# Improved convergence theorems of modulus-based matrix splitting iteration method for nonlinear complementarity problems of $H$-matrices

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**Abstract** In this paper, the convergence conditions of the modulus-based matrix splitting iteration method for nonlinear complementarity problem of $H$-matrices are weakened. The convergence domain given by the proposed theorems is larger than the existing ones. Numerical examples show the advantages of the new theorems.

**Keywords** Nonlinear complementarity problem · Modulus-based matrix splitting iteration method · $H$-matrix

**Mathematics Subject Classification** 65F10 · 90C33

## 1 Introduction

In this work, we consider the nonlinear complementarity problem (NCP($f$)) consists of finding vectors $z \in \mathbf{R}^n$ such that

$$f(z) = Az + q + \varphi(z) \geq 0, \ z \geq 0 \text{ and } z^T f(z) = 0, \tag{1}$$

where $A \in \mathbf{R}^{n \times n}$, $q \in \mathbf{R}^n$ and $\varphi(z)$ is nonlinear function.

If $\varphi(z) = 0$, (1) reduces to the linear complementarity problem (LCP($q$, $A$)), which arises in the optimal stopping in Markov chain, the economies with institutional restrictions upon prices, the network equilibrium problems, the linear and quadratic programming, the free boundary problems and the contact problems; see [1,2] for details.

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In the recent years, the modulus-based matrix splitting iteration method and its various generalizations for solving LCP($q$, $A$) were widely studied, which yields a series of iteration methods, such as modulus-based Jacobi, Gauss–Seidel, SOR and AOR iteration methods. For more works about the modulus-based matrix splitting iteration method, see [3–14]. The global convergence conditions of these methods are discussed when the system matrix is either a positive definite matrix or an $H_+$-matrix; see the references mentioned above for details.

If $\varphi(z)$ in (1) is a general function, then the problem (1) belongs to nonlinear complementarity problems, see [15,16] for their important applications in various fields. In [17] and [18], the modulus-based matrix splitting iteration method for solving a class of NCP($f$) is established. Ma and Huang present the modified modulus-based matrix splitting iteration method in [19]. The two step modulus-based matrix splitting iteration method is given in [20]. The convergence results are given when $A$ is a positive definite matrix in these works. When $A$ is an $H$-matrix, the convergence analysis is presented only in [17], where the splitting is assumed to be an $H$-compatible splitting.

In this paper, the convergence theories of modulus-based matrix splitting iteration method for NCP($f$) of $H$-matrices is improved by giving a weaker condition of splitting and a lager convergence domain.

The rest of this paper is organized as follows. In Sect. 2 we give some useful notations, definitions and lemmas. In Sect. 3 we give the new convergence analysis. Numerical tests are presented in Sect. 4 and a conclusion remark is given in the final section.

## 2 Preliminaries

The goal of this section is to introduce some notations, preliminary definitions and necessary lemmas.

For two $m \times n$ real matrices $B = (b_{ij})$ and $C = (c_{ij})$ the order $B \geq (>)C$ means $b_{ij} \geq (>)c_{ij}$ for any $i$ and $j$. Let $e$ be an $n \times 1$ vector whose elements are all equal to 1, $A = (a_{ij}) \in \mathbf{R}^{n \times n}$ and let $A = D_A = L_A - U_A = D_A - B_A$, where $D_A$, $L_A$, $U_A$ and $B_A$ are the diagonal, the strictly lower-triangular, the strictly upper-triangular and nondiagonal matrices of $A$, respectively. For $v = (v_1, ..., v_n)^T \in \mathbf{R}^n$, diag($v$) = diag($v_1, ..., v_n$) denotes a diagonal matrix with $v_i$, $i = 1, 2, \cdots, n$ as its diagonal entries.

By $|A|$ we denote $|A| = (|a_{ij}|)$ and the comparison matrix of $A$ is $\langle A \rangle = (\langle a_{ij} \rangle)$, defined by $\langle a_{ij} \rangle = |a_{ij}|$ if $i = j$ and $\langle a_{ij} \rangle = -|a_{ij}|$ if $i \neq j$. The matrix $A$ is called (e.g., see [21]) a $Z$-matrix if all of its off-diagonal entries are non-positive, an $M$-matrix if it is a nonsingular $Z$-matrix with $A^{-1} \geq 0$, and an $H$-matrix if its comparison matrix $\langle A \rangle$ is an $M$-matrix. Specially, an $H$-matrix with positive diagonal entries is called an $H_+$-matrix (e.g., see [22]).

