Strong convergence analysis of iterative solvers for random operator equations

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Abstract

For the numerical solution of linear systems that arise from discretized linear partial differential equations, multigrid and domain decomposition methods are well established. Multigrid methods are known to have optimal complexity and domain decomposition methods are in particular useful for parallelization of the implemented algorithm. For linear random operator equations, the classical theory is not directly applicable, since condition numbers of system matrices may be close to degenerate due to non-uniform random input. It is shown that iterative methods converge in the strong, i.e. L^p, sense if the random input satisfies certain integrability conditions. As a main result, standard multigrid and domain decomposition methods are applicable in the case of linear elliptic partial differential equations with lognormal diffusion coefficients and converge strongly with deterministic bounds on the computational work which are essentially optimal. This enables the application of multilevel Monte Carlo methods with rigorous, deterministic bounds on the computational work.

Keywords Strong error estimates · Multigrid methods · Domain decomposition methods · Uncertainty quantification · Random PDEs with lognormal coefficients · Multilevel Monte Carlo

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1 Introduction

Mathematical models of partial differential equations (PDEs) with random input receive increasing attention in recent years. In particular, linear diffusion equations

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with random coefficients to model random media are considered and respective system responses or quantities of interest are studied, cf. [12,13,15,16,27,36]. Generally, the random input is function valued and will be referred to as random field. In the numerical analysis of these problems constants in error estimates become random variables and may have a distribution with unbounded support. In the case that quantities of interest are moments of system responses, numerical analysis has been performed if certain integrability conditions are satisfied by the random system input. There, the case of Gaussian random fields (GRFs) as random inputs is frequently considered. Multilevel techniques such as multilevel Monte Carlo (MLMC) have been established to accelerate the approximation of moments. There, sample numbers are chosen in a greedy technique to optimize the error versus the required computational cost, cf. [3,13,21,22,24,36]. The requirement of MLMC to be efficient is a small variance between higher levels or to put it differently, the strong error between higher levels has to converge with an available rate. In the case that finite element (FE) discretizations are used and the random input is unbounded such as GRFs, the numerical analysis and numerical experiments presented in these references relied on sparse direct solvers, cf. [13,27,36]. Hence, the applicability of iterative methods which are known to be fast for deterministic problems is of natural interest, e.g., so-called full multigrid is well-known to have *optimal* complexity in the case of Poisson's equation, cf. [2].

In this paper we establish rigorously the strong convergence of a wide class of standard iterative solvers, which yields essentially optimal computational cost estimates also in the case of GRF input with low spatial regularity. As an application, MLMC is discussed with deterministic, essentially optimal estimates of the computation cost, which was previously unknown. By optimal, we mean that solutions of linear systems under consideration with dimension $\mathcal{O}(N)$ may be approximated in computational cost $\mathcal{O}(N)$ consistently with the overall discretization error. In the computational uncertainty quantification (UQ) literature, iterative solvers have been considered mostly in the context of stochastic collocation and stochastic Galerkin, cf. [18,35,38]. A particular variant of MLMC with multigrid for GRF inputs has been proposed in [30] and computational experiments have been performed.

The present manuscript analyzes the applicability and strong convergence of well established iterative methods for operator equations with unbounded random input in a general setting. Let $\mathcal{A}$ be a random, continuous linear operator from $\mathcal{V}$ to $\mathcal{V}^*$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) boundedly invertible, where $\mathcal{V}$ is a Hilbert space and $\mathcal{V}^*$ its dual space. Let its expectation $\mathbb{E}(\mathcal{A})$ be well-defined. In the present paper we are interested in the numerical analysis of approximations of the solution u to the random linear equation that for $\mathbb{P}$-almost every ($\mathbb{P}$-a.e.) $\omega \in \Omega$

$$\mathcal{A}(\omega)u(\omega) = f, \tag{1}$$

where $f \in \mathcal{V}^*$ is deterministic, in the strong sense, by iterative methods such as multigrid and domain decomposition methods. This can be rewritten in variational form to find $u : \Omega \to \mathcal{V}$ such that for $\mathbb{P}$-a.e. ω

$$a_\omega(u(\omega), v) := {}_{\mathcal{V}^*}\langle \mathcal{A}(\omega)u(\omega), v\rangle_{\mathcal{V}} = {}_{\mathcal{V}^*}\langle f, v\rangle_{\mathcal{V}} \quad \forall v \in \mathcal{V}.$$

Let us assume that there are strictly positive random variables $\hat{a}$ and $\breve{a}$ such that for $\mathbb{P}$-a.e. ω

$$\breve{a}(\omega) \; _{\mathcal{V}*}\langle \mathbb{E}(\mathcal{A})v, v\rangle_{\mathcal{V}} \leq \; _{\mathcal{V}*}\langle \mathcal{A}(\omega)v, v\rangle_{\mathcal{V}} \leq \hat{a}(\omega) \; _{\mathcal{V}*}\langle \mathbb{E}(\mathcal{A})v, v\rangle_{\mathcal{V}} \quad \forall v \in \mathcal{V}. \quad (2)$$

We will be particularly interested in the case that $\hat{a}$ and $\breve{a}^{-1}$ are unbounded random variables. This is for example the case for elliptic PDEs with lognormal coefficients. Thus, preconditioned finite dimensional discretizations suffer from random condition numbers that are unbounded and respectively iterative methods contract with random contraction numbers, which may have realizations arbitrarily close to one with positive probability. At first sight one may overcome this with random iteration numbers specified by a threshold of residual errors with the disadvantage of the occurrence of large iteration numbers, when samples of the random contraction number are close to one. Also, bounds on the computational work for such strategies would be probabilistic. A main new contribution of this paper is that deterministic iteration numbers exist that allow for strong convergence, i.e., convergence of well-known iterative methods in the $L^q(\Omega; \mathcal{V})$-norm, $q \in [1, +\infty)$, such as multigrid, multilevel preconditioned conjugate gradient, or domain decomposition. This is possible due to tail bounds of the random contraction numbers, which for example are satisfied in the important case of elliptic PDEs with lognormal coefficients. As a consequence, deterministic, essentially optimal complexity bounds are implied for the solution of resulting random linear systems when multigrid or domain decomposition methods are applied. This enables also rigorous, deterministic, essentially optimal complexity bounds for MLMC approximations of mean fields, which was previously unknown. Assumptions on the computational cost of PDE solvers that were made in previous papers [13,27,36] to obtain complexity bounds of MLMC and partly to calibrate the MLMC estimator are pervaded by the new theory presented in this manuscript. We will treat the case that $a_\omega(\cdot, \cdot)$ is symmetric for $\mathbb{P}$-a.e. ω. However, the presented theory can be extended to certain non-symmetric operators, see for example [6, Sect. 11].

In Sect. 2, we will review iterative methods as they were formulated in [39] in order to discuss various multilevel method in an unified framework. It will also be highlighted which parts in the framework and in the iterative methods are random. As a main result, we will develop integrability conditions on the random contraction numbers in Sect. 3 that result in sufficiently strong tail bounds in order to ensure strong convergence in the setting of multilevel discretizations of (1). The integrability conditions posed in Sect. 3 are analyzed for several multilevel methods such as multigrid, the so-called BPX preconditioner, cf. [10], and domain decomposition methods in Sects. 4 and 5. An important application of the presented theory are lognormal diffusion problems, which are briefly reviewed in Sect. 6. In particular deterministic bounds on the computational work without assumptions on the PDE solver of MLMC are implied, which is concluded in Sect. 7.1. Numerical experiments with GRF input are presented in Sect. 7.2 and confirm the theoretical analysis.

2 Iterative methods

In this section, we review iterative methods to approximate solutions to linear equations on a finite dimensional inner product space $(V, (\cdot, \cdot))$, where $\| \cdot \|$ denotes the norm that is induced by $(\cdot, \cdot)$. Let us consider the random linear equation that for $\mathbb{P}$-a.e. ω

$$A(\omega)u(\omega) = f, \tag{3}$$

where $A : V \to V$ is a random linear operator that is $\mathbb{P}$-a.s. symmetric positive definite with respect to $(\cdot, \cdot)$. Hence, (3) is $\mathbb{P}$-a.s. uniquely solvable. Note that we will often omit dependencies of random quantities on ω for notational convenience. Let us denote the bilinear form that is induced by A by $(\cdot, \cdot)_A$ and let $\lambda_{\max}(A)$, $\lambda_{\min}(A)$ denote the maximal and minimal eigenvalue of A. The condition number of A is denoted by $\kappa(A)$ and $\rho(A)$ denotes the spectral radius. This notation will also be used for other linear operators that occur. Note that since A is random, $\lambda_{\max}(A)$, $\lambda_{\min}(A)$, $\kappa(A)$, and $\rho(A)$ are random variables. In particular, the review article [39] enables the discussion of multigrid and domain decomposition methods in an unified framework. These methods allow in some cases for optimal preconditioning or uniform contraction numbers with respect to the dimension of V. In this section, we will mainly follow [39] and introduce abstract algorithms, which in later sections will be used as BPX or additive Schwarz preconditioner, symmetric multigrid, and overlapping domain decomposition method. We will also highlight which of the occurring objects in this review section are random.

Since A is $\mathbb{P}$-a.s. symmetric positive definite, the conjugate gradient (CG) method implies after $n \in \mathbb{N}$ iterations with initial guess U^0 the error bound that for $\mathbb{P}$-a.e. ω

$$\|u(\omega) - U^n(\omega)\|_{A(\omega)} \leq 2 \left(\frac{\sqrt{\kappa(A(\omega))} - 1}{\sqrt{\kappa(A(\omega))} + 1} \right)^n \|u(\omega) - U^0(\omega)\|_{A(\omega)}.$$

Since the random condition number $\kappa(A)$ may depend on the dimension of the linear space V, we consider the preconditioned linear system that for $\mathbb{P}$-a.e. ω

$$B(\omega)A(\omega)u(\omega) = B(\omega)f, \tag{4}$$

where the random linear operator B is chosen to be symmetric positive definite with respect to $(\cdot, \cdot)$. The random operator B shall satisfy that $\mathbb{P}$-a.s. $\kappa(BA) \leq \kappa(A)$, which then accelerates the convergence of the CG method. The combination of preconditioning and CG will be referred to as preconditioned conjugate gradient (PCG) method.

