



Explicit iterative algorithms for solving equilibrium problems

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Abstract

The paper introduces two new extragradient algorithms for solving a pseudomonotone equilibrium problem with a Lipschitz-type condition recently presented by Mastroeni in auxiliary problem principle. The algorithm uses variable stepsizes which are updated at each iteration and based on some previous iterates. The advantage of the algorithms is that they are done without the prior knowledge of Lipschitz-type constants and also without any linesearch procedure. The convergence of the algorithms is established under mild assumptions. In the case where the equilibrium bifunction is strongly pseudomonotone, the R -linear rate of convergence of the new algorithms is formulated. Several of fundamental experiments are provided to illustrate the numerical behavior of the algorithms and also to compare with others.

Keywords Equilibrium problem · Extragradient method · Pseudomonotone bifunction · Lipschitz-type condition

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1 Introduction

The paper considers some methods for approximating solutions of an equilibrium problem in a real Hilbert space H . Let C be a nonempty closed convex subset of H .

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Recall that an equilibrium problem (shortly, EP) for a bifunction $f : C \times C \rightarrow \Re$ is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \forall y \in C. \quad (\text{EP})$$

Let us denote by $EP(f, C)$ the solution set of problem (EP). Mathematically, problem (EP) can be considered as a general model in the sense that it unifies in a simple form numerous known models as optimization problems, fixed point problems, variational inequalities as well as Nash equilibrium problems [6–8,10,20,26]. This can explain why problem (EP) becomes an attractive field in recent years. Together with theoretical results of solution existence, approximation methods to solutions of problem (EP) are also interesting. The first notable method for solving problem (EP) can be the proximal point method [19,25]. The main idea of this method is to replace the original problem by a family of regularization equilibrium subproblems which can be solved more easily. The regularized solutions can converge finitely or asymptotically to some solution of the original problem.

In this paper, we are interested in the proximal-like method [9] which is also called the extragradient method [27] due to the early contributions on the saddle point problems in [21]. The convergence of the extragradient method is established in [27] under the assumptions that the bifunction is pseudomonotone and satisfies a Lipschitz-type condition presented in [24]. Another method, named the modified extragradient method in [22], can be considered as an improvement of the extragradient method in [9,27] where the complexity of the algorithm is reduced. Unlike [9,27], the modified method in [22] only requires to proceed one value of bifunction by iteration. Some other methods for solving problem (EP) can be found, for example, in the works [1–4,6,13–18,23,28–30].

At this stage, it is worth mentioning that the methods in [22,27] are used under the requirement that the Lipschitz-type constants must be known. This means that those constants must be input parameters of the algorithms. Unfortunately, Lipschitz-type constants in general are unknown or even with complex non-linear problems they are still difficult to approximate. In those cases, methods with a linesearch procedure have been considered. However, a linesearch often requires many computations at each iteration which is inherently time-consuming. Recently, the author in [11,12] has introduced some extragradient-like algorithms with previously taking suitable stepsizes to avoid the finding of the Lipschitz-type constants of bifunction. The convergence of the methods in [11,12] is proved under the assumption of strong pseudomonotonicity of bifunction. Without this assumption, those methods cannot converge, see in [11, Remark 2] or [12, Remark 4.6].

In this paper, motivated and inspired by the aforementioned results, we introduce two iterative algorithms with new stepsize rules for solving a pseudomonotone and Lipschitz-type equilibrium problem in a real Hilbert space. The variable stepsizes are generated by the algorithms at each iteration, based on some previous iterates, and without any linesearch procedure. Unlike the results in [22,27], the proposed algorithms are performed without the prior knowledge of the Lipschitz-type constants. The convergence of the new algorithms is established under some standard conditions. In the case where the bifunction is strongly pseudomonotone, the R -linear rate of

convergence of the algorithms has been proved contrary to the algorithms in [11,12]. Some experiments are performed to show the numerical behavior of the proposed algorithms, and also to compare with others.

This paper is organized as follows: Sect. 2 recalls some definitions and results used in the paper. Sect. 3 presents the new algorithms and proves their convergence while Sect. 4 deals with their rate of convergence. Several fundamental experiments are provided in Sect. 5.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H . We begin with some concepts of monotonicity of a bifunction $f : C \times C \rightarrow \Re$, see [7,26] for more details. The bifunction f is called:

(a) γ -strongly monotone on C if there exists $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2, \forall x, y \in C;$$

(b) monotone if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

(c) γ -strongly pseudomonotone on C if there exists $\gamma > 0$ such that

$$f(x, y) \geq 0 \implies f(y, x) \leq -\gamma\|x - y\|^2, \forall x, y \in C;$$

(d) pseudomonotone on C if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0, \forall x, y \in C;$$

It follows from the definitions that the following implications hold,

$$(a) \implies (b) \implies (d) \quad \text{and} \quad (a) \implies (c) \implies (d).$$

A bifunction $f : C \times C \rightarrow \Re$ satisfies a Lipschitz-type condition [23] if there exist two constants $c_1, c_2 > 0$ such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \forall x, y \in C.$$

Recall that the proximal mapping of a proper, convex and lower semicontinuous function $g : C \rightarrow \Re$ with a parameter $\lambda > 0$ is defined by

$$\text{prox}_{\lambda g}(x) = \arg \min \left\{ \lambda g(y) + \frac{1}{2}\|x - y\|^2 : y \in C \right\}, \quad x \in H.$$

The following is a property of the proximal mapping $\text{prox}_{\lambda g}$ (see, [5] for more details).

Lemma 1 For all $x \in H$, $y \in C$ and $\lambda > 0$, the following inequality holds,

$$\lambda \{g(y) - g(\text{prox}_{\lambda g}(x))\} \geq \langle x - \text{prox}_{\lambda g}(x), y - \text{prox}_{\lambda g}(x) \rangle.$$

Remark 1 From Lemma 1, it is easy to show that if $x = \text{prox}_{\lambda g}(x)$ then

$$x \in \text{Arg min } \{g(y) : y \in C\} := \left\{ x \in C : g(x) = \min_{y \in C} g(y) \right\}.$$

We also need the following lemma (see, [5, Theorem 5.5]).

Lemma 2 Let $\{x_n\}$ be a sequence in H and C be a nonempty subset of H . Suppose that $\{x_n\}$ is Fejér monotone with respect to C , i.e., $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \geq 0$ and for all $x \in C$, and that every weak sequential cluster point of $\{x_n\}$ belongs to C . Then, the sequence $\{x_n\}$ converges weakly to a point in C .

3 Explicit extragradient algorithms

In this section, we introduce two extragradient algorithms for solving pseudomonotone (EP) with a Lipschitz-type condition. The algorithms are explicit in the sense that they are done without previously knowing the Lipschitz-type constants of bifunction. This is particularly interesting in the case where these constants are unknown or even difficult to approximate. For the sake of simplicity in the presentation, we will use the notation $[t]_+ = \max\{0, t\}$ and adopt the conventions $\frac{0}{0} = +\infty$ and $\frac{a}{0} = +\infty$ ($a \neq 0$). The following is the first algorithm.

