathbf{x})) - \mathbf{f}_1(C_h^{n-1}(\mathbf{x}))) \cdot \mathbf{v}_h(\mathbf{x})d\mathbf{x}.\end{aligned} \tag{5.5}$$

It follows from the inf-sup condition (4.11) that there exists $\mathbf{v}_h$ in $\mathcal{W}_{h,1}$ such that

$$\text{div}\, \mathbf{v}_h = \rho_h(p^n) - p_h^n \quad \text{and} \quad \|\mathbf{v}_h\|_{H(\text{div},\Omega)} \le \frac{1}{\beta_1} \|\rho_h(p^n) - p_h^n\|_{L^2(\Omega)}.$$

With this $\mathbf{v}_h$, (5.5) implies (5.2) by using the properties of ρ_h.

Let us now focus on (5.3). We choose the test function $S_h = r_h^n = C_h^n - R_h C^n$ in the third equation of $(V_{h,1})$ and multiply it by the time step τ. Then, we subtract the third equation of (V) integrated over $[t_{n-1}; t_n]$. We obtain, using the definition of g^n by (4.12):

$$\Big((C_h^n - C_h^{n-1}) - (C^n - C^{n-1}), r_h^n\Big) + \alpha\Big(\tau \nabla C_h^n - \int_{t_{n-1}}^{t_n} \nabla C(t)dt, \nabla r_h^n\Big)$$
$$+ \Big(\tau \mathbf{u}_h^n \cdot \nabla C_h^n - \int_{t_{n-1}}^{t_n} \mathbf{u}(t) \cdot \nabla C(t)dt, r_h^n\Big) + r_0\Big(\tau C_h^n - \int_{t_{n-1}}^{t_n} C(t)dt, r_h^n\Big) = 0. \tag{5.6}$$

The first term in the left-hand side of (5.6) can be bounded, by inserting R_hC^n and R_hC^{n-1} and using (2.9). We obtain:

$$\left((C_h^n - C_h^{n-1}) - (C^n - C^{n-1}), r_h^n\right) = \frac{1}{2}\|r_h^n\|^2_{L^2(\Omega)} - \frac{1}{2}\|r_h^{n-1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|r_h^n - r_h^{n-1}\|^2_{L^2(\Omega)} + \left((C^{n-1} - R_hC^{n-1}) - (C^n - R_hC^n), r_h^n\right). \tag{5.7}$$

The last term of the previous equality can be bounded as follows, for any $\xi_1 > 0$, thanks to (2.8)

$$\begin{aligned}\left|\left((C^{n-1} - R_hC^{n-1}) - (C^n - R_hC^n), r_h^n\right)\right| &= \left| -\left(\int_{t_{n-1}}^{t_n} (R_hC' - C')(t)dt, r_h^n\right)\right| \\ &\le \frac{c^2h^2}{2\xi_1}\int_{t_{n-1}}^{t_n} \|C'(t)\|^2_{H^1(\Omega)}dt + \frac{\tau\xi_1}{2}\|r_h^n\|^2_{L^2(\Omega)} \\ &\le \frac{c^2h^2}{2\xi_1}\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;H^1(\Omega))} + (S_2^0)^2\frac{\tau\xi_1}{2}|r_h^n|^2_{1,\Omega}.\end{aligned} \tag{5.8}$$

By choosing $\xi_1 = \dfrac{\alpha}{25(S_2^0)^2}$, we obtain

$$\left|\left((C^{n-1}-R_hC^{n-1})-(C^n-R_hC^n), r_h^n\right)\right| \le c_1h^2\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;H^1(\Omega))} + \frac{\alpha}{50}\tau|r_h^n|^2_{1,\Omega}, \tag{5.9}$$

where c_1 is positive constant independent of h and τ.

The second term of the left-hand side of (5.6) can be bounded, by inserting $\tau(\nabla R_hC^n, \nabla r_h^n)$ and $\int_{t_{n-1}}^{t_n} (\nabla R_hC(t), \nabla r_h^n)dt$. We have:

$$\alpha\left(\tau\nabla C_h^n - \int_{t_{n-1}}^{t_n} \nabla C(t)dt, \nabla r_h^n\right) = \alpha\tau|r_h^n|^2_{1,\Omega} + \alpha\int_{t_{n-1}}^{t_n} (\nabla(R_hC^n - R_hC(t)), \nabla r_h^n)dt + \alpha\int_{t_{n-1}}^{t_n} (\nabla(R_hC(t) - C(t)), \nabla r_h^n)dt. \tag{5.10}$$

Since $C^n - C(t) = \int_t^{t_n} C'(s)ds$, then we have by using the stability of the operator R_h in $H_0^1(\Omega)$, the Fubini theorem, the Cauchy–Schwarz inequality and (2.8) for any $\xi_2 > 0$:

$$\begin{aligned}\left|\alpha\int_{t_{n-1}}^{t_n} (\nabla(R_hC^n - R_hC(t)), \nabla r_h^n)dt\right| &\le \alpha\int_{t_{n-1}}^{t_n}\int_t^{t_n} |R_hC'(s)|_{1,\Omega}|r_h^n|_{1,\Omega}dsdt \\ &\le \alpha c|r_h^n|_{1,\Omega}\int_{t_{n-1}}^{t_n} |C'(s)|_{1,\Omega}(s - t_{n-1})ds \\ &\le \frac{\alpha c^2\tau^2\xi_2}{2\sqrt{3}}\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;H^1(\Omega))} + \frac{\tau\alpha}{2\sqrt{3}\xi_2}|r_h^n|^2_{1,\Omega}.\end{aligned} \tag{5.11}$$

In addition, we apply the Cauchy–Schwarz inequality, (2.8) and (4.2) to obtain for any $\xi_3 > 0$:

$$\begin{aligned}\left|\alpha \int_{t_{n-1}}^{t_n} (\nabla(R_h C(t) - C(t)), \nabla r_h^n) dt\right| &\leq \frac{\alpha \xi_3}{2} \int_{t_{n-1}}^{t_n} |(R_h C - C)(t)|_{1,\Omega}^2 dt + \frac{\tau \alpha}{2\xi_3} |r_h^n|_{1,\Omega}^2 \\ &\leq \frac{\alpha c^2 h^2 \xi_3}{2} \|C\|_{L^2(t_{n-1},t_n;H^2(\Omega))}^2 + \frac{\tau \alpha}{2\xi_3} |r_h^n|_{1,\Omega}^2. \end{aligned} \tag{5.12}$$