Another method to be discussed is the linear iterative method of the form

$$U^{n+1}(\omega) = U^n(\omega) + B(\omega)(f - A(\omega)U^n(\omega)), \quad n \in \mathbb{N}_0, \tag{5}$$

for $\mathbb{P}$-a.e. ω, where B is a suitable random operator that is not necessarily symmetric and U^0 is given. Note that this linear iterative methods converges $\mathbb{P}$-a.s. if $\| \mathrm{Id} - BA \|_A < 1$, $\mathbb{P}$-a.s. Alternatively, one could also introduce a relaxed version of B with relaxation parameter in $(0, 2/\rho(BA))$ to guarantee convergence with random

contraction number $(\kappa(BA) - 1)/(\kappa(BA) + 1)$, cf. [39, Proposition 2.3]. Also we remark that generally the contraction number of the PCG method is smaller, cf. [39, Proposition 2.2], which is why one may say that PCG accelerates convergence.

Lemma 1 *Let $\tilde{A} : V \to V$ be a symmetric positive definite operator with respect to $(\cdot, \cdot)$ and let $\tilde{B} : V \to V$ be a symmetric positive definite preconditioner with respect to $(\cdot, \cdot)$ for $\tilde{A}$. If there exists positive random variables c_0, c_1 such that for $\mathbb{P}$-a.e. $\omega \in \Omega$*

$$c_0(\omega)(v, v)_{\tilde{A}} \leq (v, v)_{A(\omega)} \leq c_1(\omega)(v, v)_{\tilde{A}} \quad \forall v \in V,$$

then for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\kappa(\tilde{B}A(\omega)) \leq \kappa(\tilde{B}\tilde{A})\frac{c_1(\omega)}{c_0(\omega)}.$$

Proof Since $\tilde{B}\tilde{A}$ is symmetric positive definite with respect to $(\cdot, \cdot)_{\tilde{A}}$, it holds that

$$\kappa(\tilde{B}\tilde{A}) = \frac{\lambda_{\max}(\tilde{B}\tilde{A})}{\lambda_{\min}(\tilde{B}\tilde{A})}$$

and for every $v \in V$,

$$|\lambda_{\max}(\tilde{B}\tilde{A})|^{-1}(v, v)_{\tilde{A}} \leq (\tilde{B}^{-1}v, v) \leq |\lambda_{\max}(\tilde{B}\tilde{A})|^{-1}(v, v)_{\tilde{A}}.$$

Thus, by the third equivalence in [39, Lemma 2.1],

$$\lambda_{\max}(\tilde{B}\tilde{A})^{-1}(Av, v) \leq (\tilde{B}^{-1}v, v) \leq \lambda_{\min}(\tilde{B}\tilde{A})^{-1}(Av, v).$$

Then, the assumption of the lemma implies that

$$c_1^{-1}\lambda_{\max}(\tilde{B}\tilde{A})^{-1}(Av, v) \leq (\tilde{B}^{-1}v, v) \leq c_0^{-1}\lambda_{\min}(\tilde{B}\tilde{A})^{-1}(Av, v),$$

which implies the claim by the condition number estimate in [39, Lemma 2.1]. □

For $J \in \mathbb{N}$, let us assume a decomposition of V in subspaces $(V_j : j = 1, \ldots, J)$, i.e., it holds that $V_j \subset V$, $j = 1, \ldots, J$, and

$$V = \sum_{j=1}^{J} V_j. \tag{6}$$

We define the orthogonal projections $Q_j, P_j : V \to V_j$ for every $v \in V$ by

$$(Q_j v, w_j) := (v, w_j), \quad (AP_j v, w_j) := (Av, w_j) \quad \forall w_j \in V_j$$

and the operator $A_j : V_j \to V_j$ for every $v \in V_j$ by

$$(A_j v, w_j) := (Av, w_j) \quad \forall w_j \in V_j,$$

$j = 1, \ldots, J$. Consequently it holds for every $j = 1, \ldots, J$ that $A_j P_j = Q_j A$, which implies that if u is the random solution of (3), then $u_j := P_j u$ satisfied for $\mathbb{P}$-a.e. ω

$$A_j(\omega) u_j(\omega) = f_j,$$

where $f_j := Q_j f$, $j = 1, \ldots, J$. Note that A_j, P_j are random whereas Q_j is deterministic, $j = 1, \ldots, J$. Let $R_j : V_j \to V_j$, $j = 1, \ldots, J$, be random symmetric positive definite operators with respect to $(\cdot, \cdot)$, which shall *approximate the inverse of A_j* respectively. Thus,

$$B^a := \sum_{j=1}^{J} R_j Q_j \tag{7}$$

is also symmetric positive definite, cf. [39, Lemma 3.1], and shall be used as preconditioner for A.

Algorithm 1 *Apply the PCG method to* (4) *with the random preconditioner B^a defined in* (7).

A multilevel iteration can be defined under the assumption that there are nested subspaces satisfying

$$M_1 \subset M_2 \subset \cdots \subset M_J = V, \tag{8}$$

where we also define operators $\hat{Q}_j, \hat{P}_j : M_J \to M_j$ and $\hat{A}_j : M_j \to M_j$ for $j = 1, \ldots, J$ respectively. The random multilevel iterations $\hat{B}_j^s : M_j \to M_j, j = 1, \ldots, J$ with parameters $m, k \in \mathbb{N}$ will be defined iteratively. Set $\hat{B}_1^s := \hat{A}_1^{-1}$ and assume that $\hat{B}_{j-1}^s : M_{j-1} \to M_{j-1}$ is already defined. For every $g \in M_j$ the multigrid V-cycle iteration $\hat{B}_j^s g$ is defined by:

1. $v := \hat{R}_j g$
2. $w := v + \hat{B}_{j-1}^s \hat{Q}_{j-1}(g - \hat{A}_j v)$
3. $\hat{B}_j^s g := w + \hat{R}_j(g - \hat{A}_j w)$

Algorithm 2 *Let U^0 be given, then U^n is defined by the linear iteration in* (5) *with* $B = \hat{B}_J^s$.

According to [9, Eq. (2.14)] the residual operator is given by

$$\hat{E}_J^s = (\mathrm{Id} - \hat{B}_J^s \hat{A}_J) = (\mathrm{Id} - \hat{T}_J) \cdots (\mathrm{Id} - \hat{T}_1)(\mathrm{Id} - \hat{T}_1^*) \cdots (\mathrm{Id} - \hat{T}_J^*),$$

where

$$\hat{T}_j := \hat{R}_j \hat{A}_j \hat{P}_j.$$

For the convergence of Algorithm 1 we have to prove bounds of $\kappa(BA)$, whereas for Algorithm 2 we have to show that $\|\hat{E}_J^s\|_A \leq \delta$ for some random variable δ taking values in $(0, 1)$ $\mathbb{P}$-a.s. The additive preconditioner in Algorithm 1 will in later applications be the BPX, if the nested decomposition (8) is considered, or the additive Schwarz preconditioner. Algorithm 2 is multigrid by symmetric V-cycle.

In [39], assumptions are introduced involving two parameters K_0 and K_1 that allow to discuss their convergence. Here K_0 and K_1 are positive random variables. We recall the two conditions [39, Eqs. (4.2) and (4.3)]. There exists a positive random variable K_0 such that for every $v \in V$, there exists a decomposition $v = \sum_{j=1}^{J} v_j$ with $v_j \in V_j$, $j = 1, \ldots, J$, such that for $\mathbb{P}$-a.e. ω

$$\sum_{j=1}^{J} (R_j^{-1}(\omega)v_j, v_j) \le K_0(\omega)(A(\omega)v, v). \tag{A.1}$$

There exists a positive random variable K_1 such that for $\mathbb{P}$-a.e. ω and for every $S \subset \{1, \ldots, J\} \times \{1, \ldots, J\}$ and $v_j, w_j \in V$, $j = 1, \ldots, J$,

$$\sum_{(i,j)\in S} (T_i(\omega)v_i, T_j(\omega)w_j)_{A(\omega)}$$

$$\le K_1(\omega) \left(\sum_{i=1}^{J} (T_i(\omega)v_i, v_i)_{A(\omega)} \right)^{1/2} \left(\sum_{j=1}^{J} (T_j(\omega)w_j, w_j)_{A(\omega)} \right)^{1/2}. \tag{A.2}$$

Theorem 2 *Let assumptions* (A.1) *and* (A.2) *be satisfied. Let B^a be the random preconditioner given by* (7). *Then, for $\mathbb{P}$-a.e. ω*

$$\kappa(B^a(\omega)A(\omega)) \le K_0(\omega)K_1(\omega). \tag{9}$$

The residual operator $\hat{E}_J^s$ from Algorithm 2 *satisfies for $\mathbb{P}$-a.e. ω*

$$\|\hat{E}_J^s(\omega)\|_{A(\omega)} \le 1 - \frac{2 - \nu}{K_0(\omega)(1 + K_1(\omega))^2}, \tag{10}$$

where $\nu \ge \max_{j=1,\ldots,J} \{\rho(R_j(\omega)A_j(\omega))\}$.

Proof The first assertion is explicitly [39, Theorems 4.1]. The second assertion follows by [39, Theorems 4.4 and Proposition 3.5]. □

The random parameters K_0 and K_1 can be estimated in some cases with [39, Lemmas 4.5, 4.6, and 4.7]. Let us state a specific case of [39, Lemma 4.6] as the following lemma.

Lemma 3 *Let K_3 be a positive random variable that is independent of J and let $\gamma \in (0, 1)$ be deterministic. If for $\mathbb{P}$-a.e. ω and for every $v, w \in V$ and every $i, j \in \{1, \ldots, J\}$ it holds that*

$$(T_i(\omega)v, T_j(\omega)w)_{A(\omega)} \le K_3(\omega)\nu\gamma^{|i-j|}(T_i(\omega)v, v)_{A(\omega)}^{1/2}(T_j(\omega)w, w)_{A(\omega)}^{1/2}, \tag{11}$$

then

$$K_1(\omega) \le K_3(\omega)\nu\frac{2}{1-\gamma}.$$

Note that (11) is often called *strengthened Cauchy–Schwarz inequality*.

The operator $\hat{B}_J^s$ can also be used as preconditioner in a PCG method to accelerate convergence. The respective condition number can be bounded by the following proposition in combination with Theorem 2.

Proposition 4 *Let δ be a random variable taking values in $(0, 1)$. If for $\mathbb{P}$-a.e. ω,*

$$\| \operatorname{Id} - \hat{B}_J^s(\omega) A(\omega) \|_{A(\omega)} \leq \delta(\omega),$$

then $\kappa(B(\omega)A(\omega)) \leq 1/(1 - \delta(\omega))$.