Algorithm 1 Initialization Choose $x_0 \in C$ and $\lambda_0 > 0$, $\mu \in (0, 1)$.

Iterative steps Assume that $x_n \in C$ and λ_n ($n \geq 0$) are known. Compute x_{n+1} , λ_{n+1} as follows:

$$\begin{cases} y_n = \text{prox}_{\lambda_n f(x_n, \cdot)}(x_n), \\ x_{n+1} = \text{prox}_{\lambda_n f(\cdot, x_n)}(x_n), \end{cases}$$

and set

$$\lambda_{n+1} = \min \left\{ \lambda_n, \frac{\mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2[f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1})]_+} \right\}.$$

Stopping criterion If $y_n = x_n$ then stop and x_n is a solution of EP.

It is seen that the stepsize λ_n generated by the algorithm, and updated during the iteration is based on the previous iterate x_n and the two current ones y_n and x_{n+1} . This is different from the strategy to choose of diminishing and non-summable stepsizes in [11]. Comparing with linesearch algorithms, for example [27, Algorithm 2a], the stepsize in Algorithm 1 is computed more simply. Indeed, a linesearch procedure needs

many computations by iteration with some finite stopping criterion, and is of course more time-consuming. As seen below, the sequence $\{\lambda_n\}$ generated by Algorithm 1 is separated from 0. This is more useful than algorithms with diminishing stepsizes. In order to obtain the convergence of Algorithm 1, we consider the following blanket assumptions imposed on the bifunction f .

- (A1) f is pseudomonotone on C and $f(x, x) = 0$ for all $x \in C$.
- (A2) f satisfies the Lipschitz-type condition.
- (A3) $\limsup_{n \rightarrow \infty} f(x_n, y) \leq f(x, y)$ for each $y \in C$ and each $\{x_n\} \subset C$ with $x_n \rightarrow x$;
- (A4) $f(x, \cdot)$ is convex and subdifferentiable on C for every fixed $x \in C$.

Remark that under the hypothesis (A2), there exist some constants $c_1 > 0, c_2 > 0$ such that

$$\begin{aligned} f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) &\leq c_1 \|x_n - y_n\|^2 + c_2 \|x_{n+1} - y_n\|^2 \\ &\leq \max\{c_1, c_2\} (\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2). \end{aligned}$$

Thus, from the definition of $\{\lambda_n\}$, we see that this sequence is bounded from below by $\left\{ \lambda_0, \frac{\mu}{2 \max\{c_1, c_2\}} \right\}$. Moreover, the sequence $\{\lambda_n\}$ is non-increasing monotone. Thus, there exists a real number $\lambda > 0$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. In fact, from the definition of λ_{n+1} , if $f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq 0$ then $\lambda_{n+1} := \lambda_n$. We have the following first result.

Theorem 1 *Under the hypotheses (A1)–(A4), the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to some solution of problem (EP).*

Proof It follows from Lemma 1 and the definition of x_{n+1} that

$$\langle x_n - x_{n+1}, x_{n+1} - y \rangle \geq \lambda_n f(y_n, x_{n+1}) - \lambda_n f(y_n, y), \quad \forall y \in C. \tag{1}$$

From the definition of λ_{n+1} , we get

$$f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu (\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\lambda_{n+1}},$$

which, after multiplying both sides by $\lambda_n > 0$, implies that

$$\begin{aligned} \lambda_n f(y_n, x_{n+1}) &\geq \lambda_n (f(x_n, x_{n+1}) - f(x_n, y_n)) \\ &\quad - \frac{\mu \lambda_n (\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2\lambda_{n+1}}. \end{aligned} \tag{2}$$

Combining the relations (1) and (2), we obtain

$$\begin{aligned} \langle x_n - x_{n+1}, x_{n+1} - y \rangle &\geq \lambda_n \{f(x_n, x_{n+1}) - f(x_n, y_n)\} - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|x_n - y_n\|^2 \\ &\quad - \frac{\mu \lambda_n}{2\lambda_{n+1}} \|x_{n+1} - y_n\|^2 - \lambda_n f(y_n, y). \end{aligned} \tag{3}$$

Similarly, from Lemma 1 and the definition of y_n , we also obtain

$$\lambda_n(f(x_n, x_{n+1}) - f(x_n, y_n)) \geq \langle y_n - x_n, y_n - x_{n+1} \rangle. \tag{4}$$

From the relations (3) and (4), we obtain

$$\begin{aligned} \langle x_n - x_{n+1}, x_{n+1} - y \rangle &\geq \langle y_n - x_n, y_n - x_{n+1} \rangle - \frac{\mu\lambda_n}{2\lambda_{n+1}} \|x_n - y_n\|^2 \\ &\quad - \frac{\mu\lambda_n}{2\lambda_{n+1}} \|x_{n+1} - y_n\|^2 - \lambda_n f(y_n, y). \end{aligned}$$

Thus, multiplying both sides of the last inequality by 2, we come to the following estimate

$$\begin{aligned} 2 \langle x_n - x_{n+1}, x_{n+1} - y \rangle &\geq 2 \langle y_n - x_n, y_n - x_{n+1} \rangle - \frac{\mu\lambda_n}{\lambda_{n+1}} \|x_n - y_n\|^2 \\ &\quad - \frac{\mu\lambda_n}{\lambda_{n+1}} \|x_{n+1} - y_n\|^2 - 2\lambda_n f(y_n, y). \end{aligned} \tag{5}$$

We have the following facts:

$$2 \langle x_n - x_{n+1}, x_{n+1} - y \rangle = \|x_n - y\|^2 - \|x_{n+1} - x_n\|^2 - \|x_{n+1} - y\|^2. \tag{6}$$

$$2 \langle y_n - x_n, y_n - x_{n+1} \rangle = \|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_n - x_{n+1}\|^2. \tag{7}$$

Combining the relations (5)–(7), we get

$$\begin{aligned} \|x_{n+1} - y\|^2 &\leq \|x_n - y\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 \\ &\quad - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_{n+1} - y_n\|^2 + 2\lambda_n f(y_n, y), \quad \forall y \in C, \quad \forall n \geq 0. \end{aligned} \tag{8}$$

For each $x^* \in EP(f, C)$, we have that $f(x^*, y_n) \geq 0$ and by (A1) that $f(y_n, x^*) \leq 0$. Then, using $y = x^* \in C$ in relation (8), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 \\ &\quad - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_{n+1} - y_n\|^2. \end{aligned} \tag{9}$$

Let $\epsilon \in (0, 1 - \mu)$ be some fixed number. Since $\lambda_n \rightarrow \lambda > 0$,