By choosing $\xi_2 = 25/\sqrt{3}$ and $\xi_3 = 25$, we obtain, using the properties of the operator R_h and (5.10), (5.11) and (5.12)

$$\begin{aligned}&\left|\alpha\left(\tau \nabla C_h^n - \int_{t_{n-1}}^{t_n} \nabla C(t) dt, \nabla r_h^n\right) - \alpha \tau |r_h^n|_{1,\Omega}^2\right| \leq c_1 h^2 \|C\|_{L^2(t_{n-1},t_n;H^2(\Omega))}^2 \\ &\quad + c_2 \tau^2 \|\frac{\partial C}{\partial t}\|_{L^2(t_{n-1},t_n;H^1(\Omega))}^2 + \frac{\alpha}{25} \tau |r_h^n|_{1,\Omega}^2,\end{aligned} \tag{5.13}$$

where c_1 and c_2 are positive constants independent of h and τ.
Let the third term in the left-hand side of (5.6) be denoted by

$$b = \left(\tau \mathbf{u}_h^n \cdot \nabla C_h^n - \int_{t_{n-1}}^{t_n} \mathbf{u}(t) \cdot \nabla C(t) dt, r_h^n\right). \tag{5.14}$$

We insert $\left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla C(t) dt, r_h^n\right)$ and $\tau(\mathbf{u}_h^n \cdot \nabla R_h C^n, r_h^n)$ to get by noticing that $(\mathbf{u}_h^n \cdot \nabla r_h^n, r_h^n) = 0$:

$$b = b_1 + b_2 = \left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(R_h C^n - C(t)) dt, r_h^n\right) + \left(\int_{t_{n-1}}^{t_n} (\mathbf{u}_h^n - \mathbf{u}(t)) \cdot \nabla C(t) dt, r_h^n\right). \tag{5.15}$$

We insert $\pm\tau(\mathbf{u}_h^n \cdot \nabla C^n, r_h^n)$ in b_1 and we get:

$$b_1 = \left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(R_h C^n - C^n) dt, r_h^n\right) + \left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(C^n - C(t)) dt, r_h^n\right). \tag{5.16}$$

Using the L^2-L^4-L^4 generalized Cauchy–Schwarz inequality, (4.2) and Lemma 2.1, we obtain, for any $\xi_4 > 0$:

$$\begin{aligned}&\left|\left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(R_h C^n - C^n) dt, r_h^n\right)\right| \leq \tau \|\mathbf{u}_h^n\|_{L^2(\Omega)^d} \|R_h C^n - C^n\|_{W^{1,4}(\Omega)} \|r_h^n\|_{L^4(\Omega)} \\ &\quad \leq c\tau h \|\mathbf{u}_h^n\|_{L^2(\Omega)^d} \|C\|_{L^\infty(0,T;W^{2,4}(\Omega))} \|r_h^n\|_{L^4(\Omega)} \\ &\quad \leq \frac{c^2 \xi_4 h^2 \tau}{2} \|\mathbf{u}_h^n\|_{L^2(\Omega)^d}^2 \|C\|_{L^\infty(0,T;W^{2,4}(\Omega))}^2 + \frac{\tau (S_4^0)^2}{2\xi_4} |r_h^n|_{1,\Omega}^2.\end{aligned} \tag{5.17}$$

By choosing $\xi_4 = 25(S_4^0)^2/\alpha$, we obtain by using (4.13):

$$\left|\left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(R_h C^n - C^n)dt, r_h^n\right)\right| \le c_3 h^2 \tau \|C\|^2_{L^\infty(0,T;W^{2,4}(\Omega))} + \frac{\alpha}{50}\tau |r_h^n|^2_{1,\Omega}. \tag{5.18}$$

By treating as above, we have for any $\xi_5 > 0$:

$$\begin{aligned}
&\left|\left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(C^n - C(t))dt, r_h^n\right)\right| = \left|\left(\int_{t_{n-1}}^{t_n}\int_t^{t_n} \mathbf{u}_h^n \cdot \nabla C'(s)dsdt, r_h^n\right)\right| \\
&\quad = \left|\left(\int_{t_{n-1}}^{t_n}\int_s^{t_{n-1}} \mathbf{u}_h^n \cdot \nabla C'(s)dtds, r_h^n\right)\right| \\
&\quad \le \|\mathbf{u}_h^n\|_{L^2(\Omega)^d}\|r_h^n\|_{L^4(\Omega)}\int_{t_{n-1}}^{t_n}\|\nabla C'(s)\|_{L^4(\Omega)}(s - t_{n-1})ds \\
&\quad \le \frac{\tau^2\xi_5}{6}\|\mathbf{u}_h^n\|^2_{L^2(\Omega)^d}\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;W^{1,4}(\Omega))} + \frac{\tau(S_4^0)^2}{2\xi_5}|r_h^n|^2_{1,\Omega}.
\end{aligned} \tag{5.19}$$

We deduce by regrouping (5.16), (5.18) and (5.19) (for $\xi_5 = 25(S_4^0)^2/\alpha$) that

$$|b_1| \le c_3\tau h^2\|C\|^2_{L^\infty(0,T;W^{2,4}(\Omega))} + c_4\tau^2\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;W^{1,4}(\Omega))} + \frac{\alpha}{25}\tau|r_h^n|^2_{1,\Omega}. \tag{5.20}$$

We insert $\mathbf{u}^n$ in b_2 and we get, using the L^2-L^4-L^4 inequality

$$\begin{aligned}
|b_2| &= \left|\left(\int_{t_{n-1}}^{t_n}(\mathbf{u}_h^n - \mathbf{u}^n)\cdot\nabla C(t)dt, r_h^n\right) + \left(\int_{t_{n-1}}^{t_n}\int_t^{t_n}\mathbf{u}'(s)\cdot\nabla C(t)dsdt, r_h^n\right)\right| \\
&\le \tau\|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(\Omega)^d}\|C\|_{L^\infty(0,T;W^{1,4}(\Omega))}\|r_h^n\|_{L^4(\Omega)} \\
&\quad + \tau^{3/2}||C||_{L^\infty(0,T;H^1(\Omega))}||\frac{\partial \mathbf{u}}{\partial t}||_{L^2(t_{n-1},t_n;L^4(\Omega)^d)}\|r_h^n\|_{L^4(\Omega)}.
\end{aligned} \tag{5.21}$$

