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Least squares solutions to the rank-constrained matrix approximation problem in the Frobenius norm

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Abstract
In this paper, we discuss the following rank-constrained matrix approximation problem in the Frobenius norm: $\|C-AX\| = \min$ subject to $\mathrm{rk}\,(C_1 - A_1X) = b$, where b is an appropriate chosen nonnegative integer. We solve the problem by applying the classical rank-constrained matrix approximation, the singular value decomposition, the quotient singular value decomposition and generalized inverses, and get two general forms of the least squares solutions.

Keywords Matrix approximation problem · Rank-constrained matrix · SVD · Q-SVD

Mathematics Subject Classification 15A09 · 15A24

1 Introduction

In this paper, we adopt the following notation. The symbol $\mathbb{C}^{m\times n}$ ($\mathbb{U}^{m\times m}$) denotes the set of all $m \times n$ complex matrices ($m \times m$ unitary matrices), I_k denotes the $k \times k$ identity matrix, 0 denotes a zero matrix of appropriate size, and $\|\cdot\|$ stands for the matrix Frobenius norm. For $A \in \mathbb{C}^{m\times n}$, A^H and $\mathrm{rk}(A)$ stand for the conjugate transpose and the rank of A, respectively.

The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is defined as the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^H = AX,\ (4)\ (XA)^H = XA,$$

and is usually denoted by $X = A^{\dagger}$ (see [1]). The symbol $A\{i, \ldots, j\}$ is the set of matrices $X \in \mathbb{C}^{n\times m}$ which satisfies equations $(i), \ldots, (j)$ from among equations (1)–

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(4). A matrix $X \in A\{i, \ldots, j\}$ is called an $\{i, \ldots, j\}$-inverse of A, and is denoted by $A^{(i,\ldots,j)}$. It is well known that $AA^{(1,3)} = AA^{\dagger}$. Furthermore, we denote

$$P_A = A^{\dagger}A,\ E_A = I_m - AA^{\dagger} \text{ and } F_A = I_n - A^{\dagger}A.$$

Given that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $C \in \mathbb{C}^{l\times n}$, the column block matrix consisting of A and B is denoted by $\begin{pmatrix} A, & B \end{pmatrix}$, and its rank is denoted by $\text{rk}\begin{pmatrix} A, & B \end{pmatrix}$; the row block matrix consisting of A and C is denoted by $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$, and its rank is denoted by $\text{rk}\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$. It is well known that $\|\begin{pmatrix} A, & B \end{pmatrix}\|^2 = \|A\|^2 + \|B\|^2$. The two known formulas for ranks of block matrices are given in [11],

$$\text{rk}\begin{pmatrix} A, & B \end{pmatrix} = \text{rk}\,(A) + \text{rk}\,(E_A B) = \text{rk}\,(B) + \text{rk}\,(E_B A), \tag{1.1a}$$

$$\text{rk}\begin{pmatrix} A \\ C \end{pmatrix} = \text{rk}\,(A) + \text{rk}\,(CF_A) = \text{rk}\,(C) + \text{rk}\,(AF_C). \tag{1.1b}$$

In the literature, the minimum rank matrix approximations or rank-constrained matrix approximations have been widely studied [1–3,5,6,8,9,12,14–17,20,22, etc]. Recently, Friedland and Torokhti [6] studied the problem of finding least square solutions to the equation $BXC = A$ subject to $\text{rk}\,(X) \le k$ in the Frobenius norm by applying SVD; Wei and Wang [21] studied the problem of finding rank-k Hermitian nonnegative definite least squares solutions to the equation $BXB^H = A$ in the Frobenius norm and discussed the ranges of the rank k; Sou and Rantzer [15] studied the minimum rank matrix approximation problem in the spectral norm $\min\limits_{X} \text{rk}(X)$ subject to $\|A - BXC\|_2 < 1$; Wei and Shen [22] studied a more general problem $\min\limits_{X} \text{rk}(X)$ subject to $\|A - BXC\|_2 < \xi$, where $\xi \ge \theta$ and $\theta = \min\limits_{Y} \|A - BYC\|_2$, by applying SVD and R-SVD; Tian and Wang [18] gave a least-squares solution of $AXB = C$ subject to $\text{rk}(AXB - C) = \min$ in the Frobenius norm. On the other hand, the minimum rank matrix approximations or rank-constrained matrix approximations have found many applications in control theory [4,15,22], signal process [6] and numerical algebra [3,8], etc.

Note that Golub et al. [8] studied the problem of finding rank-constrained least square solutions to the equation $\begin{pmatrix} A, & X \end{pmatrix} = \begin{pmatrix} A, & B \end{pmatrix}$ subject to $\text{rk}\begin{pmatrix} A, & X \end{pmatrix} \le k$ in all unitarily invariant norms by applying SVD and QR decomposition; Demmel [3] considered the least square solutions to $\left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ for X subject to $\text{rk}\left(\begin{smallmatrix} A & B \\ C & X \end{smallmatrix}\right) \le k$ in the Frobenius norm and the 2-norm; Wang [19] studied a general problem of determining the least squares solution X of $\left(\begin{smallmatrix} X & J \\ K & L \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ subject to $\text{rk}\left(\begin{smallmatrix} X & J \\ K & L \end{smallmatrix}\right) = k$ in the Frobenius norm.

In [3,6,8], a commonly assumption is that the rank is less than or equal to k. In fact, in most situation, the rank is equal to k. For instance, consider the descriptor linear system

$$E\dot{x}(t) = Ax(t) + Bu(t). \tag{1.2}$$

Applying a full-state derivative feedback controller $u(t) = -K\dot{x}(t)$ to system (1.2), we have the closed-loop system $(E + BK)\dot{x}(t) = Ax(t)$. The dynamical order is defined to be rk $(E+BK)=p$. One of the minimum gain problems is characterize the set

$$\mathfrak{W}=\left\{K \,\middle|\, \|K\|^2=\min \ \text{ subject to } \ \mathrm{rk}\,(E+BK)=p\right\}.$$

Therefore, Duan [4] studied the problem of finding rank-k least square solutions to $BX=A$; Liu et al. [10] considered the problem $\min\limits_{\mathrm{rk}(X)=k}\|A-BXB^H\|$, in which A and X are (skew) Hermitian matrices. In this paper, we study a more general problem by applying SVD and Q-SVD. Assume that b is a prescribed nonnegative integer, $A\in\mathbb{C}^{m\times n}$, $A_1\in\mathbb{C}^{w\times n}$, $C\in\mathbb{C}^{m\times p}$ and $C_1\in\mathbb{C}^{w\times p}$ are given matrices. We now investigate the problem of determining the least squares solution X of the matrix equation $AX=C$ subject to rk $(C_1-A_1X)=b$ in the Frobenius norm. This problem can be stated as follows.

