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Characterizations of DMP inverse in a Hilbert space

Anqi Yu[1] · **Chunyuan Deng**[1]

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Abstract Let $\mathcal{H}$ be a Hilbert space. The recently introduced notions of the DMP inverse are extended from matrices to operators. The group, Moore–Penrose, Drazin inverses are integrated by DMP inverse and many closely equivalent relations among these inverses are investigated by using appropriate idempotents. Some new properties of DMP inverse are obtained and some known results are generalized.

Keywords DMP inverse · Drazin inverse · MP inverse · Idempotent

Mathematics Subject Classification 15A09 · 47A05

1 Introduction and preliminaries

Let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. $\mathcal{R}(T)$ and $\mathcal{N}(T)$ represent the range and the null space of T, respectively. We call $P \in \mathcal{B}(\mathcal{H})$ an idempotent if $P = P^2$, and an orthogonal projector if $P^2 = P = P^*$. The orthogonal projector onto a closed subspace $\mathcal{M}$ is denoted by $P_{\mathcal{M}}$. An operator S is an outer generalized inverse of T if (II) $STS = S$. Let

$$\text{(I) } TST = T, \quad \text{(III) } (TS)^* = TS, \quad \text{(IV) } (ST)^* = ST, \quad \text{(V) } TS = ST,$$
$$\text{(VI) } T^k ST = T^k.$$

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✉ Chunyuan Deng
cydeng@scnu.edu.cn; ccyydeng@gmail.com

[1] School of Mathematics Science, South China Normal University, Guangzhou 510631, People's Republic of China

The (I, II, III, IV)-inverse is called Moore–Penrose inverse (for short MP inverse), denoted by $S = T^{\dagger}$. It is well known that T has the MP inverse if and only if $\mathcal{R}(T)$ is closed and the MP inverse of T is unique (see [2,7]). And the (II, V, VI)-inverse is called Drazin inverse, denoted by $S = T^D$, where $k = i(T)$ is the Drazin index of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is Drazin invertible if and only if it has finite ascent $asc(T)$ and descent $des(T)$, which is equivalent with that 0 is a finite order pole of the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$ [16], say of order k. In such case $i(T) = asc(T) = des(T) = k$ [2,4,16]. Similarly, the (I, II, V)-inverse is called group inverse, denoted by $S = T^{\#}$ [2,12]. In the case where $i(A) \leq 1$, A^D is reduced to the group inverse $A^{\#}$ [3,6,8,19].

Recently, Baksalary and Trenkler introduced in [1] a new pseudoinverse of a matrix named core inverse. Malik and Thome in [15] generalized this definition and defined a new generalized inverse of a square matrix of an arbitrary index. They used the Drazin inverse (D) and the Moore–Penrose (MP) inverse and therefore this new generalized inverse is called the DMP-inverse (see also [5,13,14,17]).

Definition 1.1 [1,15] Let closed range operator $T \in \mathcal{B}(\mathcal{H})$ have index k. Then an operator $X \in \mathcal{B}(\mathcal{H})$ is the DMP-inverse of T, denoted by $X = T^{D,\dagger}$, if

$$XTX = X, \quad XT = TT^D \quad \text{and} \quad T^k X = T^k T^{\dagger}. \tag{1}$$

Our aim is to investigate the characterizations and the properties of DMP inverse. The matrix representation of the DMP inverse is given. We show that all kinds of general inverses and corresponding related idempotents are closed related. Some equivalent characterizations among the existence of these inverses by the existence of self-adjoint idempotents

$$P_1 = P_{\mathcal{R}(T)} = TT^{\dagger}, \quad P_2 = P_{\mathcal{R}(T^k)}, \quad P_3 = P_{\mathcal{R}(T^*)} = T^{\dagger}T$$

and idempotents

$$Q_1 = TT^D = T^{D,\dagger}T = TT^{\dagger,D}, \quad Q_2 = TT^{D,\dagger} = Q_1P_1, \quad Q_3 = T^{\dagger,D}T = P_3Q_1$$

are built.

2 Some lemmas

To prove the main results, some lemmas are needed.