By using (2.8) and Lemma 2.1 and taking $\xi = 25(S_4^0)^2/\alpha$, we obtain:

$$\begin{aligned}
|b_2| &\le c_{51}\tau\|C\|^2_{L^\infty(0,T;W^{1,4}(\Omega))}\|\mathbf{u}_h^n - \mathbf{u}^n\|^2_{L^2(\Omega)^d} \\
&\quad + c_{52}\tau^2||C||^2_{L^\infty(0,T;H^1(\Omega))}||\frac{\partial \mathbf{u}}{\partial t}||^2_{L^2(t_{n-1},t_n;L^4(\Omega)^d)} + \frac{\alpha}{25}\tau|r_h^n|^2_{1,\Omega}.
\end{aligned} \tag{5.22}$$

Finally, we combine (5.14), (5.20) and (5.22), and we deduce that

$$\begin{aligned}
&\left|\left(\tau\mathbf{u}_h\cdot\nabla C_h^n - \int_{t_{n-1}}^{t_n}\mathbf{u}\cdot\nabla C(t)dt, r_h^n\right)\right| \\
&\quad \le c_3\tau h^2\|C\|^2_{L^\infty(0,T;W^{2,4}(\Omega))} + c_4\tau^2\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;W^{1,4}(\Omega))}
\end{aligned}$$

$$+c_{51}\tau\|C\|^2_{L^\infty(0,T;W^{1,4}(\Omega))}\|\mathbf{u}_h^n-\mathbf{u}^n\|^2_{L^2(\Omega)^d}$$
$$+c_{52}\tau^2||C||^2_{L^\infty(0,T;H^1(\Omega))}||\frac{\partial \mathbf{u}}{\partial t}||^2_{L^2(t_{n-1},t_n;L^4(\Omega)^d)}+\frac{2\alpha}{25}\tau|r_h^n|^2_{1,\Omega}, \quad (5.23)$$

where c_3, c_4, c_{51} and c_{52} are positive constants independent of h and τ.

The last term in the left-hand side of (5.6) can be bounded, by inserting $\pm(\tau R_h C^n, r_h^n)$ and $\pm\tau(C^n, r_h^n)$

$$\left(\tau C_h^n-\int_{t_{n-1}}^{t_n}C(t)dt, r_h^n\right)=\tau(C_h^n-R_hC^n,r_h^n)+\tau(R_hC^n-C^n,r_h^n)$$
$$+\left(\int_{t_{n-1}}^{t_n}(C^n-C(t))dt, r_h^n\right)$$

so that

$$\left|\left(\tau C_h^n-\int_{t_{n-1}}^{t_n}C(t)dt, r_h^n\right)-\tau\|r_h^n\|^2_{L^2(\Omega)}\right|$$
$$\leq\tau\left|(R_hC^n-C^n,r_h^n)\right|+\left|\left(\int_{t_{n-1}}^{t_n}(C^n-C(t))dt, r_h^n\right)\right|. \quad (5.24)$$

Moreover, using the approximation properties of R_h, we have:

$$\tau\left|(R_hC^n-C^n,r_h^n)\right|\leq\tau\|R_hC^n-C^n\|_{L^2(\Omega)}\|r_h^n\|_{L^2(\Omega)}$$
$$\leq c\tau h\|r_h^n\|_{L^2(\Omega)}\|C\|_{L^\infty(0,T;H^1(\Omega))}$$
$$\leq\frac{c^2h^2\tau\xi_6}{2}\|C\|^2_{L^\infty(0,T;H^1(\Omega))}+\frac{\tau(S_2^0)^2}{2\xi_6}|r_h^n|^2_{1,\Omega}. \quad (5.25)$$

Furthermore,

$$\left|\left(\int_{t_{n-1}}^{t_n}(C^n-C(t))dt, r_h^n\right)\right|=\left|\left(\int_{t_{n-1}}^{t_n}\int_t^{t_n}C'(s)dsdt, r_h^n\right)\right|$$
$$\leq\frac{\tau^{3/2}}{\sqrt{3}}\|r_h^n\|_{L^2(\Omega)}\|\frac{\partial C}{\partial t}\|_{L^2(t_{n-1},t_n;L^2(\Omega))}$$
$$\leq\frac{\xi_7\tau^2}{6}\|\frac{\partial C}{\partial t}\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}+\frac{\tau(S_2^0)^2}{2\xi_7}|r_h^n|^2_{1,\Omega}. \quad (5.26)$$

Next, we combine (5.24), (5.25) and (5.26), and we choose $\xi_6=\xi_7=\dfrac{25(S_2^0)^2}{\alpha}$ to obtain:

$$\left|\left(\tau C_h^n - \int_{t_{n-1}}^{t_n} C(t)dt, r_h^n\right) - \tau \|r_h^n\|_{L^2(\Omega)}^2\right| \leq c_6 \tau h^2 \|C\|_{L^\infty(0,T;H^1(\Omega))}^2$$
$$+ c_7 \tau^2 \|\frac{\partial C}{\partial t}\|_{L^2(t_{n-1},t_n;L^2(\Omega))}^2 + \frac{\alpha}{25} \tau |r_h^n|_{1,\Omega}^2, \tag{5.27}$$

where c_6 and c_7 are positive constants independent of τ and h.
Now we use (5.6), (5.7), (5.9), (5.13), (5.23) and (5.27) and we sum over n from 1 to $m \leq N$. This leads to

$$\frac{1}{2}\|r_h^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^m \|r_h^n - r_h^{n-1}\|_{L^2(\Omega)}^2 + \alpha \sum_{n=1}^m \tau |r_h^n|_{1,\Omega}^2 + r_0 \sum_{n=1}^m \tau \|r_h^n\|_{L^2(\Omega)}^2$$
$$\leq c(h^2 + \tau^2) + c' \sum_{n=1}^m \tau \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(\Omega)^d}^2 + \frac{9\alpha}{50} \sum_{n=1}^m \tau |r_h^n|_{1,\Omega}^2.$$