Problem 1.1 *Suppose that $A\in\mathbb{C}^{m\times n}$, $A_1\in\mathbb{C}^{w\times n}$, $C\in\mathbb{C}^{m\times p}$ and $C_1\in\mathbb{C}^{w\times p}$ are given matrices. For an appropriate chosen nonnegative integer b, characterize the set*

$$\mathcal{S}=\left\{X \,\middle|\, X\in\mathbb{C}^{n\times p},\ \|C-AX\|=\min \ \textit{ subject to } \ rk\,(C_1-A_1X)=b\right\}. \tag{1.3}$$

2 Preliminaries

In this section, we mention the following results for our further discussions.

Lemma 2.1 [4] *Given that $\mathcal{X}_1\in\mathbb{C}^{s\times n_1}$ and the integer k_1 satisfying $0\le k_1\le$ $\min\{m_1,n_1\}$, then there exists $\mathcal{X}_2\in\mathbb{C}^{(m_1-s)\times n_1}$ satisfying*

$$rk\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix}=k_1$$

if and only if

$$max\,\{0,k_1-(m_1-s)\}\le\ rk\,(\mathcal{X}_1)\le\min\{s,k_1\}.$$

Lemma 2.2 *Suppose that $H\in\mathbb{C}^{m\times n}$, $\mathrm{rk}(H)=r$, l is a given nonnegative integer with $l\le r$, the decomposition of H is*

$$H=U\begin{pmatrix}0&0\\0&G\\0&0\end{pmatrix}V^H,$$

where $G\in\mathbb{C}^{m_1\times n_1}$, $\mathrm{rk}(G)=r$, $k\le m_1\le m$, $k\le n_1\le n$, U and V are unitary matrices of appropriate sizes. Then

$$S_1=S_2,$$

where

$$S_1 = \left\{\widetilde{T} \left| \widetilde{T} \in \mathbb{C}^{m\times n}, \mathrm{rk}\left(\widetilde{T}\right) = l, \left\|\widetilde{T} - H\right\| = \min \right.\right\}$$

and

$$S_2 = \left\{\widetilde{T} \left| \widetilde{T} = U \begin{pmatrix} 0 & 0 \\ 0 & T \\ 0 & 0 \end{pmatrix} V^H, T \in \mathbb{C}^{m_1\times n_1}, \mathrm{rk}\left(T\right) = l, \left\|T - G\right\| = \min \right.\right\}.$$

The following result is the classical rank-constrained matrix approximation of Eckart and Young [5], and Mirsky [12].

Lemma 2.3 *Suppose that $\mathcal{C} \in \mathbb{C}^{s\times n_1}$ with $\mathrm{rk}(\mathcal{C}) = c$, c_1 is a given nonnegative integer with $c_1 \leq c$. Let the SVD [7] of $\mathcal{C}$ be*

$$\mathcal{C} = \mathcal{U} \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \mathcal{V}^H, \tag{2.1}$$

where $\Lambda = \; diag\{\lambda_1, \ldots, \lambda_c\}$, $\lambda_1 \geq \cdots \geq \lambda_c > 0$, $\mathcal{U}$ and $\mathcal{V}$ are unitary matrices of appropriate sizes. Then

$$\min_{\mathrm{rk}(\mathcal{X})=c_1} \|\mathcal{C} - \mathcal{X}\| = \left(\sum_{i=c_1+1}^{c} \lambda_i^2\right)^{\frac{1}{2}}.$$

Furthermore, when $\lambda_{c_1} > \lambda_{c_1+1}$,

$$\mathcal{X} = \mathcal{U}\mathrm{diag}\left\{\lambda_1, \ldots, \lambda_{c_1}, 0, \ldots, 0\right\} \mathcal{V}^H;$$

when $p_2 < c_1 < p_1 \leq r$ and $\lambda_{p_2} > \lambda_{p_2+1} = \cdots = \lambda_{p_1} > \lambda_{p_1+1}$,

$$\mathcal{X} = \mathcal{U}\mathrm{diag}\left\{\lambda_1, \ldots, \lambda_{p_2}, \lambda_t \mathcal{Q}\mathcal{Q}^H, 0, \ldots, 0\right\} \mathcal{V}^H,$$

and $\mathcal{Q}$ is an arbitrary matrix satisfying $\mathcal{Q} \in \mathbb{C}^{(p_1-p_2)\times(c_1-p_2)}$ and $\mathcal{Q}^H\mathcal{Q} = I_{c_1-p_2}$.

Suppose that $\mathcal{X}_1 \in \mathbb{C}^{s\times n_1}$, $\mathcal{X}_2 \in \mathbb{C}^{(m_1-s)\times n_1}$, $\mathrm{rk}\begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} = k_1$ and $k_1 \leq \min\{m_1, n_1\}$ be a given nonnegative integer, it is easy to check that

$$\begin{cases} \mathrm{rk}\left(\mathcal{X}_1\right) > c, & \text{if } k_1 > c + (m_1 - s), \\ \mathrm{rk}\left(\mathcal{X}_1\right) \leq c, & \text{if } k_1 \leq c + (m_1 - s). \end{cases}$$

Suppose that $\mathcal{C} \in \mathbb{C}^{s\times n_1}$ with $\mathrm{rk}(\mathcal{C}) = c$. Consider the rank-constrained matrix approximation

$$\|\mathcal{C} - \mathcal{X}_1\| = \min \quad \text{subject to} \quad \mathrm{rk}\begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} = k_1, \tag{2.2}$$

under the condition $k_1 \leq \min\{c+(m_1-s), n_1\}$. We have the following Lemma 2.4 by applying Lemma 2.1 and Lemma 2.3 to (2.2).

Lemma 2.4 *Suppose that* $\mathcal{C} \in \mathbb{C}^{s\times n_1}$ *with* $\mathrm{rk}(\mathcal{C}) = c$, k_1 *is a given nonnegative integer with* $0 \leq k_1 \leq \min\{c+(m_1-s), n_1\}$, *and the SVD of* $\mathcal{C}$ *be given as in Lemma 2.3, then,*

(a) *if* $c \leq k_1 \leq \min\{c+(m_1-s), n_1\}$,

$$\min_{\mathcal{X}_1,\ rk\left(\begin{smallmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{smallmatrix}\right)=k_1} \|\mathcal{C}-\mathcal{X}_1\| = 0,$$

and

$$\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix} = \begin{pmatrix}\mathcal{C}\\ (\mathcal{X}_{21},\ \mathcal{X}_{22})\,\mathcal{V}^H\end{pmatrix},$$

where $\mathcal{X}_{21} \in \mathbb{C}^{(m_1-s)\times c}$, $\mathcal{X}_{22} \in \mathbb{C}^{(m_1-s)\times(n_1-c)}$ *and* $rk\,(\mathcal{X}_{22}) = k_1 - c$.