Proof Since the operator $\operatorname{Id} - \hat{B}_J^s A$ can be written as E^*E, where E^* denotes the adjoint operator of E, for some appropriate E that was discussed above, it holds that $((\operatorname{Id} - \hat{B}_J^s A)v, v)_A = \|E^*v\|_A^2 \geq 0$ for every $v \in V$. Hence,

$$(\hat{B}_J^s Av, v)_A \leq (v, v)_A \quad \forall v \in V. \tag{12}$$

The assumption implies that

$$(1 - \delta)(v, v)_A \leq (\hat{B}_J^s Av, v)_A \quad \forall v \in V, \tag{13}$$

which then implies the assertion. ∎

3 Strong convergence of iterative methods

We recall the possibly infinite dimensional Hilbert space $\mathcal{V}$ and let $(\mathcal{H}, (\cdot, \cdot))$ be another Hilbert space such that the embedding $\mathcal{H} \subset \mathcal{V}$ is continuous. Let $(\mathcal{V}_\ell : \ell \in \mathbb{N})$ be a nested sequence of finite dimensional subspaces of $\mathcal{V}$, i.e., $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}$. Let the finite dimensional spaces $\mathcal{V}_\ell$ have dimensions $N_\ell := \dim(\mathcal{V}_\ell)$, $\ell \in \mathbb{N}$. Similar to Sect. 2, we introduce random operators $\mathcal{A}_\ell$, $\mathcal{P}_\ell$, R_ℓ, T_ℓ, and Q_ℓ with respect to the inner product $(\cdot, \cdot)$ of $\mathcal{H}$, $\ell \in \mathbb{N}$. The inner product $(\cdot, \cdot)_{\mathbb{E}(\mathcal{A})}$ on $\mathcal{V}$ is given by

$$(v, w)_{\mathbb{E}(\mathcal{A})} := {}_{\mathcal{V}^*}\langle \mathbb{E}(\mathcal{A})v, w \rangle_{\mathcal{V}} \quad \forall v, w \in \mathcal{V}.$$

The inner product $(\cdot, \cdot)_{\mathcal{A}}$ with respect to the random symmetric operator $\mathcal{A}$ will also be considered.

For every $\ell \in \mathbb{N}$, we consider the variational form of (1) on the subspace $\mathcal{V}_\ell$. For every $\ell \in \mathbb{N}$, this gives rise to the following linear equation. We seek to find $u_\ell : \Omega \to \mathcal{V}_\ell$ such that for $\mathbb{P}$-a.e. ω

$${}_{\mathcal{V}^*}\langle \mathcal{A}_\ell(\omega) u_\ell(\omega), v_\ell \rangle_{\mathcal{V}} = {}_{\mathcal{V}^*}\langle f, v_\ell \rangle_{\mathcal{V}} \quad \forall v_\ell \in \mathcal{V}_\ell.$$

This is uniquely solvable by the Lax–Milgram lemma using (2).

We assume that the random solution $u : \Omega \to \mathcal{V}$ of the problem in (1) is approximated by the Galerkin approximations u_ℓ, $\ell \in \mathbb{N}$. such that for every $\ell \in \mathbb{N}$ and for $\mathbb{P}$-a.e. ω

$$\|u(\omega) - u_\ell(\omega)\|_{\mathcal{A}(\omega)} \leq \mathcal{C}(\omega) h_\ell^s. \tag{14}$$

We apply an iterative method such as Algorithm 1 or 2 with random contraction number δ, that is independent of ℓ, with n iterations and solve exactly on level 1, i.e., starting with $U_{\ell-1}^n$ as initial guess for level ℓ we carry out n iterations of the algorithm with a random contraction number δ that takes values in $(0, 1)$. Hence, we obtain a sequence U_ℓ^n, $\ell \in \mathbb{N}$, where we set $U_1^n := u_1$. This multilevel process was used in [2, Sect. 3] to derive optimal complexity bounds for the solution of certain linear equations and is also commonly referred to as "full multigrid".

Lemma 5 *Let us assume that $(u_\ell : \ell \in \mathbb{N})$ satisfies* (14) *for some $s > 0$ and let the sequence $(h_\ell, \ell \in \mathbb{N})$ satisfy that $h_\ell = \gamma^\ell h_0$, $\ell \in \mathbb{N}$, for some fixed $h_0 > 0$ and $\gamma \in (0, 1)$. Then, for every $\ell \in \mathbb{N}$, U_ℓ^n and for $\mathbb{P}$-a.e. ω*

$$\|u(\omega) - U_\ell^n(\omega)\|_{\mathcal{A}(\omega)} \le 2\mathcal{C}(\omega) \left(\sum_{k=0}^{\ell-1} h_{\ell-k}^s \delta(\omega)^{nk} \right)$$

$$= 2\mathcal{C}(\omega) \left(\sum_{k=0}^{\ell-1} (\gamma^{-s}\delta(\omega)^n)^k \right) h_\ell^s.$$

Proof The argument is for example given in [7, Chapter 10]. $\qquad\square$

For any Banach space $(B, \|\cdot\|_B)$ and any $p \in [1, +\infty)$, let us denote the space of strongly measurable mappings $X : \Omega \to B$ such that $\|X\|_B^p$ is integrable with respect to the probability measure $\mathbb{P}$ by $L^p(\Omega; B)$. For $B = \mathbb{R}$, we simply write $L^p(\Omega)$.

Theorem 6 *Let the assumptions of Lemma 5 be satisfied. Let us assume that $(\mathcal{C}/\sqrt{\breve{a}}) \in L^p(\Omega)$ for some $p \in [1, +\infty)$, where $\breve{a}$ is defined in* (2) *and $\mathcal{C}$ is given in* (14). *Further, assume that $1/(1 - \delta) \in L^{p'}(\Omega)$ for some $p' \in [1, +\infty)$. For every $\eta \in (0, 1)$, every deterministic number of iterations $n \in \mathbb{N}$, $q \in [1, p]$, and $r := p'(p - q)/(pq)$ it holds that for every $\ell \ge 2$*

$$\|u - U_\ell^n\|_{L^q(\Omega;\mathcal{V})} \le \mathcal{C}_r \left(h_\ell^s + n^{-r} \right),$$

where

$$\mathcal{C}_r := 2 \max \left\{ \frac{\|\mathcal{C}/\sqrt{\breve{a}}\|_{L^q(\Omega)}}{1 - \eta}, \|\mathcal{C}/\sqrt{\breve{a}}\|_{L^p(\Omega)} \frac{h_1^s}{1 - \gamma^s} \left\| \frac{1}{1 - \delta} \right\|_{L^{p'}(\Omega)}^r \log\left(\frac{1}{\eta\gamma^s} \right)^r \right\}.$$

Proof The idea of the proof is to decompose the probability space into $\Omega = \Omega_n \cup (\Omega_n)^{\complement}$, where

$$\Omega_n := \left\{ \omega \in \Omega : \delta(\omega)^n < \eta\gamma^s \right\}$$

and therefore $(\Omega_n)^{\complement} = \{\omega \in \Omega : \delta(\omega)^n \ge \eta\gamma^s\}$. Note that both sets are measurable. For notational convenience, we omit ω in the following when discussing subsets of Ω. Our goal is to show, how the probability of $(\Omega_n)^{\complement}$ tends to zero for increasing values of $n \in \mathbb{N}$, to be able to justify the applicability of a classical argument on the sets Ω_n, $n \in \mathbb{N}$. Naturally, $\Omega_{n_1} \subset \Omega_{n_2}$ for every choice of natural numbers $n_1 \le n_2$. We

recall a version of the Markov inequality, cf. [5, Eq. (2.1)], i.e., for a random variable X taking values in $(1, +\infty)$ and a non-decreasing function ϕ such that $\phi(t) > 0$ it holds that

$$\mathbb{P}(X \geq t) \leq \mathbb{P}(\phi(X) \geq \phi(t)) \leq \frac{\mathbb{E}(\phi(X))}{\phi(t)}, \quad t \in (1, +\infty).$$

We select the non-decreasing positive function $\phi(t) := t^{p'}$. Then, for $X = 1/(1 - \delta)$ and every $t \in (1, +\infty)$ it holds that

$$\mathbb{P}\left(\frac{1}{1 - \delta} \geq t\right) \leq \mathbb{E}\left(\left(\frac{1}{1 - \delta}\right)^{p'}\right) \frac{1}{t^{p'}}. \tag{15}$$

Since for every $r \in (0, 1)$, $\{\delta \geq r\} = \{1/(1 - \delta) \geq 1/(1 - r)\}$, we conclude that

$$\mathbb{P}((\Omega_n)^\complement) = \mathbb{P}\left(\delta^n \geq \eta\gamma^s\right) = \mathbb{P}\left(\delta \geq (\eta\gamma^s)^{1/n}\right) = \mathbb{P}\left(\frac{1}{1 - \delta} \geq \frac{1}{1 - (\eta\gamma^s)^{1/n}}\right).$$

We observe that the function $x \mapsto (1 - (\eta\gamma^s)^{1/x})x$ from $(1, +\infty)$ to $(0, +\infty)$ is increasing. Since the rule of L'Hospital implies that $\lim_{x\to+\infty}(1 - (\eta\gamma^s)^{1/x})x = \log(1/(\eta\gamma^s))$, we conclude that for every $n \in \mathbb{N}$

$$1 - (\eta\gamma^s)^{1/n} \leq \log\left(\frac{1}{\eta\gamma^s}\right) \frac{1}{n}.$$

For every $n \in \mathbb{N}$, we choose $t := 1/(1 - (\eta\gamma^s)^{1/n})$ in (15), and conclude that for every $n \in \mathbb{N}$ it holds that

$$\mathbb{P}((\Omega_n)^\complement) \leq \mathbb{E}\left(\left(\frac{1}{1 - \delta}\right)^{p'}\right) \log\left(\frac{1}{\eta\gamma^s}\right)^{p'} \left(\frac{1}{n}\right)^{p'}. \tag{16}$$

Hence, we have established estimates for the probability of the sets $(\Omega_n)^\complement$, $n \in \mathbb{N}$. We apply Lemma 5, (2), and decompose the probability space into $\Omega = \Omega_n \cup (\Omega_n)^\complement$ to obtain that for q as in the statement of the theorem

$$\mathbb{E}(\|u - U_\ell^n\|_{\mathbb{E}(\mathcal{A})}^q) \leq \mathbb{E}\left(\left(2\frac{\mathcal{C}}{\sqrt{\tilde{a}}} \sum_{k=0}^{\ell-1} h_{\ell-k}^s \delta^{nk}\right)^q\right)$$

$$= \mathbb{E}\left(\left(2\frac{\mathcal{C}}{\sqrt{\tilde{a}}} \sum_{k=0}^{\ell-1} (\gamma^{-s}\delta^n)^k\right)^q \mathbb{1}_{\Omega_n}\right) h_\ell^{sp}$$

$$+ \mathbb{E}\left(\left(2\frac{\mathcal{C}}{\sqrt{\tilde{a}}} \sum_{k=0}^{\ell-1} h_{\ell-k}^s \delta^{nk}\right)^q \mathbb{1}_{(\Omega_n)^\complement}\right).$$