Finally, we obtain

$$\sup_{0 \leq n \leq N} \|C_h^n - C^n\|_{L^2(\Omega)}^2 + \alpha \sum_{n=1}^N \tau |C_h^n - C^n|_{1,\Omega}^2$$
$$+ \sum_{n=1}^N \|(C_h^n - C^n) - (C_h^{n-1} - C^{n-1})\|_{L^2(\Omega)}^2$$
$$+ r_0 \sum_{n=1}^N \tau \|C_h^n - C^n\|_{L^2(\Omega)}^2 \leq c(h^2 + \tau^2) + c' \sum_{n=1}^N \tau \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(\Omega)^d}^2,$$

where c and c' are positive constants independent of h and τ. □

Remark 5.2 If the viscosity ν is constant independent of C, it suffices to take $\mathbf{u} \in L^\infty(0, T; H^1(\Omega)^d)$ in Theorem 5.1, as in this case, Eq. (5.4) becomes

$$\int_\Omega \nu(\mathbf{u}^n - \mathbf{u}_h^n)(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x} = \int_\Omega (\mathbf{f}_1(C^n(\mathbf{x})) - \mathbf{f}_1(C_h^{n-1}(\mathbf{x}))) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}. \tag{5.28}$$

Corollary 5.3 *Under the assumptions of Theorem 5.1, we have the a priori error estimates corresponding to* $(V_{h,1})$*:*

$$\begin{aligned} \sup_{0 \leq n \leq N} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d} &\leq c(h + \tau), \\ \sup_{0 \leq n \leq N} \|p^n - p_h^n\|_{L^2(\Omega)} &\leq c'(h + \tau), \\ \sup_{0 \leq n \leq N} \|C_h^n - C^n\|_{L^2(\Omega)}^2 + \alpha \sum_{n=1}^N \tau |C_h^n - C^n|_{1,\Omega}^2 &\leq c''(h^2 + \tau^2), \end{aligned} \tag{5.29}$$

where c, c' and c'' are independent of h and τ.

Proof We first consider relation (5.1). By inserting C^{n-1} and using the triangle inequality, we have

$$\sum_{n=1}^{N}\tau\|\mathbf{u}_h^n-\mathbf{u}^n\|_{L^2(\Omega)^d}^2\leq c\left(\sum_{n=1}^{N}\tau\|C_h^{n-1}-C^{n-1}\|_{L^2(\Omega)}^2+\sum_{n=1}^{N}\tau\int_\Omega\Big|\int_{t_{n-1}}^{t_n}C'(s)ds\Big|^2d\mathbf{x}\right).$$

We then deduce from (5.3) the following relation

$$\sup_{0\leq n\leq N}\|C_h^n-C^n\|_{L^2(\Omega)}^2+\alpha\sum_{n=1}^{N}\tau|C_h^n-C^n|_{1,\Omega}^2\leq c(h^2+\tau^2)+c'\sum_{n=1}^{N}\tau\|C_h^{n-1}-C^{n-1}\|_{L^2(\Omega)}^2. \tag{5.30}$$

Finally, the following inequality follows from the discrete Grönwall Lemma

$$\sup_{0\leq n\leq N}\|C_h^n-C^n\|_{L^2(\Omega)}^2+\alpha\sum_{n=1}^{N}\tau|C_h^n-C^n|_{1,\Omega}^2\leq c''(h^2+\tau^2). \tag{5.31}$$

It is crucial to note that in the right-hand side of (5.30), $\sum_{n=1}^{N}\tau$ is bounded by the final time T which is a quantity that does not depend on N; thus the constant c'' that appears in (5.31) does not depend on τ.

Inequality (5.31) and Inequalities (5.1) and (5.2) yield the desired bounds in (5.29). □

Theorem 5.4 *Let* $(\mathbf{u}, p, C)$ *be the solution of Problem* (V_a) *and* $(\mathbf{u}_h^n, p_h^n, C_h^n)$ *be the solution of Problem* $(V_{h,2})$. *If* $\mathbf{u}\in L^\infty(0,T;H^1(\Omega)^d)\cap L^\infty(0,T;L^\infty(\Omega)^d)$, $p\in L^\infty(0,T;H^2(\Omega))$, $C\in L^\infty(0,T;W^{2,4}(\Omega))$ *and* $\dfrac{\partial C}{\partial t}\in L^2(0,T;W^{1,4}(\Omega))$ *and under Assumption 3.1, there exist positive constants* c, c_1, c', c'' *depending on* $\mathbf{u}$ *and* α *such that,*

$$\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2(\Omega)^d}\leq\frac{1}{\nu_1}\big(ch+(c_{\mathbf{f}_1}^*+\lambda\|\mathbf{u}\|_{L^\infty(0,T:L^\infty(\Omega)^d)})||C^n-C_h^{n-1}||_{L^2(\Omega)}\big), \tag{5.32}$$

$$|p^n-p_h^n|_{1,\Omega}\leq c'h+\frac{\nu_2}{\beta_2}\|\mathbf{u}^n-\mathbf{u}_h^n\|_{L^2(\Omega)^d}+\frac{1}{\beta_2}(c_{\mathbf{f}_1}^*+\lambda\|\mathbf{u}\|_{L^\infty(0,T:L^\infty(\Omega)^d)})||C^n-C_h^{n-1}||_{L^2(\Omega)}, \tag{5.33}$$

and

$$\sup_{0\leq n\leq N}\|C_h^n-C^n\|_{L^2(\Omega)}^2+\alpha\sum_{n=1}^{N}\tau|C_h^n-C^n|_{1,\Omega}^2+\sum_{n=1}^{N}\|(C_h^n-C^n)-(C_h^{n-1}-C^{n-1})\|_{L^2(\Omega)}^2+r_0\sum_{n=1}^{N}\tau\|C_h^n-C^n\|_{L^2(\Omega)}^2\leq c''(h^2+\tau^2)+c_1\sum_{n=1}^{m}\tau\|\mathbf{u}_n^n-\mathbf{u}^n\|_{L^2(\Omega)^d}^2. \tag{5.34}$$

Proof Let $(\mathbf{u}, p, C)$ and $(\mathbf{u}_h^n, p_h^n, C_h^n)$ solve respectively (V_a) and $(V_{h,2})$. We shall prove first (5.32), next (5.33), and finally (5.34).