The splitting $A = M - N$ is called an $M$-splitting if $M$ is an $M$-matrix and $N$ is nonnegative; an $H$-splitting if $\langle M \rangle - |N|$ is an $M$-matrix; and an $H$-compatible splitting if $\langle A \rangle = \langle M \rangle - |N|$ (e.g., see [13]). Note that if $A = M - N$ is an $M$-splitting, then $\rho(M^{-1}N) < 1$ (e.g., see [21]). It is known that an $H$-compatible splitting of an $H$-matrix is an $H$-splitting, but not vice versa, see [8,13].

**Lemma 1** [23] *Let A be an H-matrix. Then* $|A^{-1}| \leq \langle A \rangle^{-1}$.

**Lemma 2** [24] *Let* $B \in \mathbf{R}^{n \times n}$ *be a strictly diagonal dominant matrix. Then* $\forall C \in \mathbf{R}^{n \times n}$,

$$||B^{-1}C||_\infty \leq \max_{1 \leq i \leq n} \frac{(|C|e)_i}{(\langle B \rangle e)_i}.$$

**Lemma 3** [21] *Let A be a Z-matrix. Then A is an M-matrix if and only if there exists a positive diagonal matrix D, such that AD is a strictly diagonal dominant matrix with positive diagonal entries.*

## 3 Improved convergence theorems

Let $A = M - N$ be a splitting of $A$, $h$ be a positive constant, and $\Omega$ be a positive diagonal matrix. It is known from [17] and [18] that (1) can be changed to a equivalent fixed-point equations. The next lemma is a known result.

**Lemma 4** [17] *Let* $A = M - N$ *be a splitting of the matrix* $A \in \mathbf{R}^{n \times n}$, *and* $\Omega$ *be a positive diagonal matrix. For* (1), *the following statements hold true:*
(1) *If z is a solution of* (1),*then* $x = \frac{h}{2}(z - \Omega^{-1} f(z))$ *satisfies the implicit fixed-point equation:*

$$(\Omega + M)x = Nx + (\Omega - A)|x| - h\left[q + \varphi\left(\frac{1}{h}(|x| + x)\right)\right]; \qquad (2)$$

(2) *If x satisfies* (2), *then* $z = \frac{1}{h}(|x| + x)$ *is a solution of* (1).

By (2), the modulus-based matrix splitting iteration method for solving (1) is given below:

**Method 1** *[17,18] Let* $A = M - N$ *be a splitting of A. Given an initial vector* $x^{(0)} \in \mathbf{R}^n$, *for* $k = 1, 2, \cdots$ *until the iteration sequence* $\{z^{(k)}\}_{k=1}^{+\infty} \subset \mathbf{R}^n$ *is convergent, compute* $x^{(k)} \in \mathbf{R}^n$ *by solving the linear system*

$$(\Omega+M)x^{(k)}=Nx^{(k-1)}+(\Omega - A)|x^{(k-1)}| - h\left[q + \varphi\left(\frac{1}{h}\left(|x^{(k-1)}| + x^{(k-1)}\right)\right)\right],$$
$$(3)$$

*and set*

$$z^{(k)} = \frac{1}{h}\left(|x^{(k)}| + x^{(k)}\right).$$

*Here* $\Omega$ *is an* $n \times n$ *positive diagonal matrix and h is a positive constant.*

In Method 1, taking

$$M = \frac{1}{\alpha}(D_A - \beta L_A), N = \frac{1}{\alpha}[(1 - \alpha)D_A + (\alpha - \beta)L_A + \alpha U_A],$$

we can obtain the modulus-based accelerated overrelaxation (MAOR) iteration method, which extends a series of relaxation modulus-based methods, such as the modulus-based successive overrelaxation (MSOR) iteration method, the modulus-based Gauss–Seidel (MGS) iteration method and the modulus-based Jacobi (MJ) iteration method when $\alpha = \beta$, $\alpha = \beta = 1$ and $\alpha = 1, \beta = 0$, respectively; see [17] for details.