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \epsilon > 0.$$

Thus, there exists $n_0 \geq 1$ such that

$$1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > \epsilon > 0, \quad \forall n \geq n_0. \tag{10}$$

From the relations (9) and (10), we obtain

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \epsilon \left(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 \right),$$

or

$$a_{n+1} \leq a_n - b_n, \tag{11}$$

where $a_n = \|x_n - x^*\|^2$ and $b_n = \epsilon \left(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 \right)$. Thus, the limit of $\{a_n\}$ exists and $\lim_{n \rightarrow \infty} b_n = 0$ which imply that $\{x_n\}$ is bounded and

$$\lim_{n \rightarrow \infty} \|x_n - y_n\|^2 = \lim_{n \rightarrow \infty} \|x_{n+1} - y_n\|^2 = 0. \tag{12}$$

Hence, since $\|x_{n+1} - x_n\| \leq \|x_{n+1} - y_n\| + \|x_n - y_n\|$, we also get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\|^2 = 0. \tag{13}$$

Now, we prove that each weak cluster point of $\{x_n\}$ is in $EP(f, C)$. Indeed, suppose that \bar{x} is a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence, denoted by $\{x_m\}$, of $\{x_n\}$ converging weakly to \bar{x} . Since $\|x_n - y_n\| \rightarrow \infty$, we also have that $y_m \rightarrow \bar{x}$. Passing to the limit in (8) as $n = m \rightarrow \infty$ and using hypothesis (A3), the relation (12), and the fact $\lambda_m \rightarrow \lambda > 0$, we obtain

$$f(\bar{x}, y) \geq \limsup_{m \rightarrow \infty} f(y_m, y) \geq \frac{1}{2\lambda} \limsup_{m \rightarrow \infty} \left(\|x_{m+1} - y\|^2 - \|x_m - y\|^2 \right), \quad \forall y \in C. \tag{14}$$

On the other hand, from the triangle inequality, we have

$$\left| \|x_{n+1} - y\|^2 - \|x_n - y\|^2 \right| \leq \|x_{n+1} - x_n\| (\|x_{n+1} - y\| + \|x_n - y\|).$$

Thus, from the boundedness of $\{x_n\}$ and the relation (13), we get for each $y \in C$

$$\lim_{n \rightarrow \infty} \left| \|x_{n+1} - y\|^2 - \|x_n - y\|^2 \right| = 0. \tag{15}$$

Combining the relations (14) and (15), we get $f(\bar{x}, y) \geq 0$ for all $y \in C$ or $\bar{x} \in EP(f, C)$. From the relations (9) and (10), we see that the sequence $\{x_n\}_{n \geq n_0}$ is Fejér-monotone. Thus, Lemma 2 ensures that the whole sequence $\{x_n\}$ converges weakly to \bar{x} as $n \rightarrow \infty$. This completes the proof. \square

Algorithm 1 requires to compute at each iteration the values of the bifunction f at x_n and y_n . Next, we present an algorithm, namely the explicit modified extragradient algorithm, which only needs to compute one value of f at the current approximation.

Algorithm 2 Initialization Choose $y_{-1}, y_0, x_0 \in C$ and $\lambda_0 > 0, \mu \in (0, \frac{1}{3})$.

Iterative steps Assume that $y_{n-1}, y_n, x_n \in C$ and $\lambda_n \geq 0$ ($n \geq 1$) are known, calculate x_{n+1}, y_{n+1} and λ_{n+1} as follows:

Step 1 Compute $x_{n+1} = \text{prox}_{\lambda_n f(y_n, \cdot)}(x_n)$ and set

$$\lambda_{n+1} = \min \left\{ \lambda_n, \frac{\mu(\|y_{n-1} - y_n\|^2 + \|y_n - x_{n+1}\|^2)}{2[f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})]_+} \right\}.$$

Step 2 Compute $y_{n+1} = \text{prox}_{\lambda_{n+1} f(y_n, \cdot)}(x_{n+1})$.

Stopping criterion If $x_{n+1} = y_n = x_n$ then stop and x_n is a solution of EP.

As for Algorithm 1, the sequence $\{\lambda_n\}$ generated by Algorithm 2 is non-increasing monotone and $\lim \lambda_n = \lambda > 0$. Another point of comparison with Algorithm 1 is that Algorithm 2 uses two different stepsizes λ_n and λ_{n+1} per iteration. Algorithm 2 is also weakly convergent as shown in the next theorem.

Theorem 2 Under the hypotheses (A1)–(A4), the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to some solution of problem (EP).

Proof For all $y \in C$, we have the following equality,

$$\begin{aligned} \|x_{n+1} - y\|^2 &= \|x_n - y\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 \\ &\quad + 2 \langle x_n - y_n, x_{n+1} - y_n \rangle + 2 \langle x_n - x_{n+1}, y - x_{n+1} \rangle. \end{aligned} \tag{16}$$

Lemma 1 and the definition of x_{n+1} ensure that

$$\langle x_n - x_{n+1}, y - x_{n+1} \rangle \leq \lambda_n \{f(y_n, y) - f(y_n, x_{n+1})\}, \quad \forall y \in C. \tag{17}$$

Similarly, from the definition of y_n and Lemma 1, we get

$$\langle x_n - y_n, x_{n+1} - y_n \rangle \leq \lambda_n \{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\}. \tag{18}$$

Combining the relations (16)–(18), we come to the following estimate,

$$\begin{aligned} \|x_{n+1} - y\|^2 &\leq \|x_n - y\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + 2\lambda_n f(y_n, y) \\ &\quad + 2\lambda_n \{f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})\}. \end{aligned} \tag{19}$$

From the definition of λ_{n+1} , we see that

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu(\|y_{n-1} - y_n\|^2 + \|y_n - x_{n+1}\|^2)}{2\lambda_{n+1}},$$

which together with relation (19), can be written

$$\begin{aligned} \|x_{n+1} - y\|^2 &\leq \|x_n - y\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + 2\lambda_n f(y_n, y) \\ &\quad + \frac{\lambda_n \mu}{\lambda_{n+1}} \left(\|y_{n-1} - y_n\|^2 + \|y_n - x_{n+1}\|^2 \right). \end{aligned} \tag{20}$$

Using the triangle inequality and the fact $(a + b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\|y_{n-1} - y_n\|^2 \leq (\|y_{n-1} - x_n\| + \|x_n - y_n\|)^2 \leq 2\|y_{n-1} - x_n\|^2 + 2\|x_n - y_n\|^2. \tag{21}$$

From relations (20) and (21), we have

$$\begin{aligned} \|x_{n+1} - y\|^2 &\leq \|x_n - y\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + 2\lambda_n f(y_n, y) \\ &\quad + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2 + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_n - x_n\|^2 + \frac{\lambda_n \mu}{\lambda_{n+1}} \|y_n - x_{n+1}\|^2 \\ &= \|x_n - y\|^2 - \left(1 - \frac{2\lambda_n \mu}{\lambda_{n+1}} \right) \|x_n - y_n\|^2 \\ &\quad - \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}} \right) \|y_n - x_{n+1}\|^2 + 2\lambda_n f(y_n, y) + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2. \end{aligned} \tag{22}$$