Lemma 2.1 ([7, Theorem 6]) *Let $T_{11} \in \mathcal{B}(\mathcal{H})$, $T_{22} \in \mathcal{B}(\mathcal{K})$, $T_{12} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and T_{11} be invertible. Then $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ is MP invertible if and only if $\mathcal{R}(T_{22})$ is closed, and*

$$T^{\dagger} = \begin{pmatrix} T_{11}^*\triangle & -T_{11}^*\triangle T_{12}T_{22}^{\dagger} \\ (I - T_{22}^{\dagger}T_{22})T_{12}^*\triangle & T_{22}^{\dagger} - (I - T_{22}^{\dagger}T_{22})T_{12}^*\triangle T_{12}T_{22}^{\dagger} \end{pmatrix}, \tag{2}$$

where $\triangle = \left[T_{11}T_{11}^* + T_{12}(I - T_{22}^\dagger T_{22})T_{12}^*\right]^{-1}$. *Moreover,* $TT^\dagger = \begin{pmatrix} I & 0 \\ 0 & T_{22}T_{22}^\dagger \end{pmatrix}$ *and*

$$T^\dagger T = \begin{pmatrix} T_{11}^*\triangle T_{11} & T_{11}^*\triangle T_{12}(I - T_{22}^\dagger T_{22}) \\ (I - T_{22}^\dagger T_{22})T_{12}^*\triangle T_{11} & T_{22}^\dagger T_{22} + (I - T_{22}^\dagger T_{22})T_{12}^*\triangle T_{12}(I - T_{22}^\dagger T_{22}) \end{pmatrix}. \tag{3}$$

If T has the Drazin inverse T^D with $i(T) = k$, then $\mathcal{R}(T^k)$ is an invariant subspace of T since $TT^k = T^{k+2}T^D = T^kT^2T^D$. And T has the following operator matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \tag{4}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(T^k) \oplus \mathcal{R}(T^k)^\perp$, where T_{11} is invertible and $T_{22}^k = 0$ [10].

Lemma 2.2 ([10, Theorem 2.5]) *If* $T \in \mathcal{B}(\mathcal{H})$ *is Drazin invertible with* $i(T) = k$, *then* T *has the operator matrix form* (4) *and*

$$T^D = \begin{pmatrix} T_{11}^{-1} & \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i} \\ 0 & 0 \end{pmatrix}. \tag{5}$$

Throughout this work we denote by

$$X_0 = \sum_{i=0}^{k-1} T_{11}^{i-k-1}T_{12}T_{22}^{k-1-i}, \quad \triangle = \left[T_{11}T_{11}^* + T_{12}(I - T_{22}^\dagger T_{22})T_{12}^*\right]^{-1}. \tag{6}$$

Lemma 2.3 *Let* X_0 *be defined as in* (6), *where* T_{11} *is invertible and* $T_{22}^k = 0$. *Then*

(i)

$$T_{11}X_0 - X_0T_{22} = T_{11}^{-1}T_{12}. \tag{7}$$

(ii) $X_0 = 0 \Longleftrightarrow T_{12} = 0$.

(iii) $X_0T_{22}T_{22}^\dagger = 0 \Longleftrightarrow X_0T_{22} = 0 \Longleftrightarrow T_{12}T_{22} = 0 \Longleftrightarrow X_0 = T_{11}^{-2}T_{12}$.

Proof Item (i) is clear by the definition of X_0 in (6). Item (ii) follows by the relation in (7).

(iii) If $X_0T_{22}T_{22}^\dagger = 0$, then $X_0T_{22} = X_0T_{22}T_{22}^\dagger T_{22} = 0$.

If $X_0T_{22} = 0$, by (6),

$$T_{11}^{-2}T_{12}T_{22} + T_{11}^{-3}T_{12}T_{22}^2 + \cdots + T_{11}^{-k+1}T_{12}T_{22}^{k-2} + T_{11}^{-k}T_{12}T_{22}^{k-1} = 0. \tag{8}$$

Product T_{22}^{k-2} from right in (8) we get $T_{11}^{-2}T_{12}T_{22}^{k-1} = 0$. It follows that $T_{12}T_{22}^{k-1} = 0$. In the same way, $T_{12}T_{22}^{k-2} = 0$ by production T_{22}^{k-3} from right in (8). With a step by step deduction it follows that $T_{12}T_{22} = 0$.

If $T_{12}T_{22} = 0$. then $X_0 = T_{11}^{-2}T_{12}$ by the definition of X_0.