Since on Ω_n holds that $\gamma^{-s}\delta^n < \eta$, we obtain with a geometric series argument that

$$\mathbb{E}\left(\left(2\frac{\mathcal{C}}{\sqrt{\check{a}}}\sum_{k=0}^{\ell-1}(\gamma^{-s}\delta^n)^k\right)^q \mathbb{1}_{\Omega_n}\right) h_\ell^{sq} \leq 2^q\,\mathbb{E}\left(\left(\frac{\mathcal{C}}{\sqrt{\check{a}}}\right)^q\right)\left(\frac{1}{1-\eta}\right)^q h_\ell^{sq}.$$

The relation $h_\ell = \gamma^\ell h_0, \ell \in \mathbb{N}$, implies that for every $\ell \geq 2$ it holds that $\sum_{k=0}^{\ell-1} h_{\ell-k}^s \leq \sum_{\ell \geq 1} \gamma^{\ell s} h_0 = h_1^s/(1-\gamma^s)$. The Hölder inequality with $r_1 = p/q$ and $r_2 = p/(p-q)$ implies with the tail bound of δ^n in (16) that

$$\mathbb{E}\left(\left(2\frac{\mathcal{C}}{\sqrt{\check{a}}}\sum_{k=0}^{\ell-1} h_{\ell-k}^s \delta^{nk}\right)^q \mathbb{1}_{(\Omega_n)^c}\right)$$

$$\leq 2^q \left\|\frac{\mathcal{C}}{\sqrt{\check{a}}}\right\|_{L^p(\Omega)}^q \left(\frac{h_1^s}{1-\gamma^s}\right)^q \mathbb{E}\left(\mathbb{1}_{(\Omega_n)^c}\right)^{1/r_2}$$

$$\leq 2^q \left\|\frac{\mathcal{C}}{\sqrt{\check{a}}}\right\|_{L^p(\Omega)}^q \left(\frac{h_1^s}{1-\gamma^s}\right)^q \left\|\frac{1}{1-\delta}\right\|_{L^{p'}(\Omega)}^{rq} \log\left(\frac{1}{\eta\gamma^s}\right)^{rq}\left(\frac{1}{n}\right)^{rq},$$

which finishes the proof of the theorem. $\qquad\square$

Remark 7 Let X be a random variable with values in $(0, 1)$ such that $1/(1 - X) \in L^p(\Omega)$ for some $p \in [1, +\infty)$, then (16) implies that for every $\eta < 1$ and $n \in \mathbb{N}$

$$\mathbb{P}\left(X^n \geq \eta\right) \leq \left\|\frac{1}{1-X}\right\|_{L^p(\Omega)}^p \log\left(\eta^{-1}\right)^p n^{-p}.$$

We illustrate the assumptions in Theorem 6 on $\mathcal{C}$ and $\check{a}$ in the following example.

Example 8 Consider the random coefficient function a given by $a(\omega, x) = \exp(y(\omega)$ $\psi(x))$ for a.e. $x \in D$, $\mathbb{P}$-a.e. $\omega \in \Omega$, where $\psi \in W^{1,\infty}(D)$ and y is standard normally distributed. It is easy to see that $\|\check{a}^{-1}\|_{L^p(\Omega)} \leq \|a\|_{L^p(\Omega;L^\infty(D))} \leq \exp(p\|\psi\|_{L^\infty(D)}^2/2)$ for every $p \in [1, +\infty)$, where $\check{a} = \operatorname{ess\,inf}_{x \in D}\{a(x)\}$.

Let us consider the elliptic Dirichlet problem $\mathcal{A}u := -\nabla \cdot (a\nabla u) = f$ on a convex polygon D with $f \in L^2(D)$, which has a unique solution $u : \Omega \to H_0^1(D)$ that satisfies $\|u\|_{H_0^1(D)} \leq \|f\|_{H^{-1}(D)}/\check{a}$, $\mathbb{P}$-a.s. The Sobolev spaces are denoted by $H^k(D)$, $k \in \mathbb{N}$, and $H_0^1(D) := \{v \in H^1(D) : v|_{\partial D} = 0\}$ with dual space $H^{-1}(D)$. Since $\psi \in W^{1,\infty}(D)$, we may write $-\Delta u = f/a + y\nabla\psi \cdot \nabla u$. Since the Dirichlet Laplacian is boundedly invertible from $L^2(D)$ to $H^2(D) \cap H_0^1(D)$, $\|u\|_{H^2(D)} \leq C\check{a}^{-1}(|y|\|\psi\|_{W^{1,\infty}(D)} + 1)\|f\|_{L^2(D)}$ $\mathbb{P}$-a.s., where the constant C neither depends on a nor on f. Then, Céa's lemma implies with a standard approximation property in FE spaces that (14) holds with $\mathcal{C} = C\sqrt{\|a\|_{L^\infty(D)}}\check{a}^{-1}(|y|\|\psi\|_{W^{1,\infty}(D)} + 1)\|f\|_{L^2(D)}$ and $s = 1$. By a multiple application of the Cauchy–Schwarz inequality, we conclude that $(\mathcal{C}/\sqrt{\check{a}}) \in L^p(\Omega)$ for every $p \in [1, +\infty)$.

The case $p' = +\infty$ in Theorem 6 is trivial, since then the random contraction number may be upper bounded to be uniformly strictly less than one, i.e., if $p' = +\infty$

in Theorem 6, then ess $\sup_{\omega \in \Omega} \delta(\omega) < 1$. In this case, the standard theory applies with $\overline{\delta} = \text{ess } \sup_{\omega \in \Omega} \delta(\omega) < 1$.

4 Multigrid methods

Here, we provide sufficient conditions under which the strengthened Cauchy–Schwarz inequality holds with explicit dependence on the operator $\mathcal{A}$. This will allow us to show $L^q(\Omega)$ bounds of the condition numbers and tail bounds of the random contraction number in order to apply the strong error bounds from Theorem 6.

We will provide a proof in the case of a random, symmetric elliptic differential operators. To be specific, let $H_0^1(D)$, $H^s(D)$, $s \in [-1, 2]$, be the Sobolev–Slobodeckij spaces for some polytopal domain $D \subset \mathbb{R}^d$, $d \geq 1$ arbitrary, such that $H^{-s}(D)$ is the dual space of $H^s(D)$, $s \in [0, 1]$, and $H^0(D) = L^2(D)$. The reader is referred to [23, Chapter 1] for details on Sobolev spaces. We consider the class of random symmetric operators

$$\mathcal{A} := - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right) + a : H_0^1(D) \to (H_0^1(D))^*, \qquad (17)$$

where $(a_{ij}(x))_{i,j=1,\dots,d}$ is a random symmetric matrices for a.e. $x \in D$. Let us assume that $\overline{D} = \bigcup_{k=1}^K \overline{D}_k$, where the subdomains D_k are pairwise disjoint with a polytopal boundary. Furthermore, we assume that the random fields a and a_{ij} are strongly measurable as mappings from Ω to $L^\infty(D)$, and $a_{ij}|_{D_k}$ is strongly measurable as a mapping from Ω to $W^{s,p}(D_k)$ such that $s > d/p$, $k = 1, \dots, K$, $i, j = 1, \dots, d$, where $W^{s,p}(D_k)$, $s \geq 0$, $p \in [1, +\infty)$, $k = 1, \dots, K$, denote the Sobolev–Slobodeckij spaces, cf. [23, Definition 1.3.2.1]. We assume that for $\mathbb{P}$-a.e. ω, ess $\inf_{x \in D}\{a(\omega, x)\} \geq 0$ and that there exists a strictly positive random variable $\check{a}$ such that for $\mathbb{P}$-a.e. ω

$$\underset{x \in D}{\text{ess inf}} \left\{ \sum_{i,j=1}^d a_{ij}(\omega, x) \xi_i \xi_j \right\} \geq \check{a}(\omega) |\xi|^2 \quad \forall \xi \in \mathbb{R}^d.$$

The corresponding random bilinear form $a(\cdot, \cdot)$ is for $\mathbb{P}$-a.e. ω given by

$$a_\omega(v, w) = \sum_{i,j=1}^d \int_D a_{ij}(\omega) \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_i} + a(\omega) vw \, \mathrm{d}x \quad \forall v, w \in H_0^1(D).$$

Furthermore, let $\hat{a}$ be a positive random variable such that for $\mathbb{P}$-a.e. ω

$$\check{a}(\omega) \int_D |\nabla v|^2 \mathrm{d}x \leq a_\omega(v, v) \leq \hat{a}(\omega) \int_D |\nabla v|^2 \mathrm{d}x \quad \forall v \in H_0^1(D). \qquad (18)$$

The following example is a class of random coefficients, which will be further discussed in Sects. 6 and 7.2.

Example 9 The class of lognormal random coefficient fields $a_{ij} = \exp(Z)\delta_{ij}$ and $a = 0$, where $Z : \Omega \to W^{s,p}(D)$, $s > d/p$, is a strongly measurable Gaussian random field and δ_{ij} denotes the Kronecker symbol, satisfy these conditions; see ahead in Sect. 7.2 for a class of instances of such GRFs.

The proof of the strengthened Cauchy–Schwarz inequality, see (11), draws on [9, Sects. 4 and 5] and [39, Sect. 4]. The setting in reference [9] allows for low Hölder regularity of the coefficients of elliptic operators, but does not provide a strengthened Cauchy–Schwarz inequality needed for the setting of assumptions (**A.1**) and (**A.2**). The strengthened Cauchy–Schwarz inequality proved in [39, Lemmas 6.1 and 6.3] is limited to coefficients with $W^{1,\infty}(D)$ regularity. Here, a strengthened Cauchy–Schwarz inequality will be proved with explicit dependence on the coefficients that is valid for arbitrary low Hölder regularity of the coefficients and also allows for jumps across ∂D_k (see ahead Proposition 11). We will identify estimates with explicit dependence on the random coefficients of $\mathcal{A}$.