Adding the term $\frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2$ to both sides of the inequality (22), we get

$$\begin{aligned} \|x_{n+1} - y\|^2 + \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2 &\leq \|x_n - y\|^2 + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2 \\ &\quad - \left(1 - \frac{2\lambda_n \mu}{\lambda_{n+1}} \right) \|x_n - y_n\|^2 - \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \right) \|y_n - x_{n+1}\|^2 \\ &\quad + 2\lambda_n f(y_n, y), \quad \forall y \in C, \quad n \geq 0. \end{aligned} \tag{23}$$

For each $x^* \in EP(f, C)$, we have $f(x^*, y_n) \geq 0$. Thus, $f(y_n, x^*) \leq 0$ because of the pseudomonotonicity of f . Now, using the relation (23) for $y = x^*$, we come to the following inequality,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 + \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2 &\leq \|x_n - x^*\|^2 + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2 \\ &\quad - \left(1 - \frac{2\lambda_n \mu}{\lambda_{n+1}} \right) \|x_n - y_n\|^2 - \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \right) \|y_n - x_{n+1}\|^2. \end{aligned} \tag{24}$$

Let $\epsilon \in (0, 1 - 3\mu)$ be some fixed number. Since $\lambda_n \rightarrow \lambda > 0$, we obtain

$$\lim_{n \rightarrow \infty} \left(1 - \frac{2\lambda_n\mu}{\lambda_{n+1}} \right) = 1 - 2\mu > 1 - 3\mu > \epsilon > 0,$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \right) = 1 - 3\mu > \epsilon > 0.$$

Thus, there exists $n_0 \geq 1$ such that

$$1 - \frac{2\lambda_n\mu}{\lambda_{n+1}} > \epsilon > 0, \quad \forall n \geq n_0, \tag{25}$$

$$1 - \frac{\lambda_n\mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} > \epsilon > 0, \quad \forall n \geq n_0. \tag{26}$$

Thus, from the relations (24)–(26), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 + \frac{2\lambda_{n+1}\mu}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2 &\leq \|x_n - x^*\|^2 + \frac{2\lambda_n\mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2 \\ &- \epsilon \left(\|x_n - y_n\|^2 + \|y_n - x_{n+1}\|^2 \right), \quad \forall n \geq n_0, \end{aligned} \tag{27}$$

or

$$a_{n+1} \leq a_n - b_n, \quad \forall n \geq n_0, \tag{28}$$

where

$$\begin{aligned} a_n &= \|x_n - x^*\|^2 + \frac{2\lambda_n\mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2, \\ b_n &= \epsilon \left(\|x_n - y_n\|^2 + \|y_n - x_{n+1}\|^2 \right). \end{aligned}$$

From the relation (28), we obtain that the limit of $\{a_n\}$ exists, and $\lim_{n \rightarrow \infty} b_n = 0$. Thus, from the definition of a_n and b_n , we obtain that $\{x_n\}$ is bounded and

$$\lim_{n \rightarrow \infty} \|x_n - y_n\|^2 = \lim_{n \rightarrow \infty} \|y_n - x_{n+1}\|^2 = 0. \tag{29}$$

This together with the triangle inequality implies that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\|^2 = 0. \tag{30}$$

Since $\lim_{n \rightarrow \infty} a_n \in \mathfrak{R}$, from the definition of a_n and (29), we also obtain

$$\lim_{n \rightarrow \infty} \|x_n - x^*\|^2 \in \mathfrak{R}, \quad \forall x^* \in EP(f, C). \tag{31}$$

Next, we prove that every weak cluster point of $\{x_n\}$ is in $EP(f, C)$. Indeed, suppose that \bar{x} is a weak cluster point of $\{x_n\}$, i.e., there exists a subsequence, denoted by $\{x_m\}$, of $\{x_n\}$ converging weakly to \bar{x} . From the relation (29), we also get that $y_m \rightharpoonup \bar{x}$. Now, passing to the limit in the relation (23) as $n = m \rightarrow \infty$, and using the hypothesis (A3), the relation (29), and the fact $\lambda_n \rightarrow \lambda > 0$ we obtain

$$f(\bar{x}, y) \geq \limsup_{m \rightarrow \infty} f(y_m, y) \geq \frac{1}{2\lambda} \limsup_{m \rightarrow \infty} \left(\|x_{n+1} - y\|^2 - \|x_n - y\|^2 \right), \quad \forall y \in C. \tag{32}$$

Arguing as in the proof of Theorem 1, we also obtain for each $y \in C$

$$\lim_{n \rightarrow \infty} \left| \|x_{n+1} - y\|^2 - \|x_n - y\|^2 \right| = 0. \tag{33}$$

Combining the relations (32) and (33), we get $f(\bar{x}, y) \geq 0$ for all $y \in C$ or $\bar{x} \in EP(f, C)$. To finish the proof, we show that the whole sequence $\{x_n\}$ converges weakly to \bar{x} as $n \rightarrow \infty$. Indeed, assume that $\{x_l\}$ is another subsequence of $\{x_n\}$ converging weakly to $\tilde{x} \neq \bar{x}$. As aforementioned, we see that $\tilde{x} \in EP(f, C)$. The relation (31) ensures that $\lim_{n \rightarrow \infty} \|x_n - \bar{x}\|^2 \in \mathfrak{R}$ and $\lim_{n \rightarrow \infty} \|x_n - \tilde{x}\|^2 \in \mathfrak{R}$. Using the following identity,

$$2 \langle x_n, \bar{x} - \tilde{x} \rangle = \|x_n - \tilde{x}\|^2 - \|x_n - \bar{x}\|^2 + \|\bar{x}\|^2 - \|\tilde{x}\|^2,$$

we have immediately that

$$\lim_{n \rightarrow \infty} \langle x_n, \bar{x} - \tilde{x} \rangle = K \in \mathfrak{R}. \tag{34}$$

Passing to the limit in (34) as $n = m, l \rightarrow \infty$, we obtain

$$\langle \bar{x}, \bar{x} - \tilde{x} \rangle = \lim_{m \rightarrow \infty} \langle x_m, \bar{x} - \tilde{x} \rangle = K = \lim_{l \rightarrow \infty} \langle x_l, \bar{x} - \tilde{x} \rangle = \langle \tilde{x}, \bar{x} - \tilde{x} \rangle.$$

Thus, $\|\bar{x} - \tilde{x}\|^2 = 0$ and $\bar{x} = \tilde{x}$. This completes the proof. □

4 R-linear rate of convergence

Algorithms in [11, 12] have some special advantages that they are done without the prior knowledge of the Lipschitz-type constants of bifunction. However, in the case where the bifunction f is strongly pseudomonotone (SP) and satisfies the Lipschitz-type condition (LC), the linear rate of convergence cannot be obtained for these algorithms. In this section, we will establish the R -linear rate of convergence of Algorithms 1 and 2 under the hypotheses (SP) and (LC). In addition, as in [11, 12], we assume additionally that the function $f(x, \cdot)$ is convex, lower semicontinuous and the function $f(\cdot, y)$ is hemicontinuous on C . Under these assumptions, problem (EP) has the unique solution, denoted by x^\dagger . The rate of convergence of the two proposed algorithms is ensured by the following theorem.