Let us estimate the velocity error. By taking the difference between the first equations of (V_a) and $(V_{h,2})$ and testing with $\mathbf{v} = \mathbf{v}_h \in \mathcal{V}_{h,2}$, we obtain

$$\begin{aligned}\int_\Omega \nu(C_h^{n-1}(\mathbf{x}))(\mathbf{u}^n - \mathbf{u}_h^n)(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x} &= \int_\Omega (\mathbf{f}_1(C^n(\mathbf{x})) - \mathbf{f}_1(C_h^{n-1}(\mathbf{x}))) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}\\ &+ \int_\Omega (\nu(C_h^{n-1}(\mathbf{x})) - \nu(C^n(\mathbf{x})))\mathbf{u}^n(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}\\ &- \int_\Omega \nabla(p^n - r_h(p^n)) \cdot \mathbf{v}_h d\mathbf{x}. \end{aligned} \tag{5.35}$$

By inserting $\mathcal{F}_h \mathbf{u}^n$, choosing $\mathbf{v}_h = (\mathcal{F}_h \mathbf{u}^n - \mathbf{u}_h^n)$ which belongs to $\mathcal{V}_{h,2}$ using the triangle inequality and the approximation properties of $\mathcal{F}_h$ and r_h and the bounds on ν, we obtain (5.32).

To prove the error estimate on the pressure, we take the difference between the first equations of (V_a) and $(V_{h,2})$, insert $\nabla r_h(p^n)$, test with $\mathbf{v}_h$ in $\mathcal{W}_{h,2}$, and we obtain

$$\begin{aligned}\int_\Omega \nabla(p_h^n - r_h(p^n))(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x} &= \int_\Omega \nabla(p^n - r_h(p^n))(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}\\ &+ \int_\Omega \nu(C_h^{n-1})(\mathbf{u}^n - \mathbf{u}_h^n)(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}\\ &+ \int_\Omega (\nu(C^n(\mathbf{x})) - \nu(C_h^{n-1})(\mathbf{x}))\mathbf{u}^n(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}\\ &- \int_\Omega (\mathbf{f}_1(C^n(\mathbf{x})) - \mathbf{f}_1(C_h^{n-1}(\mathbf{x}))) \cdot \mathbf{v}_h(\mathbf{x}) d\mathbf{x}. \end{aligned} \tag{5.36}$$

It follows from the inf-sup condition (4.22) and by applying the Cauchy–Schwarz inequality to the right-hand side of (5.36) and Assumption 3.1 that

$$\begin{aligned}|p_h^n - r_h(p^n)|_{1,\Omega} \le \frac{1}{\beta_2}\Big[& |p^n - r_h(p^n)|_{1,\Omega} + \nu_2 \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d}\\ & + (c_{\mathbf{f}_1}^* + \lambda \|\mathbf{u}\|_{L^\infty(0,T:L^\infty(\Omega)^d)}) || C^n - C_h^{n-1} ||_{L^2(\Omega)} \Big]. \end{aligned} \tag{5.37}$$

Inserting $r_h(p^n)$ in $|p^n - p_h^n|_{1,\Omega}$ and using a triangular inequality and (5.37) implies (5.33) by using the properties of r_h.

Now we prove (5.34). We choose the test function $S_h = r_h^n = C_h^n - R_h C^n$ in the third equation of $(V_{h,2})$ and multiply it by the time step τ. Then, we subtract the third equation of (V_a) integrated over $[t_{n-1}; t_n]$. We obtain:

$$\left((C_h^n - C_h^{n-1}) - (C^n - C^{n-1}), r_h^n\right) + \alpha\left(\tau\nabla C_h^n - \int_{t_{n-1}}^{t_n} \nabla C(t)dt, \nabla r_h^n\right)$$
$$+ \left(\tau \mathbf{u}_h^n \cdot \nabla C_h^n - \int_{t_{n-1}}^{t_n} \mathbf{u}(t)\cdot\nabla C(t)dt, r_h^n\right)$$
$$+ \frac{\tau}{2}(\text{div }\mathbf{u}_h^n C_h^n, r_h^n) + r_0\left(\tau C_h^n - \int_{t_{n-1}}^{t_n} C(t)dt, r_h^n\right) = 0. \tag{5.38}$$

All the terms of (5.38) can be treated as the proof of Theorem 5.1 (see (5.9), (5.13), (5.27)) except the non-linear one denoted by b_n.

By inserting $\pm\tau(\mathbf{u}_h^n \cdot \nabla R_h C^n, r_h^n)$, $\pm\frac{\tau}{2}(\text{div }\mathbf{u}_h^n R_h C^n, r_h^n)$, b_n becomes:

$$b_n = \tau(\mathbf{u}_h^n \cdot \nabla(C_h^n - R_h C^n), r_h^n) + \tau(\mathbf{u}_h^n \cdot \nabla R_h C^n, r_h^n) - \left(\int_{t_{n-1}}^{t_n} \mathbf{u}(t)\cdot\nabla C(t)dt, r_h^n\right)$$
$$+ \frac{\tau}{2}(\text{div }\mathbf{u}_h^n(C_h^n - R_h C^n), r_h^n) + \frac{\tau}{2}(\text{div }\mathbf{u}_h^n\, R_h C^n, r_h^n).$$