On the other hand, if $X_0 = T_{11}^{-2}T_{12}$, then

$$X_0 - T_{11}^{-2}T_{12} = T_{11}^{-3}T_{12}T_{22} + T_{11}^{-4}T_{12}T_{22}^2 + \cdots + T_{11}^{-k}T_{12}T_{22}^{k-2} + T_{11}^{-k-1}T_{12}T_{22}^{k-1} = 0.$$

Again we derive that $T_{12}T_{22} = 0$ by the above method.

If $T_{12}T_{22} = 0$, then $X_0T_{22} = 0$. If $X_0T_{22} = 0$, it is clear that $X_0T_{22}T_{22}^{\dagger} = 0$. □

The next well-known criterion (i) due to Douglas [9] (see also Fillmore and Williams [11]) about range inclusions and factorization of operators will be crucial. The criterion (ii) was given in [18] for Hilbert C*-modules.

Lemma 2.4 (i) [9] *If $A, B \in \mathcal{B}(\mathcal{H})$, there exists an operator $C \in \mathcal{B}(\mathcal{H})$ such that $A = BC$ if and only if $\mathcal{R}(A) \subset \mathcal{R}(B)$.*

(ii) [18] *If $\mathcal{L}$ and $\mathcal{M}$ are closed subspaces of $\mathcal{H}$ and $P_{\mathcal{L},\mathcal{M}}$ is an idempotent on $\mathcal{L}$ along $\mathcal{M}$, then*

$$P_{\mathcal{L},\mathcal{M}}T = T \Longleftrightarrow \mathcal{R}(T) \subset \mathcal{L}, \quad TP_{\mathcal{L},\mathcal{M}} = T \Longleftrightarrow \mathcal{N}(T) \supset \mathcal{M}.$$

3 The representation for the DMP-inverse

First we show that the solution of (1) is unique if system is consistent (see [15, Definition2.3] for the matrix case).

Theorem 3.1 *If closed range operator $T \in \mathcal{B}(\mathcal{H})$ has index k, then the DMP inverse of T is unique and $T^{D,\dagger} = T^DTT^{\dagger}$.*

Proof T is MP invertible and Drazin invertible since T is closed range operator with index k (not necessarily ≤ 1). Let $X = T^DTT^{\dagger}$. Then

$$XTX = T^DTT^{\dagger}TT^DTT^{\dagger} = T^DTT^{\dagger} = X,$$

$XT = T^DTT^{\dagger}T = T^DT$ and $T^kX = T^kT^DTT^{\dagger} = T^kT^{\dagger}$. Hence $X = T^DTT^{\dagger}$ is a solution of system (1). If X_1 and X_2 are two solutions of system (1), then

$$X_1 = X_1TX_1 = TT^DX_1 = (TT^D)^kX_1 = (T^D)^kT^kX_1 = (T^D)^kT^kT^{\dagger} = (T^D)^kT^kX_2 = T^DTX_2 = X_2TX_2 = X_2.$$

Hence system (1) has a unique solution. □

From Theorems 3.1 it follows that T is both Drazin and MP invertible if and only if T is DMP invertible. We next give the canonical form for the DMP inverse of an operator T using block operator matrix method.

Theorem 3.2 *Let closed range operator $T \in \mathcal{B}(\mathcal{H})$ have the operator matrix form* (4) *and $i(T) = k$. Then*

$$T^{D,\dagger} = \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^{\dagger} \\ 0 & 0 \end{pmatrix} \tag{9}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{R}(T^k) \oplus \mathcal{R}(T^k)^{\perp}$, where X_0 is defined in (6).

Proof By Theorem 3.1, Lemmas 2.1 and 2.2,

$$T^{D,\dagger} = T^D T T^{\dagger} = \begin{pmatrix} T_{11}^{-1} & X_0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T_{22} T_{22}^{\dagger} \end{pmatrix} = \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^{\dagger} \\ 0 & 0 \end{pmatrix}. \quad \square$$

There is another inverse associated with operator T, namely $T^{\dagger,D} = T^{\dagger} T T^D$ and its canonical form in terms of the decomposition of T in (4) is given by

$$T^{\dagger,D} = \begin{pmatrix} T_{11}^{*} \triangle & T_{11}^{*} \triangle T_{11} X_0 \\ (I - T_{22}^{\dagger} T_{22}) T_{12}^{*} \triangle & (I - T_{22}^{\dagger} T_{22}) T_{12}^{*} \triangle T_{11} X_0 \end{pmatrix}, \tag{10}$$

where X_0 and $\triangle$ are defined in (6).