(b) *if* $0 \leq k_1 < c$,

$$\min_{\mathcal{X}_1,\ rk\left(\begin{smallmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{smallmatrix}\right)=k_1} \|\mathcal{C}-\mathcal{X}_1\| = \left(\sum_{i=k_1+1}^{c} \lambda_i^2\right)^{\frac{1}{2}},$$

and

$$\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix} = \begin{pmatrix}\mathcal{U} & 0\\ 0 & I_{m_1-s}\end{pmatrix}\begin{pmatrix}\Lambda_1 & 0\\ 0 & 0\\ \mathcal{X}_{21} & 0\end{pmatrix}\mathcal{V}^H,$$

where $\mathcal{X}_{21} \in \mathbb{C}^{(m_1-s)\times c}$, *when* $\lambda_{k_1} > \lambda_{k_1+1}$,

$$\Lambda_1 = \mathrm{diag}\left\{\lambda_1, \ldots, \lambda_{k_1}\right\};$$

when $q_2 < k_1 < q_1 \leq r$ *and* $\lambda_{q_2} > \lambda_{q_2+1} = \ldots = \lambda_{q_1} > \lambda_{q_1+1}$,

$$\Lambda_1 = \mathrm{diag}\left\{\lambda_1, \ldots, \lambda_{q_2}, \lambda_{k_1}\mathcal{Q}\mathcal{Q}^H\right\},$$

in which $\mathcal{Q}$ *is an arbitrary matrix satisfying* $\mathcal{Q} \in \mathbb{C}^{(q_1-q_2)\times(k_1-q_2)}$ *and* $\mathcal{Q}^H\mathcal{Q} = I_{k_1-q_2}$.

Lemma 2.5 [13] *Suppose that* $A \in \mathbb{C}^{m\times n}$, $A_1 \in \mathbb{C}^{w\times n}$, $D^H = \left(A^H,\ A_1^H\right)$ *and* $k = \mathrm{rk}(D)$, *then there exist* $U \in \mathbb{U}^{m\times m}$ *and* $V \in \mathbb{U}^{w\times w}$ *and a nonsingular matrix* $W \in \mathbb{C}^{n\times n}$ *such that*

$$A = U\Sigma W \ \text{ and } \ A_1 = V\Sigma_1 W, \tag{2.3}$$

where $r = k - \mathrm{rk}(A_1)$, $s = \mathrm{rk}(A) + \mathrm{rk}(A_1) - k$,

$$\Sigma = \begin{array}{c} \\ r+s \\ m-r-s \end{array} \begin{array}{c} \begin{array}{ccc} r+s & k-r-s & n-k \end{array} \\ \begin{pmatrix} \Psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \Psi = \begin{array}{c} \\ r \\ s \end{array} \begin{array}{c} \begin{array}{cc} r & s \end{array} \\ \begin{pmatrix} I_r & 0 \\ 0 & S_1 \end{pmatrix} \end{array},$$

$$\Sigma_1 = \begin{array}{c} \\ w-k+r \\ k-r \end{array} \begin{array}{c} \begin{array}{ccc} r & k-r & n-k \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Psi_1 & 0 \end{pmatrix} \end{array} \quad \text{and} \quad 9_1 = \begin{array}{c} \\ s \\ k-r-s \end{array} \begin{array}{c} \begin{array}{cc} s & k-r-s \end{array} \\ \begin{pmatrix} \widehat{S}_1 & 0 \\ 0 & I_{k-r-s} \end{pmatrix} \end{array},$$

in which S_1 *and* $\widehat{S}_1$ *are both positive diagonal matrices.*

If $S_1 = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ and $\widehat{S}_1 = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_s)$ satisfy $1 > \alpha_1 \geq \ldots \geq \alpha_s > 0$, $1 > \beta_s \geq \ldots \geq \beta_1 > 0$, $\alpha_i^2 + \beta_i^2 = 1$, $i = 1, \ldots, s$, and there exists a positive diagonal matrix $\Sigma_2 = \mathrm{diag}\,(\sigma_1(D), \ldots, \sigma_k(D))$, in which $\sigma_1(D), \ldots, \sigma_k(D)$ are the positive singular values of D, and two unitary matrices $P \in \mathbb{C}^{k\times k}$ and $Q \in \mathbb{C}^{n\times n}$ satisfy

$$W = \begin{pmatrix} P^H\Sigma_2 & 0 \\ 0 & I_{n-k} \end{pmatrix} Q^H,$$

then (2.3) is the well-known Q-SVD of A and A_1.

Denoting $A^- = W^{-1}\Sigma^\dagger U^H$ and $A_1^- = W^{-1}\Sigma_1^\dagger V^H$, we know that $A^- \in A\{1, 3\}$, so it suffices to check that $AA^- = AA^\dagger$.

3 Solutions to Problem 1.1

In this section, we solve Problem 1.1 proposed in Sect. 1, and get two general forms of the least squares solutions.

Theorem 3.1 *Suppose that* $A \in \mathbb{C}^{m\times n}$, $A_1 \in \mathbb{C}^{w\times n}$, $C \in \mathbb{C}^{m\times p}$, $C_1 \in \mathbb{C}^{w\times p}$, k, r, s, *and the decompositions of* A *and* A_1 *are as in Lemma* 2.5. *Partition*

$$V^H C_1 = \begin{array}{c} \\ w-k+r \\ k-r \end{array} \begin{array}{c} p \\ \begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} \end{array}, \quad U^H C = \begin{array}{c} \\ r \\ s \\ m-r-s \end{array} \begin{array}{c} p \\ \begin{pmatrix} \widehat{C}_1 \\ \widehat{C}_2 \\ \widehat{C}_3 \end{pmatrix} \end{array}. \tag{3.1}$$

Let t *denote the rank of* C_{11}, *and let the SVD of* $C_{11} \in \mathbb{C}^{(w-k+r)\times p}$ *be*

$$C_{11} = U_1 \begin{pmatrix} \mathrm{T} & 0 \\ 0 & 0 \end{pmatrix} V_1^H, \tag{3.2}$$

where $\mathrm{T} \in \mathbb{C}^{t\times t}$ *is a nonsingular matrix,* $U_1 \in \mathbb{U}_{w-k+r}$ *and* $V_1 \in \mathbb{U}_p$*. Partition*

$$C_{12}V_1 = k-r \overset{\quad\; t \qquad\; p-t}{\begin{pmatrix} C_{121}, & C_{122} \end{pmatrix}}, \tag{3.3}$$

$$\widehat{C}_2 V_1 - \left(S_1\widehat{S}_1^{-1}, 0\right)\left(0, C_{122}\right) = s \overset{\quad t \qquad p-t}{\begin{pmatrix} \widehat{C}_{11}, & \mathcal{C} \end{pmatrix}}, \tag{3.4}$$