To present the following discussion, we assume that

$$\varphi(z) = (\varphi_1(z_1)), \varphi_2(z_2)), \cdots, \varphi_n(z_n))^T$$

is differentiable, satisfying that $0 \leq \frac{\mathrm{d}\varphi_i(z_i)}{\mathrm{d}z_i} \leq \psi_i$, where $\psi_i \in \mathbf{R}, i = 1, 2, \cdots, n$. By the mean-value theorem, there exists $\zeta_i^{(k)} \in \mathbf{R}$, such that

$$\varphi_i(z_i^{(k)}) - \varphi(z_i^*) = \frac{\mathrm{d}\varphi_i(\zeta_i^{(k)})}{\mathrm{d}z_i}(z_i^{(k)} - z_i^*), i = 1, 2, \cdots, n.$$

Let

$$\Psi^{(k)} = \mathrm{diag}\left(\frac{\mathrm{d}\varphi_1(\zeta_1^{(k)})}{\mathrm{d}z_1}, \frac{\mathrm{d}\varphi_2(\zeta_2^{(k)})}{\mathrm{d}z_2}, \cdots, \frac{\mathrm{d}\varphi_n(\zeta_n^{(k)})}{\mathrm{d}z_n}\right)$$
$$\text{and } \Psi = \mathrm{diag}(\psi_1, \psi_2, \cdots, \psi_n). \tag{4}$$

Then we have

$$\varphi(z^{(k)}) - \varphi(z^*) = \Psi^{(k)}(z^{(k)} - z^*) \text{ and } \Psi^{(k)} \leq \Psi. \tag{5}$$

**Lemma 5** *Let $\Omega$ be a positive diagonal matrix, $A \in \mathbf{R}^{n \times n}$ be an $H_+$-matrix and $A = M - N$ be an $H$-splitting of $A$. Then $\Omega + M$ is an $H$-matrix and there exists a positive diagonal matrix $D$ such that $(\langle M \rangle - |N|)D$ and $(\Omega + \langle M \rangle)D$ are two strictly diagonal dominant matrices, and $(\langle A \rangle + \langle M \rangle - |N|)De > 0$.*

*Proof* Since $A = M - N$ is an $H$-splitting of $A$, $\langle M \rangle - |N|$ and $\langle A \rangle$ are two $M$-matrices. From Lemma 3, there exists a positive diagonal matrix $D$, such that $(\langle M \rangle - |N|)D$ is a strictly diagonal dominant matrix. For $\langle M \rangle \geq \langle M \rangle - |N|$, $(\Omega + \langle M \rangle)D$ is strictly diagonal dominant too. Since

$$\langle A \rangle + \langle M \rangle - |N|$$
$$= |D_M - D_N| - |B_M - B_N| + \langle M \rangle - |N|$$
$$\geq |D_M| - |D_N| - |B_M| - |B_N| + \langle M \rangle - |N|$$
$$= 2(\langle M \rangle - |N|),$$

$(\langle A \rangle + \langle M \rangle - |N|)D$ is also a strictly diagonal dominant matrix, and then $(\langle A \rangle + \langle M \rangle - |N|)De > 0$.

**Theorem 3.1** *Let $A$ be an $H_+$-matrix, $A = M - N$ be an $H$-splitting and* (5) *holds with $\Psi^{(k)}$ and $\Psi$ given by* (4). *If $\Omega \geq D_M + \Psi$, then the iteration sequence $\{z^{(k)}\}_{k=1}^{+\infty}$ generated by Method* 1 *converges to the unique solution $z^* \in \mathbf{R}^n$ of* (1) *for any initial vector $x^{(0)} \in \mathbf{R}^n$.*

*Proof* Let $z^*$ be a solution of (1). Then we have $x^* = \frac{h}{2}(z^* - \Omega^{-1} f(z^*))$, leading to

$$(\Omega + M)x^* = Nx^* + (\Omega - A)|x^*| - h\left[q + \varphi\left(\frac{1}{h}\left(|x^*| + x^*\right)\right)\right]. \quad (6)$$

To prove $\lim\limits_{k \to +\infty} z^{(k)} = z^*$, we need only to prove $\lim\limits_{k \to +\infty} x^{(k)} = x^*$. By Lemma 5, $\Omega + M$ is a $H$-matrix. With (3), (6) and Lemma 1, we have