Let $(\mathcal{T}_\ell, \ell \in \mathbb{N})$ be a nested sequence of shape regular simplicial, uniformly refined meshes of D. Note that in one refinement step one simplex is refined into 2^d subsimplices. For every $k = 1, \ldots, K$ and $\ell \in \mathbb{N}$, we require that $\overline{D}_k = \bigcup_{\tau \in \mathcal{T}_\ell, D_k \cap \tau \neq \emptyset} \overline{\tau}$. Let $\mathcal{V}_\ell \subset \mathcal{V}$ be the space of piecewise polynomial function with respect to the mesh $\mathcal{T}_\ell, \ell \in \mathbb{N}$. For simplicity, we will consider here only first order FE, i.e., polynomial degree one. Define $-\Delta_\ell : \mathcal{V}_\ell \to \mathcal{V}_\ell$ by the bilinear form $(w_\ell, v_\ell) \mapsto \int_D \nabla w_\ell \cdot \nabla v_\ell \mathrm{d}x$ over $\mathcal{V}_\ell \times \mathcal{V}_\ell, \ell \in \mathbb{N}$. By [9, Eq. (5.1)], there exists a deterministic constant $C > 0$ such that for every $j \in \mathbb{N}$ and every $v \in \mathcal{V}_j$ with $v = \sum_{i=1}^{j} v_i$, $v_i \in \mathcal{V}_i$, such that

$$\sum_{i=1}^{j} \lambda_{\max}(-\Delta_i) \|v_i\|_{L^2(D)}^2 \le C \int_D |\nabla v|^2 \mathrm{d}x. \tag{19}$$

Moreover, the following inverse estimates hold. There exists a constant C such that for every $v \in \mathcal{V}_\ell$

$$\|v\|_{H^1(D)} \le C h_\ell^{-1} \|v\|_{L^2(D)} \quad \text{and} \quad \|v\|_{H^{1+s}(D)} \le C h_\ell^{-s} \|v\|_{H^1(D)}, \tag{20}$$

where

$$h_\ell := \max_{\tau \in \mathcal{T}_\ell} \{\mathrm{diam}(\tau)\}$$

and $s \in (0, 1/2)$, cf. [14, Theorem 3.2.6] and [11, Eq. (10.1)]. These inverse estimates are sharp, which can be seen by choosing v to be a nodal basis function of the FE space $\mathcal{V}_\ell$. Since by (18) for $\mathbb{P}$-a.e. ω

$$\check{a}(\omega)\lambda_{\max}(-\Delta_\ell) \le \lambda_{\max}(\mathcal{A}_\ell(\omega)) \le \hat{a}(\omega)\lambda_{\max}(-\Delta_\ell), \tag{21}$$

we also observe that for $\mathbb{P}$-a.e. ω

$$\lambda_{\max}(\mathcal{A}_\ell(\omega)) \ge C\check{a}(\omega)h_\ell^{-2}, \tag{22}$$

which for $-\Delta_\ell$ is a consequence of the sharpness of (20).

We require the following assumptions on the smoothers $(R_j : j = 1, \ldots, J)$. There exists a deterministic $\nu \in (0, 2)$ such that for every $j = 1, \ldots, J$, and $\mathbb{P}$-a.e. ω

$$(T_j(\omega)v, T_j(\omega)v)_{\mathcal{A}(\omega)} \leq \nu(T_j(\omega)v, v)_{\mathcal{A}(\omega)} \quad \forall v \in \mathcal{V}_j. \tag{B}$$

There exists deterministic $c_0, c_1 > 0$ such that for every $j = 1, \ldots, J$, and for $\mathbb{P}$-a.e. ω

$$c_0 \frac{\|v\|^2_{L^2(D)}}{\lambda_{\max}(\mathcal{A}_j(\omega))} \leq (R_j(\omega)v, v) \leq c_1 \frac{\|v\|^2_{L^2(D)}}{\lambda_{\max}(\mathcal{A}_j(\omega))} \quad \forall v \in \mathcal{V}_j. \tag{C}$$

Assume that there exists $\gamma \in (0, 1)$ such that for all $i, j \in \mathbb{N}$ satisfying $i \leq j$ it holds that

$$\frac{h_j}{h_i} \leq \gamma^{j-i}, \tag{D}$$

Note that (B) implies that $\rho(R_i A_i) \leq \nu$ for every i. There exist smoothers that satisfy these assumptions, cf. [6, Chapter 8] and [8,10].

Lemma 10 *Let $s \in (0, 1/2)$ and $p \in (d/s, +\infty)$, then for $\mathbb{P}$-a.e. ω, for every $\eta > 0$, $\phi \in H^1(D)$, and $\psi \in H^{1+s}(D)$ it holds that*

$$|a(\omega)(\phi, \psi)| \leq \widetilde{C}(\omega)(\eta^{-1}\|\phi\|^2_{L^2(D)} + \eta^{s/(1-s)}\|\phi\|^2_{H^1(D)})^{1/2}\|\psi\|_{H^{1+s}(D)},$$

where for a deterministic constant C independent of $(a_{ij}, a : i, j = 1, \ldots, d)$

$$\widetilde{C}(\omega) := C \left(\max_{k=1,\ldots,K} \sum_{i,j=1}^{d} \|a_{ij}(\omega)\|_{W^{s,p}(D_k)} + \|a(\omega)\|_{L^\infty(D)} \right).$$

Proof The following argument originates from the proof of [9, Lemma 4.3]. We will track the dependence on the random elliptic coefficients $(a_{ij}, a : i, j = 1, \ldots, d)$. Let us fix $k \in \{1, \ldots, K\}$. There exists a bounded linear extension operator $I_k : H^1(D_k) \to H^1(\mathbb{R}^d)$, cf. [23, Theorem 1.4.3.1]. For every function $v : D_k \to \mathbb{R}$ the zero extension to $\mathbb{R}^d$ is denoted by $\widetilde{v}$. Let $\mathscr{F}$ denote the Fourier transform on $\mathbb{R}^d$. We obtain with Plancherel's theorem

$$\int_{D_k} a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} \mathrm{d}x = \int_{\mathbb{R}^d} \frac{\partial (I_k\phi)}{\partial x_i} a_{ij} \widetilde{\frac{\partial \psi}{\partial x_j}} \mathrm{d}x$$
$$= \left(\mathscr{F}\left(\frac{\partial (I_k\phi)}{\partial x_i} \right), \mathscr{F}\left(a_{ij} \widetilde{\frac{\partial \psi}{\partial x_j}} \right) \right)_{L^2(\mathbb{R}^d)}.$$

Recall that $\|(1+|\xi|^2)^{s/2}\mathscr{F}(v)\|_{L^2(\mathbb{R}^d)}$ is the Bessel potential norm in the Hilbert case of order s of a function v. The fact that in the Hilbert case Bessel potential and Slobodeckij spaces are equal with equivalent norms, cf. [37, Definition 2.3.1(d), Theorem 2.3.2(d), Eq. 4.4.1(8)], and the boundedness of the zero extension as an operator from $H^s(D_k)$ to $H^s(\mathbb{R}^d)$, cf. [23, Corollary 1.4.4.5], imply with the Cauchy–Schwarz inequality and differentiation rules for $\mathscr{F}$ that there exists a constant C such that

$$
\int_{D_k} a_{ij} \frac{\partial\phi}{\partial x_i} \frac{\partial\psi}{\partial x_j} \mathrm{d}x
$$

$$
\leq C \left(\int_{\mathbb{R}^d} \frac{|\xi|^2}{(1+|\xi|^2)^s} \mathscr{F}(I_k\phi)\mathrm{d}\xi \right)^{1/2} \left\| a_{ij} \frac{\partial\psi}{\partial x_j} \right\|_{H^s(D_k)}
$$

$$
\leq C (\eta^{-1}\|\phi\|_{L^2(D_k)}^2 + \eta^{s/(1-s)}\|\phi\|_{H^1(D_k)}^2)^{1/2} \left\| a_{ij} \frac{\partial\psi}{\partial x_j} \right\|_{H^s(D_k)},
$$

where the inequality $|\xi|^2/(1-|\xi|^2)^s \leq \eta^{-1} + \eta^{s/(1-s)}(1+|\xi|^2)$ is derived with elementary manipulations for every $\eta > 0$ and every $\xi \in \mathbb{R}^d$. By [23, Theorem 1.4.4.2], the multiplication of elements of $W^{s,p}(D_k)$ is a bounded linear operator on $H^s(D_k)$. Thus, by summing over k and by the Cauchy–Schwarz inequality there exists a constant C such that

$$
\int_D a_{ij} \frac{\partial\phi}{\partial x_i} \frac{\partial\psi}{\partial x_j} \mathrm{d}x
$$

$$
\leq C \max_{k=1,\ldots,K} \|a_{ij}\|_{W^{s,p}(D_k)} (\eta^{-1}\|\phi\|_{L^2(D)}^2 + \eta^{s/(1-s)}\|\phi\|_{H^1(D)}^2)^{1/2} \|\psi\|_{H^{1+s}(D)}.
$$

Since it also holds that

$$
\int_D a\phi\psi\,\mathrm{d}x \leq \|a\|_{L^\infty(D)} (\eta^{-1}\|\phi\|_{L^2(D)}^2 + \eta^{s/(1-s)}\|\phi\|_{H^1(D)}^2)^{1/2} \|\psi\|_{H^{1+s}(D)},
$$

the assertion of the lemma follows. $\square$

Proposition 11 *Let Assumptions (**B**) and (**C**) be satisfied by the smoothers ($R_j : j = 1 \ldots, J$), let Assumption (**D**) hold, and let $s \in (0, 1/2)$ and $p \in (d/s, +\infty)$.*

*Then for some deterministic constant C independent of J the inequality (**A.2**) holds with the random variable*

$$
K_1 := C \left(\max_{k=1,\ldots,K} \sum_{i,j=1}^d \|a_{ij}\|_{W^{s,p}(D_k)} + \|a\|_{L^\infty(D)} \right)^2 \left(\frac{1}{\underline{a}} \right)^2.
$$