Also suppose that the SVD of $\mathcal{C}$ *is given in* (2.1)*, and denotes* $\mathrm{rk}(\mathcal{C}) = c$*. Then there exists a matrix* $X \in \mathbb{C}^{n\times p}$ *satisfying* (1.3) *if and only if*

$$t \le b \le \min\{\mathrm{rk}\,(A_1,\ \ C_1)\,, c+t+k-r-s, p\}. \tag{3.5}$$

If $c+t \le b \le \min\{\mathrm{rk}\,(A_1,\ \ C_1)\,, c+t+k-r-s, p\}$*, then*

$$\min_{rk(C_1-A_1X)=b} \|C - AX\| = \left\|\widehat{C}_3\right\|, \tag{3.6}$$

and a general form for X which satisfies (1.3) *is*

$$X = W^{-1}\begin{pmatrix} & \widehat{C}_1 & \\ \Psi_1^{-1}\left(\begin{pmatrix} \widehat{S}_1 S_1^{-1}\widehat{C}_{11} \\ \mathcal{Y} \end{pmatrix}, C_{122} + \begin{pmatrix} \widehat{S}_1 S_1^{-1}\mathcal{C} \\ \left(\mathcal{X}_{21}, \mathcal{X}_{22}\right)\mathcal{V}^H \end{pmatrix}\right) V_1^H \\ & \mathcal{Z} & \end{pmatrix}, \tag{3.7}$$

where $\mathcal{Z} \in \mathbb{C}^{(n-k)\times p}$*,* $\mathcal{Y} \in \mathbb{C}^{(k-r-s)\times t}$ *and* $\mathcal{X}_{21} \in \mathbb{C}^{(k-r-s)\times c}$ *are arbitrary matrices, and* $\mathcal{X}_{22} \in \mathbb{C}^{(k-r-s)\times(p-t-c)}$ *satisfies* $rk\,(\mathcal{X}_{22}) = b-t-c$*.*

If $t \le b < c+t$*, then*

$$\min_{rk(C_1-A_1X)=b} \|C - AX\| = \left(\left\|\widehat{C}_3\right\|^2 + \sum_{i=b-t+1}^{c} \lambda_i^2\right)^{\frac{1}{2}}, \tag{3.8}$$

and a general form for X which satisfies (1.3) *is*

$$X = W^{-1}\begin{pmatrix} & \widehat{C}_1 & \\ \Psi_1^{-1}\left(\begin{pmatrix} \widehat{S}_1 S_1^{-1}\widehat{C}_{11} \\ \mathcal{Y} \end{pmatrix}, C_{122} + \begin{pmatrix} \widehat{S}_1 S_1^{-1}\mathcal{U} & 0 \\ 0 & I_{k-r-s} \end{pmatrix}\begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \\ \mathcal{X}_{21} & 0 \end{pmatrix}\mathcal{V}^H\right) V_1^H \\ & \mathcal{Z} & \end{pmatrix}, \tag{3.9}$$

where $\mathcal{Z} \in \mathbb{C}^{(n-k)\times p}$*,* $\mathcal{Y} \in \mathbb{C}^{(k-r-s)\times t}$ *and* $\mathcal{X}_{21} \in \mathbb{C}^{(k-r-s)\times c}$ *are arbitrary matrices, and when* $\lambda_{b-t} > \lambda_{b-t+1}$*,*

$$\Lambda_1 = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_{b-t}\};$$

when $q_2 < b-t < q_1 \leq r$ *and* $\lambda_{q_2} > \lambda_{q_2+1} = \ldots = \lambda_{q_1} > \lambda_{q_1+1}$,

$$\Lambda_1 = \operatorname{diag}\left\{\lambda_1, \ldots, \lambda_{q_2}, \lambda_{b-t}\mathcal{Q}\mathcal{Q}^H\right\},$$

in which $\mathcal{Q}$ *is an arbitrary matrix satisfying* $\mathcal{Q} \in \mathbb{C}^{(q_1-q_2)\times(b-t-q_2)}$ *and* $\mathcal{Q}^H\mathcal{Q} = I_{b-t-q_2}$.

Proof Partition

$$WX = \begin{matrix} r \\ k-r \\ n-k \end{matrix}\overset{p}{\begin{pmatrix} X_1 \\ X_2 \\ \mathcal{Z} \end{pmatrix}} \text{ and } \Psi_1 X_2 V_1 = k-r \begin{pmatrix} \overset{t}{X_{21}}, & \overset{p-t}{X_{22}} \end{pmatrix}.$$

Then from (3.2) and (3.3), we have

$$\begin{aligned} C_1 - A_1 X &= V\left(\begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Psi_1 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \mathcal{Z} \end{pmatrix}\right) \\ &= V\begin{pmatrix} U_1 & 0 \\ 0 & I_{k-r} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} \mathrm{T} & 0 \\ 0 & 0 \end{pmatrix} \\ C_{12}V_1 - \Psi_1 X_2 V_1 \end{pmatrix} V_1^H \\ &= V\begin{pmatrix} U_1 & 0 \\ 0 & I_{k-r} \end{pmatrix}\begin{pmatrix} \mathrm{T} & 0 \\ 0 & 0 \\ C_{121} - X_{21} & C_{122} - X_{22} \end{pmatrix} V_1^H. \end{aligned} \tag{3.10}$$

According to (3.10),

$$\begin{aligned} t \leq b &\leq \min\{\operatorname{rk}(A_1, \ \ C_1), p\}, \\ \operatorname{rk}(C_1 - A_1 X) &= \operatorname{rk}(C_{11}) + \operatorname{rk}(C_{122} - X_{22}). \end{aligned} \tag{3.11}$$

Hence,

$$\operatorname{rk}(C_{122} - X_{22}) = b - t. \tag{3.12}$$

Denoting $k_1 = b - t$, we obtain

$$X_{22} = C_{122} + Y,$$

where $Y \in \mathbb{C}^{(k-r)\times(p-t)}$ satisfies $\operatorname{rk}(Y) = k_1$. Furthermore, a general form for X which satisfies $\operatorname{rk}(C_1 - A_1 X) = b$ is

$$X = W^{-1}\begin{pmatrix} X_1 \\ \Psi_1^{-1}\left(X_{21}, \ C_{122} + Y\right) V_1^H \\ \mathcal{Z} \end{pmatrix}, \tag{3.13}$$

where $X_1 \in \mathbb{C}^{r\times p}$, $\mathcal{Z} \in \mathbb{C}^{(n-k)\times p}$ and $X_{21} \in \mathbb{C}^{(k-r)\times t}$ are arbitrary matrices, and $Y \in \mathbb{C}^{(k-r)\times(p-t)}$ satisfies $\mathrm{rk}\,(Y) = k_1$.