$$
\begin{aligned}
&|x^{(k)} - x^*| \\
&= \left|(\Omega + M)^{-1}\left\{N(x^{(k-1)} - x^*) + (\Omega - A)(|x^{(k-1)}| - |x^*|)\right.\right. \\
&\quad \left.\left. - h\left[\varphi\left(\frac{1}{h}\left(|x^{(k-1)}| + x^{(k-1)}\right)\right) - \varphi\left(\frac{1}{h}\left(|x^*| + x^*\right)\right)\right]\right\}\right| \\
&= \left|(\Omega + M)^{-1}\left\{N(x^{(k-1)} - x^*) + (\Omega - A)(|x^{(k-1)}| - |x^*|)\right.\right. \\
&\quad \left.\left. - \Psi^{(k-1)}\left[(|x^{(k-1)}| - |x^*|) + (x^{(k-1)} - x^*)\right]\right\}\right| \\
&= \left|(\Omega + M)^{-1}\left[\left(N - \Psi^{(k-1)}\right)\left(x^{(k-1)} - x^*\right)\right.\right. \\
&\quad \left.\left. + (\Omega - A - \Psi^{(k-1)})\left(|x^{(k-1)}| - |x^*|\right)\right]\right| \\
&\leq |(\Omega + M)^{-1}|(|N - \Psi^{(k-1)}| + |\Omega - M + N - \Psi^{(k-1)}|)|x^{(k-1)} - x^*| \\
&\leq (\Omega + \langle M \rangle)^{-1}(|N - \Psi^{(k-1)}| + |\Omega - M - \Psi^{(k-1)}| + |N|)|x^{(k-1)} - x^*| \\
&\doteq \mathcal{L}_1|x^{(k-1)} - x^*|, \quad (7)
\end{aligned}
$$

where

$$\mathcal{L}_1 = \tilde{M}_1^{-1}\tilde{N}_1, \ \tilde{M}_1 = \Omega + \langle M \rangle \text{ and } \tilde{N}_1 = |N - \Psi^{(k-1)}| + |\Omega - M - \Psi^{(k-1)}| + |N|.$$

If $\Omega \geq D_M + \Psi$, we have

$$
\begin{aligned}
\tilde{M}_1 - \tilde{N}_1 &= \Omega + \langle M \rangle - |N - \Psi^{(k-1)}| - |\Omega - M - \Psi^{(k-1)}| - |N| \\
&= 2\langle M \rangle - 2|N| + |\Psi^{(k-1)}| + |N| - |N - \Psi^{(k-1)}| \\
&\geq 2\langle M \rangle - 2|N|. \quad (8)
\end{aligned}
$$

Since $A = M - N$ is an $H$-splitting, $\langle M \rangle - |N|$ is an $M$-matrix. we easily have that $\tilde{M}_1$ is an $M$-matrix too. Obviously, $\tilde{M}_1 - \tilde{N}_1$ is a $Z$-matrix. Hence by (8), $\tilde{M}_1 - \tilde{N}_1$ is

an $M$-matrix. Together with that $\tilde{M}$ is an $M$-matrix and $\tilde{N}_1 \geq 0$, we have that $\tilde{M} - \tilde{N}$ is an $M$-splitting. Then $\rho(\mathcal{L}_1) < 1$, which implies that $\lim_{k \to +\infty} x^{(k)} = x^*$. $\qquad\square$

Let $\alpha, \beta$ be the relaxation parameters used in MAOR. Since $M = \frac{1}{\alpha}(D_A - \beta L_A)$, obviously, we have the next corollary.

**Corollary 1** *With the same notations and assumptions as in Theorem* 3.1, *let* $\alpha, \beta$ *be the relaxation parameters in MAOR. If* $0 < \beta \leq \alpha \leq 1$ *and* $\Omega \geq \frac{1}{\alpha}D_A + \Psi$, *the iteration sequence* $\{z^{(k)}\}_{k=1}^{+\infty}$ *generated by MAOR converges to the unique solution* $z^* \in \mathbf{R}^n$ *of* (1) *for any initial vector* $x^{(0)} \in \mathbf{R}^n$.

*Remark 1* Clearly, the assumption that $A = M - N$ is an $H$-splitting in Theorem 3.1 and Corollary 1, is weaker than the one in Theorem 4.3 of [17], where such splitting is assumed to be an $H$-compatible splitting.

Next we give another convergence theorem which provided a larger convergence domain than that in [17].

**Theorem 3.2** *Let* $A$ *be an* $H_+$-matrix, $A = M - N$ *be an* $H$-splitting *of* $A$ *and* (5) *hold with* $\Psi^{(k)}$ *and* $\Psi$ *given by* (4). *Then the iteration sequence* $\{z^{(k)}\}_{k=1}^{+\infty}$ *generated by Method* 1 *converges to the unique solution* $z^* \in \mathbf{R}^n$ *of* (1) *for any initial vector* $x^{(0)} \in \mathbf{R}^n$ *provided that* $\Omega$ *satisfies one of the following conditions:*

(I)

$$\Omega \geq D_A + \Psi;$$

(II)

$$\left[\frac{1}{2}(|A| - \langle M \rangle + |N|) + \Psi\right] De < \Omega De \leq (D_A + \min_{1 \leq i \leq n} \psi_i I)De, \quad (9)$$

where $D$ is given by Lemma 5.