Remark (1) If $(I - T_{22}^{\dagger} T_{22}) T_{12}^{*} = 0$, then $T^{\dagger,D} = T^D$. If $i(T) \leq 1$, then $T_{22} = 0$. Hence, $T^{\#,\dagger} = T_{11}^{-1} \oplus 0$,

$$T^{\dagger} = \begin{pmatrix} T_{11}^{*} \triangle' & 0 \\ T_{12}^{*} \triangle' & 0 \end{pmatrix}, \; T^{\#} = \begin{pmatrix} T_{11}^{-1} & T_{11}^{-2} T_{12} \\ 0 & 0 \end{pmatrix}, \; T^{\dagger,\#} = \begin{pmatrix} T_{11}^{*} \triangle' & T_{11}^{*} \triangle' T_{11}^{-1} T_{12} \\ T_{12}^{*} \triangle' & T_{12}^{*} \triangle' T_{11}^{-1} T_{12} \end{pmatrix}, \tag{11}$$

where $\triangle' = (T_{11} T_{11}^{*} + T_{12} T_{12}^{*})^{-1}$.

(2) If $T^D = T^{\dagger}$, then $T_{12} = 0$ and $T_{22} = 0$ in (4). Hence

$$T^{\#} = T^{\dagger} = T^{\#,\dagger} = T^{\dagger,\#} = T_{11}^{-1} \oplus 0.$$

(3) If $T^D = T$, then $T_{22} = 0$ and $T_{11}^2 = I$ in (4). Hence, $i(T) \leq 1$ and $T = T^3$,

$$T^{\#,\dagger} = T^{\#} T T^{\dagger} = T^2 T^{\dagger} = (T^{\#})^2 T^{\dagger}, \quad T^{\#,\dagger} T = T^2, \quad T T^{\#,\dagger} = T T^{\dagger}$$

and

$$T^{\dagger,\#} = T^{\dagger} T T^{\#} = T^{\dagger} T^2 = T^{\dagger} (T^{\#})^2, \quad T^{\dagger,\#} T = T^{\dagger} T, \quad T T^{\dagger,\#} = T^2.$$

(4) If $T^{D,\dagger} = T$, then $T_{12} = 0$, $T_{22} = 0$ and $T_{11}^2 = I$ in (4). If $T^{\dagger} = T$, by (2) and (4), $(I - T_{22}^{\dagger} T_{22}) T_{12}^{*} = 0$ and $T_{22}^{\dagger} = T_{22}$. Since T_{22} is k nilpotent, $T_{22}^{\dagger} = T_{22}$ implies that $T_{22} = 0$. It follows that $T_{12} = 0$ and $T_{11}^2 = I$. Then we derive that

$$T^{D,\dagger} = T \Longleftrightarrow T^{\dagger} = T \Longleftrightarrow T_{12} = 0, \quad T_{22} = 0, \quad T_{11}^2 = I.$$

In this case,

$$\begin{aligned} T^2 &= T^{\#,\dagger}T = TT^{\#,\dagger} = TT^{\dagger} = TT^{\#} = I \oplus 0, \\ T^{\#,\dagger} &= T^{\dagger} = T = T^{\#} = T^{\dagger,\#} = T_{11} \oplus 0. \end{aligned}$$

In the following we introduce a method to obtain the DMP inverse by a different algebraic approach.

Theorem 3.3 *Let closed range operator* $T \in \mathcal{B}(\mathcal{H})$ *have index* k*. Then* $T^{D,\dagger} = (T^2T^{\dagger})^D$.

Proof Let $T \in \mathcal{B}(\mathcal{H})$ have the operator matrix form (4) and $i(T) = k$. By Theorem 3.2,

$$T^{D,\dagger} = \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^{\dagger} \\ 0 & 0 \end{pmatrix}, \quad X_0 = \sum_{i=0}^{k-1} T_{11}^{i-k-1} T_{12} T_{22}^{k-1-i}.$$

By Lemma 2.1,

$$T^2T^{\dagger} = T(TT^{\dagger}) = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T_{22}T_{22}^{\dagger} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12}T_{22}T_{22}^{\dagger} \\ 0 & T_{22}^2 T_{22}^{\dagger} \end{pmatrix}.$$

If $i(T) \leq 1$, then $T_{22} = 0$ and

$$(T^2T^{\dagger})^D = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}^D = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = T^{D,\dagger}.$$