Applying the decomposition (2.3) of A and (3.13), we gain

$$\begin{aligned} C - AX &= U\left(\begin{pmatrix}\widehat{C}_1\\ \widehat{C}_2\\ \widehat{C}_3\end{pmatrix} - \begin{pmatrix} I_r & 0 & 0 & 0\\ 0 & S_1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\left(\begin{matrix} X_1\\ \begin{pmatrix}\widehat{S}_1^{-1} & 0\\ 0 & I_{k-r-s}\end{pmatrix}\left(X_{21},\, C_{122}+Y\right)V_1^H\\ \mathcal{Z}\end{matrix}\right)\right)\\ &= U\begin{pmatrix}\widehat{C}_1 - X_1\\ \widehat{C}_2 - \left(S_1\widehat{S}_1^{-1},\, 0\right)\left(X_{21},\, C_{122}+Y\right)V_1^H\\ \widehat{C}_3\end{pmatrix}. \end{aligned} \tag{3.14}$$

Since the Frobenius norm of a matrix is invariant under unitary transformation, by applying (3.14), we obtain

$$\begin{aligned} \min_{X,\ \mathrm{rk}(C_1 - A_1X)=b} \|C - AX\|^2 = \left\|\widehat{C}_3\right\|^2 + \min_{X_1}\left\|\widehat{C}_1 - X_1\right\|^2\\ + \min_{X_{21},Y,\mathrm{rk}(Y)=k_1}\Big\|\widehat{C}_2 - \left(S_1\widehat{S}_1^{-1},\, 0\right)\\ \times\left(X_{21},\, C_{122}+Y\right)V_1^H\Big\|^2. \end{aligned} \tag{3.15}$$

It is easily to find that

$$\min_{X_1}\left\|\widehat{C}_1 - X_1\right\| = 0, \tag{3.16}$$

and the matrix X_1 satisfying (3.16) can be written uniquely as

$$X_1 = \widehat{C}_1. \tag{3.17}$$

Denote $m_1 = k - r$ and $n_1 = p - t$, and partition

$$X_{21} = \begin{matrix} \\ s\\ m_1 - s\end{matrix}\begin{matrix} t\\ \begin{pmatrix} X_{211}\\ \mathcal{Y}\end{pmatrix}\end{matrix} \quad\text{and}\quad Y = \begin{matrix} \\ s\\ m_1 - s\end{matrix}\begin{matrix} n_1\\ \begin{pmatrix} Y_1\\ Y_2\end{pmatrix}\end{matrix}.$$

Then by applying (3.4), we obtain the following identity,

$$\begin{aligned} \left\|C_2 - \left(S_1\widehat{S}_1^{-1},\, 0\right)\left(X_{21},\, C_{122}+Y\right)V_1^H\right\|^2 &= \left\|\left(\widehat{C}_{11},\, \mathcal{C}\right) - \left(S_1\widehat{S}_1^{-1},\, 0\right)\begin{pmatrix} X_{211} & Y_1\\ \mathcal{Y} & Y_2\end{pmatrix}\right\|^2\\ &= \left\|\widehat{C}_{11} - S_1\widehat{S}_1^{-1}X_{211}\right\|^2\\ &\quad + \left\|\mathcal{C} - S_1\widehat{S}_1^{-1}Y_1\right\|^2. \end{aligned} \tag{3.18}$$

Since $S_1\widehat{S}_1^{-1}$ is nonsingular,

$$\min_{X_{211}}\left\|\widehat{C}_{11}-S_1\widehat{S}_1^{-1}X_{211}\right\|=0, \tag{3.19}$$

and the matrix X_{211} satisfying (3.19) can be written uniquely as

$$X_{211}=\widehat{S}_1S_1^{-1}\widehat{C}_{11}. \tag{3.20}$$

Furthermore, we denote $\mathcal{X}_1=S_1\widehat{S}_1^{-1}Y_1$ and $\mathcal{X}_2=Y_2$. Since $S_1\widehat{S}_1^{-1}$ is nonsingular and $\mathrm{rk}(Y)=k_1$, then $\mathrm{rk}\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix}=k_1$. By applying (3.18) and (3.19), we obtain the following identity,

$$\min_{X_{21},Y,\ \mathrm{rk}(Y)=k_1}\left\|C_2-\left(S_1\widehat{S}_1^{-1},\,0\right)\left(X_{21},\,C_{122}+Y\right)V_1^H\right\|=\min_{\mathcal{X}_1,\ \mathrm{rk}\begin{pmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{pmatrix}=k_1}\|\mathcal{C}-\mathcal{X}_1\|. \tag{3.21}$$

Then applying Lemma 2.1 to the above, it produces $0\le k_1\le\min\{c+(m_1-s),n_1\}$, that is, $t\le b\le\min\{c+t+k-r-s,\,p\}$. Combining it with (3.11) leads to (3.5). Combining (3.13–3.21), we gain a general form for X which satisfies (1.3) is

$$X=W^{-1}\begin{pmatrix}\widehat{C}_1\\ \Psi_1^{-1}\left(\begin{pmatrix}\widehat{S}_1S_1^{-1}\widehat{C}_{11}\\ \mathcal{Y}\end{pmatrix},\,C_{122}+\begin{pmatrix}\widehat{S}_1S_1^{-1}Y_1\\ Y_2\end{pmatrix}\right)V_1^H\\ \mathcal{Z}\end{pmatrix}, \tag{3.22}$$

where $\mathcal{Z}\in\mathbb{C}^{(n-k)\times p}$ and $\mathcal{Y}\in\mathbb{C}^{(k-r-s)\times t}$ are arbitrary matrices, and $Y_1\in\mathbb{C}^{s\times n_1}$ and $Y_2\in\mathbb{C}^{(m_1-s)\times n_1}$ satisfy

$$\mathrm{rk}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}=b-t.$$

Applying Lemma 2.4 and (3.15–3.20) to (3.22) get (3.6–3.9). □

By applying generalized inverses, rank formulas and the above lemmas to simplify Theorem 3.1, we obtain the following theorem.

Theorem 3.2 *Suppose that $A\in\mathbb{C}^{m\times n}$, $A_1\in\mathbb{C}^{w\times n}$, $C\in\mathbb{C}^{m\times p}$, $C_1\in\mathbb{C}^{w\times p}$, k, r, s, and the decompositions of A and A_1 are given in Lemma 2.5. Denote*

$$\widehat{\mathcal{C}}=\left(P_{A_1A^-}C-AA_1^-C_1\right)F_{E_{A_1}C_1}, \tag{3.23}$$

$$c=\mathrm{rk}\left(\widehat{\mathcal{C}}\right), \tag{3.24}$$

$$t=\mathrm{rk}\left(A_1,\,C_1\right)-\mathrm{rk}\left(A_1\right),$$

$$d=\mathrm{rk}\begin{pmatrix}A_1A^-A & A_1A^-C\\ A_1 & C_1\end{pmatrix}, \tag{3.25}$$

and the SVD of $\widehat{\mathcal{C}}$ *as*

$$\widehat{\mathcal{C}} = \mathcal{U}_1 \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \mathcal{V}_1^H , \tag{2.1'}$$

where $\Lambda = \ diag\{\lambda_1, \ldots, \lambda_c\}$, $\lambda_1 \geq \cdots \geq \lambda_c > 0$, $\mathcal{U}_1$ *and* $\mathcal{V}_1$ *are unitary matrices of appropriate sizes. Then there exists a matrix* $X \in \mathbb{C}^{n\times p}$ *satisfying* (1.3) *if and only if*