*Proof* By (7), we have

$$
\begin{aligned}
|x^{(k)} - x^*| \\
&\leq |(\Omega + \langle M \rangle)^{-1}|(|N - \Psi^{(k-1)}| + |\Omega - A - \Psi^{(k-1)}|)|x^{(k-1)} - x^*| \\
&\doteq \mathcal{L}_2|x^{(k-1)} - x^*|,
\end{aligned}
$$

where

$$\mathcal{L}_2 = \tilde{M}_2^{-1}\tilde{N}_2, \ \tilde{M} = \Omega + \langle M \rangle, \ \tilde{N}_2 = |N - \Psi^{(k-1)}| + |\Omega - A - \Psi^{(k-1)}|.$$

By Lemma 2 and 5, we have

$$||D^{-1}\mathcal{L}_2 D|| = ||(\tilde{M}_2 D)^{-1}(\tilde{N}_2 D)|| \leq \max_{1 \leq i \leq n} \frac{(\tilde{N}_2 De)_i}{(\tilde{M}_2 De)_i}. \quad (10)$$

**Case (I)** If $\Omega \geq D_A + \Psi$, we have

$$
\begin{aligned}
\tilde{M}_2 & De - \tilde{N}_2 De \\
& = (\Omega + \langle M \rangle - |N - \Psi^{(k-1)}| - |\Omega - A - \Psi^{(k-1)}|) De \\
& = (\langle A \rangle + \langle M \rangle - |N| + |\Psi^{(k-1)}| + |N| - |N - \Psi^{(k-1)}|) De \\
& \geq (\langle A \rangle + \langle M \rangle - |N|) De \\
& > 0.
\end{aligned}
$$

**Case (II)** If $\left[ \frac{1}{2}(|A| - \langle M \rangle + |N|) + \Psi \right] De < \Omega De \leq (D_A + \min_{1 \leq i \leq n} \psi_i I) De$, we have

$$
\begin{aligned}
\tilde{M}_2 & De - \tilde{N}_2 De \\
& \geq (2\Omega - |A| + \langle M \rangle - |N| - 2|\Psi^{(k-1)}|) De \\
& \geq (2\Omega - |A| + \langle M \rangle - |N| - 2|\Psi|) De \\
& > 0.
\end{aligned}
$$

Summarizing Case (I) and (II), by (10), we have $\rho(\mathcal{L}_2) \leq ||D^{-1}\mathcal{L}_2 D|| < 1.$  □

*Remark 2* For MAOR, since $M = \frac{1}{\alpha}(D_A - \beta L_A)$, we have $D_M \geq D_A$. Hence the convergence domain in Theorem 3.2 is larger than that in Theorem 4.4 of [17].

## 4 Numerical examples

In this section, we give a numerical example to show the advantages of the new convergence theorems. All the computations were run by MATLAB R2016a on an Intel(R) Core(TM), where the CPU is 2.50 GHz and the memory is 4.00 GB.

In the following numerical implementations, all initial vectors are chosen to be $x^{(0)} = e$ and all iterations are terminated once

$$
\frac{|| \min(Az^{(k)} + q + \varphi(z^{(k)}), z^{(k)})||_2}{|| \min(Az^{(1)} + q + \varphi(z^{(0)}), z^{(0)})||_2} \leq 10^{-6},
$$

where the minimum is taken componentwise.

*Example 1* [18,19,25] Consider the problem arises from the discretization of the boundary problems. Let $\mathcal{D} = (0, 1) \times (0, 1)$ and function $g$ satisfy $g(0, y) = y(1 - y)$, $g(x, y) = 0$ on $y = 0$, $y = 1$ or $x = 1$. Find $u$ such that

$$
\begin{cases}
u \geq 0, & \text{in } \mathcal{D}; \\
-\Delta + f(u, x, y) - 8(y - 0.5) \geq 0, & \text{in } \mathcal{D}; \\
u(-\Delta + f(u, x, y) - 8(y - 0.5)) = 0, & \text{in } \mathcal{D}; \\
u = g, & \text{on } \partial \mathcal{D},
\end{cases}
$$

where $f(u, x, y)$ is continuously differentiable and $\frac{\partial f}{\partial u} \geq 0$ on $\bar{\mathcal{D}} \times \{u : u \geq 0\}$.