Proof The proof of this proposition merges ideas of the proofs of [9, Lemma 4.2] and [39, Lemma 6.3] to obtain a strengthened Cauchy–Schwarz inequality with explicit dependence on the coefficients ($a_{ij}, a : i, j = 1, \ldots, d$) in the setting considered here that allows for low spatial regularity of (a_{ij}), $i, j = 1, \ldots, d$. We may assume that $j \geq i$ due to the symmetry of $(\cdot, \cdot)_{\mathcal{A}}$ and let $w \in \mathcal{V}_i$ and $\phi \in \mathcal{V}_j$ be arbitrary, which

are both elements of $H^{1+s}(D)$ due to $s < 1/2$. The first inverse estimate in (20) and Lemma 10 imply that

$$|a(w, \phi)| \leq \widetilde{\mathcal{C}}(\eta^{-1} + \eta^{s/(1-s)}h_j^{-2})^{1/2}\|\phi\|_{L^2(D)}\|w\|_{H^{1+s}(D)},$$

where we tacitly absorbed the deterministic constant in (20) into $\widetilde{\mathcal{C}}$. The random variable $\widetilde{\mathcal{C}}$ is stated in Lemma 10. Then, the second inverse estimate in (20), (22), and the choice $\eta := h_j^{2(1-s)}$ results in

$$a(w, \phi) \leq \widetilde{\mathcal{C}}\sqrt{\frac{\lambda_{\max}(\mathcal{A}_j)}{\check{a}}}2\left(\frac{h_j}{h_i}\right)^s \|\phi\|_{L^2(D)}\|w\|_{H^1(D)},$$

where we again tacitly absorbed the deterministic constant in (22) into $\widetilde{\mathcal{C}}$ as well as the constant in (20). Since

$$\frac{\|\mathcal{A}_j w\|_{L^2(D)}^2}{\lambda_{\max}(\mathcal{A}_j)} = \lambda_{\max}(\mathcal{A}_j)^{-1}\left(\sup_{\phi \in \mathcal{V}_j} \frac{a(w, \phi)}{\|\phi\|_{L^2(D)}}\right)^2,$$

we conclude with (18) and the assumption of the lemma that for every $w \in \mathcal{V}_i$

$$(T_j w, w)_{\mathcal{A}} \leq c_1 \frac{\|\mathcal{A}_j w\|_{L^2(D)}^2}{\lambda_{\max}(\mathcal{A}_j)} \leq c_1 4\widetilde{\mathcal{C}}^2\left(\frac{1}{\check{a}}\right)^2\left(\frac{h_j}{h_i}\right)^{2s}(w, w)_{\mathcal{A}}. \qquad (23)$$

We argue in a similar fashion as in the second part of the proof of [39, Lemma 6.3], i.e., we conclude with the Cauchy–Schwarz inequality, (23), and with the scaling property of the smoothers in (**B**) that

$$\begin{aligned}
(T_j v, T_i w)_{\mathcal{A}} &\leq (T_j v, v)_{\mathcal{A}}^{1/2}(T_j T_i w, T_i w)_{\mathcal{A}}^{1/2} \\
&\leq \widetilde{\mathcal{C}}^2\left(\frac{1}{\check{a}}\right)^2 (T_j v, v)_{\mathcal{A}}^{1/2}\left(\frac{h_j}{h_i}\right)^{2s}(T_i w, T_i w)_{\mathcal{A}}^{1/2} \\
&\leq \widetilde{\mathcal{C}}^2 \eta\left(\frac{1}{\check{a}}\right)^2\left(\frac{h_j}{h_i}\right)^{2s}(T_j v, v)_{\mathcal{A}}^{1/2}(T_i w, w)_{\mathcal{A}}^{1/2},
\end{aligned}$$

where we again absorbed deterministic constants into $\widetilde{\mathcal{C}}$. Since $h_j/h_i \leq \gamma^{j-i}$ by assumption, the assertion of the proposition follows with Lemma 3. $\qquad\square$

Proposition 12 *Let the smoothers $(R_j : j = 1, \ldots, J)$ satisfy Assumption (**C**) with a deterministic constant c_0. There exists a deterministic constant C independent of j such that inequality (**A.1**) holds with the random variable*

$$K_0 := C\frac{\hat{a}}{\check{a}}.$$

Proof Since the assumption implies that $(R_i^{-1} v_i, v_i) \leq \lambda_{\max}(\mathcal{A}_i)/c_0 \ (v_i, v_i)$, the assertion of the proposition follows with (18), (19), and (21). □

5 Domain decomposition methods

We will consider overlapping domain decomposition methods in the setting of assumptions (**A.1**) and (**A.2**) and for the random elliptic operator defined in (17). We denote by $\mathcal{V}_0 \subset \mathcal{V}$ a *coarse* first order FE space with mesh width h_0 of the type introduced in Sect. 4. The first order FE space with a *fine* grid is denoted by $\widetilde{\mathcal{V}} \subset \mathcal{V}$. For a given set of overlapping subdomains $(D_j : j = 1, \ldots, J)$ of D such that $\overline{D} = \bigcup_{j=1}^{J} \overline{D}_j$. These subdomains result for example by extending a given disjoint set of subdomains by a multiple of h_0 in each spatial direction such that the union of its closures contains D. Also we assume that the boundary ∂D_j aligns with the considered mesh, $j = 1, \ldots, J$. The FE spaces $\mathcal{V}_j$, $j = 1, \ldots, J$, are subspaces of $\widetilde{\mathcal{V}}$ and are defined by

$$\mathcal{V}_j := \{v \in \widetilde{\mathcal{V}} : v(x) = 0 \ \forall x \in D \backslash D_j\}, \quad j = 1, \ldots, J$$

So, we consider the redundant space decomposition

$$\widetilde{\mathcal{V}} = \sum_{j=0}^{J} \mathcal{V}_j.$$

We consider the case that symmetric multigrid solvers from Sect. 4 (Assumptions (**B**), (**C**), and (**D**) are satisfied) are used as so-called *subspaces solvers* ($R_j : j = 0, \ldots, J$), which are random here. Therefore, suppose that the spaces $\mathcal{V}_j$ have nested subspaces $\mathcal{M}_{j,1} \subset \ldots \subset \mathcal{M}_{j,J'(j)} = \mathcal{V}_j$, $j = 0, \ldots, J$. Naturally, only few levels are used on the subspace $\mathcal{V}_0$, i.e. $J'(0) = \mathcal{O}(1)$. As in Sect. 4, we seek for random variables K_0 and K_1 with explicit dependence on the random operator $\mathcal{A}$ in (17), in order to obtain $L^q(\Omega)$-estimates for the condition numbers using additive Schwarz preconditioners.

Proposition 13 *There exists a deterministic constant $C > 0$ that is independent of J and $J'(j)$, $j = 0, \ldots, J$, such that inequality* (**A.1**) *holds with the random variable*

$$K_0 := C \left(\frac{\hat{a}}{\check{a}}\right)^4 \frac{(\hat{a})^2}{(\check{a})^6} \left(1 + \left(\max_{k=1,\ldots,K} \sum_{i,j=1}^{d} \|a_{ij}\|_{W^{s,p}(D_k)} + \|a\|_{L^\infty(D)}\right)^4\right).$$

Proof By (18) and [39, Lemmas 4.5 and 7.1],

$$K_0 \leq \frac{\hat{a}}{\check{a}} \frac{1}{\min_{j=0,\ldots,J} \lambda_{\min}(R_j \mathcal{A}_j)}.$$

Since the R_j's are chosen to be symmetric multigrid solvers, Propositions 12 and 11, Theorem 2, and (13) imply there exists a deterministic constant $c > 0$ such that for every $j = 0, \ldots, J$,

$$\lambda_{\min}(R_j \mathcal{A}_j) \geq c \frac{(\check{a})^5}{\hat{a}} \left(1 + \left(\max_{k=1,\ldots,K} \sum_{i,j=1}^{d} \|a_{ij}\|_{W^{s,p}(D_k)} + \|a\|_{L^\infty(D)} \right)^4 \right)^{-1}.$$

This implies the assertion of the proposition. $\square$

Proposition 14 *Inequality (A.2) holds with the deterministic number*

$$K_1 := (1 + |\{(i, j) \in \{1, \ldots, J\}^2 : D_i \cap D_j\}|)$$

Proof The assertion will follow by [39, Lemma 4.7] after we show an estimate of the form of Assumption (B). By (12), it holds that $\lambda_{\max}(R_j \mathcal{A}_j) \leq 1$, $j = 0, \ldots, J$. Let $j = 0, \ldots, J$ be arbitrary. Since R_j is a symmetric multigrid solver, $R_j \mathcal{A}_j$ is symmetric and positive definite with respect to $(\cdot, \cdot)_\mathcal{A}$. There exists an orthonormal basis of eigenvectors of $R_j \mathcal{A}_j$ with respect to $(\cdot, \cdot)_\mathcal{A}$ such that $R_j \mathcal{A}_j v_i = \lambda_i v_i$. Hence, for every $v \in \mathcal{V}_j$, $v = \sum_i (v, v_i)_\mathcal{A} v_i$ and

$$(R_j \mathcal{A}_j v, R_j \mathcal{A}_j v)_\mathcal{A} = \sum_i \lambda_i^2 ((v, v_i)_\mathcal{A})^2$$

$$\leq \lambda_{\max}(R_j \mathcal{A}_j) \sum_i \lambda_i ((v, v_i)_\mathcal{A})^2 = (R_j \mathcal{A}_j v, v)_\mathcal{A},$$

which implies the estimate of the proposition with [39, Lemma 4.7]. $\square$

6 Application to lognormal diffusion problems

The presented theory in Sects. 3, 4, and 5 is in particular applicable to lognormal diffusion problems. Let Z be a GRF on D that takes values in Hölder spaces $C^t(\overline{D})$ such that for some $t \in (0, 1]$

$$Z \in L^q(\Omega; C^t(\overline{D})) \quad \text{and} \quad \exp(Z) \in L^q(\Omega; C^0(\overline{D})) \quad \forall q \in [1, +\infty), \quad \textbf{(E)}$$

see ahead Sect. 7.2 for a class of instances of such GFRs. Since for every $v \in C^t(\overline{D})$, $t \in (0, 1]$, $\|\exp(v)\|_{C^t(\overline{D})} \leq \|\exp(v)\|_{C^0(\overline{D})} (1 + \|v\|_{C^t(\overline{D})})$, the assumption in (E) implies by the Cauchy–Schwarz inequality

$$\exp(Z) \in L^q(\Omega; C^t(\overline{D})) \quad \forall q \in [1, +\infty)$$

For the lognomal coefficient $a := \exp(Z)$, we consider the elliptic diffusion problem with Dirichlet boundary conditions in variational form to find $u : \Omega \to \mathcal{V}$ such that for $\mathbb{P}$-a.e. ω

$$a_\omega(u(\omega), v) = \int_D a(\omega)\nabla u(\omega) \cdot \nabla v = {}_{\mathcal{V}^*}\langle f, v\rangle_{\mathcal{V}} \quad \forall v \in \mathcal{V}. \tag{24}$$

where $\mathcal{V} = H^1_0(D)$. Well-posedness and approximation by Finite Elements is well-known, cf. [12,13,36]. We use the FE spaces $\mathcal{V}_\ell$ from Sect. 4 with maximal mesh width h_ℓ of $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$, and remark that for each ℓ, the space $\mathcal{V}_\ell$ may have the additional structure for overlapping domain decomposition methods with multigrid subspace solvers as introduced in Sect. 5.