Then, we insert $\pm\tau(\mathbf{u}_h^n\cdot\nabla C^n, r_h^n)$, $\pm\left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n\cdot\nabla C(t)dt, r_h^n\right)$ and $\pm\frac{\tau}{2}(\text{div }\mathbf{u}_h^n\, C^n, r_h^n)$ to get by noticing that $\tau(\mathbf{u}_h^n \cdot \nabla(C_h^n - R_h C^n), r_h^n) + \frac{\tau}{2}(\text{div }\mathbf{u}_h^n(C_h^n - R_h C^n), r_h^n) = 0$:

$$b_n = \tau(\mathbf{u}_h^n \cdot \nabla(R_h C^n - C^n), r_h^n) + \left(\int_{t_{n-1}}^{t_n} \mathbf{u}_h^n \cdot \nabla(C^n - C(t))dt, r_h^n\right)$$
$$+ \int_{t_{n-1}}^{t_n} ((\mathbf{u}_h^n - \mathbf{u}(t))\cdot\nabla C(t)dt, r_h^n)$$
$$+ \frac{\tau}{2}(\text{div }\mathbf{u}_h^n(R_h C^n - C^n), r_h^n) + \frac{\tau}{2}(\text{div }\mathbf{u}_h^n\, C^n, r_h^n). \tag{5.39}$$

The sum of the first two terms in the right-hand side of (5.39) is exactly the expression of b_1 in (5.16). The third term in the right-hand side of (5.39) is exactly b_2 as defined in (5.15). Therefore, these terms can be treated exactly as in the proof of theorem 5.1. The last two terms, denoted below as $b_{n,4}$ and $b_{n,5}$ can be treated as follows:

$$\begin{aligned} b_{n,4} &= \frac{\tau}{2}(\text{div }\mathbf{u}_h^n(R_h C^n - C^n), r_h^n) \\ &= -\frac{\tau}{2}(\mathbf{u}_h^n \cdot \nabla(R_h C^n - C^n), r_h^n) - \frac{\tau}{2}(\mathbf{u}_h^n \cdot \nabla r_h^n, (R_h C^n - C^n)) \\ &= b_{n,4}^1 + b_{n,4}^2, \end{aligned}$$

where, using Lemma 2.1, (2.8) and (4.2), we have, for any $\bar{\xi}_1 > 0$

$$\begin{aligned} |b_{n,4}^1| &\le \frac{\tau}{2}\|\mathbf{u}_h^n\|_{L^2(\Omega)^d}\|\nabla(R_h C^n - C^n)\|_{L^4(\Omega)}\|r_h^n\|_{L^4(\Omega)} \\ &\le \frac{ch^2\tau\bar{\xi}_1}{8}\|\mathbf{u}_h^n\|^2_{L^2(\Omega)^d}\|C\|^2_{L^\infty(0,T;W^{2,4}(\Omega))} + \frac{\tau(S_4^0)^2}{2\bar{\xi}_1}|r_h^n|^2_{1,\Omega} \end{aligned}$$

and where, using (4.2) and (4.3) with $d \leq 3$ we have, for any $\bar{\xi}_2 > 0$

$$
\begin{aligned}
|b_{n,4}^2| &\leq \frac{\tau}{2} \|\mathbf{u}_h^n\|_{L^4(\Omega)^d} \|\nabla r_h^n\|_{L^2(\Omega)} \|R_h C^n - C^n\|_{L^4(\Omega)} \\
&\leq \frac{ch^2}{2} \tau \|\mathbf{u}_h^n\|_{L^2(\Omega)^d} \times h^{-\frac{d}{4}} |r_h^n|_{1,\Omega} \|C^n\|_{W^{2,4}(\Omega)} \\
&\leq \frac{c\bar{\xi}_2}{8} h^{\frac{8-d}{2}} \tau \|\mathbf{u}_h^n\|^2_{L^2(\Omega)^d} \|C\|^2_{L^\infty(0,T;W^{1,4}(\Omega))} + \frac{\tau}{2\bar{\xi}_2} |r_h^n|^2_{1,\Omega}.
\end{aligned}
$$

Note that in the bounds of $|b_{n,4}^1|$ and $|b_{n,4}^2|$, the $L^2(\Omega)$ norm of $\mathbf{u}_h^n$ can be bounded by a constant depending only on the data of the problem through Theorem 4.2 using both (4.23) and (4.24). Note also that $\frac{8-d}{2} \geq 2$ for $d \leq 3$, which is sufficient for our proof.

Moreover, $b_{n,5}$ can be decomposed in the same way and treated as follows (where we use that div $\mathbf{u}^n = 0$):

$$
\begin{aligned}
b_{n,5} &= \frac{\tau}{2} (\text{div } (\mathbf{u}_h^n - \mathbf{u}^n) C^n, r_h^n) \\
&= -\frac{\tau}{2} ((\mathbf{u}_h^n - \mathbf{u}^n) \cdot \nabla C^n, r_h^n) - \frac{\tau}{2} ((\mathbf{u}_h^n - \mathbf{u}^n) \cdot \nabla r_h^n, C^n) \\
&= b_{n,5}^1 + b_{n,5}^2.
\end{aligned}
$$

Using Lemma 2.1 and (2.8) we have, for any $\bar{\xi}_3 > 0$

$$
\begin{aligned}
|b_{n,5}^1| &= |\frac{\tau}{2} ((\mathbf{u}_h^n - \mathbf{u}^n) \cdot \nabla C^n, r_h^n)| \\
&\leq \frac{\tau}{2} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(\Omega)^d} \|C\|_{L^\infty(0,T;W^{1,4}(\Omega))} \|r_h^n\|_{L^4(\Omega)} \\
&\leq \frac{\tau\bar{\xi}_3}{8} \|\mathbf{u}_h^n - \mathbf{u}^n\|^2_{L^2(\Omega)^d} \|C\|^2_{L^\infty(0,T;W^{1,4}(\Omega))} + \frac{\tau (S_4^0)^2}{2\bar{\xi}_3} |r_h^n|^2_{1,\Omega}
\end{aligned}
$$

and for any $\bar{\xi}_4 > 0$

$$
\begin{aligned}
|b_{n,5}^2| &= |\frac{\tau}{2} ((\mathbf{u}_h^n - \mathbf{u}^n) \cdot \nabla r_h^n, C^n)| \\
&\leq \frac{\tau}{2} \|\mathbf{u}_h^n - \mathbf{u}^n\|_{L^2(\Omega)^d} \|C^n\|_{L^\infty(\Omega)} |r_h^n|_{1,\Omega} \\
&\leq \frac{\tau\bar{\xi}_4}{8} \|\mathbf{u}_h^n - \mathbf{u}^n\|^2_{L^2(\Omega)^d} \|C\|^2_{L^\infty([0,T]\times\Omega)} + \frac{\tau}{2\bar{\xi}_4} |r_h^n|^2_{1,\Omega}.
\end{aligned}
$$