By discreting the problem using the five-point difference scheme with mesh-step size $h = \frac{1}{2^\varsigma}$, we can get the nonlinear complementarity problem (1). More concretely, the matrix $A$ is given by the block tridiagonal matrix

$$
A = \begin{pmatrix}
S & -I & & & & \\
-I & S & -I & & & \\
& -I & S & \ddots & & \\
& & \ddots & \ddots & -I & \\
& & & -I & S & -I \\
& & & & -I & S
\end{pmatrix} \in \mathbf{R}^{n \times n},
$$

with $S = \text{tridiag}(-1, 4, -1) \in \mathbf{R}^{m \times m}$ and $I \in \mathbf{R}^{m \times m}$ is the identity matrix, where $n = m^2$ and $m = \frac{1}{h} - 1$.

Clearly, $A$ is a symmetry positive definite $H_+$-matrix. Let $f(u, x, y)$ be $u^2$, $u - sinu$, $ln(1 + x)$ and $arctanu$, respectively. If we take $\Psi = 2I$, it is easy to see that (5) holds. We present two experiments with different parameters.

**Experiment (I)** Take $\Omega = D_A + 2I$. Let $\varsigma = 8$ and $\varsigma = 9$, respectively. Consider the splitting $A = M_1 - N_1$, where

$$
M_1 = \begin{pmatrix}
\hat{S} & & & \\
& \hat{S} & & \\
& & \ddots & \\
& & & \hat{S}
\end{pmatrix} \quad \text{and} \quad \hat{S} = \text{tridiag}(1, 4, 0). \tag{11}
$$

**Experiment (II)** Let $\Omega = D_A$ and $\Omega = D_A - 0.9I$, respectively. Take $\varsigma = 9$ and consider the MSOR ($\alpha = 1.2$).

All numerical results are shown in Table 1, where Method 1 converges for all cases. The number of iteration steps is denoted by 'IT' and the elapsed CPU time in seconds is denoted by 'CPU'.

In Experiment (I), since $\langle M_1 \rangle - |N_1| \neq \langle A \rangle$, the splitting (11) is not an $H$-compatible splitting, which leads to that the convergence results of Method 1 in [17] cannot be used here. On the other hand, for $\langle M_1 \rangle - |N_1|$ is an $M$-matrix, the splitting

**Table 1** Numerical results of Example 1

| $f(u, x, y)$ | Experiment (I) | | | | Experiment (II) | | | |
|---|---|---|---|---|---|---|---|---|
| | $\varsigma = 8$ | | $\varsigma = 9$ | | $\Omega = D_A$ | | $\Omega = D_A - 0.9I$ | |
| | IT | CPU | IT | CPU | IT | CPU | IT | CPU |
| $u^2$ | 44 | 0.4936 | 46 | 2.3146 | 22 | 1.0523 | 42 | 2.1301 |
| $u - sinu$ | 161 | 1.7996 | 164 | 8.6693 | 89 | 4.4786 | 76 | 4.0518 |
| $ln(1 + u)$ | 47 | 0.6779 | 49 | 2.8868 | 25 | 1.3970 | 39 | 2.2961 |
| $arctanu$ | 44 | 0.4976 | 45 | 2.302 | 26 | 1.3271 | 50 | 2.6430 |

(11) is an $H$-splitting. The results of Experiment (I) show the advantage of Theorem 3.1.

In Experiment (II), $\Omega = D_A$ and $\Omega = D_A - 0.9I$ are in the convergence range (9), which is not the convergence condition of Theorem 4.4 of [17]. The results of Experiment (II) also show the advantage of Theorem 3.2.

## 5 Conclusion

Theorems 3.1 and 3.2 give the convergence theories of modulus-based matrix splitting iteration methods for NCP($f$) under the weaker condition that $A = M - N$ is an $H$-splitting. Since an $H$-splitting of an $H$-matrix is not necessarily an $H$-compatible splitting, we have more choices for the splitting $A = M - N$ which makes the modulus-based matrix splitting iteration methods converge than [17]. Furthermore, comparing to the convergence domain in [17], a larger one is given by Theorems 3.2. Therefore, our convergence theories extend the scope of modulus-based matrix splitting iteration methods for NCP($f$) in applications.

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