Theorem 3 Under the hypotheses (SP) and (LC), the sequence $\{x_n\}$ generated by Algorithm 1 (or Algorithm 2) converges R -linearly to the unique solution x^\dagger of problem (EP).

Proof The proof is divided into two cases following that Algorithms 1 or 2 is considered.

Case 1 The sequence $\{x_n\}$ is generated by Algorithm 1. In this case, using the relation (8) with $y = x^\dagger$, we obtain

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 &\leq \|x_n - x^\dagger\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 \\ &\quad - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_{n+1} - y_n\|^2 + 2\lambda_n f(y_n, x^\dagger), \quad \forall n \geq 0. \end{aligned} \tag{35}$$

Since $f(x^\dagger, y_n) \geq 0$, we get that $f(y_n, x^\dagger) \leq -\gamma \|y_n - x^\dagger\|^2$ by assumption (SP), where γ is some positive real number. Thus, from relation (35), we obtain

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 &\leq \|x_n - x^\dagger\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 \\ &\quad - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_{n+1} - y_n\|^2 - 2\gamma\lambda_n \|y_n - x^\dagger\|^2. \end{aligned} \tag{36}$$

Since $\{\lambda_n\}$ is non-increasing monotone and $\lambda_n \rightarrow \lambda$, we have that $\lambda_n \geq \lambda_\infty = \lambda$ for all $n \geq 0$. Thus, from (36), we get

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 &\leq \|x_n - x^\dagger\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_{n+1} \\ &\quad - y_n\|^2 - 2\gamma\lambda \|y_n - x^\dagger\|^2. \end{aligned} \tag{37}$$

Let ϵ be some fixed number in the interval $\left(0, \frac{1-\mu}{2}\right)$. We find that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 2\epsilon > 0.$$

Thus, there exists $n_0 \geq 1$ such that

$$1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > 2\epsilon > 0, \quad \forall n \geq n_0. \tag{38}$$

It follows from the relations (37) and (38) that, for all $n \geq n_0$

$$\begin{aligned}
 \|x_{n+1} - x^\dagger\|^2 &\leq \|x_n - x^\dagger\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 - 2\gamma\lambda \|y_n - x^\dagger\|^2 \\
 &\leq \|x_n - x^\dagger\|^2 - 2\epsilon \|x_n - y_n\|^2 - 2\gamma\lambda \|y_n - x^\dagger\|^2 \\
 &\leq \|x_n - x^\dagger\|^2 - \min\{\epsilon, \gamma\lambda\} \left\{2\|x_n - y_n\|^2 + 2\|y_n - x^\dagger\|^2\right\} \\
 &\leq \|x_n - x^\dagger\|^2 - \min\{\epsilon, \gamma\lambda\} \|x_n - x^\dagger\|^2 \\
 &= r \|x_n - x^\dagger\|^2,
 \end{aligned}
 \tag{39}$$

where $r = 1 - \min\{\epsilon, \gamma\lambda\} \in (0, 1)$. From the relation (39), we obtain by induction that

$$\|x_{n+1} - x^\dagger\|^2 \leq r^{n-n_0+1} \|x_{n_0} - x^\dagger\|^2, \quad \forall n \geq n_0,$$

or $\|x_{n+1} - x^\dagger\|^2 \leq Mr^n$ for all $n \geq n_0$, where $M = r^{1-n_0} \|x_{n_0} - x^\dagger\|^2$. This implies the desired conclusion.

Case 2 The sequence $\{x_n\}$ is generated by Algorithm 2. Now, using the relation (22) with $y = x^\dagger$ and noting that $f(y_n, x^\dagger) \leq -\gamma \|y_n - x^\dagger\|^2$, we obtain

$$\begin{aligned}
 \|x_{n+1} - x^\dagger\|^2 &\leq \|x_n - x^\dagger\|^2 - \left(1 - \frac{2\lambda_n\mu}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 \\
 &\quad - \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}}\right) \|y_n - x_{n+1}\|^2 \\
 &\quad - 2\lambda_n\gamma \|y_n - x^\dagger\|^2 + \frac{2\lambda_n\mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2.
 \end{aligned}
 \tag{40}$$

Let θ and ρ be two real numbers such that

$$\theta \in \left(0, \frac{1 - 2\mu}{2}\right) \quad \text{and} \quad 1 < \rho < \frac{1}{2} \left(\frac{1}{\mu} - 1\right).
 \tag{41}$$

Adding the term $\frac{2\lambda_{n+1}\mu\rho}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2$ to both sides of (40), we get

$$\begin{aligned}
 \|x_{n+1} - x^\dagger\|^2 + \frac{2\lambda_{n+1}\mu\rho}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2 &\leq \|x_n - x^\dagger\|^2 + \frac{2\lambda_n\mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2 \\
 &\quad - \left(1 - \frac{2\lambda_n\mu}{\lambda_{n+1}}\right) \|x_n - y_n\|^2 - \left(1 - \frac{\lambda_n\mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1}\mu\rho}{\lambda_{n+2}}\right) \|y_n - x_{n+1}\|^2 \\
 &\quad - 2\lambda_n\gamma \|y_n - x^\dagger\|^2, \quad \forall y \in C, \quad n \geq 0.
 \end{aligned}
 \tag{42}$$

Since $\lambda_n \rightarrow \lambda > 0$, we obtain

$$\lim_{n \rightarrow \infty} \left(1 - \frac{2\lambda_n \mu}{\lambda_{n+1}} \right) = 1 - 2\mu > 2\theta > 0,$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda_n \mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1} \mu \rho}{\lambda_{n+2}} \right) = 1 - (1 + 2\rho)\mu > 0.$$

Thus, there exists $n_0 \geq 1$ such that

$$1 - \frac{2\lambda_n \mu}{\lambda_{n+1}} > 2\theta > 0, \quad \forall n \geq n_0, \tag{43}$$

$$1 - \frac{\lambda_n \mu}{\lambda_{n+1}} - \frac{2\lambda_{n+1} \mu \rho}{\lambda_{n+2}} > 0, \quad \forall n \geq n_0. \tag{44}$$

From the relations (42)–(43) and the fact $2\lambda_n \gamma \geq 2\lambda \gamma$, we have