$$t \leq b \leq \min\{\mathrm{rk}\,(A_1,\ \ C_1)\,, d-s,\, p\}. \tag{3.5'}$$

If $d + r - k \leq b \leq \min\{\mathrm{rk}\,(A_1,\ \ C_1)\,, d-s,\, p\}$, *then*

$$\min_{rk(C_1 - A_1 X) = b} \|C - AX\| = \|E_A C\|, \tag{3.6'}$$

and a general form for X which satisfies (1.3) *is*

$$\begin{aligned} X = & \left(A^- - A_1^- A_1 A^-\right) C + \left(I - D^- D\right) \widehat{\mathcal{Z}} \\ & + A_1^- A_1 A^- C P_{E_{A_1} C_1} + \left(A_1^- - A^- A A_1^-\right) \widehat{\mathcal{Y}} P_{E_{A_1} C_1} + A_1^- C_1 F_{E_{A_1} C_1} \\ & + A_1^- A_1 A^- \widehat{\mathcal{C}} F_{E_{A_1} C_1} + \left(A_1^- - A^- A A_1^-\right) \widehat{\mathcal{X}}_2 F_{E_{A_1} C_1} \ , \end{aligned} \tag{3.7'}$$

where $\widehat{\mathcal{Z}} \in \mathbb{C}^{n\times p}$ *and* $\widehat{\mathcal{Y}} \in \mathbb{C}^{w\times p}$ *are arbitrary matrix, and* $\widehat{\mathcal{X}}_2 \in \mathbb{C}^{w\times p}$ *satisfies*

$$\mathrm{rk} \begin{pmatrix} A_1^- A_1 A^- \widehat{\mathcal{C}} F_{E_{A_1} C_1} \\ \left(A_1^- - A^- A A_1^-\right) \widehat{\mathcal{X}}_2 F_{E_{A_1} C_1} \end{pmatrix} = b - t\ . \tag{3.26}$$

If $t \leq b < d + r - k$, *then*

$$\min_{rk(C_1 - A_1 X) = b} \|C - AX\| = \left(\|E_A C\|^2 + \sum_{i=b-t+1}^{c} \lambda_i^2 \right)^{\frac{1}{2}} , \tag{3.8'}$$

and a general form for X which satisfies (1.3) *is*

$$\begin{aligned} X = & \left(A^- - A_1^- A_1 A^-\right) C + \left(I - D^- D\right) \widehat{\mathcal{Z}} \\ & + A_1^- A_1 A^- C P_{E_{A_1} C_1} + \left(A_1^- - A^- A A_1^-\right) \widehat{\mathcal{Y}} P_{E_{A_1} C_1} + A_1^- C_1 F_{E_{A_1} C_1} \\ & + A_1^- A_1 A^- \widehat{\mathcal{X}}_1 F_{E_{A_1} C_1} + \left(A_1^- - A^- A A_1^-\right) \widehat{\mathcal{X}}_2 F_{E_{A_1} C_1} \ , \end{aligned} \tag{3.9'}$$

where $\widehat{\mathcal{Z}} \in \mathbb{C}^{n\times p}$ *and* $\widehat{\mathcal{Y}} \in \mathbb{C}^{w\times p}$ *are arbitrary matrices, and* $\widehat{\mathcal{X}}_1 \in \mathbb{C}^{w\times p}$ *and* $\widehat{\mathcal{X}}_2 \in \mathbb{C}^{w\times p}$ *satisfy*

$$\widehat{\mathcal{X}}_1 = \mathcal{U}_1 \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{V}_1^H, \tag{3.27}$$

and

$$\mathrm{rk}\begin{pmatrix} A_1^- A_1 A^- \widehat{\mathcal{X}}_1 F_{E_{A_1} C_1} \\ \left(A_1^- - A^- A A_1^-\right) \widehat{\mathcal{X}}_2 F_{E_{A_1} C_1} \end{pmatrix} = b - t, \tag{3.28}$$

when $\lambda_{b-t} > \lambda_{b-t+1}$,

$$\Lambda_1 = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_{b-t}\};$$

when $q_2 < b - t < q_1 \le r$ *and* $\lambda_{q_2} > \lambda_{q_2+1} = \cdots = \lambda_{q_1} > \lambda_{q_1+1}$,

$$\Lambda_1 = \mathrm{diag}\left\{\lambda_1, \ldots, \lambda_{q_2}, \lambda_{b-t} \mathcal{Q}\mathcal{Q}^H\right\},$$

in which $\mathcal{Q}$ *is an arbitrary matrix satisfying* $\mathcal{Q} \in \mathbb{C}^{(q_1-q_2)\times(b-t-q_2)}$ *and* $\mathcal{Q}^H\mathcal{Q} = I_{b-t-q_2}$.

Proof From (2.3) and $A_1 A_1^- = A_1 A_1^\dagger$, it is easy to find that

$$I_w - A_1 A_1^\dagger = V \begin{pmatrix} I_{w-k+r} & 0 \\ 0 & 0 \end{pmatrix} V^H,$$

and

$$E_{A_1} C_1 = \left(I_w - A_1 A_1^\dagger\right) C_1 = V \begin{pmatrix} C_{11} \\ 0 \end{pmatrix}. \tag{3.29}$$

It follows that $\mathrm{rk}\,(C_{11}) = \mathrm{rk}\left(\left(I_w - A_1 A_1^\dagger\right) C_1\right) = \mathrm{rk}\left(A_1,\ C_1\right) - \mathrm{rk}\,(A_1) = t$.

From (2.3), $A^- = W^{-1}\Sigma^\dagger U^H$ and $A_1^- = W^{-1}\Sigma_1^\dagger V^H$, we obtain

$$A_1 A^- = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \widehat{S}_1 S_1^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^H \text{ and } A A_1^- = U \begin{pmatrix} 0 & 0 & 0 \\ 0 & S_1 \widehat{S}_1^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^H. \tag{3.30}$$

This gives

$$\left(A_1 A^-\right) A A_1^- A_1 = A_1 A^- A. \tag{3.31}$$

Applying (3.31), (1.1a) and (1.1b) to (3.24), we obtain

$$\begin{aligned} c &= \operatorname{rk}\left(\left(P_{A_1A^-}C - AA_1^-C_1\right)F_{E_{A_1}C_1}\right) \\ &= \operatorname{rk}\begin{pmatrix} \left(P_{A_1A^-}C - AA_1^-C_1\right) \\ E_{A_1}C_1 \end{pmatrix} - \operatorname{rk}\left(E_{A_1}C_1\right) \\ &= \operatorname{rk}\begin{pmatrix} 0 & P_{A_1A^-}C - AA_1^-C_1 \\ A_1 & C_1 \end{pmatrix} - \operatorname{rk}(A_1) - t \\ &= \operatorname{rk}\begin{pmatrix} \left(A_1A^-\right)AA_1^-A_1 & A_1A^-C \\ A_1 & C_1 \end{pmatrix} - k + r - t \\ &= \operatorname{rk}\begin{pmatrix} A_1A^-A & A_1A^-C \\ A_1 & C_1 \end{pmatrix} - k + r - t \\ &= d - t - k + r. \end{aligned}$$

Furthermore, applying (2.3) and (3.1–3.4) to (3.23), we obtain

$$\widehat{\mathcal{C}} = U\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C} \\ 0 & 0 \end{pmatrix}V_1^H. \tag{3.32}$$

Thus, $\operatorname{rk}(\widehat{\mathcal{C}}) = \operatorname{rk}(\mathcal{C}) = c = d - t - k + r$. Hence (3.5′) follows from (3.5).