By using the above bounds with a suitable choice of $\bar{\xi}_i$, $i = 1, \ldots, 4$, and summing over n from 1 to $m \leq N$, we get

$$\frac{1}{2}\|r_h^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\sum_{n=1}^{m}\|r_h^n - r_h^{n-1}\|_{L^2(\Omega)}^2 + \alpha\sum_{n=1}^{m}\tau|r_h^n|_{1,\Omega}^2 + r_0\sum_{n=1}^{m}\tau\|r_h^n\|_{L^2(\Omega)}^2$$
$$\leq c(h^2+\tau^2) + c\sum_{n=1}^{m}\tau\|\mathbf{u}_n^n - \mathbf{u}^n\|_{L^2(\Omega)^d}^2,$$

and we deduce finally (5.34). □

Corollary 5.5 *Under the assumption of Theorem 5.4, we have the a priori error estimates corresponding to* $(V_{h,2})$*:*

$$\begin{aligned}\sup_{0\leq n\leq N}\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2(\Omega)^d} &\leq c(h+\tau),\\ \sup_{0\leq n\leq N}|p^n - p_h^n|_{H^1(\Omega)} &\leq c'(h+\tau),\\ \sup_{0\leq n\leq N}\|C_h^n - C^n\|_{L^2(\Omega)}^2 + \alpha\sum_{n=1}^{N}\tau|C_h^n - C^n|_{1,\Omega}^2 &\leq c''(h^2+\tau^2),\end{aligned} \tag{5.40}$$

where c, c' *and* c'' *are independent of* h *and* τ.

6 Numerical results

To validate the theoretical results, we perform several numerical simulations using Freefem++ (see [18]). We consider a square domain $\Omega =]0,1[^2$. Each edge is divided into N equal segments so that Ω is divided into $2N^2$ triangles. For the numerical tests, we consider $\alpha = 1$, $r_0 = 1$, $\mathbf{f}_1(C) = C+1$ and $\nu(C) = \sin(C)+2$. We choose the right-hand sides $\mathbf{f}_0$ and g so that the exact solution is given by $(\mathbf{u}, p, C) = (e^{-t/4}curl\psi, p, C)$ where ψ, p and C are defined by

$$\begin{aligned}\psi(x,y) &= e^{-100\left((x-\frac{1}{2})^2+(y-\frac{1}{2})^2\right)},\\ p(x,y,t) &= (t+1)\cos(\pi x)\cos(\pi y),\\ C(x,y,t) &= \sin t\, x^2(x-1)^2y^2(y-1)^2.\end{aligned} \tag{6.1}$$

We define the following total relative error between the exact and numerical solutions:

$$err = \left(\frac{\sup_{1\leq n\leq N}|\mathbf{u}_h^n - \mathbf{u}^n|_{L^2(\Omega)^2}^2 + \sup_{1\leq n\leq N}|p_h^n - p^n|_{L^2(\Omega)}^2 + \sum_{n=1}^{N}\tau|C_h^n - C^n|_{H^1(\Omega)}^2}{\sup_{1\leq n\leq N}|\mathbf{u}^n|_{L^2(\Omega)^2}^2 + \sup_{1\leq n\leq N}|p^n|_{L^2(\Omega)}^2 + \sum_{n=1}^{N}\tau|C^n|_{H^1(\Omega)}^2}\right)^{1/2}. \tag{6.2}$$

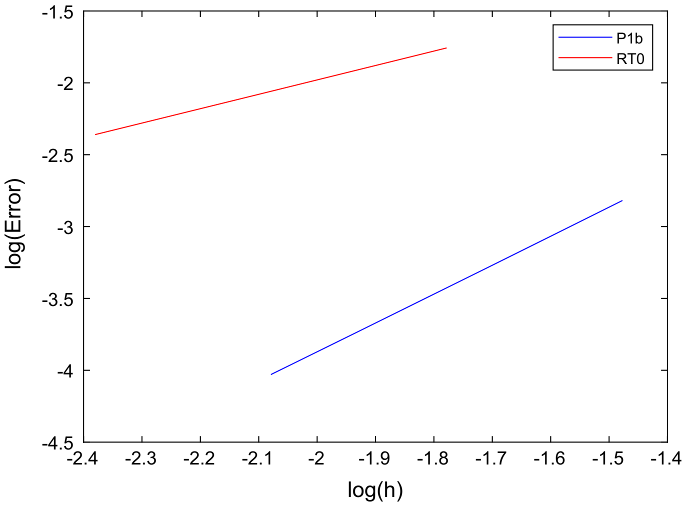

Fig. 1 Pressure error with respect to the mesh size

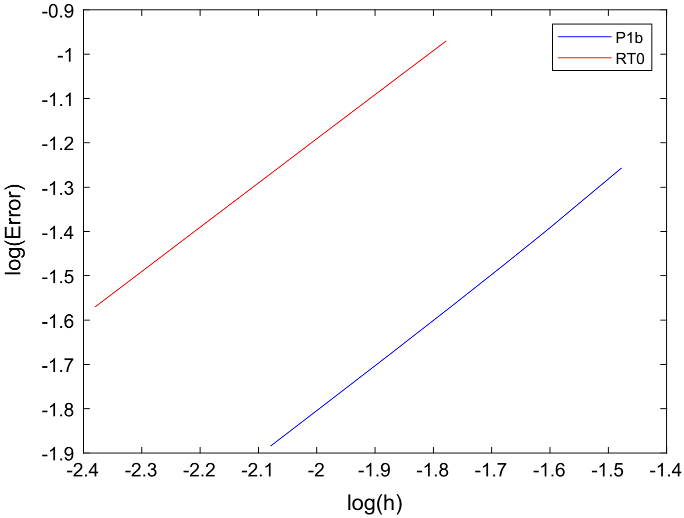

Fig. 2 Velocity error with respect to the mesh size

We test the algorithms for N ranging from 60 to 120, by a step of 10 with $T = 1, h = \dfrac{1}{N}$ and $\tau = h$. Moreover, in order to be able to compare the accuracy of the schemes on similar number of unknowns and non-zero entries in the matrices associated to the linear systems, we also use $N = 30$ for the second scheme and $N = 240$ for the first one.