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 + \frac{2\lambda_{n+1} \mu \rho}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2 &\leq \|x_n - x^\dagger\|^2 + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2 \\ &\quad - \left(2\theta \|x_n - y_n\|^2 + 2\lambda \gamma \|y_n - x^\dagger\|^2 \right), \quad n \geq n_0. \end{aligned} \tag{45}$$

We also have

$$\begin{aligned} 2\theta \|x_n - y_n\|^2 + 2\lambda \gamma \|y_n - x^\dagger\|^2 &\geq \min \{ \theta, \lambda \gamma \} \left(2\|x_n - y_n\|^2 + 2\|y_n - x^\dagger\|^2 \right) \\ &\geq \min \{ \theta, \lambda \gamma \} \|x_n - x^\dagger\|^2. \end{aligned} \tag{46}$$

Combining the relations (45) and (46), we obtain

$$\begin{aligned} \|x_{n+1} - x^\dagger\|^2 + \frac{2\lambda_{n+1} \mu \rho}{\lambda_{n+2}} \|y_n - x_{n+1}\|^2 &\leq (1 - \min \{ \theta, \lambda \gamma \}) \|x_n - x^\dagger\|^2 \\ &\quad + \frac{2\lambda_n \mu}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2, \quad n \geq n_0. \end{aligned} \tag{47}$$

Set $a_n = \|x_n - x^\dagger\|^2$ and $b_n = \frac{2\lambda_n \mu \rho}{\lambda_{n+1}} \|y_{n-1} - x_n\|^2$. From the relation (47), we get

$$a_{n+1} + b_{n+1} \leq (1 - \min \{ \theta, \lambda \gamma \}) a_n + \frac{b_n}{\rho} \leq r(a_n + b_n), \quad n \geq n_0, \tag{48}$$

where $r = \max \left\{ 1 - \min \{ \theta, \lambda \gamma \}, \frac{1}{\rho} \right\} \in (0, 1)$. Thus, by induction, we obtain that $a_{n+1} + b_{n+1} \leq r^{n-n_0+1} (a_{n_0} + b_{n_0})$ for all $n \geq n_0$, which can be reduced to $a_{n+1} = \|x_{n+1} - x^\dagger\|^2 \leq M r^n$ with $M = r^{1-n_0} (a_{n_0} + b_{n_0})$. This finishes the proof. \square

5 Computational experiments

This section presents some fundamental experiments on a model which generalizes the Nash-Cournot oligopolistic equilibrium model [10]. For a readable purpose, we present briefly this model as follows: Suppose that there are m firms producing a common homogeneous commodity. Let p_j be the price of goods produced by firm j depending on the commodities of all firms $j, j = 1, 2, \dots, m$, i.e., $p_j = p_j(x_1, x_2, \dots, x_m)$. Denote $c_j(x_j)$ the cost of firm j which only depends on its produced level x_j . Then the profit made by company j is given by $f_j(x_1, x_2, \dots, x_m) = x_j p_j(x_1, x_2, \dots, x_m) - c_j(x_j)$. Let C_j be the strategy set of firm j . Then $C = C_1 \times C_2 \times \dots \times C_m$ is the strategy of the model. Actually, each firm j seeks to maximize its own profit by choosing the corresponding production level x_j under the presumption that the production of the other firms is a input parameter. A commonly used approach to the model is based on the famous Nash equilibrium concept.

Let $x = (x_1, x_2, \dots, x_m) \in C$ and recall that a point $x^* = (x_1^*, x_2^*, \dots, x_m^*) \in C$ is called an equilibrium point of the model if

$$f_j(x_1^*, x_2^*, \dots, x_m^*) \geq f_j(x_1^*, \dots, x_{j-1}^*, y_j, x_{j+1}^*, \dots, x_m^*) \quad \forall y_j \in C_j, \quad \forall j = 1, 2, \dots, m,$$

By setting $\psi(x, y) := -\sum_{j=1}^m f_j(x_1^*, \dots, x_{j-1}^*, y_j, x_{j+1}^*, \dots, x_m^*)$ and $f(x, y) := \psi(x, y) - \psi(x, x)$, the problem of finding a Nash equilibrium point of the model can be shortly formulated as:

$$\text{Find } x^* \in C : f(x^*, x) \geq 0 \quad \forall x \in C. \tag{EP}$$

For the purpose of experiment, we assume that the price $p_j(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^m x_j$, i.e., $p_j(s) = a_j - b_j s$, where $a_j > 0, b_j > 0$, and that the function $c_j(x_j)$ is increasing and affine for every j . This assumption means that the cost for producing a unit is increasing as the quantity of the production gets larger. In that case, the bifunction f can be formulated in the form $f(x, y) = \langle Px + Qy + q, y - x \rangle$, where $q \in \mathfrak{R}^m$ and P, Q are two matrices of order m such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite. The bifunction f satisfies the Lipschitz-type condition and is pseudomonotone. The feasible set C has the form $C = \{x \in \mathfrak{R}^m : Ax \leq b\}$ where A is a random matrix of size $l \times m$ ($l = 10$ and $m = 200$ or $m = 300$) and $b \in \mathfrak{R}^l$ such that $x_0 = y_{-1} = y_0 = (1, 1, \dots, 1) \in C$. The two matrices P, Q are generated randomly¹. All the optimization problems can be solved effectively by the function *quadprog* in Matlab Optimization Toolbox. All the programs are written in Matlab 7.0 and computed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50 GHz, RAM 2.00 GB.

Experiment 1 In this experiment, we study the numerical behavior of Algorithm 1 (shortly, EEGA) and Algorithm 2 (EMEGA). We take $\mu = 0.33$ and

¹ We randomly choose $\lambda_{1k} \in [-2, 0], \lambda_{2k} \in [0, 2], k = 1, \dots, m$. We set $\widehat{Q}_1, \widehat{Q}_2$ as two diagonal matrixes with eigenvalues $\{\lambda_{1k}\}_{k=1}^m$ and $\{\lambda_{2k}\}_{k=1}^m$, respectively. Then, we construct a positive semidefinite matrix Q and a negative definite matrix T by using random orthogonal matrixes with \widehat{Q}_2 and \widehat{Q}_1 , respectively. Finally, we set $P = Q - T$