From (2.3) and (3.1), we obtain

$$\left(I - AA^-\right)C = E_AC = U\begin{pmatrix} 0 \\ 0 \\ \widehat{C}_3 \end{pmatrix}. \tag{3.33}$$

Hence (3.6′) follows from (3.6) and (3.33).

Since (3.32), $\mathcal{C}$ and $\widehat{\mathcal{C}}$ have the same singular values. Hence (3.8′) follows from (3.8) and (3.33).

Using (2.3), (3.1) (3.2) and (3.29), we obtian

$$P_{E_{A_1}C_1} = V_1\begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix}V_1^H \text{ and } F_{E_{A_1}C_1} = V_1\begin{pmatrix} 0 & 0 \\ 0 & I_{p-t} \end{pmatrix}V_1^H. \tag{3.34}$$

From (2.3),(3.30) and (3.34),it is easy to find that

$$\begin{aligned} \left(A^- - A_1^-A_1A^-\right)C &= W^{-1}\begin{pmatrix} \widehat{C}_1 \\ 0 \\ 0 \end{pmatrix}, \\ \left(I - D^-D\right)\widehat{\mathcal{Z}} &= W^{-1}\begin{pmatrix} 0 \\ 0 \\ \mathcal{Z} \end{pmatrix}, \end{aligned}$$

$$A_1^- A_1 A^- C P_{E_{A_1} C_1} = W^{-1} \begin{pmatrix} 0 \\ \left(\begin{pmatrix} S_1^{-1} \widehat{C}_{11} \\ 0 \end{pmatrix}, 0 \right) V_1^H \\ 0 \end{pmatrix},$$

$$\left(A_1^- - A^- A A_1^- \right) \widehat{\mathcal{Y}} P_{E_{A_1} C_1} = W^{-1} \begin{pmatrix} 0 \\ \left(\begin{pmatrix} 0 \\ \mathcal{Y} \end{pmatrix}, 0 \right) V_1^H \\ 0 \end{pmatrix},$$

$$A_1^- C_1 F_{E_{A_1} C_1} = W^{-1} \begin{pmatrix} 0 \\ \Psi_1^{-1} \left(0, C_{122} \right) V_1^H \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} & W^{-1} \begin{pmatrix} \widehat{C}_1 \\ \Psi_1^{-1} \left(\begin{pmatrix} \widehat{S}_1 S_1^{-1} \widehat{C}_{11} \\ \mathcal{Y} \end{pmatrix}, C_{122} \right) V_1^H \\ \mathcal{Z} \end{pmatrix} \\ & \quad = \left(A^- - A_1^- A_1 A^- \right) C + \left(I - D^- D \right) \widehat{\mathcal{Z}} + A_1^- A_1 A^- C P_{E_{A_1} C_1} \\ & \qquad + \left(A_1^- - A^- A A_1^- \right) \widehat{\mathcal{Y}} P_{E_{A_1} C_1} + A_1^- C_1 F_{E_{A_1} C_1}, \end{aligned} \tag{3.35}$$

where $\widehat{\mathcal{Z}} \in \mathbb{C}^{n \times p}$, $\widehat{\mathcal{Y}} \in \mathbb{C}^{w \times p}$, $\mathcal{Y} \in \mathbb{C}^{(k-r-s) \times t}$ and $\mathcal{Z} \in \mathbb{C}^{(n-k) \times p}$ are arbitrary matrices. Furthermore, using (3.30), (3.32) and (3.34), we obtain

$$A_1^- A_1 A^- \widehat{\mathcal{C}} F_{E_{A_1} C_1} = W^{-1} \begin{pmatrix} 0 \\ \Psi_1^{-1} \left(0, \begin{pmatrix} \widehat{S}_1 S_1^{-1} \mathcal{C} \\ 0 \end{pmatrix} \right) V_1^H \\ 0 \end{pmatrix}. \tag{3.36}$$

Since $\widehat{\mathcal{X}}_2 \in \mathbb{C}^{w \times p}$ satisfies (3.26), we obtain

$$\left(A_1^- - A^- A A_1^- \right) \widehat{\mathcal{X}}_2 F_{E_{A_1} C_1} = W^{-1} \begin{pmatrix} 0 \\ \Psi_1^{-1} \left(0, \begin{pmatrix} 0 \\ \mathcal{X}_2 \end{pmatrix} \right) V_1^H \\ 0 \end{pmatrix}, \tag{3.37}$$

where $\mathcal{X}_2 \in \mathbb{C}^{(k-r-s) \times (p-t)}$ satisfies

$$\operatorname{rk} \begin{pmatrix} \mathcal{C} \\ \mathcal{X}_2 \end{pmatrix} = b - t.$$

Hence (3.7′) follows from (3.22) and (3.35–3.37).

Since $\mathcal{C}$ and $\widehat{\mathcal{C}}$ have the same singular values, by applying Lemma 2.2, (2.1), (2.1′), (3.28), (3.30) and (3.34), we obtain

$$A_1^- A_1 A^- \widehat{\mathcal{X}}_1 F_{E_{A_1} C_1} = W^{-1} \begin{pmatrix} 0 \\ \Psi_1^{-1} \left(0, \begin{pmatrix} \widehat{S}_1 S_1^{-1} \mathcal{X}_1 \\ 0 \end{pmatrix} \right) V_1^H \\ 0 \end{pmatrix}, \tag{3.38}$$

where $\mathcal{X}_1 \in \mathbb{C}^{s \times (p-t)}$ satisfies $\|\mathcal{C} - \mathcal{X}_1\| = \min$ subject to $\mathrm{rk}\begin{pmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{pmatrix} = k_1$. Hence (3.9′) follows from (3.22), (3.35) and (3.38). □

We provide an example to illustrate that Theorem 3.2 is feasible.