The result holds. If $k \geq 2$, then $(T_{22}^2 T_{22}^{\dagger})^{k-1} = T_{22}^k T_{22}^{\dagger} = 0$ since $T_{22}^k = 0$. By Lemma 2.2,

$$(T^2T^{\dagger})^D = \begin{pmatrix} T_{11} & T_{12}T_{22}T_{22}^{\dagger} \\ 0 & T_{22}^2 T_{22}^{\dagger} \end{pmatrix}^D = \begin{pmatrix} T_{11}^{-1} & Y \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} Y &= \sum_{i=0}^{k-2} T_{11}^{i-k} T_{12} T_{22} T_{22}^{\dagger} (T_{22}^2 T_{22}^{\dagger})^{k-2-i} \\ &= T_{11}^{-2} T_{12} T_{22} T_{22}^{\dagger} + T_{11}^{-3} T_{12} T_{22} T_{22}^{\dagger} T_{22}^2 T_{22}^{\dagger} + T_{11}^{-4} T_{12} T_{22} T_{22}^{\dagger} T_{22}^3 T_{22}^{\dagger} \\ &\quad + \cdots + T_{11}^{-k} T_{12} T_{22} T_{22}^{\dagger} T_{22}^{k-1} T_{22}^{\dagger} \\ &= \Big[T_{11}^{-2} T_{12} + T_{11}^{-3} T_{12} T_{22} + T_{11}^{-4} T_{12} T_{22}^2 + \cdots + T_{11}^{-k} T_{12} T_{22}^{k-2} \end{aligned}$$

$$+T_{11}^{-k-1}T_{12}T_{22}^{k-1}\Big]T_{22}T_{22}^{\dagger}$$
$$= X_0 T_{22}T_{22}^{\dagger}.$$

Hence, $T^{D,\dagger} = (T^2T^{\dagger})^D$. □

4 Characterizations of relating idempotents

Suppose that $T \in \mathcal{B}(\mathcal{H})$ is a closed range operator with index k. Then $\mathcal{R}(T^{\dagger}) = \mathcal{R}(T^*)$ and $\mathcal{N}(T^{\dagger}) = \mathcal{N}(T^*)$. By Lemma 2.4,

$$\mathcal{R}(T^k) = \mathcal{R}(T^D) = \mathcal{R}(T^DTT^{\dagger}TT^D) \subseteq \mathcal{R}(T^DTT^{\dagger}) = \mathcal{R}(T^{D,\dagger}) \subseteq \mathcal{R}(T^D)$$

and

$$\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}T^D) \supseteq \mathcal{N}(T^D) = \mathcal{N}(T^D(T^D)^kT^k) \supseteq \mathcal{N}(T^k).$$

The outer generalized inverse $T^{(2)}_{\mathcal{S},\mathcal{T}}$ of $T \in \mathcal{B}(\mathcal{H})$ is the operator $X \in \mathcal{B}(\mathcal{H})$ satisfying $XTX = X$, $\mathcal{R}(X) = \mathcal{S}$ and $\mathcal{N}(X) = \mathcal{T}$ [2,4]. Hence, $T^{\dagger}$, T^D and $T^{\#}$ are outer generalized inverse $T^{(2)}_{\mathcal{S},\mathcal{T}}$ with prescribed range $\mathcal{S}$ and null space $\mathcal{T}$:

$$T^{\dagger} = T^{(2)}_{\mathcal{R}(T^*),\mathcal{N}(T^*)}, \quad T^D = T^{(2)}_{\mathcal{R}(T^k),\mathcal{N}(T^k)}, \quad T^{\#} = T^{(2)}_{\mathcal{R}(T),\mathcal{N}(T)}.$$

The DMP-inverse is one kind of outer generalized inverses $T^{D,\dagger} = T^{(2)}_{\mathcal{R}(T^k),\mathcal{N}(T^DTT^{\dagger})}$. If $i(T) \leq 1$, then

$$\mathcal{N}(T^*) = \mathcal{N}(T^{\dagger}) = \mathcal{N}(T^{\dagger}TT^{\#}TT^{\dagger}) \supseteq \mathcal{N}(T^{\#}TT^{\dagger}) = \mathcal{N}(T^{\#,\dagger}) \supseteq \mathcal{N}(T^{\dagger})$$

and $T^{\#,\dagger} = T^{(2)}_{\mathcal{R}(T),\mathcal{N}(T^*)}$. Since $T^{D,\dagger} = T^DTT^{\dagger}$ and $T^{\dagger,D} = T^{\dagger}TT^D$,

$$TT^{\dagger,D} = T^{D,\dagger}T = TT^D, \quad T^{\dagger,D}TT^{\dagger,D} = T^{\dagger,D}, \quad T^{D,\dagger}TT^{D,\dagger} = T^{D,\dagger}.$$