Figures 1, 2, and 3 show, in logarithmic scale, the curves of the errors on the pressure, the velocity and the concentration, respectively, according to the mesh step h for the first and second schemes $(V_{h,1})$ and $(V_{h,2})$. Figure 4 represents, in logarithmic

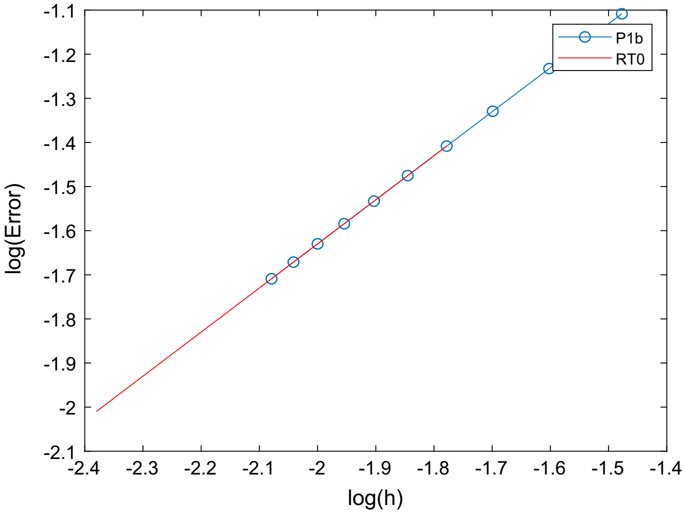

Fig. 3 Concentration error with respect to the mesh size

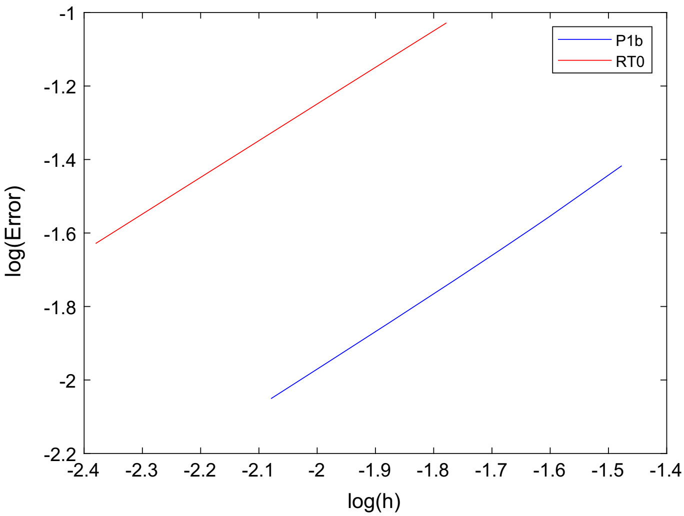

Fig. 4 Total relative error with respect to the mesh size

scale, the curve of the total relative error according to the mesh step h for the first and second discrete schemes $(V_{h,1})$ and $(V_{h,2})$. Their respective slopes are 1.0013 and 1.0142 which is in conformity with the theoretical order of convergence of both schemes. We also note that the second scheme, which uses higher order polynomials for the pressure than the first scheme, yields a second order accuracy on this variable.

In order to include a more complete comparison of the accuracy of the two schemes with respect to their numerical complexity, we show on Figs. 4, 5 and 6 their total relative errors, not only with respect to the mesh size, but also with respect to the

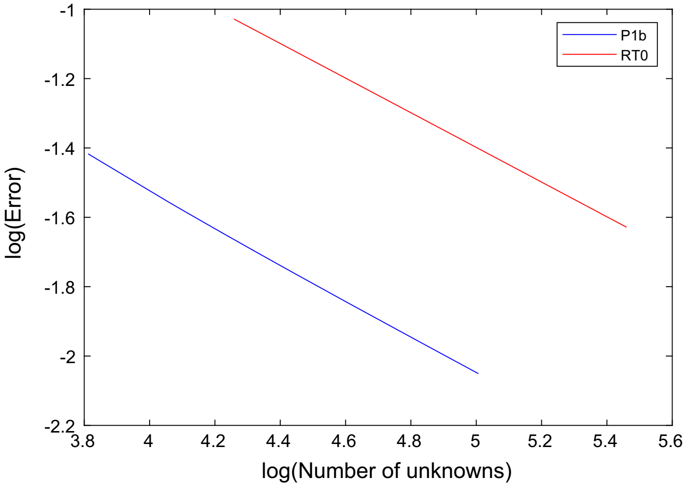

Fig. 5 Total relative error with respect to the number of unknowns

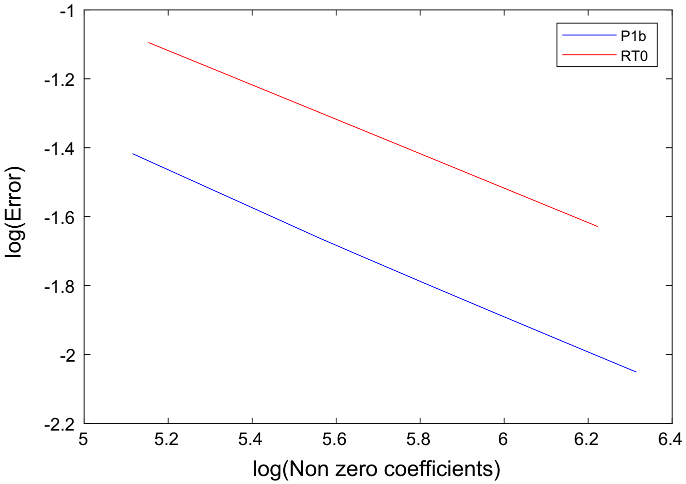

Fig. 6 Total relative error with respect to the number of non-zero coefficients in the matrices

number of unknowns and with respect to the number of non-zero coefficients in the associated matrices. In all three figures, the curve of the second scheme is below the one of the first scheme, which means that, for a given complexity, the second scheme is more accurate than the first one for this particular example. It is probable that this advantage is linked to the high stiffness of the exact solution, which is better captured by the second scheme which uses higher-order polynomials.

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