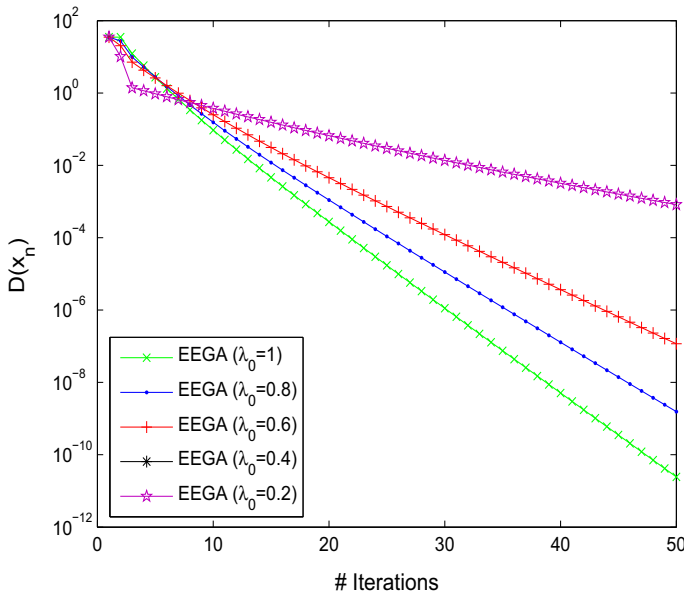


Fig. 1 Algorithm 1 (EEGA) in \mathfrak{R}^{200} . The execution time is 5.97, 5.57, 5.71, 5.82, 5.25, respectively

$\lambda_0 \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ and perform the experiments in \mathfrak{R}^{200} or \mathfrak{R}^{300} . Observe the characteristic of the solutions of problem (EP) that $x \in EP(f, C)$ if and only if $x = \text{prox}_{\lambda f(x, \cdot)}(x)$ for some $\lambda > 0$, we use the function $D(x) = \|x - \text{prox}_{\lambda f(x, \cdot)}(x)\|^2$ to show the numerical behavior of algorithms EEGA and EMEGA. It is of course that $D(x) = 0$ iff $x \in EP(f, C)$. The convergence of $\{D(x_n)\}$ generated by each algorithm to 0 can be considered as the sequence $\{x_n\}$ converges to the solution of the problem. The numerical results are shown in Figs. 1, 2, 3 and 4.

In view of these figures, we see that the proposed algorithms work better when λ_0 is larger. Moreover, Algorithm 2 (EMEGA) is better than Algorithm 1 (EEGA) in both the execution time and the obtained error after the same number of iterations.

Experiment 2 The purpose of this experiment is to present a comparison between the proposed algorithms (EEGA and EMEGA) and two other algorithms, namely the linesearch extragradient algorithm (LEGA) presented in [27, Algorithm 2a] and the ergodic algorithm (ErgA) proposed in [3]. We choose these algorithms to compare because they have a common feature that the Lipschitz-type constants are not the input parameters of the algorithms. The control parameters are $\mu = 0.33$ and $\lambda_0 = 1$ (for EEGA and EMEGA); $\alpha = 0.5, \theta = 0.5, \rho = \gamma = 1$ (for LEGA); and $\lambda_n = \frac{1}{n}$ (for the ErgA). The results are shown in Figs. 5, 6, 7 and 8. In each figure, the x -axis represents either the number of iterations or the execution time elapsed in second while the y -axis gives the value of $D(x_n)$ generated by each algorithm.

The numerical results here have illustrated that the proposed algorithms work better than other ones in both the number of iterations and the execution time. Besides, Algorithm 2 (EMEGA) is also better than Algorithm 1 (EEGA). The advantage of Algorithm 2 over Algorithm 1 is only to compute a value of bifunction at each iteration.

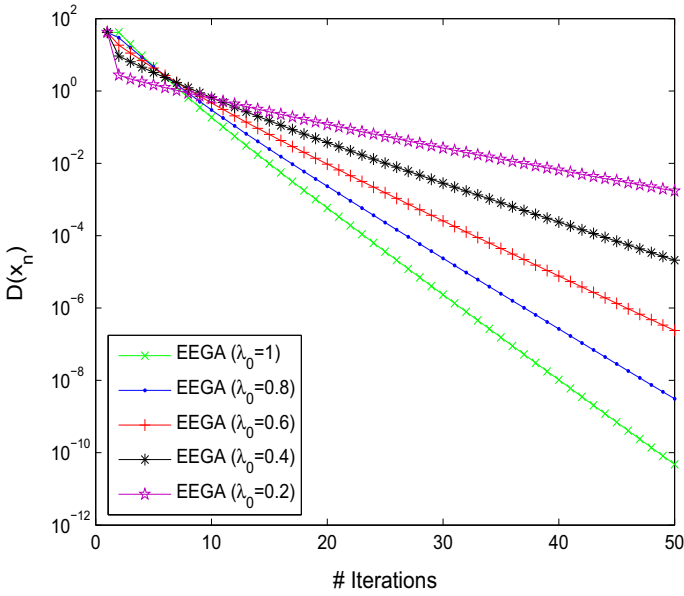


Fig. 2 Algorithm 1 (EEGA) in \mathfrak{R}^{300} . The execution time is 23.84, 23.17, 23.66, 23.63, 23.34, respectively

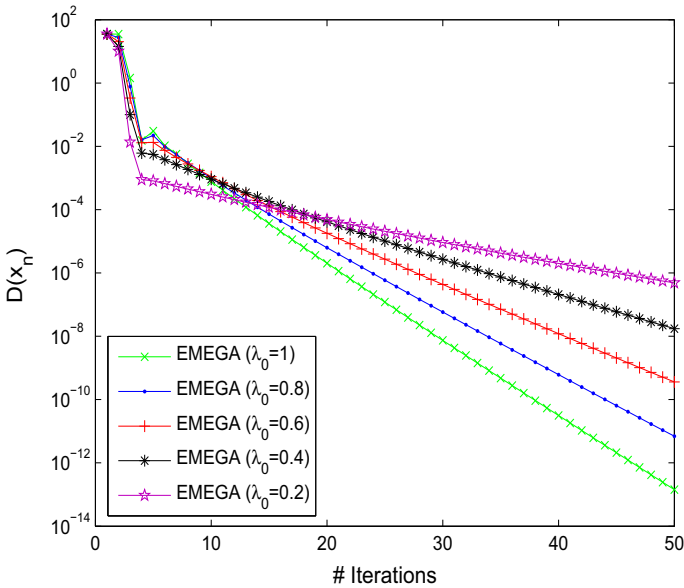


Fig. 3 Algorithm 2 (EMEGA) in \mathfrak{R}^{200} . The execution time is 5.19, 5.16, 5.22, 5.12, 5.04, respectively

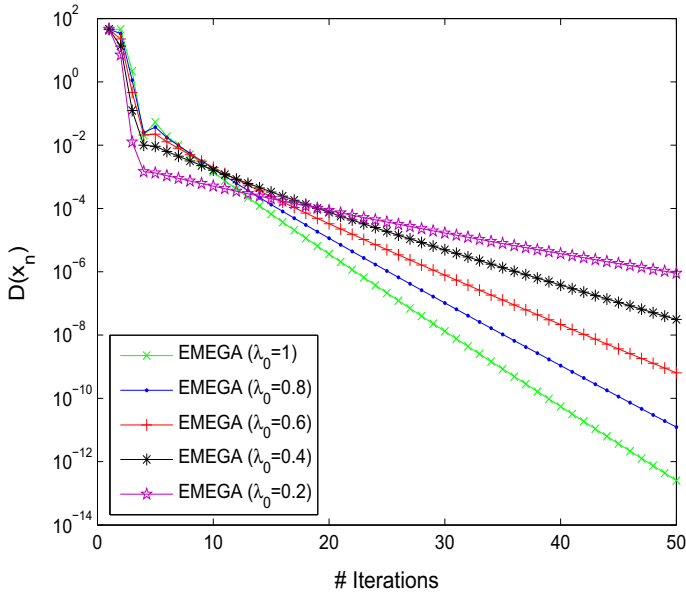


Fig. 4 Algorithm 2 (EMEGA) in \mathfrak{R}^{300} . The execution time is 22.05, 21.87, 22.69, 22.92, 21.93, respectively