Elements of $H^{1+s}(D)$, $s \in [0, 1]$, can be approximated by functions in $\mathcal{V}_\ell$, cf. [14, Theorem 3.2.1], i.e., there exists a deterministic constant $C > 0$ such that for every $v \in H^{1+s}(D)$ there is $w_\ell \in \mathcal{V}_\ell$ such that

$$\|v - w_\ell\|_{\mathcal{V}} \le Ch^s_\ell\|v\|_{H^{1+s}(D)}. \tag{25}$$

Note that the approximation property stated in [14, Theorem 3.2.1] can be interpolated to also hold for non-integer order Sobolev spaces. The following regularity estimate makes the dependence on the coefficient a explicit. For every $s \in [0, \min\{t, t_{-\Delta}\})\backslash\{1/2\}$ there exists a deterministic constant $C > 0$ such that for $\mathbb{P}$-a.e. ω

$$\|u(\omega)\|_{H^{1+s}(D)} \le C \frac{\|a(\omega)\|_{C^0(\overline{D})}\|a(\omega)\|^2_{C^t(\overline{D})}}{(\min_{x\in\overline{D}} a(\omega, x))^4}\|f\|_{H^{-1+s}(D)}, \tag{26}$$

where $t_{-\Delta}$ is the maximal value such that the inverse of the Dirichlet Laplacian satisfies $(-\Delta)^{-1} : H^{-1+t_{-\Delta}}(D) \to \mathcal{V} \cap H^{1+t_{-\Delta}}(D)$ is bounded. Recall that $D \subset \mathbb{R}^d$ is a polytope. For $d = 2$, the estimate (26) is due to [36, Lemma 5.2] (for $d = 3$, the reader is referred to [36, Remark 5.2(c)]). The solution u can be approximated in $\mathcal{V}_\ell$ by the FE approximation denoted by u_ℓ, in the $L^q(\Omega, \mathcal{V})$-norm, $\ell \in \mathbb{N}$. Specifically, by Céa's lemma, (25), and (26), there exists a deterministic constant $C > 0$ that is independent of a such that for $\mathbb{P}$-a.e. ω and every $\ell \in \mathbb{N}$

$$\|u(\omega) - u_\ell(\omega)\|_{L^q(\Omega;\mathcal{V})} \le C \left\|\frac{\|a(\omega)\|^2_{C^0(\overline{D})}\|a(\omega)\|^2_{C^t(\overline{D})}}{(\min_{x\in\overline{D}} a(\omega, x))^5}\right\|_{L^q(\Omega)} \|f\|_{H^{-1+s}(D)}\, h^s_\ell.$$

Finiteness of the right hand side follows by the Cauchy–Schwarz inequality using the Assumption (**E**). Since the embedding $C^t(\overline{D}) \subset W^{s,p}(D)$ is continuous for every $0 < s < t$ and every $p \in [1, +\infty)$, the conditions from Sects. 4 and 5 are satisfied. Let

$$U^n_\ell : \Omega \to \mathcal{V}_\ell \tag{27}$$

be the result of $n \in \mathbb{N}$ iterations of an iterative algorithm introduced in the previous sections, with initial guess $U^n_{\ell-1}$, $2 \le \ell \in \mathbb{N}$, and $U^n_1 = u_1$. The iterative methods are symmetric multigrid (see Algorithms 2) or PCG using the BPX preconditioner (see Algorithm 1) in Sect. 4. In the setting of Sect. 5, $U^n_\ell : \Omega \to \mathcal{V}_\ell$ in (27) may result from $n \in \mathbb{N}$ iterations of PCG using the additive Schwarz preconditioner (see Algorithm 1), where symmetric multigrid (see Algorithm 2) is used as subspace solvers.

Theorem 15 *For $0 < s < t \le 1$ and every $q, r \in [1, +\infty)$, there exists a constant $C > 0$ such that for every number of iterations $n \in \mathbb{N}$ and $\ell \ge 2$,*

$$\|u - U_\ell^n\|_{L^q(\Omega; \mathcal{V})} \le C(h_\ell^s + n^{-r}).$$

Proof By the Cauchy–Schwarz inequality, $\|a\|_{C^0(\overline{D})}^2 \|a\|_{C^t(\overline{D})}^2 (\min_{x \in \overline{D}} a(x))^{-5} \in L^p(\Omega)$ for every $p \in [1, +\infty)$, which is one of the conditions of Theorem 6.

It remains to verify the needed properties of the random contraction number in the conditions of Theorem 6. In the framework of assumptions (**A.1**) and (**A.2**), by Theorem 2 the random contraction number δ satisfies for the linear iteration $1/(1-\delta) \le K_0(1+K_1)^2/(2-\nu)$. In the case of multigrid, Propositions 11 and 12 and the Cauchy–Schwarz inequality imply that $1/(1 - \delta) \in L^{p'}(\Omega)$ for every $p' \in [1, +\infty)$. For overlapping domain decomposition methods, this statement is due to Propositions 13 and 14. If the additive preconditioner is applied with PCG, the random contraction number δ satisfies by Theorem 2, $1/(1 - \delta) \le \sqrt{K_0 K_1} + 1$. By the same argument, $1/(1 - \delta) \in L^{p'}(\Omega)$ for every $p' \in [1, +\infty)$. Hence, the parameter r in Theorem 6 may be arbitrarily large, which implies the assertion. □

Corollary 16 *In the setting of Theorem 15, let U_ℓ^n result from $n \in \mathbb{N}$ iterations of PCG with a deterministic preconditioner $\widetilde{B}_j$ such that $\kappa(\widetilde{B}_j \mathbb{E}(\mathcal{A}_j))$ is bounded uniformly in j. Then, the strong convergence estimate of Theorem 15 also holds.*

Proof By Lemma 1, for $\mathbb{P}$-a.e. ω, $\kappa(\widetilde{B}_j \mathcal{A}_j(\omega)) \le \hat{a}(\omega)/\check{a}(\omega)\kappa(\widetilde{B}_j \mathbb{E}(\mathcal{A}_j))$. Since $\hat{a}/\check{a} \in L^{q'}(\Omega)$ for every $q' \in [1, +\infty)$, the claim follows as in the proof of Theorem 15. □

Corollary 17 *In the setting of Theorem 15, for every $\varepsilon > 0$ there exists a constant $C_{q,\varepsilon,s} > 0$ that is independent of h_ℓ such that for every $\ell \ge 2$*

$$\|u - U_\ell^n\|_{L^q(\Omega; \mathcal{V})} \le C_{q,\varepsilon,s} \; h_\ell^s.$$

with a deterministic number of iterations given by $n = \lceil n_0 h_\ell^{-\varepsilon} \rceil$ for a deterministic constant $n_0 > 0$.

The cost for one sample of U_ℓ^n is

$$\mathcal{O}(h_\ell^{-d-\varepsilon})$$

with deterministic *constants that are independent of h_ℓ, $\ell \ge 0$.*

Proof The cost of one iteration is $\mathcal{O}(h_\ell^{-d})$, since the matrix vector product has cost $\mathcal{O}(h_{\ell'}^{-d})$ for the sparse stiffness matrices that result taking the nodal basis of $\mathcal{V}_{\ell'}$, $\ell' \le \ell$.

The error contributions in the estimate of Theorem 15 are equilibrated for this choice of iteration number, since r in Theorem 15 can be chosen arbitrarily large, i.e., $r = s/\varepsilon$ is admissible. □

Remark 18 The established theory in this paper is also applicable for deterministic preconditioners that do not imply uniform condition numbers. A possible class of

examples are so-called "algebraic multigrid" (AMG) preconditioners for which a multilevel convergence theory does not seem to be available, cf. [40, Sect. 5]. However, if such a deterministic symmetric AMG preconditioner could be tuned to $\mathbb{E}(\mathcal{A}_j)$ or any operator which is spectrally equivalent in the sense of (2), the respective iterative method also converges strongly as in Theorem 15 by a similar argument applying Corollary 16. Note that in the example (24) (see also Example 9), if Z is stationary, then $\mathbb{E}(\mathcal{A})$ is the Dirichlet Laplacian multiplied by $\mathbb{E}(\exp(Z))$, which is constant with respect to $x \in D$.

7 Application to multilevel Monte Carlo

Multilevel Monte Carlo methods make the numerical approximation of moments of random quantities feasible where sampling is computationally costly. The analysis of the computational cost versus accuracy of MLMC that does not rely on additional assumptions on the used PDE solver is an application of the presented theory in previous sections. The MLMC estimator with $L \in \mathbb{N}$ levels to approximate the mean field $\mathbb{E}(u)$ is given by

$$
E_L^{\mathrm{ML}}(U_L^{n_L}) := \sum_{\ell=1}^{L} E_{M_\ell}(U_\ell^{n_\ell} - U_{\ell-1}^{n_{\ell-1}}),
$$

where $(E_{M_\ell} : \ell = 1, \ldots, L)$ are Monte Carlo estimators that are mutually independent. We used the convention that $U_0^{n_0} := 0$. Sample numbers $M_\ell, \ell = 1, \ldots, L$ are chosen to optimize the accuracy versus the computational cost.

7.1 Computational cost versus accuracy

In the literature [13,22], generic asymptotic bounds of the required computational cost for a certain accuracy with MLMC are given that pose assumptions on the behavior of $\|u - u_\ell\|_{L^2(\Omega;\mathcal{V})} = \mathcal{O}(h_\ell^s)$ and on the required computational cost to sample u_ℓ. In [13], the assumption on the decay of $\|u - u_\ell\|_{L^2(\Omega;\mathcal{V})} = \mathcal{O}(h_\ell^s)$ was verified without investigating the computational cost to sample u_ℓ. In the present paper this is achieved by Corollary 17, i.e., $\|u - U_\ell^{n_\ell}\|_{L^2(\Omega;\mathcal{V})} = \mathcal{O}(h_\ell^s)$ with computational cost $\mathcal{O}(h_\ell^{-d-\varepsilon})$ to sample $U_\ell^{n_\ell}$ for any $\varepsilon > 0$, where the implied constants depend on ε. Thus, by [13, Theorem 4.1], an error threshold $0 < \mathrm{TOL}$, i.e.,

$$
\| \mathbb{E}(u) - E_L^{\mathrm{ML}}(U_L^{n_L})\|_{L^2(\Omega;\mathcal{V})} = \mathcal{O}(\mathrm{TOL}),
$$

can be achieved with computational cost

$$
\mathrm{work}_L = \begin{cases} \mathcal{O}(\mathrm{TOL}^{-2}) & \text{if } 2s > d + \varepsilon, \\ \mathcal{O}(\mathrm{TOL}^{-(d+\varepsilon)/s}) & \text{if } 2s < d + \varepsilon, \end{cases} \tag{28}
$$

where $d = 1, 2, 3$ is the dimension of the domain D. We have assumed that a fast method is available to sample the random coefficient a. This is for example the case for certain stationary Gaussian random fields (GRFs), which can be sampled by truncated Karhunen Loève series or circulant embedding which are fast, FFT based techniques, cf. [34, Chapter 7].