Hence $T^{\dagger,D}T$, $TT^{\dagger,D}$, $TT^{D,\dagger}$ and $T^{D,\dagger}T$ are idempotents. Define the self-adjoint idempotents

$$\begin{aligned} P_1 &= P_{\mathcal{R}(T)} = TT^{\dagger}, \\ P_2 &= P_{\mathcal{R}(T^k)}, \\ P_3 &= P_{\mathcal{R}(T^*)} = T^{\dagger}T \end{aligned} \tag{12}$$

and the idempotents

$$\begin{aligned} Q_1 &= TT^D = T^{D,\dagger}T = TT^{\dagger,D}, \\ Q_2 &= TT^{D,\dagger} = Q_1P_1, \end{aligned}$$

$$Q_3 = T^{\dagger,D}T = P_3Q_1. \tag{13}$$

We have the following equivalent relations.

Theorem 4.1 *Let closed range operator $T \in \mathcal{B}(\mathcal{H})$ have index k. Let Q_i, $i = 1, 2, 3$, be defined as in rm (13). Then*

(i) $Q_1 = Q_2 \Longleftrightarrow \mathcal{N}(T^*) \subseteq \mathcal{N}(T^k) \Longleftrightarrow T^D = T^{D,\dagger}$.
(ii) $Q_1 = Q_3 \Longleftrightarrow \mathcal{R}(T^k) \subseteq \mathcal{R}(T^*) \Longleftrightarrow T^D = T^{\dagger,D}$.
(iii) $Q_2 = Q_3 \Longleftrightarrow \mathcal{N}(T^*) \subseteq \mathcal{N}(T^k)$ *and* $\mathcal{R}(T^k) \subseteq \mathcal{R}(T^*) \Longleftrightarrow T^{D,\dagger} = T^D = T^{\dagger,D}$.

Proof (i) Since $Q_1 = T^{D,\dagger}T = TT^D$ and $Q_2 = TT^{D,\dagger} = TT^DTT^\dagger$,

$$\begin{aligned} Q_1 = Q_2 &\Longleftrightarrow TT^DTT^\dagger = TT^D \Longleftrightarrow TT^D(I - TT^\dagger) = 0 \\ &\Longleftrightarrow \mathcal{N}(T^*) = \mathcal{N}(TT^\dagger) = \mathcal{R}(I - TT^\dagger) \subseteq \mathcal{N}(TT^D) = \mathcal{N}(T^D) = \mathcal{N}(T^k) \\ &\Longleftrightarrow T^D(I - TT^\dagger) = 0 \Longleftrightarrow T^D = T^{D,\dagger}. \end{aligned}$$

(ii) Since $Q_3 = T^{\dagger,D}T = T^\dagger TT^DT$,

$$\begin{aligned} Q_1 = Q_3 &\Longleftrightarrow T^\dagger TT^DT = T^DT \Longleftrightarrow (I - T^\dagger T)TT^D = 0 \\ &\Longleftrightarrow \mathcal{R}(T^k) = \mathcal{R}(T^D) = \mathcal{R}(TT^D) \subseteq \mathcal{N}(I - T^\dagger T) = \mathcal{R}(T^\dagger T) = \mathcal{R}(T^*) \\ &\Longleftrightarrow (I - T^\dagger T)T^D = 0 \Longleftrightarrow T^D = T^{\dagger,D}. \end{aligned}$$

(iii) We only show that $Q_2 = Q_3 \Longrightarrow \mathcal{N}(T^*) \subseteq \mathcal{N}(T^k)$ and $\mathcal{R}(T^k) \subseteq \mathcal{R}(T^*)$. The rest results are obvious by items (i) and (ii). If $Q_2 = Q_3$, then $TT^DTT^\dagger = T^\dagger TT^DT$. First, product T from left we get $T^2T^DTT^\dagger = T^2T^D$. So $T^2T^D(I - TT^\dagger) = 0$, which implies that $\mathcal{N}(T^*) = \mathcal{N}(TT^\dagger) = \mathcal{R}(I - TT^\dagger) \subseteq \mathcal{N}(T^2T^D) = \mathcal{N}(T^D) = \mathcal{N}(T^k)$. Second, product T from right we get $T^DT^2 = T^\dagger TT^DT^2$. So $(I - T^\dagger T)T^2T^D = 0$, which implies that $\mathcal{R}(T^k) = \mathcal{R}(T^2T^D) \subseteq \mathcal{N}(I - T^\dagger T) = \mathcal{R}(T^*)$. □