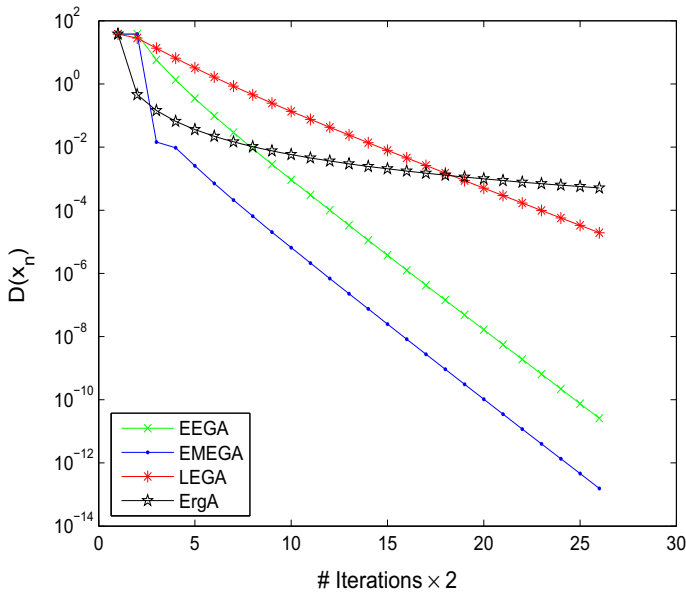


Fig. 5 D_n and # iterations (in \mathfrak{R}^{200})

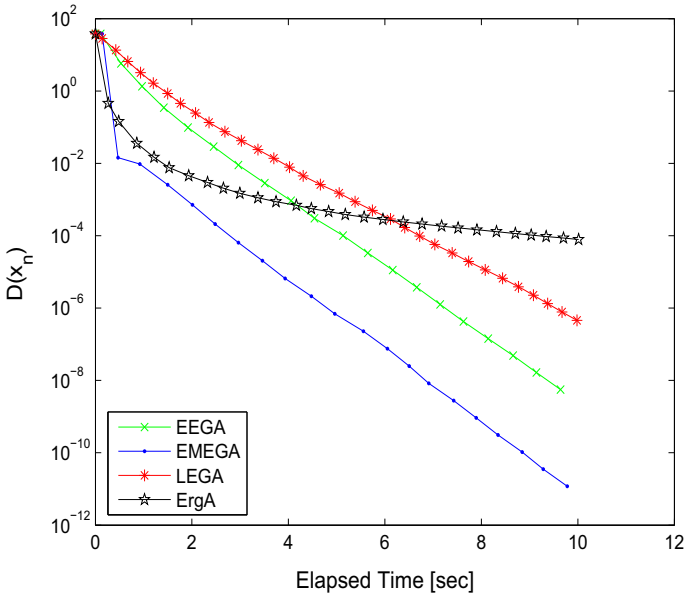


Fig. 6 D_n and time (in \mathfrak{R}^{200})

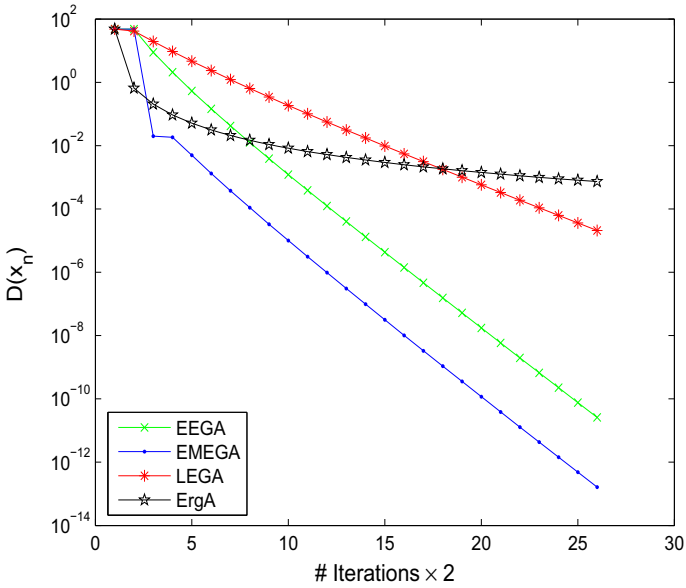


Fig. 7 D_n and # iterations (in \mathfrak{R}^{300})

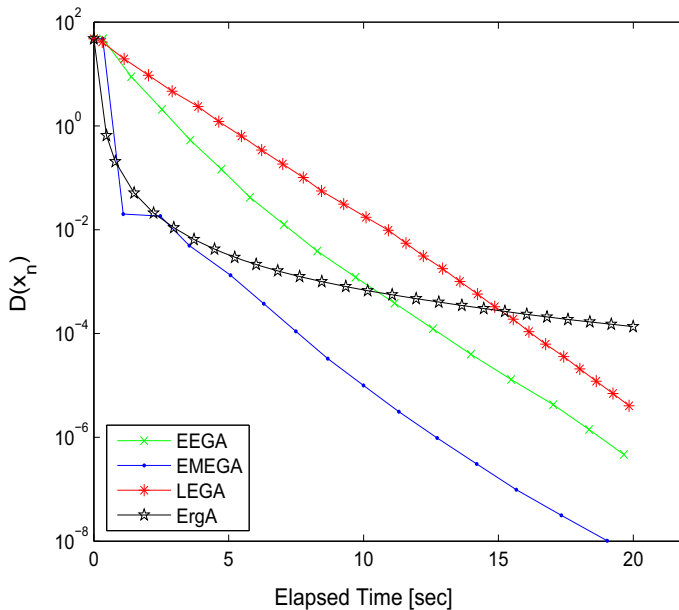


Fig. 8 D_n and time (in \mathfrak{N}^{300})

6 Conclusions

The paper has proposed the two explicit extragradient-like algorithms for solving an equilibrium problem involving a pseudomonotone and Lipschitz-type bifunction in a real Hilbert space. A new stepsize rule has been introduced which is not based on the information of the Lipschitz-type constants. The convergence as well as the R -linear rate of convergence of the algorithms have been established. Several experiments are reported to illustrate the numerical behavior of our two algorithms, and to compare them with other ones well known in the literature.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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