Example 3.1 Take

$$A = \begin{pmatrix} 1.16 & 0.8 & 1.96 & 0 & 1.16 \\ 0 & 0.8 & 0 & 0.8 & 1.6 \\ 0 & 0 & 0 & 0 & 0 \\ -0.12 & -0.6 & -0.72 & 0 & -0.12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0.36 & 0 & 0.36 & 0.72 \\ -0.224 & 0 & 0.736 & 0.96 & -0.224 \\ 0.768 & 0 & 1.048 & 0.28 & 0.768 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.48 & 0 & 0.48 & 0.96 \end{pmatrix},$$

$$C = \begin{pmatrix} 1.2 & 8.6 & 0.8 & 3 & 14 \\ 1 & 1 & 1 & 0 & 1 \\ 7.6 & 0.56 & 9.6 & 9.6 & 2.8 \\ 1.6 & -5.2 & -0.6 & 4 & 2 \\ 8.2 & 1.92 & -2.8 & -2.8 & 9.6 \end{pmatrix} \quad \text{and}$$

$$C_1 = \begin{pmatrix} 11.12 & -3.84 & 6 & 48 & 6 \\ 0 & 6.8 & 6.8 & -2.8 & 9.6 \\ 0 & 12.4 & 12.4 & 9.6 & 2.8 \\ -4.8 & 3.6 & 0 & 80 & 0 \\ 4.16 & 2.88 & 8 & -36 & 8 \end{pmatrix}.$$

Then $r=1, s=2, k=4, m=n=w=p=5, t=2, \mathrm{rk}\,(A_1,\ C_1)=5$,

$$W=\begin{pmatrix}1&1&2&0&1\\0&1&0&1&2\\1&0&1&0&1\\0&0&1&1&0\\0&1&1&0&1\end{pmatrix},\ S_1=\begin{pmatrix}0.8&0\\0&0.6\end{pmatrix},\ \Psi_1=\begin{pmatrix}0.6&0&0\\0&0.8&0\\0&0&1\end{pmatrix},$$

$$\widehat{S}_1=\begin{pmatrix}0.6&0\\0&0.8\end{pmatrix},$$

$$\begin{pmatrix}C_{11}\\C_{12}\end{pmatrix}=\begin{pmatrix}6.4&-4.8&0&60&0\\-4.8&3.6&0&80&0\\\hdashline 10&0&10&0&10\\0&10&10&10&0\\0&10&10&0&10\end{pmatrix},\ \begin{pmatrix}\widehat{C}_1\\\widehat{C}_2\\\widehat{C}_3\end{pmatrix}=\begin{pmatrix}0&10&1&0&10\\\hdashline 1&1&1&0&1\\2&1&0&5&10\\\hdashline 5&0&10&10&0\\10&2&0&0&10\end{pmatrix},$$

$$U_1=\begin{pmatrix}0.8&0.6\\-0.6&0.8\end{pmatrix},\ V_1=\begin{pmatrix}0.8&0&0.6&0&0\\-0.6&0&0.8&0&0\\0&0&0&0.96&0.28\\0&1&0&0&0\\0&0&0&-0.28&0.96\end{pmatrix},$$

$$T=\begin{pmatrix}10&0\\0&100\end{pmatrix},$$

and

$$(C_{121},\ C_{122})=\left(\begin{array}{cc|ccc}8&0&6&6.8&12.4\\-6&10&8&9.6&2.8\\-6&0&8&6.8&12.4\end{array}\right),$$

$$(\widehat{C}_{11},\ \mathcal{C})=\frac{1}{3}\left(\begin{array}{cc|ccc}0.6&0&-19.8&-25.16&-45.88\\3&15&-12&-30&22.5\end{array}\right).$$

Compute the SVD of $\mathcal{C}$ by Matlab7 on a personal computer

$$\mathcal{U}=\begin{pmatrix}-0.9997&0.0252\\0.0252&0.9997\end{pmatrix},\ \Lambda=\begin{pmatrix}18.6519&0\\0&13.1201\end{pmatrix}\ \text{and}$$

$$\mathcal{V}=\begin{pmatrix}0.3483&-0.3175&-0.8820\\0.4360&-0.7781&0.4523\\0.8298&0.5420&0.1326\end{pmatrix}.$$

Thus, by Theorem 3.1, there exists a rank-constrained least squares solution X to Problem 1.1 if and only if $2\le b\le 5$. When $b=4$,

$$\min_{\mathrm{rk}(C_1-A_1X)=4}\|C-AX\|=429^{\frac{1}{2}}, \tag{3.39}$$

and a general form for X satisfying (3.39) is given as follows

$$X = \frac{1}{3}\begin{pmatrix} 1.5 & 0.5 & -0.5 & -0.5 & -2 \\ 1.5 & 0.5 & -1.5 & -0.5 & -1 \\ -0.5 & -0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -0.5 & 0.5 & -1 \\ -1 & 0 & 1 & 0 & 1 \end{pmatrix} \left(\begin{matrix} 0 \\ 3.75 \\ 10 \\ 14.4+0.8y_1+0.2090x_1-0.1905x_2 \\ z_1 \end{matrix} \right| \\ \left. \begin{matrix} 30 & 3 & 0 & 30 \\ 3.75 & 3.75 & 0 & 3.75 \\ 5 & 0 & 25 & 50 \\ 19.2-0.6y_1+0.2787x_1-0.2540x_2 & 30+0.6509x_1-0.5952x_2 & y_2 & 30+0.6746x_1+0.7382x_2 \\ z_2 & z_3 & z_4 & z_5 \end{matrix} \right),$$

where x_i, y_j and z_l are arbitrary, $i = 1, 2$, $j = 1, 2$ and $l = 1, \ldots, 5$.

When $b = 2$,

$$\min_{\mathrm{rk}(C_1 - A_1 X) = 2} \|C - AX\|^2 = 949.03 \tag{3.40}$$

and a general form for X satisfying (3.40) is given as follows

$$X = \frac{1}{3}\begin{pmatrix} 1.5 & 0.5 & -0.5 & -0.5 & -2 \\ 1.5 & 0.5 & -1.5 & -0.5 & -1 \\ -0.5 & -0.5 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & -0.5 & 0.5 & -1 \\ -1 & 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 30 & 3 & 0 & 30 \\ 18.6 & 23.55 & 50 & 0 & 50 \\ 22 & 21 & 37.5 & 25 & 0 \\ 14.4+0.8y_1 & 19.2-0.6y_1 & 30 & y_2 & 30 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix},$$

where y_j and z_l are arbitrary, $j = 1, 2$ and $l = 1, \ldots, 5$.

Remark 3.1 By applying SVD and Q-SVD, we get two general forms of the least squares solutions of $AX = C$ subject to $\mathrm{rk}\,(C_1 - A_1 X) = b$. One thing worthy of note is that it seems hard to obtain one general form of the least squares solutions of $AXB = C$ subject to $\mathrm{rk}\,(C_1 - A_1 X B_1) = b$.

Investigate its reason, it is the matrix decomposition that is key tool to prove processing of Theorems 3.1 and 3.2. Thus we will focus on introducing a corresponding matrix decomposition in further study.

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Compliance with ethical standards

Conflict of interest No potential conflict of interest was reported by the author.

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