7.2 Numerical experiments

We consider a class of centered, stationary GRFs that are solutions to the white noise stochastic PDE

$$(-\Delta + \kappa^2)^{\alpha/2} Z = W \quad \text{on } \widetilde{D}, \tag{29}$$

where $\widetilde{D}$ is a superset of the domain D, W is spatial white noise on $\widetilde{D}$ (cf. [1]), $\nu = \alpha - d/2 > 0$, and $\kappa > 0$. It is well-known that for $\widetilde{D} = \mathbb{R}^d$, the GRF Z has so-called Matérn covariance with smoothness parameter ν and length scale parameter $\lambda = \sqrt{\nu}/\kappa$, cf. [33].

In our numerical experiments, we will choose $d = 2$ and $D = [0, 1]^2$. A stationary GRF as a solution to (29) results by restricting Z to the domain D. It is convenient to choose $\widetilde{D} = [-1/2, 3/2]^2$ with periodic boundary conditions. Note that $\text{dist}(D, \partial\widetilde{D}) = 1/2$, and we consider values of the correlation length, which are smaller or equal than this *window size*, cf. [29]. The solution Z to (29) can be obtained by a spectral Galerkin method using the eigenfunctions of the Laplacian with periodic boundary conditions on $\widetilde{D}$ normalized in $L^2(\widetilde{D})$. Since these eigenfunctions separate, we denote them with double indices $(\psi_{k_1,k_2})_{k_1,k_2 \in \mathbb{N}_0}$. We observe that white noise W applied to any ONB yields a sequence of independent standard normally distributed random variables. Since this ONB diagonalizes the operator $(-\Delta + \kappa^2)^{\alpha/2}$, the stationary GRF Z can be explicitly expanded with respect to this basis. The random coefficients with respect to this ONB are given by

$$\frac{1}{(\pi^2(k_1^2 + k_2^2) + \kappa^2)^{\alpha/2}} y_{k_1,k_2}, \quad k_1, k_2 \in \mathbb{N}_0,$$

where $(y_{k_1,k_2})_{k_1,k_2 \in \mathbb{N}_0}$ is a sequence of independent, standard normally distributed random variables. For $\sigma > 0$, to be determined later, the GRF Z will be rescaled such that the pointwise variance satisfies

$$\mathbb{E}(Z(x)^2) = \sigma^2, \quad \forall x \in D.$$

The expansion of Z can be truncated for numerical purposes and efficiently implemented with FFT. For any $\widetilde{k} \in \mathbb{N}$, let us denote the truncation of the expansion of Z to the terms such that $k_1, k_2 \leq \widetilde{k}$ by $Z^{\widetilde{k}}$. By an argument similar to the proof of [27, Theorem 2.2], for every $t \in (0, \nu)$, $\varepsilon \in (0, \nu - t)$, and every $q \in [1, +\infty)$, there exists a constant $C > 0$ such that for every truncation $\widetilde{k} \in \mathbb{N}$

$$\|Z - Z^{\widetilde{k}}\|_{L^q(\Omega; C^t(\widetilde{D}))} \leq C\widetilde{k}^{-(\nu-t-\varepsilon)}.$$

The additional error introduced by truncating the expansion of Z is consistent with the FE discretization error if $\widetilde{k}$ is chosen level-dependently such that $\widetilde{k}_\ell = \lceil \widetilde{k}_1 h_\ell^{-\nu} \rceil$ for some $\widetilde{k}_1 > 0$ at our disposal and FE mesh width h_ℓ. The reader is referred to the discussion in [27, Sects. 3 and 5]. As a consequence of Fernique's theorem, we may conclude similarly as in [12, Proposition 3.10]. that $\exp(Z) \in L^q(\Omega; C^0(\overline{D}))$ for every $q \in [1, +\infty)$; see also [27, Proposition B.1]. Thus, the assumptions in (E) are verified by the GRF Z and the developed theoretical error estimates of this paper, in particular Corollary 17 and (28), hold.

The GRF Z is taken as a random input and we seek to approximate the expectation of the solution to (24) with a MLMC estimator, where $a = \exp(Z)$ and right hand side $f(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$. We use the sample numbers according to [26, Eqs. (44) and (47)], i.e., for $M^* \in \mathbb{N}$

$$M_\ell = \lceil M_1 h_\ell^{(2s+d+\varepsilon)/2} \rceil, \quad \ell = 2, \ldots, L, \tag{30}$$

and

$$M_1 = M^* \begin{cases} \lceil 2^{sL/2} \rceil & \text{if} \quad d + \varepsilon < 2s, \\ \lceil 2^{(2s+d+\varepsilon)L/2} \rceil & \text{if} \quad d + \varepsilon > 2s, \end{cases} \tag{31}$$

here with $M^* = 5$ and $L = 1, \ldots, 8$ levels. For the level dependent truncation of the expansion of the GRF, we use $\widetilde{k}_\ell = \lceil \widetilde{k}_1 h_\ell^{-\nu} \rceil$ with $\widetilde{k}_1 = 2$. We consider a full multigrid cycle with PCG using BPX with Jacobi smoothers and a direct solver on the lowest level. We recall that the iteration numbers depend mildly on the level of the MLMC estimator and will be chosen by $n_\ell = \lceil n_0 h_\ell^{-\varepsilon} \rceil$ for $0 < \varepsilon \ll 1$. We use a triangulation of D, which is uniformly refined, resulting in FE spaces $\mathcal{V}_\ell \subset \mathcal{V}$, $\ell \geq 1$, with maximal mesh width h_ℓ, which incorporate Dirichlet boundary conditions. The FE spaces are spanned by a nodal basis. The implementation uses the FE C++ library BETL, cf. [28], the GRF is implemented using FFTW, cf. [17], and the execution of MLMC is parallelized using the MPI-based wrapper gMLQMC, cf. [19]. Since the GRF Z may be periodically extended, $[0, 2]^2$ can be used as a computational domain, which eases implementation. The $L^2(\Omega; \mathcal{V})$-norm will be estimated by $\| \mathbb{E}(u) - E_L^{ML}(U_L^{n_L}) \|_{L^2(\Omega; \mathcal{V})}^2 \approx \sum_{j=1}^R \| \mathbb{E}(u) - \widehat{E_L^{ML}(U_L^{n_L})}^{(j)} \|_{\mathcal{V}}^2 / (R - 1)$ where R i.i.d. realizations of $E_L^{ML}(U_L^{n_L})$ are used. The reference solution is approximated by the average of $R = 20$ realizations of $E_L^{ML}(U_L^{n_L})$ with $L = 9$, sample numbers (30) and (31) with $M^* = 40$, and a sparse direct solver.

In Fig. 1a, b, the error is plotted versus the degrees of freedom in the FE space of the highest level that is active in the MLMC estimator. The empirical rate is computed by least squares taking into account the five data pairs corresponding to finer resolution. The fitted lines are shifted down for better visability. We remark that the relation to the total required work is asymptotically $\text{work}_L = \mathcal{O}(h_L^{-d-\varepsilon})$. Thus, the rate implied by (28) is for $\nu = 0.5$ approximately 0.25. We observe that the performance of the iterative solver is in the required range of accuracy as good as a sparse direct solver. In Fig. 1a, b, the rate seems to depend on the value of the correlation length. Sometimes for practitioners, a smaller value for the correlation length is of interest. However, the performance of the iterative solver is still as good as the sparse direct
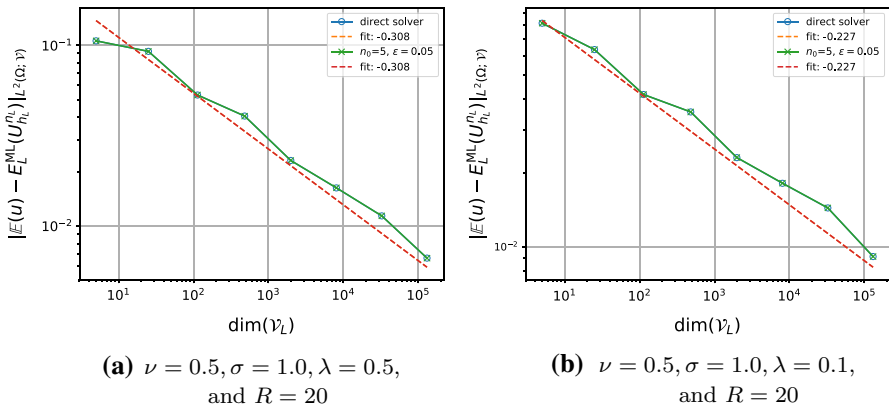
Fig. 1 Comparison of full multigrid cycle using PCG with BPX with $n_0 = 5$, $\varepsilon = 0.05$ to a sparse direct solver

solver. These numerical tests were performed with iteration numbers $n_\ell = \lceil n_0 h_\ell^{-\varepsilon} \rceil$ for $0 < \varepsilon \ll 1$. The presented theory also underpins the strategy to apply PCG with a stopping criterion and use n_ℓ as an upper bound of the iteration numbers. This way, one may benefit from certain superlinear convergence effects of CG, cf. [4], which were for ease of exposition not accounted for in the presented theory.

8 Conclusions and extensions

In the study of linear random operator equations with non-uniform input a rigorous framework has been established to verify strong convergence of a wide range of iterative solvers. This includes standard solvers such as multigrid, domain decomposition, and preconditioned conjugate gradient. In the case of lognormal random input, essentially optimal, deterministic complexity bounds are implied. This offers an alternative to direct solvers for this type of problems. For MLMC, we concluded deterministic, essentially optimal computational cost versus accuracy estimates, which is a novel result. Numerical experiments with CG preconditioned by BPX confirmed these asymptotic bounds and we conclude that standard iterative solvers are applicable in the case of lognormal Hölder continuous input. Assumptions on the computational cost of PDE solvers are also common in the context of multilevel quasi-Monte Carlo methods (MLQMC) for PDE problems, cf. [20,26,31,32]. Applicability of iterative solvers for MLQMC in a certain setting has been analyzed by the author in [25].

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