Theorem 4.2 *Let closed range operator $T \in \mathcal{B}(\mathcal{H})$ have index k. Let P_i and Q_i, $i = 1, 2, 3$, be defined as in* (12) *and* (13)*, respectively. Then*

(i)

$$\begin{aligned} P_2 = Q_1 &\Longleftrightarrow P_2 = Q_3 \Longleftrightarrow [T, P_2] =: TP_2 - P_2T = 0 \\ &\Longleftrightarrow T_{12} = 0. \quad (\textit{see } (4) \textit{ for } T_{i2}, i = 1, 2) \end{aligned}$$

(ii) $P_2 = Q_2 \Longleftrightarrow P_2T(I - P_2)T = 0 \Longleftrightarrow T_{12}T_{22} = 0$.
(iii) $P_1 = P_2 \Longleftrightarrow P_1 = Q_2 \Longleftrightarrow i(T) \leq 1$ *(i.e., T is group invertible)* $\Longleftrightarrow T_{22} = 0$.
(iv)

$$\begin{aligned} P_1 = P_3 &\Longleftrightarrow P_2 = P_3 \Longleftrightarrow P_1 = Q_1 \Longleftrightarrow P_1 = Q_3 \\ &\Longleftrightarrow P_3 = Q_1 \Longleftrightarrow P_3 = Q_2 \Longleftrightarrow P_3 = Q_3 \\ &\Longleftrightarrow i(T) \leq 1 \quad \textit{and} \quad \mathcal{R}(T) = \mathcal{R}(T^*) \quad (\textit{i.e., } T \textit{ is EP}) \\ &\Longleftrightarrow T_{12} = 0 \quad \textit{and} \quad T_{22} = 0. \end{aligned}$$

Proof We give the proof only for the items (i) and (ii); the other items may be proved in the same manner.

(i) By (3)–(6), we know

$$P_2 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} I & T_{11}X_0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q_3 = \begin{pmatrix} T_{11}^* \triangle T_{11} & T_{11}^* \triangle T_{11}^2 X_0 \\ (I - T_{22}^\dagger T_{22}) T_{12}^* \triangle T_{11} & (I - T_{22}^\dagger T_{22}) T_{12}^* \triangle T_{11}^2 X_0 \end{pmatrix}.$$

Then $P_2 = Q_1 \iff X_0 = 0 \iff T_{12} = 0$ by Lemma 2.3 $\iff P_2 = Q_3 \iff [T, P_2] =: TP_2 - P_2T = 0$.

(ii) By (3)–(6), we know

$$Q_2 = \begin{pmatrix} I & T_{11}X_0T_{22}T_{22}^\dagger \\ 0 & 0 \end{pmatrix}.$$

Then $P_2 = Q_2 \iff X_0T_{22}T_{22}^\dagger = 0 \iff T_{12}T_{22} = 0$ by Lemma 2.3 $\iff P_2T(I - P_2)T = 0$. □

Theorem 4.3 *Let closed range operator $T \in \mathcal{B}(\mathcal{H})$ have index k. Let P_i and Q_i, $i = 1, 2, 3$, be defined as in (12) and (13), respectively. Then*

(i) $P_1T^D = T^DP_3 = T^D$.
(ii) $P_1Q_1 = Q_1 = Q_1P_3$ *and* $P_2P_1 = P_2 = P_1P_2$.
(iii) $T^{D,\dagger} = T^DP_1 = Q_1T^DP_1$ *and* $T^{\dagger,D} = P_3T^D = P_3T^DQ_1$.
(vi) $Q_1TP_3 = P_1TQ_1 = Q_1TQ_1 = TQ_1 = T^2T^D = (T^D)^D$.
(v) $T^2\left[T^{D,\dagger}\right]^2 = TT^{D,\dagger} = Q_2$ *and* $\left[T^{D,\dagger}\right]^2 T^2 = T^{D,\dagger}T = T^DT = Q_1$.

Proof We only give the proof of the items (i) and (v). The other items can be checked by the definitions in (12) and (13).

(i) Since $\mathcal{R}(T^D) = \mathcal{R}(T^k) \subseteq \mathcal{R}(T)$ and $\mathcal{N}(T^D) = \mathcal{N}(T^k) \supseteq \mathcal{N}(T)$, by Lemma 2.4,

$$P_1T^D = TT^\dagger T^D = T^D = T^DT^\dagger T = T^DP_3.$$

(iv) $T^2\left[T^{D,\dagger}\right]^2 = T^2T^DTT^\dagger TT^DT^\dagger = T^2T^DTT^DT^\dagger = T^2T^DT^\dagger = TT^{D,\dagger} = Q_2$ and $\left[T^{D,\dagger}\right]^2 T^2 = T^DTT^\dagger T^DTT^\dagger T^2 = T^DT = T^DTT^\dagger T = T^{D,\dagger}T = Q_1$. □

It is well known that $(T^D)^D = T^2T^D$ and $(T^\dagger)^\dagger = T$. The relations for $(T^{D,\dagger})^{D,\dagger}$, $(T^D)^{D,\dagger}$ and $(T^{D,\dagger})^D$ are given as follows.

Theorem 4.4 *Let closed range operator $T \in \mathcal{B}(\mathcal{H})$ have index k. Let P_2 be defined as in (12). Then*

(i) $(T^{D,\dagger})^{D,\dagger} = (T^D)^{D,\dagger} = TP_2$.
(ii) $\left[(T^D)^{D,\dagger}\right]^2 T^{D,\dagger} = T^2 T^{D,\dagger} = (T^{D,\dagger})^D$.

Proof (i) Let $T \in \mathcal{B}(\mathcal{H})$ have the operator matrix form (4) and $i(T) = k$. By Lemma 2.2 and Theorem 3.2,

$$(T^{D,\dagger})^D = \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix}^D = \begin{pmatrix} T_{11} & T_{11}^2 X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix}.$$

By Lemma 2.1 and Theorem 3.2,

$$(T^{D,\dagger})^\dagger = \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} (T_{11}^{-1})^* \Delta'' & 0 \\ (X_0 T_{22} T_{22}^\dagger)^* \Delta'' & 0 \end{pmatrix},$$

where $\Delta'' = \left[T_{11}^{-1}(T_{11}^{-1})^* + X_0 T_{22} T_{22}^\dagger (X_0 T_{22} T_{22}^\dagger)^*\right]^{-1}$. So,

$$\begin{aligned}
(T^{D,\dagger})^{D,\dagger} &= \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix}^{D,\dagger}, \quad X_0 = \textstyle\sum_{i=0}^{k-1} T_{11}^{i-k-1} T_{12} T_{22}^{k-1-i}. \\
&= \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix}^D \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix}^\dagger \\
&= \begin{pmatrix} T_{11} & T_{11}^2 X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (T_{11}^{-1})^* \Delta'' & 0 \\ (X_0 T_{22} T_{22}^\dagger)^* \Delta'' & 0 \end{pmatrix} \\
&= \begin{pmatrix} I & T_{11} X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (T_{11}^{-1})^* \Delta'' & 0 \\ (X_0 T_{22} T_{22}^\dagger)^* \Delta'' & 0 \end{pmatrix} \\
&= \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

In the same vein, we obtain that

$$(T^D)^{D,\dagger} = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} = (T^{D,\dagger})^{D,\dagger} = TP_2.$$

Hence (i) holds.

(ii) By the proof of item (i),

$$\begin{aligned}
\left[(T^D)^{D,\dagger}\right]^2 T^{D,\dagger} &= \begin{pmatrix} T_{11}^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} T_{11} & T_{11}^2 X_0 T_{22} T_{22}^\dagger \\ 0 & 0 \end{pmatrix} = (T^{D,\dagger})^D
\end{aligned}$$

and

$$T^2 T^{D,\dagger} = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}^2 \begin{pmatrix} T_{11}^{-1} & X_0 T_{22} T_{22}^{\dagger} \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} T_{11} & T_{11}^2 X_0 T_{22} T_{22}^{\dagger} \\ 0 & 0 \end{pmatrix} = (T^{D,\dagger})^D.$$

□

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